CERTAIN CLASSES OF ANALYTIC FUNCTIONS AND SOME INEQUALITIES IN THE ALGEBRA $C(T)$

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ABSTRACT: The main object of this paper is to generalize some classes of holomorphic functions in a certain Banach algebra $C(T)$ concerning the Fréchet (or F-)derivative. Several inequalities in this algebra are also proven.

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1. INTRODUCTION AND DEFINITIONS

Following the usual notations concerning *univalent functions*, by $\mathcal{A}$ we denote the class of functions $u(z)$ *normalized* by

$$u(z) = z + \sum_{p=2}^{\infty} a_p \, z^p, \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and by $\mathcal{S}$ the class of functions defined by (1) which are *univalent* in the disk $\mathcal{U}$ ([1] to [5], [7], [10], and [12]). Let $C(T)$ denote the Banach algebra, with *sup norm* $\|\cdot\|$, of continuous complex-valued functions defined on a compact metric space T. By $B(f; r)$ we denote an open ball in $C(T)$ centered at $f \in C(T)$ of radius $r > 0$, and by $\mathcal{U}(a; r)$ the open ball in the complex plane $\mathbb{C}$ centered at $a \in \mathbb{C}$ and of radius $r > 0$ (a neighborhood of a).

In terms of a certain *fractional derivative*, Chen *et al.* [2] and Cho *et al.* [3] obtained interesting generalizations of some familiar inequalities for univalent and convex functions. Srivastava [10], on the other hand, obtained several interesting inequalities involving generalized hypergeometric functions. Nikić [9] considered Koebe's and Bieberbach's inequalities ([1], [4], [5], and [7]) in the algebra $C(T)$; the derivative in Koebe's inequality in [9] happens to be Lorch's L-derivative ([6] and [8]).

Definition 1. Let $L(C(T);\, C(T))$ denote the space of bounded linear operators $\lambda : C(T) \to C(T)$. Let G be an open non-empty subset of $C(T)$. A function $\psi : G \to C(T)$ is said to be *F-differentiable* at a point $f \in G$ if there exists $\lambda \in L(C(T);\, C(T))$ and a map η defined in a ball $B(0;r)$ with values in $C(T)$ such that

$$\lim_{h \to 0} \frac{\eta(h)}{\|h\|} = 0$$

and such that

$$\psi(f+h) - \psi(f) = \lambda h + \eta(h)$$

for all $h \in B(0;r)$. We call λ the *F-derivative* of ψ at f and denote it by $\psi'_F(f)$. If ψ is *F-differentiable* at every point of G, we say that ψ is *F-analytic* (or *F-holomorphic*) in G. The set of all functions, F-analytic in G, is denoted by $FH(G)$.

It is easy to verify that the F-derivative is uniquely determined. The inclusion of the letter F is in honor of M. Fréchet (1878-1973) (*cf.* [6]).

If $\lambda \in C(T)$, then the product λh ($h \in C(T)$) determines an element of $L(C(T);\, C(T))$. Hence an *L-differentiable* function $\psi : G \to C(T)$ at $f \in G$ is also *F-differentiable* at f, with $\psi'_L(f) = \psi'_F(f)$, where $\psi'_L(f)$ is the *L-derivative* of ψ at f; therefore, $LH(G) \subset FH(G)$, where (by analogy with the aforementioned definition) $LH(G)$ denotes the set of all functions which are L-analytic (or L-holomorphic) in G. Thus we have Koebe's inequality concerning the *F-derivative* for the class $\mathcal{S}_C$ (*cf.* [9]).

Theorem 1. *Let $B_0 = B(0;1)$. If $g \in B_0$ and $\psi \in \mathcal{S}_C$, then*

$$\frac{1-\|g\|}{(1+\|g\|)^3} \le \|\psi'_F(g)\| \le \frac{1+\|g\|}{(1-\|g\|)^3}. \tag{2}$$

The equality holds true in the upper bound for $g \in B_0 \setminus \{0\}$ only if ψ is a suitable rotation of the Koebe function in $C(T)$. If the equality occurs in the lower bound for $g \in B_0 \setminus \{0\}$, then the function ψ is a suitable rotation of the Koebe function in $C(T)$.

Proof. The inequality (2) is an immediate consequence of the inequality (K_C) in [9], because $\psi'_F(g) = \psi'_L(g)$ and $\mathcal{S}_C \subset LH(B_0)$. But the fact that $\mathcal{S}_C \subset LH(B_0)$

is deduced from Theorem 1 in [9], keeping in mind that $\psi'_L(g) = \phi' \circ g$, where ϕ is an analytic function in an open non-empty set $D \subset \mathbb{C}$ and $\circ$ stands for the composition.

In order to show that the upper bound of $\|\psi'_F(g)\|$, given by (2), are the best possible for suitable rotations of the Koebe function in $C(T)$, observe that by

$$\kappa_\tau(g) = \frac{g}{(1 - e^{i\tau} g)^2} \qquad (g \in B_0;\ \ \tau \in \mathbb{R})$$

we mean a rotation of the Koebe function

$$\kappa(g) = \frac{g}{(1-g)^2} \qquad (g \in B_0)$$

in $C(T)$. Now let $\eta \in T$ and $\tau_1 \in \mathbb{R}$ such that

$$e^{i\tau_1}\, g(\eta) = |g(\eta)| = \|g\|.$$

Since $\kappa_{\tau_1}(g) = k_{\tau_1} \circ g$, where k_{τ_1} is a rotation of the classical Koebe function (*cf.* [1], [4], [5], [7], and [10]), it follows that $\kappa_\tau \in S_C$ and that, from Theorem 1 in [9], $\kappa'_{\tau_1}(g) = k'_{\tau_1} \circ g$; that is,

$$\kappa'_{\tau_1}(g) = \frac{1 + e^{i\tau_1}\, g}{\left(1 - e^{i\tau_1}\, g\right)^3},$$

where $\kappa'_{\tau_1}(g)$ is F-derivative of κ_{τ_1} at g. Therefore,

$$\frac{1 + \|g\|}{\left(1 - \|g\|\right)^3} = \frac{1 + |g(\eta)|}{\left(1 - |g(\eta)|\right)^3} = \left|\left(\kappa'_{\tau_1}(g)\right)(\eta)\right|. \tag{3}$$

Now let $\xi \in T$ with $\left|\left(\kappa'_{\tau_1}(g)\right)(\xi)\right| = \|\kappa'_{\tau_1}(g)\|$. Then

$$\left|\left(\kappa'_{\tau_1}(g)\right)(\eta)\right| \leq \|\kappa'_{\tau_1}(g)\|. \tag{4}$$

Since

$$\|\kappa'_{\tau_1}(g)\| \leq \frac{1 + \|g\|}{(1 - \|g\|)^3},$$

from (3) and (4) we have

$$\frac{1 + \|g\|}{\left(1 - \|g\|\right)^3} = \|\kappa'_{\tau_1}(g)\|.$$

Finally, suppose that the equality occurs, say, in the lower bound. Then

$$\frac{1 - |g(\eta)|}{\left(1 + |g(\eta)|\right)^3} = \frac{1 - \|g\|}{\left(1 + \|g\|\right)^3} = \|\psi'_F(g)\| \text{ and } \|\psi'_F(g)\| = |(\psi'_F(g))(\zeta)| \ \ (\zeta \in T). \tag{5}$$

Noting that $\psi'_F(g) = \phi' \circ g$ $(\phi \in \mathcal{S})$, we have

$$\|\psi'_F(g)\| = |\phi'(g(\zeta))|.$$

Since $g \in B_0 \setminus \{0\}$, that is, $g(\eta) \in \mathcal{U}$, we have

$$\frac{1-|g(\eta)|}{(1+|g(\eta)|)^3} \leq |\phi'(g(\eta))|$$

(*cf.* [1], [4], [5], [7], and [9]). Hence, noting further that $|\phi'(g(\eta))| \leq \|\psi'_F(g)\|$, we find from (5) that

$$\frac{1-|g(\eta)|}{(1+|g(\eta)|)^3} = |\phi'(g(\eta))|.$$

From this, because $g \in B_0 \setminus \{0\}$, that is, $g(\eta) \in \mathcal{U} \setminus \{0\}$, we conclude that ϕ is a suitable rotation of the classical Koebe function (*cf.* [4]). Whence ψ is a suitable rotation of the Koebe function in $C(T)$.

Similarly, we can show that ψ is a suitable rotation of the Koebe function in $C(T)$, assuming that the equality occurs for $g \in B_0 \setminus \{0\}$ in the upper bound.

For simplicity, we shall henceforth write ψ' instead of ψ'_F.

2. THE CLASSES $\mathcal{A}_C$, $\mathcal{S}_C^F$, AND $\mathcal{S}_{C(T)}$

Definition 2. Let G and D be open non-empty subsets of $C(T)$ and $\mathbb{C}$, respectively, with $f(T) \subset D$ for every $f \in G$. Let $\psi \in FH(G)$. If there exists an analytic function $\phi : D \to \mathbb{C}$ (*i.e.*, $\phi \in H(D)$) such that $\psi(f) = \phi \circ f$ for every $f \in G$, where (as before) $\circ$ stands for the composition, we say that the function ψ is *induced* by the function ϕ.

Since $LH(G) \subset FH(G)$, from the first part of Theorem 1 in [9] we have

Theorem 2. *Let D be an open non-empty subset of $\mathbb{C}$ and let*

$$G(D) = \{f : f \in C(T) \text{ and } f(T) \subset D\}.$$

If $\phi \in H(D)$, then the function $\psi : G(D) \to C(T)$, defined by $\psi(f) = \phi \circ f$, is induced by the function ϕ and $\psi'(f) = \phi' \circ f$.

Theorem 3. *Let D be an open non-empty subset of $\mathbb{C}$. If the function $\psi(f) = \phi \circ f$ is F-analytic in $G(D)$, where ϕ is defined on D with values in $\mathbb{C}$ (i.e., $\phi : D \to \mathbb{C}$), then the function ψ is induced by ϕ.*

Proof. Let $w \in D$ be chosen arbitrarily. Then there exist $f \in G(D)$ and $x \in T$ such that $f(x) = w$. Also, there exists a ball $B(0;r) \subset C(T)$ and a function $\theta : B(0;r) \to C(T)$ such that

$$\phi \circ (f+h) - \phi \circ f = (\psi'(f))\,(h) + h\theta(h), \tag{6}$$

for every $h \in B(0;r)$, and

$$\lim_{h \to 0} \theta(h) = 0.$$

Hence

$$\phi\,(w + h(x)) - \phi(w) = ((\psi'(f))\,(h))\,(x) + h(x)\,(\theta(h))\,(x). \tag{6'}$$

Let $\epsilon > 0$. According to (6) and (6′), there exists δ, with $0 < \delta < r$, such that $\|\theta(h)\| < \epsilon$ if $\|h\| < \delta$; therefore, $|\theta(h)(x)| < \epsilon$ if $\|h\| < \epsilon$. Then, for $z \in \mathcal{U}(w;\delta)$ we take a constant function $h \in B(0;r)$ with $h(y) = z - w$ for all $y \in T$. From (6′) we then obtain

$$\phi(z) - \phi(w) = ((\psi'(f))\,(z-w))\,(x) + (z-w)\,(\theta(h))\,(x). \tag{6''}$$

Now, let $j : T \to T$ be the constant function $j(y) = 1$ for all $y \in T$. Then $h = hj$, and so

$$(\psi'(f))\,(h) = (\psi'(f))\,(hj).$$

Therefore

$$(\psi'(f))\,(z-w) = (\psi'(f))\,((z-w)j)\,.$$

Since $\psi'(f) \in L\,(C(T);\; C(T))$, we see that

$$(\psi'(f))\,((z-w)\,j) = (z-w)\,(\psi'(f))\,(j),$$

and, consequently,

$$((\psi'(f))\,(z-w))\,(x) = (z-w)\,(\psi'(f))\,(x).$$

From this, in view of (6″), we obtain

$$\phi(z) - \phi(w) = (z-w)\,((\psi'(f))\,(j))\,(x) + (z-w)\,(\theta(h))\,(x).$$

But $(\psi'(f))\,(j) \in C(T)$ and $x \in T$, and hence $((\psi'(f))\,(j))\,(x) \in \mathbb{C}$.

Since

$$|\theta(h)(x)| < \epsilon$$

if $z \in U(w;\delta)$, we infer that $\phi'(w)$ exists (and $\phi'(w) = ((\psi'(f))(j))(x))$. This implies that $\phi \in H(D)$, because w is an arbitrary element of D. Therefore, the function ψ is induced by ϕ.

Corollary 1. *If the function $\psi : G(D) \to C(T)$ is induced by ϕ, then*

(a)
$$\phi'(f(x)) = ((\psi'(f))(j))(x)$$

and

(b)
$$\psi'(f) = \psi'_L(f) = \phi' \circ f.$$

Definition 3. Let $\psi \in FH(G)$. If there exists $(\psi')'(f)$ at $f \in G$, then

$$\psi^{(2)}(f) = (\psi')'(f)$$

is called the *second* F-derivative of the function ψ at f; in general,

$$\psi^{(n)}(f) = (\psi^{(n-1)})'(f) \qquad (n \geq 2)$$

is called n^{th} *F-derivative* at $f \in G$ if $\psi^{(n-1)} \in FH(G)$. For convenience, we write $\psi^{(0)} = \psi$. We say that the function ψ is n *F-differentiable* at $f \in G$ (in the open set G) if there exists $\psi^{(n)}(f)$ (at every $f \in G$). The function ψ is called *infinite F-differentiable* in the open set G if ψ is n F-differentiable in G for every positive integer n (*i.e.*, for every $n \in \mathbb{N}$).

Theorem 4. *Let*

$$\psi(f) = \sum_{p=0}^{\infty} a_p\, f^p \qquad (f \in B(0;r);\ a_p \in \mathbb{C}). \tag{7}$$

Then $\psi \in FH\left(B(0;r)\right)$ and

$$\psi^{(n)}(f) = \sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)\, a_p\, f^{p-n} \quad (f \in B(0;r);\ a_p \in \mathbb{C};\ n \in \mathbb{N}). \tag{8}$$

Proof. The series

$$\sum_{p=0}^{\infty} a_p\, z^p \qquad (z \in \mathbb{C})$$

converges in the neighborhood $V = U(0; r)$ and, hence, the function

$$\phi(z) = \sum_{p=0}^{\infty} a_p z^p$$

is analytic in V, that is, $\phi \in H(V)$. Since

$$B(0; r) = G(V) \text{ and } \psi(f) = \phi \circ f$$

with $f \in B(0; r)$, it follows from Theorem 2 that ψ is induced by ϕ; whence $\psi \in FH\,(B(0; r))$.

Because of the fact that

$$\phi^{(n)}(z) = \sum_{p=n}^{\infty} p(p-1)\cdots(p-n+1)\, a_p\, z^{p-n} \qquad (z \in V;\ n \in \mathbb{N}),$$

from $\psi^{(n)}(f) = \phi^{(n)} \circ f$ we obtain (8).

Corollary 1. *The function* ψ given by (7) *is infinite* F*-differentiable in the ball* $B(0; r)$.

Definition 4. Let $B_0 = B(0; 1)$. By $\mathcal{A}_C$ we denote class of all functions

$$\psi : B_0 \to C(T)$$

defined by

$$\psi(f) = f + \sum_{p=2}^{\infty} a_p f^p \qquad (f \in B_0;\ a_p \in \mathbb{C}). \tag{9}$$

By $\mathcal{S}_C^F$ we denote the class of all functions in $\mathcal{A}_C$ which are injective in B_0. The class $\mathcal{S}_C^F$ is called the class of *normalized univalent functions* with constant coefficients in the algebra $C(T)$. A function $\psi \in \mathcal{S}_C^F$ is said to be *C-normalized univalent* in the ball B_0.

By $\mathcal{S}_{C(T)}$ we denote the class of all injective functions $\psi : B_0 \to C(T)$ defined by

$$\psi(f) = f + \sum_{p=2}^{\infty} a_p f^p \qquad (f \in B_0; \quad a_p \in C(T)).$$

The class $\mathcal{S}_{C(T)}$ is called the class of *normalized univalent functions* in the algebra $C(T)$. A function $\psi \in \mathcal{S}_{C(T)}$ is said to be *normalized univalent* in the ball B_0.

Theorem 5. $\mathcal{A}_C \subset LH(B_0)$ *and* $\psi'(f) = \psi'_L(f)$, *with* $f \in B_0$ *and* $\psi \in \mathcal{A}_C$.

Proof. Let $\psi \in \mathcal{A}_C$. Then ψ is induced by the function $\phi : \mathcal{U} \to \mathbb{C}$, where

$$\phi(z) = z + \sum_{p=2}^{\infty} a_p z^p \qquad (z \in \mathcal{U}).$$

From Theorem 1 in [9] we conclude that $\psi \in LH(B_0)$. Hence $\psi'(f) = \psi'_L(f)$, because of the uniqueness of the F-derivative.

Corollary 1. $\mathcal{S}_C = \mathcal{S}_C^F$.

Corollary 2. *If $g \in B_0$ and $\psi \in \mathcal{S}_C^F$, then the inequality* (2) *holds true.*

Corollary 3. *If $g \in B_0$ and $\psi \in \mathcal{S}_C^F$, then the inequality* (B_C) *in* [9] *holds true.*

3. LANDAU'S AND GOODMAN'S INEQUALITIES IN $C(T)$

Let $\mathcal{K}$ denote the class of functions defined by (1) which are analytic in the open unit disk $\mathcal{U}$ and satisfy the inequality:

$$\Re\left(1 + \frac{zu''(z)}{u'(z)}\right) > 0 \qquad (z \in \mathcal{U}),$$

that is, $\mathcal{K}$ is the subclass of $\mathcal{A}$ consisting of convex functions in $\mathcal{U}$.

Theorem 6. (a) *If $\phi \in \mathcal{S}$ and $z \in \mathcal{U}$, then*

$$\text{(L)} \qquad |\phi^{(n)}(z)| \leq \frac{n!(n + |z|)}{(1 - |z|)^{n+2}} \qquad (n \in \mathbb{N});$$

(b) *If $\phi \in \mathcal{K}$ and $z \in \mathcal{U}$, then*

$$\text{(G)} \qquad |\phi^{(n)}(z)| \leq \frac{n!}{(1 - |z|)^{n+1}} \qquad (n \in \mathbb{N}).$$

For each $z \in \mathcal{U} \setminus \{0\}$, the equality holds true in (L) *and* (G) *if ϕ is, respectively, a suitable rotation of the Koebe function and a suitable rotation of the Löwner function.*

The inequalities (L) and (G) are well-known as Landau's and Goodman's inequalities, respectively (*cf.* [2], [3], [4], and [5]). In terms of a certain fractional derivative operator (*cf.*, *e.g.*, [11]), Cho *et al.* [3] obtained a generalization of the inequality (L), while Chen *et al.* [2] obtained a generalization of the inequality (G).

Definition 5. $\mathcal{K}_C$ is the subclass of $\mathcal{A}_C$ consisting of functions which satisfy the following inequality on T:

$$\Re\left(1 + f\,\frac{\psi''(f)}{\psi'(f)}\right) > 0 \qquad (f \in B_0). \tag{10}$$

Theorem 7. *If $f \in B_0$ and $\psi \in \mathcal{S}_C$, then*

$$(\mathrm{L}_C) \qquad \|\psi^{(n)}(f)\| \le \frac{n!(n + \|f\|)}{(1 - \|f\|)^{n+2}} \qquad (n \in \mathbb{N}).$$

If $f \in B_0 \setminus \{0\}$, then there exists a rotation of the Koebe function in $C(T)$ for which the equality holds true for each $n \in \mathbb{N}$.

Proof. The function ψ is defined by (9); so the function

$$\phi(z) = z + \sum_{p=2}^{\infty} a_p\, z^p \tag{9'}$$

is analytic in the open unit disk $\mathcal{U}$. Observe that $B_0 = G(\mathcal{U})$ and $\psi(f) = \phi \circ f$. Following the proof of Theorem 1 in [9], we conclude that $\phi \in \mathcal{S}$. Let $n \in \mathbb{N}$. From Theorem 2 and Theorem 4 we have $\psi'(f) = \phi' \circ f$ and, in general, $\psi^{(n)}(f) = \phi^{(n)} \circ f$ $(n \in \mathbb{N})$. The function $\phi^{(n)} \circ f : T \to \mathbb{C}$ is continuous, and, hence, there exists $\xi \in T$ such that

$$\sup_T |\phi^{(n)} \circ f| = |(\phi^{(n)} \circ f)(\xi)| = |\phi^{(n)}\,(f(\xi))\,|.$$

But

$$\|\psi^{(n)}(f)\| = \|\phi^{(n)} \circ f\| = \sup_T |\phi^{(n)} \circ f|,$$

and so

$$\|\psi^{(n)}(f)\| = |\phi^{(n)}\,(f(\xi))\,|. \tag{11}$$

Since $\phi \in \mathcal{S}$ and $f(\xi) \in \mathcal{U}$, from Landau's inequality (L) we have

$$|\phi^{(n)}\,(f(\xi))\,| \le \frac{n!\,(n + |f(\xi)|)}{(1 - |f(\xi)|)^{n+2}} \qquad (n \in \mathbb{N}). \tag{12}$$

Because the function $f : T \to \mathbb{C}$ is continuous, there exists $\eta \in T$ such that

$$\sup_T |f| = |f(\eta)|.$$

Thus

$$|f(\xi)| \le |f(\eta)| = \|f\|. \tag{13}$$

From the fact that the function

$$\alpha(t) = \frac{n+t}{(1-t)^{n+2}} \qquad (n \in \mathbb{N})$$

is increasing in the subinterval [0,1), in view of (11), (12), and (13), we obtain (L_C).

To show that the equality in (L_C) holds true, for $f \in B_0 \setminus \{0\}$ and $n \in \mathbb{N}$, if ψ is a suitable rotation of the Koebe function in $C(T)$, let $\eta \in T$ and $\tau \in \mathbb{R}$ such that

$$e^{i\tau} f(\eta) = |f(\eta)| = \|f\|.$$

From Theorem 2 and Theorem 4 we see that

$$\kappa_\tau^{(n)}(f) = k_\tau^{(n)} \circ f \qquad (n \in \mathbb{N}).$$

Thus

$$\begin{aligned} \frac{n!\,(n+\|f\|)}{(1-\|f\|)^{n+2}} &= \frac{n!\,(n+|f(\eta)|)}{(1-|f(\eta)|)^{n+2}} \\ &= \left|\left(\kappa_\tau^{(n)}(f)\right)(\eta)\right| \\ &\leq \|\kappa_\tau^{(n)}(f)\|. \end{aligned} \tag{14}$$

But, for any given $\xi \in T$, we have

$$|n + e^{i\tau} f(\eta)| \geq |n + e^{i\tau} f(\xi)|$$

and

$$|1 - e^{i\tau} f(\eta)| \leq |1 - e^{i\tau} f(\xi)|.$$

It then follows from (14) that $\left|\left(\kappa_\tau^{(n)}(f)\right)(\eta)\right| = \|\kappa_\tau^{(n)}(f)\|$. Hence the equality in (L_C) holds true for all $n \in \mathbb{N}$ if $\psi = \kappa_\tau \quad (\tau \in \mathbb{R})$.

Theorem 8. *If $f \in B_0$ and $\psi \in \mathcal{K}_C$, then*

$$(G_C) \qquad \|\psi^{(n)}(f)\| \leq \frac{n!}{(1-\|f\|)^{n+1}} \qquad (n \in \mathbb{N}).$$

If $f \in B_0 \setminus \{0\}$, then there exists a rotation of the Löwner function in $C(T)$ for which the equality holds true for all $n \in \mathbb{N}$.

Proof. Let the function ψ, defined by (9), satisfy the inequality (10). Then the function ψ is induced by the function ϕ, defined by (9′), which is analytic in the open unit disk $\mathcal{U}$. Observe that

$$B_0 = G(\mathcal{U}) \text{ and } \psi(f) = \phi \circ f.$$

From Theorem 2, we then have $\psi'(f) = \phi' \circ f$, and, therefore,

$$\psi^{(n)}(f) = \phi^{(n)} \circ f \qquad (f \in B_0 \text{ and } n \in \mathbb{N}).$$

Then, evidently,

$$1 + f\frac{\psi''(f)}{\psi'(f)} = 1 + f\frac{\phi'' \circ f}{\phi' \circ f}. \tag{15}$$

Now, we can prove that $\phi \in \mathcal{K}$. Indeed, if this were not true, there would exist $w \in \mathcal{U}$ such that

$$\Re\left(1 + w\,\frac{\phi''(w)}{\phi'(w)}\right) \leq 0. \tag{16}$$

Let $v = w$ be constant function on T. Then $v \in B_0$. It now follows from the inequality (10) that

$$\Re\left(1 + v\,\frac{\psi''(v)}{\psi'(v)}\right) > 0,$$

and consequently, in view of (15),

$$\Re\left(1 + v\,\frac{\phi'' \circ v}{\phi' \circ v}\right) > 0;$$

that is,

$$\Re\left(1 + w\,\frac{\phi''(w)}{\phi'(w)}\right) > 0, \tag{17}$$

because

$$v\,\frac{\phi'' \circ v}{\phi' \circ v} = w\,\frac{\phi''(w)}{\phi'(w)}.$$

Therefore, we arrive at a contradiction, since (16) is contrary to (17).

Next we observe that there exists $\xi \in T$ such that (11) is valid. Since $\phi \in \mathcal{K}$ and $f(\xi) \in \mathcal{U}$, from the inequality (G) we have

$$\left|\phi^{(n)}\left(f(\xi)\right)\right| \leq \frac{n!}{\left(1 - |f(\xi)|\right)^{n+1}} \qquad (n \in \mathbb{N}). \tag{18}$$

Now, let $\eta \in T$ such that

$$|f(\eta)| = \sup_T |f|.$$

Then

$$|f(\xi)| \leq |f(\eta)| = \sup_T |f| = \|f\|. \tag{19}$$

Since the function

$$\beta(t) = \frac{1}{(1-t)^{n+1}} \qquad (n \in \mathbb{N})$$

is increasing in the subinterval [0,1), from (18) and (19), and noting that

$$\|\psi^{(n)}(f)\| = \|\phi^{(n)} \circ f)\| = |\phi^{(n)}(f(\xi))|,$$

we obtain the inequality (G_C).

It remains to prove that the equality in (G_C) holds true, for $f \in B_0 \setminus \{0\}$ and $n \in \mathbb{N}$, if ψ is a suitable rotation of the Löwner function in $C(T)$. For this purpose, observe that we mean by $\lambda_\tau(f) = \ell_\tau \circ f$ a rotation (defined for $\tau \in \mathbb{R}$) of the Löwner function in $C(T)$, where ℓ_τ is a rotation of the classical Löwner function (*cf.* [2])

$$\ell(z) = \frac{z}{1-\zeta} = \sum_{p=1}^{\infty} z^p \qquad (z \in \mathcal{U}).$$

Because $\ell_\tau \in \mathcal{K}$, it follows that $\lambda_\tau \in \mathcal{K}_C$. On the other hand, from $\lambda_\tau(f) = \ell_\tau \circ f$, in view of Theorem 2 and Theorem 4, we obtain $\lambda_\tau^{(n)}(f) = \ell_\tau^{(n)} \circ f$ for all $n \in \mathbb{N}$. Let $\eta \in T$ such that $|f(\eta)| = \|f\|$ and assume that $\tau \in \mathbb{R}$ such that $e^{i\tau} f(\eta) = |f(\eta)|$. Since

$$\frac{n!}{(1-\|f\|)^{n+1}} = \left|\left(\lambda_\tau^{(n)}(f)\right)(\eta)\right| \leq \|\lambda_\tau^{(n)}(f)\| \qquad (n \in \mathbb{N}),$$

it is easily seen, in view of $|1 - e^{i\tau} f(\eta)| \leq |1 - e^{i\tau} f(\xi)|$ for all $\xi \in T$, that

$$\frac{n!}{(1-\|f\|)^{n+1}} = \|\lambda_\tau^{(n)}(f)\| \qquad (n \in \mathbb{N}).$$

Theorem 9. *If $\psi \in \mathcal{K}_C$, then the set $\psi(B_0)$ is convex in $C(T)$ (that is, $\mathcal{K}_C$ is a class of convex functions in B_0).*

Proof. Let $g_1 \in \psi(B_0)$ and $g_2 \in \psi(B_0)$ be arbitrarily given. We need to prove that the linear segment

$$I = \{tg_1 + (1-t)\,g_2 : t \in [0,1]\}$$

lies entirely in $\psi(B_0)$. For this purpose, observe that the function ψ is induced by a function $\phi \in H(\mathcal{U})$. Following the proof of Theorem 8, we conclude that $\phi \in \mathcal{K}$. Now, let $g \in I$. Then $g = t(\phi \circ f_1) + (1-t)(\phi \circ f_2)$, where $f_1 \in B_0$ and $f_2 \in B_0$. Therefore, $g \in C(T)$ and, since the set $\phi(\mathcal{U})$ is convex, $g(x) \in \phi(\mathcal{U})$ for all $x \in T$. Thus we have the function $f : T \to \mathcal{U}$ such that $f(x) = z$ with $\phi(z) = g(x)$ (*cf.* [2], [4], and [5]). Hence, $\phi \circ f = g$. Finally, since $g \in C(T)$ and $\phi \in \mathcal{K}$, we infer that $f \in C(T)$. Hence $f \in B_0$, and so $g \in \psi(B_0)$ because $\psi(f) = \phi \circ f$.

This evidently completes the proof of Theorem 9.

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