Tunneling model in Kruskal-Szekeres coordinates and information paradox

by

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B.Sc., Inha University, 1999
M.Sc., University of Victoria, 2005

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in the Department of Physics and Astronomy

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ABSTRACT

In recent work by Kraus and Wilczek, it is first uncovered that small deviations from exact thermality in Hawking radiation have the capacity to carry off the maximum information content of a black hole. It is summarized, simplified and extended in this dissertation. This goes a considerable way toward resolving a long-standing "information loss paradox."

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Chapter 1

Introduction

A black hole is a very simple object. It has only three macroscopic parameters - mass, charge and angular momentum. When Hawking's area theorem was discovered, Bekenstein noted the relation between the mass and the area of the black hole horizon, $dM = \kappa dA/(8\pi)$ where the surface gravity κ represents the strength of the gravitational field at the horizon. This form of the equation resembles the entropy equation in thermal equilibrium, $Tds = đE$. As Hawking's area theorem tells us that the area of the black hole never decreases in any case, similarly the entropy of the thermal system never decreases. On this basis Bekenstein proposed that a black hole has an entropy that is proportional to the area of the black hole.

At first it was thought to be just a formal analogy, because according to the classical relativity theory black hole can only absorb and never emit anything, implying zero temperature for the black hole. Even Hawking didn't at first believe Bekenstein's proposal, but Hawking's discovery of black hole radiation played a key role in the general acceptance of Bekenstein's idea. Hawking applied quantum field theory in curved spacetime in a fixed background geometry. According to Hawking's results, black holes not only absorb materials but also emit particles just like black body radiation with temperature $\kappa/2\pi$ and the radiation spectrum is purely thermal. Thus we could assign a temperature to black holes and make Bekenstein's proposal consistent. Thus a black hole has been taken as a thermal object with temperature.

With this temperature for the black hole, Bekenstein's conjecture suggested that a black hole has entropy of $A/4$. For many years people could not understand the microscopic nature of this black hole entropy. However, an even worse problem came from the fact that the black hole radiation process violates the fundamental principle of quantum mechanics.

Suppose a system composed of a large collection of books has collapsed gravitationally and formed a black hole. In quantum mechanical language, the state of the system before collapse is a pure state. But the black hole radiation has only one characteristic feature - temperature. If the radiation is pure thermal there is no correlation between the emitted particles. Hence the thermal nature of the Hawking radiation produces information loss of the original matter that collapsed to form the black hole. After collapse we have lost all information about the system. If this black hole evaporates in a completely thermal process as in Hawking's statement, after complete evaporation of the mass the final state would be a totally mixed state, meaning lost information. But the basic principle of quantum mechanics does not allow any transition from a pure state to a mixed state. If the black hole radiation is completely thermal, we cannot bypass violating quantum mechanics. In terms of entropy, the system evolves from a zero entropy state in a specific pure state to a large entropy state in thermal mixed state. If the evolution was unitary, then the entropy before must be equal to the entropy after.

Hawking's response to that was that somehow quantum mechanics breaks down in the black hole formation and evaporation process. People could not accept Hawking's advocation. For decades many ideas have been suggested to recover the information.

Opinions about the significance of this problem have also differed:

> I believe that in time, when the repercussions are fully understood, physicists will recognize it as the beginning of a great scientific revolution. It is too early to know exactly how that revolution will play out, but it will touch on the deepest issues: the nature of space and time, the meaning of elementary particles, and the mysteries of the origin of the universe. Susskind[17], p174(2008)

> Black hole radiance was originally derived [16] in an approximation where the background geometry was given, by calculating the response of quantum fields to this (collapse) geometry. In this approximation the radiation is thermal, and much has been made both of the supposed depth of this result and the paradoxes that ensue if it is taken literally. For if the radiation is accurately thermal there is no connection between what went into the hole and what comes out, a possibility which is difficult to reconcile with unitary evolution in quantum theory . or, more simply, with the idea that there are equations uniquely connecting the past with the

> future. To address such questions convincingly, one must go beyond the approximation of treating the geometry as given, and treat it too as a quantum variable. This is not easy $\cdots$ Kraus and Wilczek [2] (1995)

One of the most promising attempts was to take into account the effect of back reaction of the emitted particles on the background geometry. It might drive a deviation from pure thermality of the radiation and may solve the information paradox. Hawking's derivation of radiation did not take account of this self-gravitational correction to the radiation.

In 1995, Kraus and Wilczek [2] counted this back reaction effect and quantized the tunneling scalar field. To preserve spherical symmetry, they worked in an s-wave approximation, which models the emitted particles as thin spherical shells. (Even with these simplifying assumptions, their intricate analysis occupies 17 pages.) They developed the action for a self gravitating shell and managed to quantize it. One important (but perhaps not unexpected) outcome of this quantum analysis was that, to leading order, the main effect of back-reaction on the geometry is the purely classical one of mass loss by the black hole as a results of the emission. A shorter derivation, taking account of this effect only, was given six years later by Parikh and Wilczek[2] (Chapter 10). They made an important contribution because the results show a deviation from perfect thermal radiation.

The original derivation of Hawking radiation contains long and intricate calculations making it hard to get an intuitive picture. A simpler derivation of Hawking radiation employing a tunneling picture with WKB approximation was presented later [3][4].

Parikh and Wilczek [2] calculated Hawking radiation using the Hamilton-Jacobi tunneling formula for null geodesics across the event horizon. They also took back reaction into account and the Hawking radiation was modified by that. The radiation spectrum was not precisely thermal implying possible unitarity of the underlying quantum process. However their attempt to show a correlation between emissions was not successful.

But Parikh's conclusion was based on a misapplication of the statistical test for correlations (explained in more detail in Chapter 12). This oversight passed unnoticed until four Chinese physicists corrected it in March 2009 (Zhang et al[5]). They concluded that up to $\exp S_{BH}$ bits of information can be carried off in the correlations, which can include all of the information in the hole if, as seems reasonable, the Bekenstein-Hawking entropy S_{BH} (equal to a quarter of the horizon area) is a measure of the hole's *information capacity*, in the sense that $\exp S_{BH}$ is the maximum number of bits that can be accommodated in a black hole formed by an astrophysical collapse.

Kraus and Wilczek's key result [2] was their arithmetic-mean prescription (AMP) for the effective action (including back-reaction) of a massive particle (modelled as a spherical shell in the s-wave approximation) tunneling out of a spherical black hole.

The much simpler and more transparent treatment introduced here calls upon the general-relativistic dynamics of thin shells, together with the analytic properties of Schwarzschild's time co-ordinate t over the extended Kruskal manifold, and occupies just a few lines (chapter 11). Moreover, this extends immediately to charged evaporation and brings new aspects of the problem into focus – breakdown of AMP when interactions with non gravitational forces are introduced.

By way of introduction to this somewhat novel treatment of the black hole tunneling problem, chapter 9 rederives the classic Schwinger formula for charged pair creation by an electric field, using the same (essentially geometrical) approach. Comparison is instructive, revealing both the resemblance and at least one sharp difference between the two cases.

Chapter 12 briefly reviews the key question on which the current literature makes a confused impression – does Hawking radiation with Kraus-Wilczek deviations from thermality have information-holding correlations? – and reaffirms the positive answer given by Zhang et al [5]. Sec. 13 concludes the paper with some open questions.

A shorter account of this research is published in Phys.Rev. *D82*, 124036 (2010) and available online at at arXiv.org, hep-th/1009.0879.[18]

Chapter 2

Pure state, Mixed state, Density matrix, Principle of quantum evolution

In this chapter we will clarify the meaning of pure and mixed states and show the unitary quantum evolution principle does not allow a change of pure state to mixed state.

Let's take an example of spin states. The most general state ket of a spin half system is

$$| \alpha\rangle = c_+ \, |\uparrow_z\rangle + c_- \, |\downarrow_z\rangle \tag{2.1}$$

(2.1) represents the pure spin state in some definite direction. (2.1) cannot describe a mixed state of spin such as a collection of atoms with random spin orientations. To describe this mixed state, we introduce a fractional population or probability weight specified by w_i which is a real number. There is no information on the relative phase between the states which would specify a quantum superposition of states.

We define the density matrix ρ by

$$\rho \equiv \sum_i w_i \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid \tag{2.2}$$

where $| \alpha^{(i)}\rangle$ means a pure quantum state and w_i is a probability weight for that pure state. The mixing of $| \alpha^{(i)}\rangle$ with w_i as shown by (2.2) constitutes an ensemble of pure states and is called a mixed state.

The $| \alpha^{(i)}\rangle$ do not need to orthogonal and i can run through more than the dimension of the ket space.

2.1 Density matrix ρ

The ensemble average of an observable A in the mixed state $\sum_i w_i \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid$ can be found by

$$\begin{aligned}
[A] &= \sum_i w_i \langle\alpha^{(i)} \mid A \mid \alpha^{(i)}\rangle = \sum_i w_i \langle\alpha^{(i)} \mid \underbrace{}_{\sum_k |b_k\rangle\langle b_k|} A \underbrace{}_{\sum_j |b_j\rangle\langle b_j|} \mid \alpha^{(i)}\rangle \\
&= \sum_k \sum_j \sum_i w_i \underbrace{\langle b_j \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid b_k\rangle}_{\rho^{(i)}_{jk}} \langle b_k \mid A \mid b_j\rangle \qquad (2.3) \\
&= \sum_k \sum_j \sum_i w_i \rho^{(i)}_{jk} A_{kj} = \sum_k \sum_j \rho_{jk} A_{kj} \qquad (2.4) \\
&= Tr(\rho A) \qquad (2.5)
\end{aligned}$$

where $\{\mid b_j\rangle\}$ is an arbitrary orthonormal basis. Note that we separated the information about the quantum state ρ_{jk} from the observable itself A_{kj} in (2.4). This is the main motivation of defining density matrix in quantum statistics.

The trace of the density matrix is 1.

$$\begin{aligned}
Tr(\langle j \mid \rho \mid k\rangle) &= Tr(\langle j \mid \sum_i w_i \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid k\rangle) = \sum_j \langle j \mid \sum_i w_i \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid j\rangle \\
&= \sum_{ij} w_i \langle j \mid \alpha^{(i)}\rangle\langle\alpha^{(i)} \mid j\rangle = \sum_i \sum_j w_i \langle\alpha^{(i)} \mid j\rangle\langle j \mid \alpha^{(i)}\rangle \\
&= \sum_i w_i \langle\alpha^{(i)} \mid \alpha^{(i)}\rangle = \sum_i w_i = 1
\end{aligned}$$

Suppose a complex $n \times n$ matrix $B = \langle\varphi_i \mid \rho \mid \varphi_j\rangle$ is represented in the $\{\varphi_i\}$ basis. In another basis B would be represented by unitary-similarity transformation UBU^{-1}

$$Tr(UBU^{-1}) = Tr(BU^{-1}U) = TrB \quad \cdots \text{Trace invariance}$$

We can prove that a pure state stays pure under unitary evolution

$$\begin{aligned}
\rho &= \mid \alpha\rangle\langle\alpha \mid \\
\rho_U &= U \mid \alpha\rangle\langle\alpha \mid U^\dagger \\
\rho_U^2 &= \Big(U \mid \alpha\rangle\langle\alpha \mid U^\dagger\Big)\Big(U \mid \alpha\rangle\langle\alpha \mid U^\dagger\Big) = U \mid \alpha\rangle\langle\alpha \mid U^\dagger = \rho_U
\end{aligned}$$

Chapter 3

Schwarzschild solution

Let's derive the metric of the spherical geometry. A general form of spherical symmetric geometry is

$$ds^2 = -e^{a(r,t)}dt^2 + e^{b(r,t)}dr^2 + r^2 d\Omega^2 \tag{3.1}$$

Let's find Einstein tensor $G^{\alpha}{}_{\beta}$. The metric (3.1) generates following Einstein tensors

$$\begin{aligned}
G_{00} &= \frac{1}{r^2}e^{a-b}(rb_r + e^b - 1) \\
G_{11} &= \frac{1}{r^2}(a_r r + 1 - e^b) \\
G_{10} &= G_{01} = \frac{b_t}{r}
\end{aligned}$$

where the subscripts r and t represent partial differentiation with respect to r and t.

$$\begin{aligned}
r^2 G^0{}_0 &= r^2 g^{00} G_{00} = r^2(-e^{-a})\frac{1}{r^2}e^{a-b}(rb_r + e^b - 1) = e^{-b}(1 - rb_r) - 1 \\
r^2 G^1{}_1 &= r^2 g^{11} G_{11} = r^2 e^{-b}\frac{1}{r^2}(a_r r + 1 - e^b) = e^{-b}(ra_r + 1) - 1 \\
e^b G^1{}_0 &= e^b g^{11} G_{10} = e^b e^{-b}\frac{b_t}{r} = \frac{b_t}{r} \\
-e^a G^0{}_1 &= -e^a g^{00} G_{01} = (-e^a)(-e^{-a})\frac{b_t}{r} = \frac{b_t}{r} \\
G^2{}_2 &= G^3{}_3 \cdots \text{ complicated second order expression}
\end{aligned}$$

where $G^2{}_2 = G^3{}_3$ by spherical symmetry.

Applying to Einstein field equations

$$G^{\alpha}{}_{\beta} = 8\pi T^{\alpha}{}_{\beta} \qquad (G = c = 1) \tag{3.2}$$

results in

$$r^2 G^0{}_0 = \frac{\partial}{\partial r}(re^{-b} - r) = 8\pi r^2 T^0{}_0 \tag{3.3a}$$

$$r^2 G^1{}_1 = e^{-b}(ra_r + 1) - 1 = 8\pi r^2 T^1{}_1 \tag{3.3b}$$

$$-e^a G^0{}_1 = e^b G^1{}_0 = \frac{b_t}{r} = 8\pi e^b T^1{}_0 \tag{3.3c}$$

Since $-T^0_0$ is local energy density ρ, (3.3a) can be written by

$$\frac{\partial}{\partial r}(re^{-b} - r) \quad = \quad 8\pi r^2(-\rho) \tag{3.4}$$

$$\frac{\partial}{\partial r}\Big(\underbrace{\frac{r}{2}(1 - e^{-b})}_{=M(r,t)}\Big) \quad = \quad 4\pi r^2 \rho \tag{3.5}$$

$$e^{-b} = 1 - \frac{2M(r,t)}{r} \tag{3.6}$$

(3.3c) can be rewritten by

$$\frac{-(e^{-b})_t}{r} \quad = \quad 8\pi T^1{}_0 \tag{3.7}$$

$$\frac{2M_t(r,t)}{r^2} \quad = \quad 8\pi T^1{}_0 \tag{3.8}$$

$$\frac{\partial M(r,t)}{\partial t} \quad = \quad 4\pi r^2 T^1{}_0 \tag{3.9}$$

$$\frac{\partial M(r,t)}{\partial t} \quad = \quad -4\pi r^2 e^{a-b} T^0{}_1 \tag{3.10}$$

This implies that T^1_0 is the local energy flux.

−(3.3a)+(3.3b) gives

$$
\begin{aligned}
-\frac{\partial}{\partial r}(re^{-b}-r)+e^{-b}(ra_r+1)-1 &= 8\pi r^2(-T^0_{\ 0}+T^1_{\ 1}) && (3.11)\\
e^{-b}r(a_r+b_r) &= 8\pi r^2(\rho+T^1_{\ 1}) && (3.12)\\
e^{-b}(\underbrace{a+b}_{\equiv s(r,t)})_r &= 8\pi r(\rho+T^1_{\ 1}) && (3.13)\\
\Big(1-\frac{2M(r,t)}{r}\Big)\frac{\partial s(r,t)}{\partial r} &= 8\pi r(\rho+T^1_{\ 1}) && (3.14)
\end{aligned}
$$

Thus general form of spherical symmetric metric (3.1) can be written by Einstein field equation (3.2)

$$
\begin{aligned}
ds^2 &= -e^{a(r,t)}dt^2+e^{b(r,t)}dr^2+r^2d\Omega^2 && (3.15)\\
&= -e^{s(r,t)}\Big(1-\frac{2M(r,t)}{r}\Big)dt^2+\frac{dr^2}{1-\frac{2M(r,t)}{r}}+r^2d\Omega^2 && (3.16)
\end{aligned}
$$

plus Bianchi identity $G^{\alpha}_{\ \beta|\alpha}=0$ gives all equations of general relativity.

Let us find the vacuum form of spherical solution (3.16) of Einstein field equation . In vacuum $T^{\alpha}_{\ \beta}=0$, (3.9) implies $M(r,t)\to M(r)$ and (3.14) implies $s(r,t)\to s(t)$. Then the vacuum solution (3.16) is

$$
ds^2=-e^{s(t)}\Big(1-\frac{2M}{r}\Big)dt^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2d\Omega^2
$$

Define $dt'\equiv e^{s(t)/2}dt$ then,

$$
ds^2=-\Big(1-\frac{2M}{r}\Big)dt'^2+\frac{dr^2}{1-\frac{2M}{r}}+r^2d\Omega^2 \tag{3.17}
$$

This is the Schwarzschild solution.

Find the spherical solution for a charged black hole.

The stress-energy tensor for electro-magnetic field is

$$
T_{\alpha}^{\ \beta}=\frac{1}{4\pi}(F_{\alpha\mu}F^{\beta\mu}-\frac{1}{4}\delta^{\beta}_{\alpha}F^{\mu\nu}F_{\mu\nu}) \tag{3.18}
$$

where $F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu}$ and A^μ is the four vector potential of electro-magnetic field.

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

Let's find ${T_\alpha}^\beta$ at a point on x axis of rectangular coordinate whose origin is the position of the charged black hole. Charged black hole has only radial directional electric field. Thus that point has only E_x component of the electro-magnetic field tensor $F^{\mu\nu}$. By $F_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}$

$$E_x = F^{01} = -F^{10} = F_{10} = -F_{01}$$

and all other components of $F^{\mu\nu}$ and $F_{\mu\nu}$ are zeros. Detail calculation of each T^α_β shows

$$\begin{aligned} T^0_{\ 0}(=-\rho) &= \frac{1}{4\pi}(F_{0\mu}F^{0\mu} - \frac{1}{4}\delta^0_0 F^{\mu\nu}F_{\mu\nu}) \\ &= \frac{1}{4\pi}(F_{01}F^{01} - \frac{1}{4}(F^{01}F_{01} + F^{10}F_{10})) \\ &= \frac{1}{4\pi}(-E^2 - \frac{1}{4}(-2E^2)) = -\frac{E^2}{8\pi} \end{aligned}$$

$$T^1_{\ 1} = \frac{1}{4\pi}(F_{1\mu}F^{1\mu} - \frac{1}{4}\delta^1_1 F^{\mu\nu}F_{\mu\nu}) = -\frac{E^2}{8\pi}$$

$$T^2_{\ 2} = \frac{1}{4\pi}(F_{2\mu}F^{2\mu} - \frac{1}{4}\delta^2_2 F^{\mu\nu}F_{\mu\nu}) = \frac{E^2}{8\pi}$$

$$T^3_{\ 3} = \frac{1}{4\pi}(F_{3\mu}F^{3\mu} - \frac{1}{4}\delta^3_3 F^{\mu\nu}F_{\mu\nu}) = \frac{E^2}{8\pi}$$

$$T^1_{\ 0} = \frac{1}{4\pi}(F_{0\mu}F^{1\mu} - \frac{1}{4}\delta^1_0 F^{\mu\nu}F_{\mu\nu}) = 0$$

Hence (3.9) tells us that $M(r,t) \to M(r)$ and by (3.5)

$$\frac{\partial M}{\partial r} = 4\pi r^2 \rho = 4\pi r^2 \left(\frac{E^2}{8\pi}\right) = \frac{r^2 E^2}{2} = \frac{e^2}{2r^2}$$

where we used $E = \frac{e}{r^2}$ which can be found by $F^{\mu\nu}{}_{|\nu} = 4\pi J^\mu$. Therefore

$$M(r) = C_1 - \frac{e^2}{2r}$$

$\frac{e^2}{2r}$ represents the static electric field energy outside r; The Schwarzschild observer measures the energy $M(r \to \infty) = C_1$. Let's denote this Schwarzschild energy by m. Then the resulting metric for charged black hole is by (3.17)

$$\begin{aligned} ds^2 &= -\Big(1 - \frac{2M(r)}{r}\Big)dt^2 + \frac{dr^2}{1 - \frac{2M(r)}{r}} + r^2 d\Omega^2 && (3.19) \\ &= -\Big(1 - \frac{2m}{r} + \frac{e^2}{r^2}\Big)dt^2 + \frac{dr^2}{1 - \frac{2m}{r} + \frac{e^2}{r^2}} + r^2 d\Omega^2 && (3.20) \end{aligned}$$

This solution for charged black hole is called Reissner-Nordström solution.

Chapter 4

Rotating black hole

The vacuum solution of rotating black hole is called the Kerr metric as following,

$$\begin{aligned}
ds^2 &= -(dt - a\sin^2\theta d\phi)^2\frac{\Delta}{\Sigma} + \Big(adt - (\Sigma + a^2\sin^2\theta)d\phi\Big)^2\frac{\sin^2\theta}{\Sigma} \\
&\quad +\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 && (4.1)\\
&= -\Big(1 - \frac{2Mr}{\Sigma}\Big)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \\
&\quad +\Big(r^2 + a^2 + \frac{2Mr}{\Sigma}a^2\sin^2\theta\Big)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 && (4.2)\\
&= -\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)dt^2 - \frac{2a\sin^2\theta(r^2 + a^2 - \Delta)}{\Sigma}dtd\phi \\
&\quad +\Big(\frac{(r^2 + a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 && (4.3)
\end{aligned}$$

where we define $a \equiv J/M$, an angular momentum per unit mass of the black hole and

$$\Sigma \equiv r^2 + a^2\cos^2\theta$$

$$\Delta \equiv r^2 - 2Mr + a^2$$

The event horizon is at $g^{rr} = 0$,

$$\Delta(r = r_\pm) = 0 \rightarrow r_\pm = M \pm \sqrt{M^2 - a^2} \tag{4.4}$$

The surface area ($dt = dr = 0$) at the horizon($\Delta = 0$) is

$$A = \int (r_\pm^2 + a^2)\sin\theta d\theta d\phi = 4\pi(r_\pm^2 + a^2) \tag{4.5}$$

The angular velocity ω of zero angular momentum observer (ZAMO) is

$$\omega = \frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}}$$

The four velocity of static observer (zero angular velocity) [1] is proportional to the Killing vector $t^\alpha = \frac{\partial x^\alpha}{\partial t}$,

$$u^\alpha \equiv \frac{dx^\alpha}{d\lambda} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial x^\alpha}{\partial t}\frac{dt}{d\lambda} \tag{4.6}$$

This u^α limits to lightlike at the points where $g_{tt} = 0$. This place r_{sl} is called static limit.

$$r_{sl} = M + \sqrt{M^2 - a^2\cos^2\theta} \tag{4.7}$$

The static observer at r_{sl} has to travel at the speed of light opposite to the direction of the hole's rotation to overcome the frame dragging and to maintain constant value of ϕ. The observer in $r < r_{sl}$ cannot remain static. He has to have positive angular velocity by frame dragging.

An observer who observes time independent value of metric is called a stationary observer. In the case of a rotating black hole, by rotational symmetry, a stationary observer can rotate in the ϕ direction with an arbitrary, uniform angular velocity $\Omega \equiv \frac{d\phi}{dt}$. His four velocity u^α is

$$\begin{aligned} u^\alpha = \frac{dx^\alpha}{d\lambda} &= \frac{\partial x^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = \frac{\partial x^\alpha}{\partial t}\frac{dt}{d\lambda} + \frac{\partial x^\alpha}{\partial \phi}\frac{d\phi}{d\lambda} \\ &= \frac{dt}{d\lambda}\Big(\frac{\partial x^\alpha}{\partial t} + \frac{d\phi}{dt}\frac{\partial x^\alpha}{\partial \phi}\Big) = \frac{dt}{d\lambda}(t^\alpha + \Omega\phi^\alpha) \end{aligned} \tag{4.8}$$

Because of the independence of metric with respect to (t, ϕ), both t^α and ϕ^α are Killing vectors and their linear combination $\xi^\alpha \equiv t^\alpha + \Omega\phi^\alpha$ is also a Killing vector if Ω is constant.

[1] For massive observer $d\lambda = d\tau$

Since the stationary observer must be timelike,

$$(t^\alpha + \Omega\phi^\alpha)(t_\alpha + \Omega\phi_\alpha) < 0 \tag{4.9}$$

where t^α is the time like Killing vector and ϕ^α is the spacelike Killing vector. Thus we can imagine the value of Ω would determine whether $t^\alpha + \Omega\phi^\alpha$ is timelike or spacelike.

$$\begin{aligned}
(t^\alpha + \Omega\phi^\alpha)(t_\alpha + \Omega\phi_\alpha) &= g_{\alpha\beta}(t^\alpha + \Omega\phi^\alpha)(t^\beta + \Omega\phi^\beta) && (4.10)\\
&= g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} && (4.11)\\
&= g_{\phi\phi}\Big(\Omega^2 + 2\frac{g_{t\phi}}{g_{\phi\phi}}\Omega + \frac{g_{tt}}{g_{\phi\phi}}\Big) && (4.12)\\
&= g_{\phi\phi}\Big(\Omega^2 - 2\omega\Omega + \frac{g_{tt}}{g_{\phi\phi}}\Big) && (4.13)
\end{aligned}$$

$$\Omega_\pm = \omega \pm \sqrt{\omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} = \omega \pm \frac{\sqrt{\Delta}\Sigma}{[(r^2+a^2)^2 - a^2\Delta\sin^2\theta]\sin\theta} \tag{4.14}$$

Thus (4.9) tells us that the stationary observer can have angular velocity only within (Ω_-, Ω_+). Since only positive angular velocity observers are allowed inside static limit, we can understand that Ω_- changes its sign at static limit. As the observer goes further inside the static limit, Ω_+ decreases and Ω_- increases. Finally at one point $\Omega_+ = \Omega_- = \Omega_H$. By (4.14) at this point all stationary observers have the same angular velocity ω. This happens when $\Delta = 0$.

$$\Delta = 0 \rightarrow r_+ = M + \sqrt{M^2 - a^2} \tag{4.15}$$

We call $\omega_+ = \omega(r_+)$ the angular velocity of black hole.

$$\omega_+ = \Omega(r_+) = -\frac{g_{t\phi}}{g_{\phi\phi}}\Big|_{r=r_+} = \frac{(r_+^2+a^2)a}{(r_+^2+a^2)^2} = \frac{a}{r_+^2+a^2} \tag{4.16}$$

where we used the fact $\Delta = r_+^2 - 2Mr_+ + a^2 = 0$ at the horizon. Then by (4.13) we can check out $t^\alpha + \Omega_H\phi^\alpha$ is a null vector at r_+. Since r_+ is the timelike limit of stationary observer, we call r_+ is the stationary limit. We can check this stationary limit coincides with the event horizon of a rotating black hole. The region between the static and stationary limits, $r_+ < r < r_{sl}$, is called the ergo-region. In the ergo-region $t^\alpha \equiv \frac{\partial x^\alpha}{\partial t}$ is a spacelike vector. Thus the energy of the particle in ergo-region can be either sign. This allows the Penrose process which enables to extract rotational positive energy from the hole.

Let us derive the area theorem for a sequence of stationary rotating black holes.

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H \delta J \tag{4.17}$$

where A, M and J are the area, energy and angular momentum of the black hole. The area A of the horizon of rotating black hole is given by (4.5)

$$A = 4\pi(r_+^2 + a^2)$$

We would like to calculate the change of this area by the change of mass and angular momentum via emitting or absorbing of particles. The change of area can be written as

$$\delta A = 8\pi(r_+\delta r_+ + a\delta a) \tag{4.18}$$

We want to express this in terms of δM and δJ. From the definition of a, $J = Ma$, we get the expression of δa,

$$\delta a = \frac{\delta J - a\delta M}{M}$$

Since we are dealing with the area at the horizon, we can use the relation

$$r_+^2 - 2Mr_+ + a^2 = 0$$

$$(r_+ - M)\delta r_+ = r_+\delta M - a\delta a$$

$$(r_+ - M)\delta r_+ = r_+\delta M - a\frac{a\delta M - \delta J}{M} = \frac{Mr_+\delta M + a^2\delta M - a\delta J}{M}$$

Substituting these δa and δr_+ into (4.18)

$$\frac{r_+ - M}{r_+^2 + a^2}\frac{\delta A}{8\pi} = \delta M - \frac{a}{r_+^2 + a^2}\delta J$$

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H \delta J \tag{4.19}$$

In order to extract energy from a rotating black hole we inject a particle with energy E and angular momentum J into the black hole. The Killing vector $\xi^\alpha = t^\alpha + \Omega_H \phi^\alpha$ is a timelike vector just outside the horizon. p^α of injected particle is timelike vector.

Moreover $p_\alpha \xi^\alpha$ is conserved for the free falling particle. Thus

$$-p_\alpha \xi^\alpha = E - \Omega_H J > 0$$

These E and J are the change of energy and angular momentum of the *black hole*. Thus right hand side of (4.19) must be positive. Then the area theorem (4.17) follows in (4.19) for a rotating black hole.

4.1 Kerr-Newman metric

The solution of the Eistein's field equations for charged and rotating black hole is called Kerr-Newman solution and its metric can be represented as the following,

$$\begin{aligned} ds^2 &= -(dt - a\sin^2\theta d\phi)^2 \frac{\Delta}{\Sigma} + \Big(adt - (r^2+a^2)d\phi\Big)^2 \frac{\sin^2\theta}{\Sigma} \\ &\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \qquad (4.20) \\ &= -\Big(1 - \frac{2Mr - Q^2}{\Sigma}\Big)dt^2 - \frac{(2Mr-Q^2)2a\sin^2\theta}{\Sigma}dtd\phi \\ &\quad + \Big(r^2 + a^2 + \frac{2Mr-Q^2}{\Sigma}a^2\sin^2\theta\Big)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \qquad (4.21) \\ &= -\Big(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\Big)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dtd\phi \\ &\quad + \Big(\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\Big)\sin^2\theta d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \qquad (4.22) \end{aligned}$$

$$\Sigma \equiv r^2 + a^2\cos^2\theta$$

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2$$

$$\Delta(r = r_\pm) = 0 \rightarrow r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} \qquad (4.23)$$

The surface area ($dt = dr = 0$) at the horizon($\Delta = 0$) is

$$A = \int (r_\pm^2 + a^2)\sin\theta d\theta d\phi = 4\pi(r_\pm^2 + a^2) \qquad (4.24)$$

Just as in the Kerr metric, (4.13), (4.14), (4.16) are still valid in the Kerr-Newman metric.

We can derive the area theorem for the Kerr-Newman metric in a similar way as we did for rotating black hole.

$$\delta A = 8\pi(r_+\delta r_+ + a\delta a) \tag{4.25}$$

We would like to express this in terms of δM, δJ, δQ. $\Delta = 0$ gives the relation

$$r_+^2 - 2Mr_+ + a^2 + Q^2 = 0$$

$$\delta r_+ = \frac{Mr_+\delta M + a^2\delta M - a\delta J - MQ\delta Q}{M(r_+ - M)} \tag{4.26}$$

Substituting δa and δr_+ to (4.25) results in

$$\frac{r_+ - M}{r_+^2 + a^2}\frac{\delta A}{8\pi} = \delta M - \frac{a}{r_+^2 + a^2}\delta J - \frac{Qr_+}{r_+^2 + a^2}\delta Q \tag{4.27}$$

$$\frac{\kappa}{8\pi}\delta A = \delta M - \Omega_H\delta J - \Phi_H\delta Q \tag{4.28}$$

where Ω_H is an angular velocity of a black hole and Φ_H is a electric potential at the horizon.

Chapter 5

Surface gravity

Black hole thermodynamics is characterized by the temperature of the black hole. Because the temperature of a black hole is determined by the surface gravity of a black hole, it is useful to define it. Suppose a rest mass m_0 is static at r in Schwarzschild geometry. Because of red shift effect the energy observed by Schwarzschild observer at infinity E_∞ is

$$E_\infty = m_0\sqrt{f(r)}$$

where $f(r) = 1 - \frac{2M}{r}$ with Schwarzschild mass M.

If we lower it quasi-statically to horizon of the Schwarzschild space time, all of the gravitational potential energy of m_0 is extracted as work.

The surface gravity $\kappa(r_H)$ is defined as work done (calibrated for ∞ observer) to raise unit mass m_0 at the horizon r_H through unit proper distance.

$$\kappa(r_H) \equiv \frac{\partial E_\infty}{f^{-\frac{1}{2}}\partial r}\Big|_{r=r_H} = \frac{1}{2}\frac{\partial f(r)}{\partial r}\Big|_{r=r_H} \tag{5.1}$$

Surface gravity κ for a charged black hole is by (5.1) with $f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$

$$\kappa(r_+) = \frac{\sqrt{m^2 - e^2}}{r_+^2} \tag{5.2}$$

5.1 Surface gravity using general formula

We can show $t^\alpha \equiv \frac{\partial x^\alpha}{\partial t}$ satisfies

$$t_{\alpha;\beta} + t_{\beta;\alpha} = 0 \tag{5.3}$$

given $g_{\alpha\beta,t} = 0$. Thus t^α is a timelike Killing vector.

Because a time coordinate vector become lightlike at the horizon, $-t^\alpha t_\alpha = 0$ at the horizon. The $\phi = -t^\alpha t_\alpha = 0$ surface turn out to be the same as the event horizon. Because t^α is lightlike at the horizon, t^α is parallel and orthogonal to t^α and orthogonal to ϕ.

$$\phi_{,\alpha} \propto t_\alpha$$

Define surface gravity κ by

$$(-t^\alpha t_\alpha)_{;\mu} = 2\kappa t_\mu \tag{5.4}$$

where t_μ is the null Killing vector at the horizon. In Schwarzschild geometry

$$(-t^\alpha t_\alpha)_{;\mu} = \Big(- g_{\alpha\beta} \frac{\partial x^\alpha}{\partial t} \frac{\partial x^\beta}{\partial t} \Big)_{;\mu} = 2\kappa g_{\mu\nu} \frac{\partial x^\nu}{\partial t}$$

κ cannot be defined for $\mu = r$ because this is $0 = 0 \cdot \kappa$. We need better coordinates. The advanced time coordinate v defined by

$$dv \equiv dt + \frac{dr}{f}$$

fills the required role.

$$\begin{aligned} ds^2 &= -f dt^2 + \frac{dr^2}{f} = f\Big(- dt + \frac{dr}{f} \Big) \underbrace{\Big(dt + \frac{dr}{f} \Big)}_{dv} \\ &= f\Big(- dv + \frac{2dr}{f} \Big) dv = -f dv^2 + 2dr dv \end{aligned}$$

We can calculate (5.4) in these coordinates.

$$(-t^\mu t_\mu)_{,\alpha} dx^\alpha = 2\kappa t_\alpha dx^\alpha$$

$$-t^\mu t_\mu = -g_{\mu\nu} \frac{\partial x^\mu}{\partial t} \frac{\partial x^\nu}{\partial t} = -g_{vv} = f$$

$$\begin{aligned} t_\alpha dx^\alpha &= g_{\alpha\beta} t^\beta dx^\alpha = g_{\alpha\beta} \frac{\partial x^\beta}{\partial t} dx^\alpha = g_{\alpha v} dx^\alpha \\ &= g_{vv} dv + g_{rv} dr = dr, \qquad g_{vv} = 0 \text{ at the horizon} \end{aligned}$$

Thus

$$f_{,\alpha} dx^\alpha = 2\kappa dr$$

$$\kappa = \frac{1}{2} \frac{df}{dr}$$

We can derive alternative definition of surface gravity using anti-symmetric property (5.3) of a Killing vector,

$$(-t^\beta t_\beta)_{;\alpha} = -t^\beta{}_{;\alpha} t_\beta - t^\beta t_{\beta;\alpha} = -2t_{\beta;\alpha} t^\beta = 2t_{\alpha;\beta} t^\beta$$

This must be the same as $2\kappa t_\alpha$ by original definition (5.4). Hence we get the alternative definition of κ as following

$$t^\alpha{}_{;\beta} t^\beta = \kappa t^\alpha \tag{5.5}$$

Let us derive the surface gravity κ of rotating black hole using the definition

$$(-\xi^\alpha \xi_\alpha)_{;\mu} = 2\kappa \xi_\mu \qquad (5.4)$$

where ξ^α is a null Killing vector at the horizon.

$$\begin{aligned} \xi^\alpha \xi_\alpha &= g_{\alpha\beta}(t^\alpha + \Omega\phi^\alpha)(t^\beta + \Omega\phi^\beta) = g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} \\ &= g_{\phi\phi}\Big(\Omega + \frac{g_{t\phi}}{g_{\phi\phi}}\Big)^2 + g_{tt} - \frac{(g_{t\phi})^2}{g_{\phi\phi}} \end{aligned} \tag{5.6}$$

We don't need to care about the first term. Let's calculate the rest of terms.

$$\begin{aligned}
g_{tt} - \frac{(g_{t\phi})^2}{g_{\phi\phi}} &= -\frac{\Delta - a^2\sin^2\theta}{\Sigma} - \frac{\left(-\frac{a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\right)^2}{\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta} \\
&= \frac{a^2\sin^2\theta - \Delta}{\Sigma} - \frac{(a\sin\theta)^2(r^2+a^2-\Delta)^2}{\Sigma[(r^2+a^2)^2-\Delta a^2\sin^2\theta]} \\
&= \frac{(a^2\sin^2\theta-\Delta)[(r^2+a^2)^2-\Delta a^2\sin^2\theta] - (a\sin\theta)^2(r^2+a^2-\Delta)^2}{\Sigma[(r^2+a^2)^2-\Delta a^2\sin^2\theta]}
\end{aligned}$$

$$\begin{aligned}
g_{tt} - \frac{(g_{t\phi})^2}{g_{\phi\phi}} &= -\frac{(r^2+a^2\cos^2\theta)^2(r^2-2Mr+a^2)}{\Sigma[(r^2+a^2)^2-\Delta a^2\sin^2\theta]} \\
&= -\frac{\Sigma\Delta}{(r^2+a^2)^2-\Delta a^2\sin^2\theta} \qquad (5.7)
\end{aligned}$$

Hence

$$(-\xi^\alpha\xi_\alpha)_{;\mu} = \frac{\Sigma\Delta_{,\mu}}{(r^2+a^2)^2-\Delta a^2\sin^2\theta} \qquad (5.8)$$

at the horizon. We have

$$\Delta = 0, \qquad \Delta_{,\mu} = 2(r_+ - M)r_{,\mu}$$

and

$$\xi_\mu = (1 - a\Omega_H\sin^2\theta)r_{,\mu} \qquad (5.9)$$

We cannot derive $\xi_\mu = (1 - a\Omega_H\sin^2\theta)r_{,\mu}$ using Kerr metric (4.1), (4.2) or (4.3) because $\xi_r = 0$ in that metric. We need better coordinate for our purpose. Define

$$v \equiv t + r_m, \qquad \Psi \equiv \phi + r_n \qquad (5.10)$$

where

$$r_m \equiv \int \frac{r^2+a^2}{\Delta}dr = r + \frac{Mr_+}{\sqrt{M^2-a^2}}\ln\left|\frac{r}{r_+}-1\right| - \frac{Mr_-}{\sqrt{M^2-a^2}}\ln\left|\frac{r}{r_-}-1\right| \qquad (5.11)$$

and

$$r_n \equiv \int \frac{a}{\Delta}dr = \frac{a}{2\sqrt{M^2-a^2}}\ln\left|\frac{r-r_+}{r-r_-}\right| \qquad (5.12)$$

Then a straightforward substitution converts Kerr metric (4.22) into the following form of metric

$$\begin{aligned} ds^2 &= -\Big(1-\frac{2Mr}{\Sigma}\Big)dv^2 + 2dvdr - 2a\sin^2\theta dr d\Psi \\ &\quad -\frac{4Mar\sin^2\theta}{\Sigma}dvd\Psi + \frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\Psi^2 + \rho^2 d\theta^2 \end{aligned} \quad (5.13)$$

In this form of metric we can derive (5.9).

Thus by (5.4),

$$\frac{\Sigma 2(r_+ - M)r_{,\mu}}{(r_+^2+a^2)^2} = 2\kappa(1-a\Omega_H\sin^2\theta)r_{,\mu}$$

$$\kappa = \frac{r_+ - M}{r_+^2+a^2} = \frac{\sqrt{M^2-a^2}}{r_+^2+a^2} \quad (5.14)$$

5.2 Area theorem of a charged black hole

As stated in the introduction, Hawking's area theorem led Bekenstein to propose that a black hole has an entropy which is proportional to the area of the black hole.

The Reissner-Nordström metric for a charged black hole is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad (5.15)$$

with $f(x) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$. The horizon of this metric is at

$$f(r) = 0, \quad r = M \pm \sqrt{M^2-e^2}$$

Outer horizon is at $r_+ = M + \sqrt{M^2-e^2}$. By (5.1) the surface gravity of it is

$$\kappa_+ \equiv \kappa(r_+) = \frac{1}{2}f'(r_+) = \frac{M}{r_+^2} - \frac{e^2}{r_+^3} = \frac{M - \frac{e^2}{r_+}}{r_+^2}$$

Or

$$\kappa_+ = \frac{\sqrt{M^2-e^2}}{r_+^2}$$

Add some charge de to a charged(e) black hole. Then the mass of the black hole changes at least by the work done on de.

$$dM = \frac{ede}{r_+} + dE^{dissipative}$$

where $\frac{ede}{r_+}$ is a pure electric work which can be negative or positive depending on sign of ede. $dE^{dissipative}$ represents all energy change except electric work. We assume it is positive for non-quantum field consideration. It includes rest mass, kinetic energy of charge de and electro magnetic and gravitational waves absorbed by the black hole. Change of dM, de results in dr_+

$$\begin{aligned} dr_+ &= dM + \frac{MdM - ede}{\sqrt{M^2 - e^2}} = \frac{(\sqrt{M^2 - e^2} + M)dM - ede}{\sqrt{M^2 - e^2}} \\ &= \frac{r_+ dM - ede}{\kappa_+ r_+^2} = \frac{dE^{dissipative}}{\kappa_+ r_+} \geq 0 \end{aligned}$$

Thus $dr_+ \geq 0$ always.

$$\kappa_+ r_+ dr_+ = dM - \underbrace{\frac{ede}{r_+}}_{\phi_+ de} = dE^{dissipative} \tag{5.16}$$

$$A \equiv 4\pi r_+^2 \qquad dA = 8\pi r_+ dr_+$$

$$\frac{\kappa_+}{8\pi} dA = \kappa_+ r_+ dr_+ = dM - \phi_+ de = dE^{dissipative} \tag{5.17}$$

dM can be positive or negative. However, since $dr_+ \geq 0$ always, $dA \geq 0$ always. Comparing with the typical thermodynamic relation

$$\begin{gathered} TdS = dE + PdV - \mu dN = dQ \\ \mu : \text{Chemical potential} \\ N : \text{Number of particles} \end{gathered}$$

implies Bekenstein's proposal.

Chapter 6

Unruh effect

6.1 Uniformly accelerated observer

The Minkowski metric of (1+1) dimensional space time is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + dx^2 \tag{6.1}$$

The four-velocity u^β and four-acceleration a^α are related by

$$\eta_{\alpha\beta} a^\alpha u^\beta = 0 \tag{6.2}$$

The inertial observer co-moving with the accelerated observer measures his four velocity

$$u^\alpha = (1, 0) \tag{6.3}$$

Thus by (6.2) and in order to make $\eta_{\alpha\beta} a^\alpha a^\beta = a^2$, the four-acceleration of the accelerated observer must be

$$a^\alpha(\tau) = (0, a) \tag{6.4}$$

$$\eta_{\alpha\beta} \ddot{x}^\alpha \ddot{x}^\beta = \eta_{\alpha\beta} a^\alpha(\tau) a^\beta(\tau) = a^2 \tag{6.5}$$

where dots represent the covariant derivative with respect to τ.

Define light-cone coordinates (U, V) as

$$U = t - x, \qquad V = t + x \tag{6.6}$$

Then the metric (6.1) is expressed by

$$ds^2 = g_{UV}dUdV = -dUdV \tag{6.7}$$

where

$$g_{UV} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \tag{6.8}$$

$$\dot{U}\dot{V} = 1 \tag{6.9}$$

The covariant expression (6.5) in terms of (U, V) is

$$-\ddot{U}\ddot{V} = a^2 \tag{6.10}$$

We can solve the differential equations (6.9) and (6.10). By setting some integral constants, we get

$$U(\tau) = -\frac{1}{a}e^{-a\tau}, \qquad V(\tau) = \frac{1}{a}e^{a\tau} \tag{6.11}$$

Let us find the metric for the frame (q^0, q^1) co-moving with the accelerated observer. We want the metric in (q^0, q^1) coordinates to be conformally flat.

$$ds^2 = H(q^0, q^1)\Big[-(dq^0)^2 + (dq^1)^2\Big] \tag{6.12}$$

Define the alternative coordinates (u, v) by

$$u \equiv q^0 - q^1, \qquad v \equiv q^0 + q^1 \tag{6.13}$$

The world line of the accelerated observer is static in (q^0, q^1) coordinates,

$$q^0 = \tau, \qquad q^1 = 0$$

$$u(\tau) = v(\tau) = \tau \tag{6.14}$$

Since (6.1), (6.7) and (6.12) describe the same Minkowski space time,

$$ds^2 = -dUdV = H(q^0, q^1)\Big[-(dq^0)^2 + (dq^1)^2\Big] = -H(u, v)dudv \tag{6.15}$$

Now we have the expressions (6.11) and (6.14) for the orbit of uniform accelerated observer in two different coordinates. Hence we can find the relation between two coordinate systems.

$$\frac{dU}{d\tau} = \frac{du}{d\tau}\frac{dU}{du}, \qquad \frac{dV}{d\tau} = \frac{dv}{d\tau}\frac{dV}{dv}$$

$$-adu(\tau) = d\ln U(\tau) \qquad adv(\tau) = d\ln V(\tau) \tag{6.16}$$

$$U(\tau) = C_1 e^{-au(\tau)}, \qquad V(\tau) = C_2 e^{av(\tau)} \tag{6.17}$$

We know the background space time is flat everywhere. We can make it look especially clear along the world line of the accelerated observer,

$$\begin{aligned} ds^2 &= -dU(\tau)dV(\tau) = a^2 C_1 C_2 du dv \\ &= -a^2 C_1 C_2 \Big[-(dq^0)^2 + (dq^1)^2 \Big] \end{aligned}$$

by setting $-a^2 C_1 C_2 = 1$ or $C_1 = -1/a$, $C_2 = 1/a$. Then

$$U = -\frac{1}{a}e^{-au}, \qquad V = \frac{1}{a}e^{av} \tag{6.18}$$

The metric (6.15) can be expressed by

$$ds^2 = -dUdV = -e^{a(v-u)}dudv = e^{2aq^1}\Big[-(dq^0)^2 + (dq^1)^2 \Big]$$

$$\begin{aligned} t(q^0, q^1) &= \frac{V+U}{2} = \frac{1}{2a}(e^{av} - e^{-au}) = \frac{1}{2a}\Big[e^{a(q^0+q^1)} - e^{-a(q^0-q^1)} \Big] \\ &= \frac{e^{aq^1}}{a}\frac{e^{aq^0} - e^{-aq^0}}{2} = \frac{e^{aq^1}}{a}\sinh aq^0 \end{aligned}$$

$$x(q^0, q^1) = \frac{V-U}{2} = \frac{e^{aq^1}}{a}\cosh aq^0$$

6.2 Unruh effect

The action of a massless scalar field in (1+1) dimension is

$$I(\phi) = -\frac{1}{2}\int \phi_{,\mu}\phi_{,\nu}g^{\mu\nu}\sqrt{-g}d^2x \tag{6.19}$$

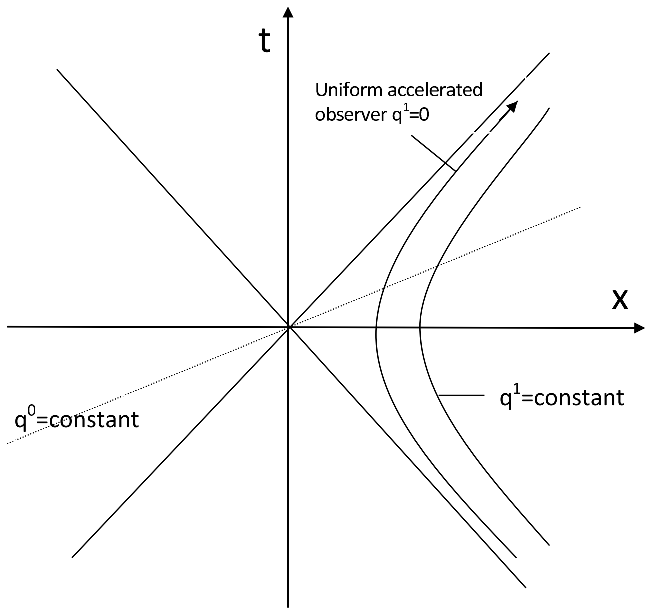

Figure 6.1: (q^0, q^1) coordinate system of uniform accelerated observer in Minkowski space time. $x^2 - t^2 = \frac{e^{2aq^1}}{a^2}$
.

In 1+1 dimensions (6.19) is conformally invariant. Thus

$$\begin{aligned} I(\phi) &= \frac{1}{2}\int\Big[(\partial_t\phi)^2-(\partial_x\phi)^2\Big]dxdt \\ &= \frac{1}{2}\int\Big[(\partial_{q^0}\phi)^2-(\partial_{q^1}\phi)^2\Big]dq^0dq^1 \end{aligned} \tag{6.20}$$

By (6.6)

$$I(\phi) = 2\int \partial_U\phi\partial_V\phi dUdV \tag{6.21}$$

$$= 2\int \partial_u\phi\partial_v\phi dudv \tag{6.22}$$

This yields the field equations

$$\partial_U\partial_V\phi = 0, \qquad \partial_u\partial_v\phi = 0 \tag{6.23}$$

The solution can be expressed by a linear combination of positive and negative mode functions in right and left motion. With a proper normalization factor, the mode expansion of the massless scalar field operator $\hat{\phi}$ is

$$\hat{\phi} = \int_0^\infty \frac{1}{\sqrt{2\pi}}\frac{d\Omega}{\sqrt{2\Omega}}(e^{-i\Omega U}\hat{a}^-_\Omega + e^{i\Omega U}\hat{a}^+_\Omega) + (\text{left moving modes}) \tag{6.24a}$$

$$\hat{\phi} = \int_0^\infty \frac{1}{\sqrt{2\pi}}\frac{d\omega}{\sqrt{2\omega}}(e^{-i\omega u}\hat{b}^-_\omega + e^{i\omega u}\hat{b}^+_\omega) + (\text{left moving modes}) \tag{6.24b}$$

where the $\hat{a}$ and $\hat{b}$ operators satisfy

$$[\hat{a}^-_\Omega, \hat{a}^+_{\Omega'}] = \delta(\Omega-\Omega'), \quad [\hat{b}^-_\omega, \hat{b}^+_{\omega'}] = \delta(\omega-\omega') \tag{6.25}$$

We define the Hartle Hawking vacuum by

$$\hat{a}^-_\Omega \mid 0\rangle_{HH} = 0 \tag{6.26}$$

and the Boulware vacuum by

$$\hat{b}^-_\omega \mid 0\rangle_B = 0 \tag{6.27}$$

Calculate ${}_{HH}\langle 0 \mid (\partial_U \hat{\phi})^2 \mid 0\rangle_{HH}$ for (6.24).

$$\hat{\phi} = \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{d\Omega}{\sqrt{2\Omega}} (e^{-i\Omega U} \hat{a}_\Omega^- + e^{i\Omega U} \hat{a}_\Omega^+) + (\text{left moving modes})$$

$$\partial_U \hat{\phi} = \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{d\Omega}{\sqrt{2\Omega}} (-i\Omega e^{-i\Omega U} \hat{a}_\Omega^- + i\Omega e^{i\Omega U} \hat{a}_\Omega^+) + (\text{left moving modes})$$

$$\begin{aligned}
{}_{HH}\langle 0 \mid (\partial_U \hat{\phi})^2 \mid 0\rangle_{HH} &= \frac{1}{2\pi} {}_{HH}\langle 0 \mid \int_0^\infty \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} \frac{d\Omega'}{\sqrt{2\Omega'}} (-i\Omega)(i\Omega') \\
&\quad \times e^{-i(\Omega-\Omega')U} \hat{a}_\Omega^- \hat{a}_{\Omega'}^+ \mid 0\rangle_{HH} \\
&= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \frac{d\Omega}{\sqrt{2\Omega}} \frac{d\Omega'}{\sqrt{2\Omega'}} \Omega\Omega' e^{-i(\Omega-\Omega')U} \delta(\Omega - \Omega') \\
&= \frac{1}{4\pi} \int_0^\infty d\Omega\Omega \qquad (6.28)
\end{aligned}$$

(6.28) is the vacuum expectation value of 'energy density − energy flux' of the scalar field. It diverges. But in ordinary quantum field theory we set its value to zero. By the same method we can show

$$ {}_B\langle 0 \mid (\partial_u \hat{\phi})^2 \mid 0\rangle_B = \frac{1}{4\pi} \int_0^\infty d\omega\omega \qquad (6.29)$$

Thus

$$ {}_{HH}\langle 0 \mid (\partial_U \hat{\phi})^2 \mid 0\rangle_{HH} = {}_B\langle 0 \mid (\partial_u \hat{\phi})^2 \mid 0\rangle_B \qquad (6.30)$$

$$\begin{aligned}
{}_B\langle 0 \mid \left(\frac{\partial \phi}{\partial U}\right)^2 \mid 0\rangle_B &= \frac{1}{a^2 U^2} {}_B\langle 0 \mid \left(\frac{\partial \phi}{\partial u}\right)^2 \mid 0\rangle_B \qquad (6.31) \\
&= \frac{1}{a^2 U^2} {}_{HH}\langle 0 \mid \left(\frac{\partial \phi}{\partial U}\right)^2 \mid 0\rangle_{HH} \qquad (6.32)
\end{aligned}$$

This shows that $(T_{00} - T_{01})$ observed by Hartle-Hawking observer ,$\left(\frac{\partial \phi}{\partial U}\right)^2$, in the Boulware vacuum state is infinite at the horizon $U = 0$. This non-fictitious quantity cannot be infinite. Thus Boulware vacuum state is not physically acceptable state. Hartle-Hawking observer will observe no particle in Hartle-Hawking ground state.

$$ {}_{HH}\langle 0 \mid \hat{a}_\Omega^+ \hat{a}_\Omega^- \mid 0\rangle_{HH} = 0$$

Could Boulware observer observe particles in Hartle-Hawking ground state?

$$N_\omega^{HH} = {}_{HH}\langle 0 \mid \hat{b}_\omega^+ \hat{b}_\omega^- \mid 0\rangle_{HH} = ? \tag{6.33}$$

From the general expansion of $\hat{\phi}$ in (6.24), we expect general form of Bogolyubov transformation

$$\hat{b}_\omega^- = \int_0^\infty d\Omega(\alpha_{\omega\Omega}\hat{a}_\Omega^- - \beta_{\omega\Omega}\hat{a}_\Omega^+) \tag{6.34}$$

If $\beta_{\omega\Omega} = 0$ then $N_\omega^{HH} = 0$. For uniform accelerated observer

$$|\beta_{\omega\Omega}|^2 = e^{-\frac{2\pi\omega}{a}}|\alpha_{\omega\Omega}|^2 \tag{6.35}$$

Let us derive (6.35). First of all put Bogolyubov transformation (6.34) into (6.24b). Then compare the coefficients of $\hat{a}_\Omega^-$ and $\hat{a}_\Omega^+$ in (6.24a). Then we get the relations between two mode functions $e^{\pm i\Omega U}$ and $e^{\pm i\omega u}$.

$$\begin{aligned}
\rightarrow \quad \frac{e^{-i\Omega U}}{\sqrt{2\Omega}} &= \int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}}(e^{-i\omega' u}\alpha_{\omega'\Omega} - e^{i\omega' u}\beta_{\omega'\Omega}^*) \\
\frac{e^{i\Omega U}}{\sqrt{2\Omega}} &= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}}(e^{i\omega u}\alpha_{\omega\Omega}^* - e^{-i\omega u}\beta_{\omega\Omega})
\end{aligned}$$

Integrating both sides by $\int du e^{i\omega u}$ results in a unique integral relation between $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$.

$$\begin{aligned}
\int du \frac{e^{-i\Omega U}}{\sqrt{2\Omega}} e^{i\omega u} &= \int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}} \int du e^{i(\omega-\omega')u}\alpha_{\omega'\Omega} - \int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}} \int du e^{i(\omega+\omega')u}\beta_{\omega'\Omega}^* \\
&= \int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}} 2\pi\delta(\omega-\omega')\alpha_{\omega'\Omega} = \frac{2\pi}{\sqrt{2\omega}}\alpha_{\omega\Omega}
\end{aligned}$$

$$\begin{aligned}
\int du \frac{e^{i\Omega U}}{\sqrt{2\Omega}} e^{i\omega u} &= \int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}} \int du(-e^{-i\omega' u + i\omega u})\beta_{\omega'\Omega} = -\int_0^\infty \frac{d\omega'}{\sqrt{2\omega'}} 2\pi\delta(\omega'-\omega)\beta_{\omega'\Omega} \\
&= -\frac{2\pi}{\sqrt{2\omega}}\beta_{\omega\Omega}
\end{aligned}$$

Apply $U = -\frac{1}{a}e^{-au}$ (6.18) to get the integral expression of $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ involving

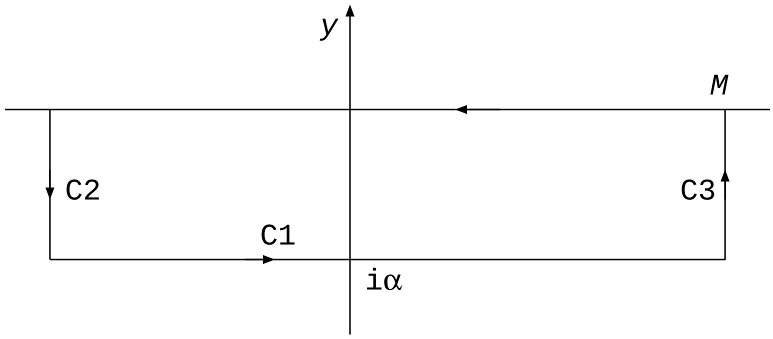

Figure 6.2: Contour path

only the u variable. Then define $F(\Omega,\omega)$ as follows,

$$\sqrt{\frac{\Omega}{\omega}}\alpha_{\omega\Omega} = \int_{-\infty}^{\infty}\frac{du}{2\pi}e^{-i\Omega U+i\omega u} = \int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\left(i\omega u+\frac{i\Omega}{a}e^{-au}\right) \equiv F(\Omega,\omega) \tag{6.36}$$

$$\sqrt{\frac{\Omega}{\omega}}\beta_{\omega\Omega} = -\int_{-\infty}^{\infty}\frac{du}{2\pi}e^{i\Omega U+i\omega u} = -\int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\left(i\omega u-\frac{i\Omega}{a}e^{-au}\right) \equiv -F(-\Omega,\omega) \tag{6.37}$$

The relation between $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ is equivalent to the relation between $F(\Omega,\omega)$ and $-F(-\Omega,\omega)$. Thus to get the relation between $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$, find the relation between $F(\Omega,\omega)$ and $-F(-\Omega,\omega)$.

$$F(\Omega,\omega) \equiv \int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\left(i\omega u+\frac{i\Omega}{a}e^{-au}\right) =?$$

Since the integrand of $F(\Omega,\omega)$ has no pole, we can use a contour integral technique. Take the contour path as figure 6.2.

$$\oint\frac{dz}{2\pi}\exp\left(i\omega z+\frac{i\Omega}{a}e^{-az}\right) = 0$$

Let us introduce convergence factor ϵ which is real and has a small positive value.

Then we can express $F(\Omega,\omega)$ by introducing a real and small constant ϵ,

$$\begin{aligned} F(\Omega,\omega) &\equiv \int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\Big(i\omega u+\frac{i\Omega}{a}e^{-au}\Big) \\ &= \int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\Big(-\epsilon u^2+i\omega u+\frac{i\Omega}{a}e^{-au}\Big) \end{aligned} \tag{6.38}$$

$$\oint\frac{dz}{2\pi}\exp\Big(-\epsilon z^2+i\omega z+\frac{i\Omega}{a}e^{-az}\Big)=0$$

$$\begin{aligned} &\int_{\infty}^{-\infty}\frac{du}{2\pi}\exp\Big(-\epsilon u^2+i\omega u+\frac{i\Omega}{a}e^{-au}\Big) \\ &+\int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\Big(-\epsilon(u-i\alpha)^2+i\omega(u-i\alpha)+\frac{i\Omega}{a}e^{-a(u-i\alpha)}\Big) \\ &+\underbrace{\lim_{M\to-\infty}\int_{0}^{-\alpha}\frac{idy}{2\pi}\exp\Big(-\epsilon(M+iy)^2+i\omega(M+iy)+\frac{i\Omega}{a}e^{-a(M+iy)}\Big)}_{C2} \\ &+\underbrace{\lim_{M\to\infty}\int_{-\alpha}^{0}\frac{idy}{2\pi}\exp\Big(-\epsilon(M+iy)^2+i\omega(M+iy)+\frac{i\Omega}{a}e^{-a(M+iy)}\Big)}_{C3}=0 \end{aligned}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\Big(-\epsilon u^2+i\omega u+\frac{i\Omega}{a}e^{-au}\Big) \\ &=\int_{-\infty}^{\infty}\frac{du}{2\pi}\exp\Big(-\epsilon u^2+i2\epsilon u\alpha+\epsilon\alpha^2+i\omega u+\omega\alpha+\frac{i\Omega}{a}e^{-au}e^{ia\alpha}\Big) \\ &+C2+C3 \end{aligned} \tag{6.39}$$

If we assume $C2$ and $C3$ vanish, the value of $F(\Omega,\omega)$ is the same with the integral along C1 to any value of α. In order for (6.39) to be a relation between $F(\Omega,\omega)$ and $F(-\Omega,\omega)$ with small ϵ, $e^{ia\alpha}$ must be -1. Namely $a\alpha=\cdots,-3\pi,-\pi,\pi,3\pi,\cdots$. Then

$$F(\Omega,\omega)=F(-\Omega,\omega)e^{\omega\alpha} \tag{6.40}$$

Let us see how $C2$ and $C3$ vanish.

$$\lim_{M\to\pm\infty}\int_0^{-\alpha}\frac{idy}{2\pi}\exp\Big(-\epsilon(M+iy)^2+i\omega(M+iy)+\frac{i\Omega}{a}e^{-a(M+iy)}\Big)$$
$$=\lim_{M\to\pm\infty}\int_0^{-\alpha}\frac{idy}{2\pi}\exp\Big(-\epsilon M^2-i2\epsilon My+\epsilon y^2+i\omega M-\omega y$$
$$+\frac{i\Omega}{a}e^{-aM}\cos ay+\frac{\Omega}{a}e^{-aM}\sin ay\Big)$$
$$=\lim_{M\to\pm\infty}\int_0^{-\alpha}\frac{idy}{2\pi}\exp\Big(-\epsilon M^2+\frac{\Omega}{a}e^{-aM}\sin ay\Big)$$
$$\times\exp\Big(i2\epsilon My+\epsilon y^2+i\omega M-\omega y+\frac{i\Omega}{a}e^{-aM}\cos ay\Big)$$

In order for this to vanish for $M\to-\infty$, the integral range ay must be between $-\pi$ and 0. This sets $a\alpha=\pi$. Then this integral also vanish for $M\to\infty$. Therefore $C2$ and $C3$ vanish and we get final result (6.40) if we set $a\alpha=\pi$.

From (6.36), (6.37) and (6.40) with $\alpha=\pi/a$ we get the desired relation between $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$

$$|\alpha_{\omega\Omega}|^2=|\beta_{\omega\Omega}|^2e^{\frac{2\pi\omega}{a}}$$

for a uniformly accelerated observer.

For a non accelerated observer $|\beta_{\omega\Omega}|=0$ results in $N_\omega^{HH}=0$ in (6.33). If $a\neq 0$ we expect $|\beta_{\omega\Omega}|\neq 0$, $N_\omega^{HH}\neq 0$. For that purpose derive

$$\int_0^\infty d\Omega(\alpha_{\omega\Omega}\alpha^*_{\omega'\Omega}-\beta_{\omega\Omega}\beta^*_{\omega'\Omega})=\delta(\omega-\omega') \tag{6.41}$$

using (6.25)

$$\begin{aligned}
\delta(\omega-\omega') &= [\hat{b}^-_\omega,\hat{b}^+_{\omega'}] \qquad (6.25)\\
&= \Big[\int_0^\infty d\Omega(\alpha_{\omega\Omega}\hat{a}^-_\Omega-\beta_{\omega\Omega}\hat{a}^+_\Omega),\int_0^\infty d\Omega'(\alpha^*_{\omega'\Omega'}\hat{a}^+_{\Omega'}-\beta^*_{\omega'\Omega'}\hat{a}^-_{\Omega'})\Big]\\
&= \int_0^\infty\int_0^\infty d\Omega d\Omega'\alpha_{\omega\Omega}\alpha^*_{\omega'\Omega'}[\hat{a}^-_\Omega,\hat{a}^+_{\Omega'}]+\int_0^\infty\int_0^\infty d\Omega d\Omega'\beta_{\omega\Omega}\beta^*_{\omega'\Omega'}[\hat{a}^+_\Omega,\hat{a}^-_{\Omega'}]\\
&= \int_0^\infty d\Omega(\alpha_{\omega\Omega}\alpha^*_{\omega'\Omega}-\beta_{\omega\Omega}\beta^*_{\omega'\Omega})
\end{aligned}$$

For $\omega=\omega'$

$$\int_0^\infty d\Omega(|\alpha_{\omega\Omega}|^2-|\beta_{\omega\Omega}|^2)=\delta(0) \tag{6.42}$$

$$\begin{aligned}
N_\omega^{HH} &= {}_{HH}\langle 0 \mid \hat{b}_\omega^{+}\hat{b}_\omega^{-} \mid 0\rangle_{HH} = {}_{HH}\langle 0 \mid \int_0^\infty d\Omega'(\alpha^*_{\omega\Omega'}\hat{a}^{+}_{\Omega'} - \beta^*_{\omega\Omega'}\hat{a}^{-}_{\Omega'}) \\
&\quad \times \int_0^\infty d\Omega(\alpha_{\omega\Omega}\hat{a}^{-}_{\Omega} - \beta_{\omega\Omega}\hat{a}^{+}_{\Omega}) \mid 0\rangle_{HH} \\
&= \int_0^\infty d\Omega|\beta_{\omega\Omega}|^2
\end{aligned} \tag{6.43}$$

Applying (6.35) to (6.42) gives

$$\int_0^\infty d\Omega\Big(e^{\frac{2\pi\omega}{a}}|\beta_{\omega\Omega}|^2 - |\beta_{\omega\Omega}|^2\Big) = \delta(0) \tag{6.44}$$

$$N_\omega^{HH} = \frac{\delta(0)}{e^{\frac{2\pi\omega}{a}} - 1} \tag{6.45}$$

here $\delta(0)$ represents the volume of the system. Thus the expectation value of number density n_ω^{HH} is

$$n_\omega^{HH} = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \tag{6.46}$$

(6.46) tells us that an accelerated observer in Hartle-Hawking (i.e., Minkowski) vacuum state observes a thermal distribution of particles with temperature corresponding $\beta = \frac{2\pi}{a}$. This is called the Unruh effect. The stress-energy associated with Unruh effect exerts no gravitational force, and in that sense it is "fictitious" like centrifugal force.

Chapter 7

Hawking radiation

7.1 Kruskal-Szekeres coordinates

The Schwarzschild solution for a spherical symmetric geometry is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{7.1}$$

where $f(r) = 1 - \frac{2M}{r}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We set $G = \hbar = c = 1$. For simplicity, consider the (1+1) dimensional Schwarzschild geometry

$$ds^2 = -f dt^2 + \frac{dr^2}{f} = -f\Big(dt - \frac{dr}{f}\Big)\Big(dt + \frac{dr}{f}\Big) \tag{7.2}$$

Define coordinates (u, v) and (U, V) for an arbitrary constant κ by

$$du \equiv -\frac{1}{\kappa} d\ln U \equiv dt - \frac{dr}{f} \tag{7.3a}$$

$$dv \equiv \frac{1}{\kappa} d\ln V \equiv dt + \frac{dr}{f} \tag{7.3b}$$

$$\begin{aligned}\int \frac{dr}{f} &= \int \frac{dr}{1 - \frac{r_0}{r}} = \int \Big(1 + \frac{r_0}{r - r_0}\Big) dr && (7.4)\\ &= r + r_0 \ln(r - r_0) + C && (7.5)\end{aligned}$$

From (7.3)

$$-\frac{1}{\kappa}\ln U = t - r - r_0 \ln(r - r_0) + C$$

$$U = C_1 e^{-\kappa(t-r)}(r-r_0)^{\kappa r_0} \tag{7.6}$$

$$\frac{1}{\kappa}\ln V = t + r + r_0\ln(r-r_0) + C$$

$$V = C_2 e^{\kappa(t+r)}(r-r_0)^{\kappa r_0} \tag{7.7}$$

$$ds^2 = -f\Big(-\frac{1}{\kappa}\Big)d\ln U\Big(\frac{1}{\kappa}\Big)d\ln V = \frac{f}{\kappa^2 UV}dUdV \tag{7.8}$$

$$= \frac{\frac{r-r_0}{r}}{\kappa^2 C_{12}e^{2\kappa r}(r-r_0)^{2\kappa r_0}}dUdV$$

$$= \frac{1}{C_{12}\kappa^2 r}(r-r_0)^{1-2\kappa r_0}e^{-2\kappa r}dUdV \tag{7.9}$$

To avoid the singularity set $\kappa = \frac{1}{2r_0}$. Then

$$ds^2 = \frac{1}{C_{12}\kappa^2 r}e^{-\frac{r}{r_0}}dUdV \tag{7.10}$$

This is the Kruskal-Szekeres metric. On the other hand, from the differential equations between (U, V) and (u, v)

$$U = -\frac{1}{\kappa}e^{-\kappa u}, \qquad V = \frac{1}{\kappa}e^{\kappa v} \tag{7.11}$$

The energy observed by (u, v) observer is the same energy observed by (r, t) observer. From (7.3)

$$\frac{V}{U} = e^{2\kappa t} \tag{7.12}$$

From (7.12) we can read that $V = 0$ corresponds to the past horizon and $U = 0$ corresponds to the future horizon and $t =$ constant corresponds to $V =$ constant $\cdot\, U$.

Define a timelike coordinate T and space like coordinate R as

$$U = T - R, \qquad V = T + R \tag{7.13}$$

It says that a null geodesic is $T = \pm R$ as shown in figure 7.1. From (7.3) $r =$ constant corresponds to $UV =$ constant. Curves $r =$constant for $r > r_0$ sit on the left and right hand sectors in figure 7.1. There is a physical singularity at $r = 0$ and the space time cannot be extended beyond this singularity.

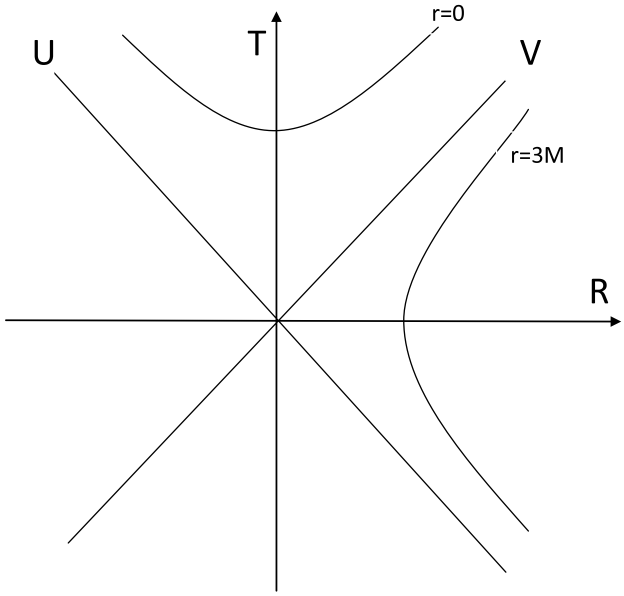

Figure 7.1: Kruskal-Szekeres coordinates

7.2 Hawking radiation

Let us apply quantization to the massless scalar field in a Schwarzschild geometry. We assume a classical background which has no or negligible back reaction. The spherical symmetric metric in Kruskal-Szekeres coordinates truncates in 2 dimensions to

$$ds^2 = -f(r)dudv = \frac{e^{-\frac{r}{r_0}}}{C_{12}\kappa^2 r}dUdV \tag{7.14}$$

The action for a massless scalar field is

$$I(\phi) = -\frac{1}{2}\int \phi_{,\mu}\phi_{,\nu}g^{\mu\nu}\sqrt{-g}d^2x \tag{6.19}$$

In Kruskal-Szekeres coordinates (U, V),

$$I(\phi) = \int \phi_{,U}\phi_{,V}dUdV \tag{7.15}$$

$$I(\phi) = \int \phi_{,u}\phi_{,v}dudv \tag{7.16}$$

comparing between (6.21), (6.22) and (7.15), (7.16), we see that the relation between (U, V) and (u, v) is exactly the same as before. Therefore the mode expansions of $\hat{\phi}(x)$ and quantization conditions are the same as before (6.24a), (6.24b) and (6.25), where (u, v) and (U, V) are related to (r, t) by (7.3). (7.3) implies that the energy measured by Schwarzschild observer at infinity is the same as the energy measured by (u, v) Boulware observer but different to the energy measured by (U, V) Hartle-Hawking observer. Thus we expect that the Hartle-Hawking vacuum measured by Boulware observer or Schwarzschild observer would be non-vacuum. If the Boulware vacuum has quantum fluctuation of infinite energy density at the horizon, this back reaction makes it hard to apply quantum field theory for an unperturbed background geometry.

Calculate the average number of particles in Hartle-Hawking vacuum measured by the Boulware observer or Schwarzschild observer.

$$N_\omega^{HH} = {}_{HH}\langle 0 \mid \hat{b}_\omega^+ \hat{b}_\omega^- \mid 0\rangle_{HH} =? \tag{7.17}$$

As before, the general form of Bogolyubov transformation can be written as

$$\hat{b}_{\omega}^{-} = \int_{0}^{\infty} d\Omega(\alpha_{\omega\Omega}\hat{a}_{\Omega}^{-} - \beta_{\omega\Omega}\hat{a}_{\Omega}^{+}) \qquad (6.34)$$

$$|\beta_{\omega\Omega}|^2 = e^{-\frac{2\pi\omega}{a}}|\alpha_{\omega\Omega}|^2 \qquad (6.35)$$

was derived by applying (6.18) to (6.34) and mode expansions (6.24a), (6.24b). We can also derive the similar result as (6.35) by applying (7.11) to (6.34) and the same mode expansions (6.24a), (6.24b).

The resulting relation between $\alpha_{\omega\Omega}$ and $\beta_{\omega\Omega}$ must be

$$|\beta_{\omega\Omega}|^2 = e^{-\frac{2\pi\omega}{\kappa}}|\alpha_{\omega\Omega}|^2 \tag{7.18}$$

Hence the calculation of N_{ω}^{HH} in Schwarzschild geometry would be (6.45), (6.46) replacing a by κ.

$$n_{\omega}^{HH} = \frac{1}{e^{\frac{2\pi}{\kappa}\omega} - 1} \tag{7.19}$$

Therefore the average number of particles in the Hartle-Hawking vacuum measured by Boulware observer or Schwarzschild observer shows a thermal distribution of particles with temperature corresponding to surface gravity κ. This temperature $\frac{\kappa}{2\pi}$ is called the Hawking temperature and this radiation is called Hawking radiation. We showed this only for outgoing (right-moving) modes.

Chapter 8

Tunneling probability formula

In quantum mechanics the term tunneling means the quantum mechanical penetration of a system though a barrier which is not possible classically. Most of the papers[2][3][4] explaining Hawking radiation as a tunneling effect use the famous tunneling formula

$$P \sim \exp\Big(-\frac{2}{\hbar}\mathrm{Im}\ W\Big)$$

where W is the Jacobi action. Thus it would be instructive to know the derivation of this formula.

Fourier pair of wave function $\psi(x_2, t_2)$ is

$$\begin{aligned}\psi(x_2, t_2) &= \int e^{-\frac{i}{\hbar}E_2 t_2}\phi(x_2, E_2)\frac{dE_2}{2\pi\hbar} \\ \phi(x_2, E_2) &= \int e^{\frac{i}{\hbar}E_2 t_2}\psi(x_2, t_2)dt_2\end{aligned}$$

Using Fourier pair and Feynman kernel $K(x_2, E_2; x_1, E_1)$, we get

$$\phi(x_2, E_2) = \int e^{\frac{i}{\hbar}E_2 t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(x_2, t_2; x_1, t_1)\int e^{-\frac{i}{\hbar}E_1 t}\phi(E_1, x)\frac{dE_1}{2\pi\hbar}dx_1 dt_1 dt_2 \quad (8.1)$$

$$= \int\int\int_{t_1}^{\infty} e^{\frac{i}{\hbar}E_2 t_2}K(x_2, t_2; x_1, t_1)e^{-\frac{i}{\hbar}E_1 t_1}dt_2 dt_1\phi(E_1, x_1)\frac{dE_1}{2\pi\hbar}dx_1 \quad (8.2)$$

(8.2) implies that the Feynman kernel for given initial and final energy is

$$K(x_2, E_2; x_1, E_1) = \int_{-\infty}^{\infty}\int_{t_1}^{\infty} e^{\frac{i}{\hbar}E_2 t_2}K(x_2, t_2; x_1, t_1)e^{-\frac{i}{\hbar}E_1 t_1}dt_2 dt_1 \quad (8.3)$$

We can easily find that for a time-independent Hamiltonian, the integral of t_1

results in energy conservation.

$$\begin{aligned}
K(x_2,E_2;x_1,E_1) &= \int\int e^{\frac{i}{\hbar}E_2t_2}K(x_2,t_2;x_1,t_1)e^{-\frac{i}{\hbar}E_1t_1}dt_2dt_1 \\
&= \int_{-\infty}^{\infty}\int_{t_1}^{\infty} e^{-\frac{i}{\hbar}E_2t_2}\sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-\frac{i}{\hbar}E_n(t_2-t_1)}e^{-\frac{i}{\hbar}E_1t_1}dt_2dt_1
\end{aligned}$$

For given t_1 define $t_2' \equiv t_2 - t_1$. Then

$$\begin{aligned}
K(x_2,E_2;x_1,E_1) &= \int_{-\infty}^{\infty}\int_{0}^{\infty} e^{\frac{i}{\hbar}E_2(t_2'+t_1)}\sum_n \phi_n(x_2)\phi_n^*(x_1)e^{-\frac{i}{\hbar}E_nt_2'}e^{-\frac{i}{\hbar}E_1t}dt_2'dt_1 \\
&= \sum_n\int_0^{\infty} e^{\frac{i}{\hbar}E_2t_2'}e^{-\frac{i}{\hbar}E_nt_2'}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}(E_2-E_1)t_1}dt_1dt_2'\phi_n(x_2)\phi_n^*(x_1) \\
&= 2\pi\hbar\delta(E_2-E_1)\sum_n\int_0^{\infty} e^{\frac{i}{\hbar}(E_2-E_n)t_2'}dt_2'\phi_n(x_2)\phi_n^*(x_1) \qquad (8.4)
\end{aligned}$$

(8.4) explicitly shows the conservation of energy for a time-independent Hamiltonian. We can continue from (8.4) to get the Jacobi action expression of $K(x_2,E_2;x_1,E_1)$. Bring (8.4) back to the expression of t_2.

$$\begin{aligned}
K(x_2,E_2;x_1,E_1) &= 2\pi\hbar\delta(E_2-E_1)\sum_n\int_{t_1}^{\infty} e^{\frac{i}{\hbar}(E_2-E_n)(t_2-t_1)}dt_2\phi_n(x_2)\phi_n^*(x_1) \\
&= 2\pi\hbar\delta(E_2-E_1)\sum_n\int_{t_1}^{\infty} e^{\frac{i}{\hbar}E_2(t_2-t_1)}e^{-\frac{i}{\hbar}E_n(t_2-t_1)}dt_2\phi_n(x_2)\phi_n^*(x_1) \\
&= 2\pi\hbar\delta(E_2-E_1)\int_{t_1}^{\infty} e^{\frac{i}{\hbar}E_2(t_2-t_1)}\underbrace{\sum_n e^{-\frac{i}{\hbar}E_n(t_2-t_1)}\phi_n(x_2)\phi_n^*(x_1)}_{\langle x_2,t_2|x_1,t_1\rangle}\,dt_2 \\
&= 2\pi\hbar\delta(E_2-E_1)\int_{t_1}^{\infty} e^{\frac{i}{\hbar}E_2(t_2-t_1)}\langle x_2,t_2 \mid x_1,t_1\rangle dt_2
\end{aligned}$$

We can set $t_1 = 0$. Then,

$$\begin{aligned}
K(x_2,E_2;x_1,E_1) &= 2\pi\hbar\delta(E_2-E_1)\int_0^{\infty} e^{\frac{i}{\hbar}E_2t_2}\langle x_2,t_2 \mid x_1,t_1=0\rangle dt_2 \qquad (8.5) \\
&= 2\pi\hbar\delta(E_2-E_1)\int_0^{\infty}\sum_{\{x(t)\}} e^{\frac{i}{\hbar}\int_{t_1}^{t_2}E_1dt}e^{\frac{i}{\hbar}\int_{t_1}^{t_2}Ldt}dt_2 \qquad (8.6)
\end{aligned}$$

where $\{x(t)\}$ represents the collection of all possible paths. From

$$\int_{t_1}^{t_2}(E_1+L)dt=\int_{t_1}^{t_2}p\dot{x}dt=\int_{x_1}^{x_2}pdx=W|_{x_1}^{x_2}$$

where $p\equiv\frac{\partial L}{\partial\dot{x}}$ and W represent the generalized momentum and the Jacobi action respectively, we get

$$K(x_2,E_2;x_1,E_1)=2\pi\hbar\delta(E_2-E_1)\int_0^\infty\sum_{\{x(t)\}}e^{\frac{i}{\hbar}W|_{x_1}^{x_2}}dt_2 \tag{8.7}$$

For a classical object($W\gg\hbar$) or for WKB approximation [7]

$$\sum_{\{x(t)\}}e^{\frac{i}{\hbar}W|_{x_1}^{x_2}}\sim e^{\frac{i}{\hbar}W[x_{cl}(t)]_{x_1}^{x_2}} \tag{8.8}$$

where $x_{cl}(t)$ represents the classical path satisfying the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial\dot{x}}\Big)-\frac{\partial L}{\partial x}=0 \tag{8.9}$$

Thus

$$K(x_2,E;x_1,E)\sim e^{\frac{i}{\hbar}W[x_{cl}(t)]_{x_1}^{x_2}} \tag{8.10}$$

For a tunneling particle

$$K(x_2,E;x_1,E)\sim e^{-\frac{1}{\hbar}\mathrm{Im}\ W[x_{cl}(t)]_{x_1}^{x_2}} \tag{8.11}$$

Hence the tunneling probability P is

$$\begin{aligned}P &= K^*(x_2,E;x_1,E)K(x_2,E;x_1,E)\\ &\sim \exp\Big(-\frac{2}{\hbar}\mathrm{Im}\ W[x_{cl}(t)]_{x_1}^{x_2}\Big)\end{aligned} \tag{8.12}$$

For real time t, a tunneling particle does not have a path $x_{cl}(t)$ to satisfy (8.9). In other words, a path-integral evaluation of the transition amplitude is hampered by the fact that the sum-over-paths is not dominated by any single(real) path. In this case, the classical path $x_{cl}(t)$ means the path satisfying (8.9) through imaginary time. In the resulting Euclidean-signature spacetime a classical tunneling route often exists and dominates the sum-over-paths. The procedure is analogous to the steepest-descent method for evaluating a real definite integral by diverting the integration contour through a saddle point in the complex plane. (cf [9]) In our paper, as this formula implies, we are going to take tunneling across the horizon as the analytic continuation of imaginary time path.

Chapter 9

Pair creation in a uniform electric field revisited

In black hole tunneling picture suggested by Parikh and Wilczek [2] does not have a *finite width* potential barrier for tunneling. In that sense black hole tunneling looks nothing to do with pair creation picture of electron and positron by strong electric field. In this chapter we will show how we can derive the pair creation formula of charged particles by a strong electric field using the tunneling formula (8.12) by an analytic continuation technique. Later on we will apply this technique to the black hole tunneling situation.

Figure 9.1 illustrates the two kinds of interpretation of Schwinger mechanism of pair creation by strong electric field. The direction of the electric field is from negative x to positive x. An electron is accelerated by the electric field with acceleration $a = \frac{eE}{m}$. Figure 9.1(a) represents the two particle (+e,−e) interpretation for pair creation. An electron traveling from left to right is decelerated by the electric field and finally stops at some point and accelerates in the opposite direction. A positron traveling from right to left also shows a similar pattern. The position of the turning point depends on the energy of the particle. Tunneling takes place more probably where the tunneling particle approaches the minimum energy state. This mechanism corresponds to figure 9.2(a) with the sense of segment AB reversed so as to be forward in time from B to A. Or this corresponds to figure 9.2(a) with AB flipped and the arrow goes forward in time. This cannot be the tunneling picture. Thus we can see that tunneling point is $\pm 1/a$.

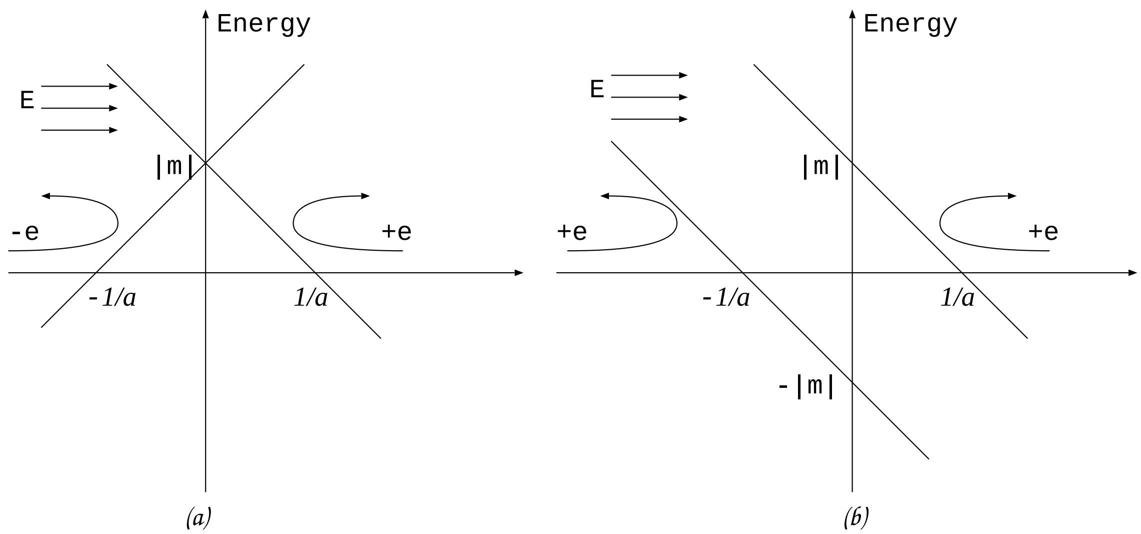

Figure 9.1: (a) Two particle (+e,−e) interpretation. (b) One particle(+e) interpretation.

Figure 9.1(b) represents the one-particle (+e) interpretation for the pair creation mechanism. Again +e traveling from right to left decelerated by the electric field. +e on the left side traveling from left to right also decelerated by the electric field. This picture corresponds to figure 9.2(a). Figure 9.1(b) mechanism corresponds to tunneling mechanism. In quantum field theory vacuum is the state in which pair creation and annihilation are continuous. If we supply strong enough electric field to this vacuum, the pair created particles will not reunite to annihilate each other. As a result we get two real particles. It can be interpreted that the virtual particle e increase its effective mass from $-m$ to $+m$ by extracting energy $2m$ from the strong electric field during the tunneling.

The resulting tunneling probability discovered by Schwinger up to leading term is [6]

$$P \sim e^{-\frac{\pi m^2}{eE}}$$

In this chapter we are going to derive the same results by the analytic continuation of imaginary time path of tunneling particle.

A positron e under uniform electric field E, experiences the acceleration $a = \frac{eE}{m}$. A particle experiencing uniform acceleration satisfies the following equation of motion. See chapter 6.1.

$$x^2 - t^2 = \frac{1}{a^2} \tag{9.1}$$

Employ the parameter τ defined by

$$ax = \cosh a\tau \tag{9.2a}$$

$$at = \sinh a\tau \tag{9.2b}$$

This shows that τ is the proper time of the uniformly accelerating particle.

Let's express the motion of this particle in figure 9.2 by the change of τ.
Along CD:
In order to satisfy

$$(x, t) = \left(\frac{1}{a}, 0\right) \text{ and } (x, t) = (\infty, \infty)$$

set

$$a\tau : 0 \to \infty$$

Along AB:

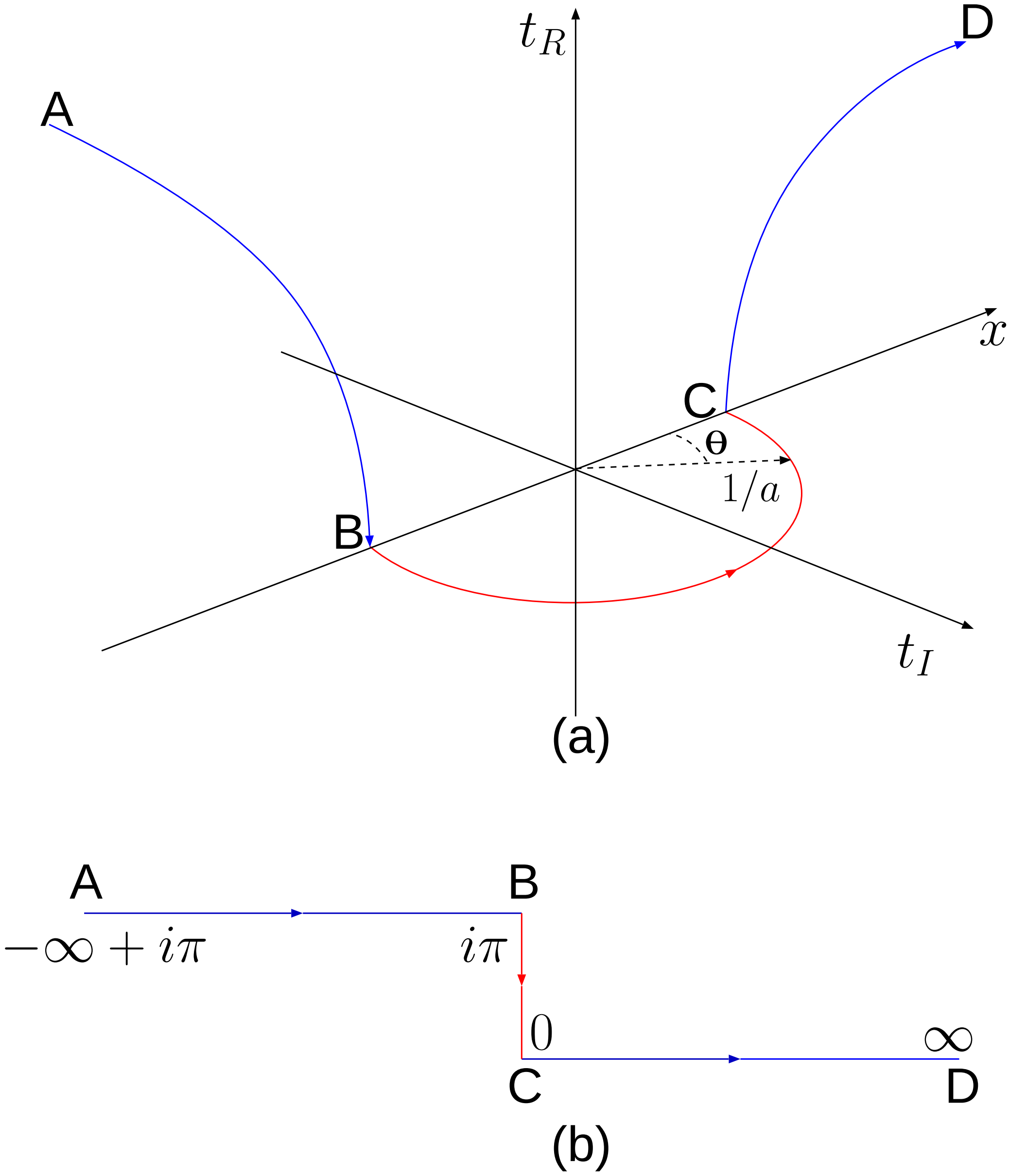

Figure 9.2: (a) Electron pair creation as tunneling. (b) March of $a\tau$

In order to satisfy

$$(x,t) = (-\infty, \infty) \text{ and } (x,t) = \Big(-\frac{1}{a}, 0\Big)$$

There is no real value of $a\tau$ to satisfy this requirement. We need complex value of $a\tau$ for that. Set

$$a\tau : -\infty \pm i\pi \to \pm i\pi$$

Or in order to satisfy

$$(x,t) = (-\infty, -\infty) \text{ and } (x,t) = \Big(-\frac{1}{a}, 0\Big)$$

set

$$a\tau : \infty \pm i\pi \to \pm i\pi$$

Along BC:
In order to satisfy

$$(x,t) = \Big(-\frac{1}{a}, 0\Big) \text{ and } (x,t) = \Big(\frac{1}{a}, 0\Big)$$

set

$$a\tau : \pm i\pi \to 0$$

This change of $a\tau$ results in the analytic continuation of time t via the imaginary time axis. The line BC in figure 9.2 shows these features clearly. Integration along this imaginary time coordinate results in the imaginary Jacobi action Im W which is needed to calculate tunneling probability (8.12).

Let's calculate the energy of the tunneling particle

$$\begin{aligned}
H &\equiv p_i \frac{dx^i}{dt} - L = m\Big(\frac{dx}{d\tau}\Big)^2 \frac{d\tau}{dt} - \Big(-m\frac{d\tau}{dt} + eA_t\frac{dx^t}{dt}\Big) \\
&= m\Big(\frac{dx}{d\tau}\Big)^2 \frac{d\tau}{dt} + m\frac{d\tau}{dt} - eEx \\
&= m\sinh^2 a\tau \frac{1}{\cosh a\tau} + m\frac{1}{\cosh a\tau} - m\cosh a\tau \\
&= m\frac{\sinh^2 a\tau + 1 - \cosh^2 a\tau}{\cosh^2 a\tau} = 0
\end{aligned} \tag{9.3}$$

where electric potential of uniform electric field E is $A_t = -\varphi = Ex$

Then from the definition of the Hamiltonian, Jacobi action W can be expressed by

$$\begin{aligned} W &\equiv \int p_i dx^i = \int L dt = \int \Big(-m\frac{d\tau}{dt} + eA_t \frac{dx^t}{dt}\Big) dt \qquad (9.4) \\ &= \int \Big(-m + eEx\frac{dt}{d\tau}\Big) d\tau = \int \Big(-m + max\frac{dt}{d\tau}\Big) d\tau \\ &= m\int (-1 + \cosh a\tau \cosh a\tau) d\tau = \frac{m}{a}\int_{BC} \sinh^2 a\tau d(a\tau) \\ &= \frac{m}{a}\int_{\pm i\pi}^{0} \sinh^2 a\tau d(a\tau) = \pm\frac{i\pi m}{2a} \end{aligned}$$

Tunneling probability P of (8.12) is

$$P = e^{-2\mathrm{Im}\, W} = e^{\mp\frac{\pi m}{a}} \qquad (9.5)$$

Because zero electric field should accompany with zero tunneling probability, we take only negative sign in the exponential. This is the same result as Swinger's pair creation probability function[6]. We could calculate it in much simpler way using tunneling formula (8.12) via analytic continuation through imaginary time. Here we used the fact that the energy of the tunneling particle is zero and the definition of the Hamiltonian. As a result we could integrate Lagrangian instead of calculating Jacobi action directly. In fact we could have done this calculation in an easier way by calculating the Jacobi action directly and applying the tunneling formula (8.12).

In a different Lorentz frame with relative velocity to our frame, electric field is not static and introduction of a magnetic field makes the analysis complicated. So in figure 9.2(a) we don't change to different Lorentz frame.

Our treatment of gravitational tunneling in the following section will closely follow these lines, but with one key difference. In the electric tunneling phase BC, work done on the rest mass by the field reversed its sign from $-m$ to $+m$, or (equivalently) reversed the sense of proper time τ. The effect was a sign reversal of the inertial term $-\int m d\tau$ in the action (9.4), which was implemented by complexifying τ. In contrast, for a particle moving under gravity the gravitational force vanishes in its rest-frame by the equivalence principle and cannot affect its rest mass. Thus the inertial action and proper time remain real during the tunneling phase. With this lesson in mind, we can apply this analytic continuation technique to calculate tunneling probability of black hole radiation.

Before we calculate tunneling probability using the analytic continuation technique, let me introduce the calculation by Parikh using Painlevè-Gullstrand coordinates.

Chapter 10

Tunneling in Painlevè-Gullstrand coordinates

Painlevè-Gullstrand coordinates are defined by a coordinate transformation $t \rightarrow t+b(r)$ in Schwarzschild coordinates, where $b(r)$ is a function to be found by demanding that constant time slices be flat. The resulting metric is

$$ds^2 = -\Big(1 - \frac{2M}{r}\Big)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2 d\Omega^2 \tag{10.1}$$

This metric (and its inverse) has no singularity at the horizon and $dt = 0$ slices are just ordinary Euclidean flat space.

For emission of a small pulse of energy ε

$$\text{Im } W(\varepsilon) = \text{Im} \int \varepsilon dt = \varepsilon \text{Im} \int dt \tag{10.2}$$

where t is Painlevè time and ε is the energy corresponding to that.

For an out-going light-like particle

$$\text{Im } W(\varepsilon) = \varepsilon \text{Im} \int_{r_1}^{r_2} \frac{dr}{1 - \sqrt{\frac{2M}{r}}} \tag{10.3}$$

Using

$$h(r) = h(r_0) + h'(r_0)(r - r_0) + \frac{h''(r_0)}{2!}(r - r_0)^2 + \cdots$$

near horizon

$$1-\sqrt{\frac{2M}{r}} \approx \frac{1}{2r_0}(r-r_0) = \kappa(r-r_0)$$

where the surface gravity κ is defined by $\kappa \equiv \frac{1}{2}f'(r_0) = \frac{1}{2r_0}$. r_1 and r_2 near horizon can be written as

$$r_1 = r_0 + \epsilon, \qquad r_2 = r_0 - \epsilon$$

or

$$r = r_0 + \epsilon e^{i\theta} \qquad \text{with } \theta_1 = 0, \quad \theta_2 = \pi \tag{10.4}$$

Thus tunneling near horizon $r_2 < r < r_1$ would give

$$\begin{aligned}
\text{Im } W(\varepsilon) &\approx \varepsilon \text{Im} \int_{r_1}^{r_2} \frac{dr}{\kappa(r-r_0)} && (10.5)\\
&= \frac{\varepsilon}{\kappa}\text{Im } \ln(r-r_0)\Big|_{r_1}^{r_2} = \frac{\varepsilon}{\kappa}\text{Im } \ln e^{i\theta}\Big|_0^{\pi} && (10.6)\\
&= \frac{\varepsilon}{\kappa}\pi = 4\pi M\varepsilon = \frac{\varepsilon}{2}\frac{1}{T_2} && (10.7)
\end{aligned}$$

where $T_2 = \frac{1}{8\pi M}$ is the Hawking temperature for Schwarzschild mass M. If we integrate this up to a substantial amount of energy E,

$$\begin{aligned}
\text{Im } W(E) &= 4\pi(M-\varepsilon)\varepsilon + 4\pi(M-2\varepsilon)\varepsilon + 4\pi(M-3\varepsilon)\varepsilon + \cdots && (10.8)\\
&= 4\pi M\varepsilon N - \sum_{j=1}^{N} 4\pi j\varepsilon\varepsilon = 4\pi M\varepsilon N - 4\pi\frac{(N+1)N}{2}\varepsilon^2
\end{aligned}$$

For $N \to \infty$,

$$\begin{aligned}
\text{Im } W(E) &= 4\pi M\varepsilon N - 4\pi\frac{N^2\varepsilon^2}{2} = 4\pi ME - 4\pi\frac{E^2}{2}\\
&= 4\pi ME\Big(1-\frac{E}{2M}\Big) && (10.9)\\
&= \frac{\pi}{2}E\Big(4(M-E)+4M\Big) = \frac{\pi}{2}E\Big(\frac{1}{2\pi T_-}+\frac{1}{2\pi T_+}\Big)\\
&= \frac{E}{4}\Big(\frac{1}{T_-}+\frac{1}{T_+}\Big) && (10.10)\\
&= \frac{E}{2}\overline{\Big(\frac{1}{T}\Big)} && (10.11)
\end{aligned}$$

where T_+ means the black hole temperature before radiation and T_- means the black hole temperature after radiation E; Hence the tunneling probability $P(E)$ of radiation with energy E is by (8.12)

$$P(E) = e^{-2\mathrm{Im}\ W(E)} = e^{-8\pi ME(1-E/2M)} = e^{-E\overline{1/T}} \tag{10.12}$$

Related physical arguments about this result will be discussed in chapter 12.

Chapter 11

Tunneling out of a spherical black hole.

In section 7.1 we introduced Kruskal-Szekeres coordinates for a spherical geometry. Because of the conformal relation between (U, V) and (u, v) coordinates, we could get the same form of basic mode functions of each observer. The key step to get Im W via the analytic continuation along the horizon is to get the proper coordinate which enable us to bypass the singularity at the horizon.

In this chapter we will use the Kruskal-Szekeres coordinates and follow the analytic continuation along t to calculate Im W.

11.1 L→R tunneling in an eternal black hole.

In chapter 9 we studied the pair creation of charged particles in the presence of a strong electric field in terms of the tunneling picture. The tunneling particle has zero total energy as calculated in (9.3). The most probable tunneling route was to a configuration in which the emerging particle has zero kinetic energy as illustrated in figure 9.2. The mechanism is the same for eternal black hole tunneling picture in figure 11.1. That is, we treat transitions between static or momentarily static configurations on opposite sides of the horizon. A virtual particle (world line AB in Figure 11.1) starts from rest in sector L, enters a tunneling mode at B, then circulates in complex time around the semi-circle to C, whence it emerges in sector R as a real static particle. Meanwhile, the hole's mass and charge have been reduced, and the

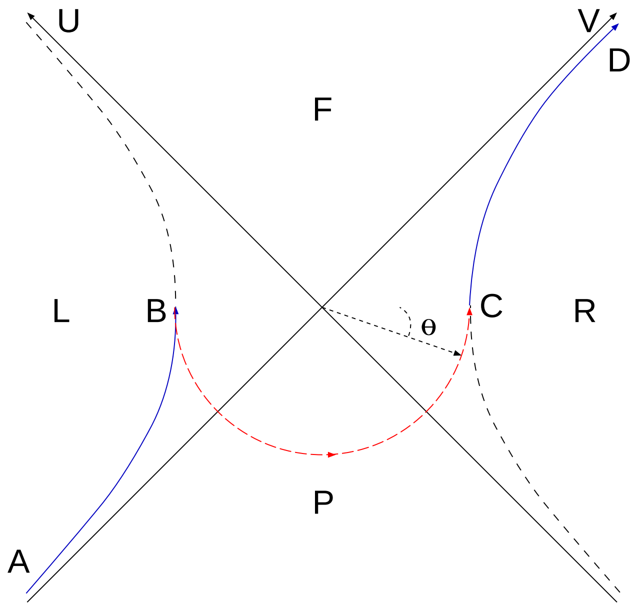

Figure 11.1: L $\rightarrow$ R tunneling in an eternal black hole

surface gravity κ correspondingly increased, by loss of this particle - from κ_L in sector L before the tunneling, to κ_R in sector R after the particle has escaped. Thus the point B represents a bigger Schwarzschild radius than that of point C in 11.1. The background geometry also changes from that due to the old mass M to the new mass $M - \varepsilon$. (The precise way this change occurs does not affect the result.)

This picture closely resembles the Schwinger tunneling picture of Figure 9.2.

Eternal black holes are not formed by real astrophysical collapse. Hence the initial tunneling state in sector L of an eternal black hole has no counterpart in real astrophysical collapse. So this tunneling model is further removed from reality than the picture of direct tunneling from F to R that we shall consider in the next chapter. However, it can be formulated completely and precisely, and most importantly allows effects of back-reaction to be easily accommodated. The extra stretch of the tunneling route from L into F route to R adds no imaginary part to the action, since falling from L into F is classically allowable.

The spherical geometries that we consider in this chapter have metrics of the general form

$$ds^2 = \frac{dr^2}{f(r)} + r^2 d\Omega^2 - f(r)dt^2$$

This includes Schwarzschild and Reissner-Nordström black holes imbedded in flat, de Sitter or AdS backgrounds. At the horizon $f(r_0) = 0$ and the surface gravity is $\kappa = \frac{1}{2}f'(r_0)$ (assumed non vanishing).

Advanced and retarded Eddington-Finkelstein coordinates u, v and Kruskal-Szekeres coordinates U, V are defined by

$$\begin{cases} du = -\frac{dU}{\kappa U} = dt - \frac{dr}{f(r)} \\ dv = \frac{dV}{\kappa V} = dt + \frac{dr}{f(r)} \end{cases} \tag{11.1}$$

$$2dt = \frac{1}{\kappa} d(\ln V - \ln U)$$

The two coordinate systems are related by

$$U = -\frac{1}{\kappa} e^{-\kappa u}, \qquad V = \frac{1}{\kappa} e^{\kappa v}$$

In theses coordinates, the metric is

$$ds^2 = \frac{f(r)}{\kappa^2 UV} dU dV + r^2(U, V) d\Omega^2$$

and is manifestly regular for all points(U,V) where $f(r)$ is regular, including the two horizon sheets $U=0$ and $V=0$.

Schwarzschild time t is initially defined only over the R-sector of the extended Kruskal manifold of Figure 11.1. It will be useful to extend it analytically to the full space in such a way that $e^{-i\omega t}$ is positive-frequency with respect to the globally regular times U and V. This requires that $e^{-i\omega t}$ be regular and bounded in the lower halves of the complex U and V planes. A definition which achieves this is

$$\int d\ln V = \int d\ln(|V|e^{i\theta}) = \ln|V| + i\theta + C = \ln|V| + \frac{i\pi}{2}\epsilon(V) \tag{11.2}$$

where $\epsilon(V)$ is the sign projection function.

$$\begin{aligned}\int dt &= \frac{1}{2\kappa}\Big[\Big(\ln|V| + \frac{i\pi}{2}\epsilon(V)\Big) - \Big(\ln|U| + \frac{i\pi}{2}\epsilon(U)\Big)\Big] && (11.3)\\ &= \frac{1}{2\kappa}\ln\left|\frac{V}{U}\right| + \frac{i\pi}{2\kappa}\epsilon\begin{cases}\epsilon=-1 & \text{for LH}\\ \epsilon=+1 & \text{for RH}\end{cases} && (11.4)\end{aligned}$$

in which we have selected that branch of $\ln V$ which is regular on the lower-half V-plane, and real on the lower imaginary axis, so that its values, e.g for real V,

$$\ln V \equiv \ln|V| + \frac{i\pi}{2}\epsilon(V) \tag{11.5}$$

are left-right symmetric. (The imaginary constant thereby added to t in (11.2) does not affect the time-differences we shall be concerned with.) $\ln V$ is discontinuous along the real axis of V at the origin.

In an s-wave approximation the tunneling (charged) particle is modeled as a spherical shell. According to (B.43) in Appendix, the appropriate(Jacobi) action is

$$W = -\int M d\tau + \int E d\bar{t} - \int \overline{q\frac{e}{r}dt} \tag{11.6}$$

Schwarzschild time t is regular along the entire tunneling path ABCD(figure 11.1) and the background geometry continuously changes so that κ evolves from the initial κ_L to the final κ_R. (The final result (11.7) is independent of precisely how this change takes place.)

$$\text{Im } t_B = -\frac{\pi}{2\kappa_L}, \qquad \text{Im } t_C = +\frac{\pi}{2\kappa_R} \tag{11.7}$$

valid for t_+, t_- and $\bar{t}$.

Thus, for small radiation with Schwarzschild mass-energy ε

$$\begin{aligned}
\text{Im } W(\varepsilon) &= \text{Im} \int \varepsilon dt = \varepsilon \text{ Im} \int dt = \varepsilon\Big(\frac{\pi}{2\kappa_R} - \Big(-\frac{\pi}{2\kappa_L}\Big)\Big) && (11.8)\\
&= \frac{\pi}{2}\varepsilon\Big(\frac{1}{\kappa_R} + \frac{1}{\kappa_L}\Big) = \varepsilon\pi\overline{\Big(\frac{1}{\kappa}\Big)} && (11.9)\\
&= \frac{\pi}{2}\varepsilon(2r_2 + 2r_1) = 2\pi\varepsilon\Big((M-\varepsilon) + M\Big) && (11.10)\\
&= 4\pi M\varepsilon\Big(1 - \frac{\varepsilon}{2M}\Big) && (11.11)
\end{aligned}$$

where t is Schwarzschild time and ε is the energy corresponding to that. For E which is the successive radiation energy summed over ε,

$$\begin{aligned}
\text{Im } W(E) &= 2\pi\varepsilon\Big((M-\varepsilon) + M + (M-2\varepsilon) + (M-\varepsilon) + (M-3\varepsilon) + (M-2\varepsilon)\\
&\quad + \cdots (M - N\varepsilon) + (M - (N-1)\varepsilon)\Big)
\end{aligned}$$

$$\begin{aligned}
\text{Im } W(E) &= 2\pi\varepsilon\Big(2NM - \sum_{j=1}^{N} j\varepsilon - \sum_{j=1}^{N-1} j\varepsilon\Big) && (11.12)\\
&= 2\pi\varepsilon\Big(2NM - \frac{N(N+1)}{2}\varepsilon - \frac{(N-1)N}{2}\varepsilon\Big)\\
&\quad \text{As } N \to \infty\\
&= 2\pi\Big(2\varepsilon NM - \frac{N^2\varepsilon^2}{2} - \frac{N^2\varepsilon^2}{2}\Big) = 2\pi(2EM - E^2)\\
&= 4\pi ME\Big(1 - \frac{E}{2M}\Big) && (11.13)
\end{aligned}$$

We calculated (11.13) assuming all emissions ε_i have the same energy ($\varepsilon_i = \varepsilon_j, i \neq j$). We get the same result (11.13) even we assume all the emissions have different energies ($\varepsilon_i \neq \varepsilon_j, i \neq j$) as long as each emission is very small compare to total emission energy E so that each of them is counted as infinitesimal quantity compare to E. Even if we start with finite radiation energy E instead of infinitesimal radiation energy ε in (11.8) we can get the same result of Im $W(E)$. Thus, for an emission with Schwarzschild mass-energy E

$$\text{Im } W(E) = \frac{\pi}{2}E\Big(\frac{1}{\kappa_R} + \frac{1}{\kappa_L}\Big) = E\pi\overline{\Big(\frac{1}{\kappa}\Big)} \qquad (11.14)$$

stemming from the imaginary-time path BC. This agrees with the Parikh-Wilczek

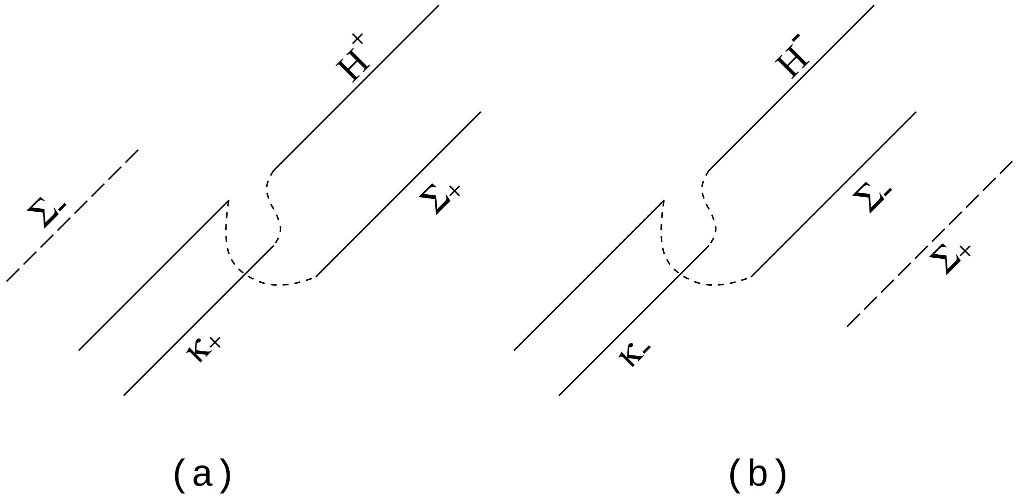

Figure 11.2: Shell-tunneling through the future horizon. First (a), the outer face Σ_+ tunnels through the initial location H^+ of the apparent horizon. Finally (b), the inner face Σ_- tunnels through the shrunken horizon H^-.

result [2] derived in Painlevè-Gullstrand coordinates. Hence the tunneling probability for emission of Schwarzschild mass-energy E from a non-charged black hole is, by (8.12),

$$P(E) = e^{-2\text{Im } W} = e^{-\beta(E)E}, \quad \beta \equiv \overline{\left(\frac{2\pi}{\kappa}\right)} \tag{11.15}$$

in which the effective Hawking temperature $T_H = 1/\beta$ is given as the harmonic mean of surface gravities κ_L and κ_R before and after emission.

The energy distribution (11.15) was first derived by Kraus and Wilczek [2] by a more elaborate route. Since β depends on E, (11.15) is no longer a simple Boltzmann exponential. As shown by Zhang et al [5](and as we shall outline in chapter 12) correlations present in these deviations from thermality have the capacity to carry off the maximum information content of the hole.

11.2 F→R tunneling:tunneling through the future horizon of an astrophysical black hole

In a real black hole formed by collapse, the L-sector of the L→R tunneling scenario does not exist. A picture that corresponds more closely to our intuitive idea of Hawking evaporation is shown in Figure 11.2. A virtual particle, represented by a thin or thick spherical shell formed just beneath the future horizon H^+, tunnels through the horizon from sector F to sector R and escapes. The attendant loss of mass causes the horizon to shrink from H^+ to H^-. (In Figs 11.2(a) and (b) the two face histories Σ_+ and Σ_- are shown separately for clarity.) (One way to visualize the tunneling for a thin shell is to imagine the pre- and post- tunneling histories reslotted so that the $U_+ = 0$ and $U_- = 0$ axes are aligned. This permits both histories to be mapped onto a single Kruskal diagram.)

As we have seen in the pair creation case (chapter 9), the most probable tunneling event takes place when the particle has just barely enough kinetic energy to tunnel through. This requires that both pre- and post-tunneling histories should be very nearly surfaces of constant retarded time u. Tunneling thus takes place between two given values of u, inside and outside that horizon, where

$$u_\epsilon = -\frac{1}{\kappa_\epsilon} \ln U_\epsilon \quad (\epsilon = \pm)$$

and κ_+ and κ_- are the surface gravities before and after the shell has escaped. In the tunneling formula (8.12), we are interested in only the change of *imaginary* value of u during the tunneling process, which is $\overline{\frac{\pi}{\kappa}}$.

The Schwarzschild time t may be subjected to arbitrary space-dependent translations

$$t \to t_{gen} = t + \psi(r) \tag{11.16}$$

without affecting the definition of conjugate variable $E(= m)$, the energy measured at infinity. The Jacobi action in these coordinates is (11.6),

$$W = -\int M d\tau + \int\int dm d\bar{t}_{gen} - \int\int \overline{de\frac{e}{r}dt_{gen}} \tag{11.17}$$

For the tunneling episode, the natural choice of parameter is $t_{gen} = u$. This yields for the imaginary part of the shell action

$$\text{Im } W(\varepsilon) = \text{Im } \int \varepsilon_u du = \varepsilon_u \text{Im } \Delta u = \varepsilon_u \overline{\left(\frac{\pi}{\kappa}\right)} \tag{11.18}$$

So in the same way as (11.12) we get the imaginary action for finite energy E_u

$$\text{Im } W(E_u) = E_u \overline{\left(\frac{\pi}{\kappa}\right)} \tag{11.19}$$

For a charged non-rotating black hole

$$\begin{aligned}
\text{Im } W(dm, de) &= -\text{Im} \int M d\tau + \text{Im} \int\int dm du - \text{Im} \int\int de \frac{\overline{e}}{r} du && (11.20)\\
&= \text{Im} \left(dm - de \frac{\overline{e}}{r_H} \right) \int du = \left(dm - de \frac{\overline{e}}{r_H} \right) \text{Im } \Delta u && (11.21)
\end{aligned}$$

where r_H represents the position of the horizon. Note that we got r_H because the integration is at the horizon.

$$\begin{aligned}
\text{Im } \Delta u &= \frac{\pi}{2} \Big(\frac{1}{\kappa_+} + \frac{1}{\kappa_-} \Big)\\
&\quad \text{up to first order of } dm \text{ and } de,\\
&= \frac{\pi}{2} \Big[2 \frac{(M_b + \sqrt{M_b^2 - e^2})^2}{\sqrt{M_b^2 - e^2}} - \frac{e^3}{(M_b^2 - e^2)^{3/2}} de\\
&\quad + \Big(\frac{M_b (M_b + \sqrt{M_b^2 - e^2})^2}{(M_b^2 - e^2)^{3/2}} - \frac{2(M_b + \sqrt{M_b^2 - e^2})^2}{M_b^2 - e^2} \Big) dm \Big]
\end{aligned}$$

where M_b is the mass of the black hole and dm is the mass of the tunneling shell. Then

$$\begin{aligned}
\text{Im } W(dm, de) &= \frac{\pi}{2} \Big(dm - \frac{\overline{e}}{r_H} de \Big) \Big(\frac{2}{\kappa_+} - \frac{e^3}{(M_b^2 - e^2)^{3/2}} de && (11.22)\\
&\quad + \frac{1}{\kappa_+} \frac{M_b - 2\sqrt{M_b^2 - e^2}}{M_b^2 - e^2} dm \Big) && (11.23)
\end{aligned}$$

We need to sum over all $dm_i \; de_i$ to get Im $W(\Delta m, \Delta e)$ for finite emission of $\Delta m, \Delta e$ as we did in (11.12). However this summation looks non-unique and path dependent unless $e = 0$. However if we assume further that the evaporation is, to a sufficient approximation, quasi-stationary – i.e., that we can neglect dissipative effects like gravitational radiation as the hole settles into its new configuration – then it follows from the first law of black hole mechanics [12] that

$$-2\pi đE/\kappa(E) = \frac{1}{4} dA = dS \tag{11.24}$$

the (negative) changes of horizon area and Bekenstein-Hawking entropy, which *are* exact differentials. Thus for quasi-stationary evaporation of a charged black hole, tunneling probability and the change of entropy have a simple relation,

$$P(\Delta m, \Delta e) = e^{\Delta S} \tag{11.25}$$

When rotation as well as charge is considered, we have a Lagrangian of the shell in the following form,

$$Ldt = -M d\tau - q\Phi dt$$

thus by $Ldt = pdq - Edt$, the required Jacobi action dW is

$$dW = pdq = -M d\tau + Edt - q\Phi dt$$

This leads to

$$\mathrm{Im}\ W(dm, de) = -\mathrm{Im}\ \int M d\tau + \mathrm{Im}\ \int\int dm du - \mathrm{Im}\ \int\int de \frac{\overline{e}}{r} du$$

While the dissipative energy of the charged rotating shell is

$$dE^{diss} = dm - \frac{ede}{r_+} - \Omega_+ dJ$$

where Ω_H is the angular velocity of the rotating black hole. Thus we don't have the simple relation like (11.25) for a rotating black hole.

11.2.1 Thin shell formula

We have derived the tunneling formulae (10.11), (11.13), (11.14), (11.19), (11.25) for $P(\Delta m, \Delta e)$ by summing over the effects of small radiation dm, de (compare to $\Delta m, \Delta e$). These formulae are called thick shell formulae in the sense the we get the tunneling effect of $\Delta m, \Delta e$ by summing over the effects of small radiation dm, de. This technique has been adopted in [3] (see (10.8)).

If the tunneling of m, e is abrupt through the horizon (e.g, tunneling by thin shell), in general, we may get different formula for tunneling probability because there is no physical reason that the tunneling probability for abrupt change and that for gradual change are the same. For L→R picture for uncharged black hole,

$$\begin{aligned} \mathrm{Im}\ W(E) &= \mathrm{Im}\int Edt = E\ \mathrm{Im}\int dt = E\Big(\frac{\pi}{2\kappa_R} - \Big(-\frac{\pi}{2\kappa_L}\Big)\Big) && (11.26)\\ &= \frac{\pi}{2}E\Big(\frac{1}{\kappa_R}+\frac{1}{\kappa_L}\Big) = E\pi\overline{\Big(\frac{1}{\kappa}\Big)} && (11.27)\end{aligned}$$

In the F→R picture for an uncharged black hole,

$$\mathrm{Im}\ W(E_u) = \mathrm{Im}\int E_u du = E_u \mathrm{Im}\ \Delta u = E_u\overline{\Big(\frac{\pi}{\kappa}\Big)} \qquad (11.28)$$

As we can check, the thin shell formulae (11.27) and (11.28) are the same as the thick shell formulae (11.14) and (11.19) for tunneling out of an uncharged black hole.

For tunneling out of a charged black hole in the F→R picture,

$$\begin{aligned} \mathrm{Im}\ W^{thin}(\Delta m, \Delta e) &= \mathrm{Im}\int \Delta m du - \mathrm{Im}\int \Delta e\frac{\overline{e}}{r}du && (11.29)\\ &= \Big(\Delta m - \Delta e\overline{\frac{e}{r_H}}\Big)\mathrm{Im}\ \Delta u = \Big(\Delta m - \Delta e\overline{\frac{e}{r_H}}\Big)\overline{\Big(\frac{\pi}{\kappa}\Big)} && (11.30)\end{aligned}$$

where Δm and Δe are the mass and charge of the thin shell radiation and $\overline{e}$ is the average of the inner and outer surface charges. Hence we get the tunneling probability of a charged thin shell,

$$P^{thin}(\Delta m, \Delta e) = e^{-2\mathrm{Im}\ W} = \exp\Big[\Big(\Delta m - \Delta e\overline{\frac{e}{r_H}}\Big)\overline{\Big(-\frac{2\pi}{\kappa}\Big)}\Big] \qquad (11.31)$$

This tunneling probability formula for a charged thin shell is not identical to the formula for a thick shell (11.25) based on a continuum model. Explicit calculation shows that,

$$\Big(\Delta m - \Delta e\overline{\frac{e}{r_H}}\Big)\overline{\Big(-\frac{2\pi}{\kappa}\Big)} = \pi\overline{\Big(\frac{r_H^2}{\sqrt{m^2-e^2}}\Big)}\Delta r_H, \quad \Delta S = 2\pi\overline{r_H}\Delta r_H \qquad (11.32)$$

for the thin- and thick-shell cases respectively. For uncharged black holes these two expressions agree, but they begin to diverge for non-vanishing charge. We may be able to reduce the thick shell mechanism to the thin shell mechanism by decreasing the thickness of thick shell to zero while keeping the mass constant. In this way we might expect the thin shell formula to follow from the thick shell formula for the tunneling probability. If that were true the thin shell formula and thick shell formula for tunneling probability should be consistent each other. But as we can see in (11.32) thin shell probability and thick shell probability are not consistent with each other for tunneling in a charged black hole. Mathematically, the source of this discrepancy is the difference between the electric interaction term in the original 3+1 action (11.20) used in the thick-shell formulation and that used in the effective 1+1 action (11.26) of the thin-shell version.

It will require a deeper investigation to decide which of these expressions is more correct, or (more likely) to bring to light some more complex formula which amalgamates features of both. Both depend only, as they should, on the observables, i.e. the states before and after emission.

One expects discontinuous aspects of the quantized emission to be more marked at low temperatures, and here the thin-shell formula (11.31), which predicts zero emission probability for zero temperature ($\kappa_+ = 0$), accords well with expectations.

Chapter 12

Promising solution to the information paradox

As Kraus and Wilczek first proved which is followed by Parikh [2] and as verified in (11.13), if back-reaction of Hawking radiation to the background geometry is taken into account, the probability that a Schwarzschild black hole of initial mass M emitting a quantum of energy ω is given by

$$P = e^{-8\pi\omega(M-\frac{\omega}{2})} \tag{12.1}$$

However according to Parikh, even this result could not show a correlation between emissions [3].

$$P(E_1+E_2) = \exp\left(-8\pi E_1\left(M-\frac{E_1}{2}\right)-8\pi E_2\left(M-E_1-\frac{E_2}{2}\right)\right) = P(E_1)P(E_2) \tag{12.2}$$

Generally, one defines the correlation coefficient $C(a, b)$ between two events a and b by

$$C(a, b) = \ln\frac{P(a, b)}{P(a)P(b)} \tag{12.3}$$

where $P(a, b)$ is the probability of both a and b, and $P(b) = \sum_a P(a, b)$ the probability of b. The *conditional* probability of b (given that a has already occurred) is

$$P(a; b) = \frac{P(a, b)}{P(a)} \tag{12.4}$$

Thus according to Parikh, (12.2) does not show the correlations.

$$C(E_1, E_2) = \ln \frac{P(E_1 + E_2)}{P(E_1)P(E_2)} = 0$$

Zhang et al [5] pointed out that the deviation from pure thermality derived by Parikh and Wilczek actually shows a correlation between emissions. They found that the last equation of (12.2) is not correct which must be replaced by conditional probability. In other words Hawking radiation does correlate successive events. It means information leakage is possible through Hawking radiation.

We can express (12.1) as follows

$$P(M; E) = \exp\left(- 8\pi E\Big(M - \frac{E}{2}\Big)\right) \tag{12.5}$$

From the tunneling probability (12.1) we have

$$P(M; E_1)P(M - E_1; E_2) = P(M; E_1 + E_2) \tag{12.6}$$

$$P(M; E_1)P(M - E_1; E_2)P(M - E_1 - E_2; E_3) = P(M; E_1 + E_2 + E_3) \tag{12.7}$$

$$\begin{aligned} P(E_1, E_2, \cdots E_n) &= P(M; E_1)P(M - E_1; E_2)P(M - E_1 - E_2; E_3) \\ &\quad \cdots P(M - E_1 - E_2 - \cdots - E_{n-1}; E_n) \qquad (12.8) \\ &= P(M; E_1 + E_2 + \cdots + E_n) \\ &= P(M; M) \\ &= e^{-8\pi M\left(M - \frac{M}{2}\right)} = e^{-4\pi M^2} \qquad (12.9) \end{aligned}$$

This shows all possible combinations of exhaustion of energy $P(E_1, E_2, \cdots E_n)$ are equally probable. (We have seen this property of tunneling probability when we had calculated the tunneling probability of thick shell by summing over all possible infinitesimal energy ε_i ($\varepsilon_i = \varepsilon_j, i \neq j$). The resulting tunneling probability is independent of how each infinitesimal emission (small compared to the total energy radiation E) are different from each other.) Hence if we interpret the entropy of the black hole microscopically as all possible number of combinations of exhaustion of energy then,

$$S \equiv -\sum_i P_i \ln P_i = -\ln P_i = 4\pi M^2 = \frac{A}{4} \tag{12.10}$$

where we use the fact that all possible combinations of exhaustion of energy $P(E_1, E_2, \cdots E_n)$ are equally probable. (12.10) shows that the entropy carried away by evaporation is the same as the entropy of the black hole. From (12.10)

$$P = \frac{1}{N} = e^{-S}, \qquad N = e^S, \qquad \frac{N_2}{N_1} = e^{S_2 - S_1} \tag{12.11}$$

This equates transition probability to a statistical factor e^{S_2}/e^{S_1}, equal to the ratio of the number of final states to initial states.

Thus, the degrees of freedom in the outgoing radiation equal the maximum information capacity of the hole, as measured by the Bekenstein-Hawking entropy. This provides evidence, purely on the basis of counting, that unitarity could be preserved and that the radiation has enough room to accommodate all of the information.

Chapter 13

Conclusions

Using the tunneling formula derived from the path integral method, we have shown the derivation of the Hawking radiation formula can be simplified through analytic continuation in Kruskal-Szekeres coordinates. By this method we formulated thin- and thick-shell formulae which differ from each other for the charged black hole case. Further discussion maybe required about this.

Counting Hawking radiation as a tunneling process leads to non-thermal aspects of black hole radiation due to energy conservation. This raises hope that black hole evaporation may, after all, be a unitary process.

Appendix A

Variational principles

In this chapter we will discuss how the variational principle is applied to general relativity. Variation of metric on Einstein Hilbert action will result in Einstein's field equation. And we will see how the variation of scalar field and fluid can be applied in curved space time. First of all let us derive the basic formulae used for variation in general relativity. Using

$$g^{ac}g_{bc} = \delta^a_b$$

$$\delta(g^{ac}g_{bc}) = 0$$

$$\delta g^{ac}g_{bc} + g^{ac}\delta g_{bc} = 0 \tag{A.1}$$

Multiplying by g_{ad} yields

$$\delta g_{ab} = -g_{a\mu}g_{b\nu}\delta g^{\mu\nu} \tag{A.2}$$

Multiplying (A.1) by g^{bd} gives

$$\delta g^{ab} = -g^{a\mu}g^{b\nu}\delta g_{\mu\nu} \tag{A.3}$$

Next, prove that the difference of two affine connections at one point is a tensor. That is $(\Gamma_{(2)} - \Gamma_{(1)})^c_{ab}(x)$ is a tensor, where $\Gamma_{(1)}, \Gamma_{(2)}$ correspond to two different geometries. Let's prove this: For a vector A^a at one point in two different geometries expressed by the same coordinate system.

$$\begin{aligned}\overset{(1)}{\nabla}_b A^a &= A^a{}_{,b} + A^c \Gamma^a_{(1)cb} \\ \overset{(2)}{\nabla}_b A^a &= A^a{}_{,b} + A^c \Gamma^a_{(2)cb}\end{aligned}$$

Subtraction yields

$$(\overset{(2)}{\nabla}_b - \overset{(1)}{\nabla}_b) A^a = A^c (\Gamma_{(2)} - \Gamma_{(1)})^a_{cb} \tag{A.4}$$

Since the left hand side is a tensor, by quotient rule $(\Gamma_{(2)} - \Gamma_{(1)})^a_{cb}(x)$ is a tensor.

What would be the change of Γ^a_{ab} by the δg^{ab}? Define a small variation h^{ab},

$$\underbrace{\delta g^{ab}}_{\equiv -h^{ab}} = -g^{a\mu} g^{b\nu} \delta g_{\mu\nu} \tag{A.5}$$

$$\begin{aligned} h_{\mu\nu} &\equiv \overline{g}_{\mu\nu} - g_{\mu\nu} \\ h^{ab} &= g^{a\mu} g^{b\nu} h_{\mu\nu} \end{aligned} \tag{A.6}$$

For small variation of δg^{ab}

$$h^{ab} = g^{ab} - \overline{g}^{ab}$$

Under small variation

$$g_{ab} \to \overline{g}_{ab} = g_{ab} + \delta g_{ab} \qquad \Gamma \to \overline{\Gamma} = \Gamma + \delta\Gamma$$

We have

$$\delta \Gamma^c_{ab} \equiv \overline{\Gamma}^c_{ab} - \Gamma^c_{ab} = \frac{1}{2}(h^c{}_{a|b} + h^c{}_{b|a} - h_{ab}{}^{|c})$$

Proof:

Since both sides are tensor, we need check only in one specific coordinate system with $(\Gamma^c_{ab})_o = 0$, $(g_{ab,c})_o = 0$ but $(\delta\Gamma^c_{ab})_o \neq 0$ at point o. Then

$$\begin{aligned}(\delta\Gamma^c_{ab})_o &= (\overline{\Gamma}^c_{ab})_o = \frac{\overline{g}^{cd}}{2}(\overline{g}_{ad,b} + \overline{g}_{bd,a} - \overline{g}_{ab,d}) \\ &= \frac{(g^{cd} - h^{cd})}{2}(h_{ad,b} + h_{bd,a} - h_{ab,d}) \\ &\quad \text{For small variation } h^{cd} \to 0 \\ &= \frac{g^{cd}}{2}(h_{ad,b} + h_{bd,a} - h_{ab,d})\end{aligned}$$

Since h_{ab} follows tensor transformation (A.6)

$$(\delta\Gamma^c_{ab})_o = \frac{1}{2}(h^c{}_{a|b} + h^c{}_{b|a} - h_{ab}{}^{|c})$$

Prove the Palatini identity: $\delta R_{ab} = (\delta\Gamma^c_{ab})_{|c} - (\delta\Gamma^c_{ac})_{|b}$

Proof:

Both sides are tensors, so we need to prove this only for one (Riemannian) coordinate system $(\Gamma)_o = 0$.

$$R_{ab} = \partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} \underbrace{-\Gamma^c_{db}\Gamma^d_{ac} + \Gamma^c_{dc}\Gamma^d_{ab}}_{(*)}$$

$$\delta R_{ab} = (\partial_c\delta\Gamma^c_{ab} - \partial_b\delta\Gamma^c_{ac}) \underbrace{-\delta\Gamma^c_{db}\Gamma^d_{ac} - \Gamma^c_{db}\delta\Gamma^d_{ac} - \cdots}_{(*)}$$

$$\to \delta R_{ab} = (\delta\Gamma^c_{ab})_{|c} - (\delta\Gamma^c_{ac})_{|b} \tag{A.7}$$

Finally we need the expression of $\delta\sqrt{-g}$ by small δg. Prove that for small variation δg

$$\delta\sqrt{-g} = \sqrt{-g}\frac{1}{2}g^{ab}\delta g_{ab} = -\sqrt{-g}\frac{1}{2}g_{ab}\delta g^{ab} \tag{A.8}$$

Proof:

$$\underbrace{g^{-1}}_{g^{ji}} = \frac{1}{\det g}(\text{cofactor of } g)^T$$

$$g^{ij} = \frac{1}{\det g}(\text{cofactor of } g_{ij})$$

For small variation of the elements A_{ij}

$$\delta\det(A_{ij}) = \delta A_{ij} \times (\text{cofactor of } A_{ij})$$

With this result, we can calculate $\delta\sqrt{-g}$

$$\delta g (= \delta\det(g_{ij})) = \delta g_{ij} \times (\text{cofactor of } g_{ij}) = \delta g_{ij} \times (g g^{ij})$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-1)\delta g = \frac{1}{2\sqrt{-g}}(-g)g^{ij}\delta g_{ij} = \frac{\sqrt{-g}}{2}g^{ij}\delta g_{ij} = -\frac{\sqrt{-g}}{2}g_{ij}\delta g^{ij}$$

A.1 Einstein Hilbert action

Equipped with all these tools , we can derive the Einstein field equation from the Einstein Hilbert action,

$$I_{EH} = -\frac{1}{16\pi}\int \sqrt{-g}(R-2\Lambda)d^4x \tag{A.9}$$

For a small variation of $g_{ab} \to g_{ab} + \delta g_{ab}$

$$\begin{aligned}
\sqrt{-g}g^{ab}\delta R_{ab} &= \sqrt{-g}g^{ab}((\delta\Gamma^c_{ab})_{|c} - (\delta\Gamma^c_{ac})_{|b}) \\
&= \sqrt{-g}\{(g^{ab}\delta\Gamma^c_{ab})_{|c} - (g^{ab}\delta\Gamma^c_{ac})_{|b}\} \\
& \quad \text{Remind } \sqrt{-g}A^c_{|c} = \partial_c(\sqrt{-g}A^c) \\
&= \{\partial_c(\sqrt{-g}g^{ab}\delta\Gamma^c_{ab}) - \partial_b(\sqrt{-g}g^{ab}\delta\Gamma^c_{ac})\}
\end{aligned}$$

$$\int \sqrt{-g}g^{ab}\delta R_{ab}d^4x = \int \{\partial_c(\sqrt{-g}g^{ab}\delta\Gamma^c_{ab}) - \partial_b(\sqrt{-g}g^{ab}\delta\Gamma^c_{ac})\}d^4x \tag{A.10}$$

This results is zero, if $\delta\Gamma = 0$ on the boundary. $\delta\Gamma = 0$ is reasonable for the boundary infinitely far from the source of the gravity.

$$\begin{aligned}
\delta I_{EH} &= -\frac{1}{16\pi}\int_\infty \delta\sqrt{-g}(R-2\Lambda) + \sqrt{-g}\delta(R-2\Lambda)d^4x \\
&= -\frac{1}{16\pi}\int_\infty -\frac{1}{2}\sqrt{-g}g_{ab}\delta g^{ab}(R-2\Lambda) + \sqrt{-g}\delta R d^4x \\
& \quad \delta R = \delta(g^{ab}R_{ab}) = \delta g^{ab}R_{ab} + g^{ab}\delta R_{ab} \\
&= -\frac{1}{16\pi}\int_\infty \sqrt{-g}\underbrace{\delta g^{ab}}_{arbitrary}(R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda)d^4x \underbrace{- \frac{1}{16\pi}\int_\infty \sqrt{-g}g^{ab}\delta R_{ab}d^4x}_{=0 \text{ by } (A.10)}
\end{aligned}$$

To make $\delta I_{EH} = 0$ for an arbitrary δg^{ab}, $R_{ab} - \frac{1}{2}g_{ab}R + g_{ab}\Lambda$ must be zero. This is the Einstein field equation in vacuum

$$G_{ab} + g_{ab}\Lambda = 0$$

A.2 Einstein-Hilbert action for gravity+matter.

Consulting the Einstein field equation, we can define the energy momentum tensor T^{ab} using Einstein-Hilbert action for gravity+matter.

$$\begin{aligned} I_{EHM} &\equiv I_{grav}[g] + I_{mat}[g, \Psi] \qquad \text{(Note: } I_{grav}[g] = -I_{EH}[g]) \\ &= \int \mathcal{L}_{grav}(g_{ab}) d^4x + \int \mathcal{L}_{mat}(g_{ab}, \Psi) d^4x \end{aligned}$$

where

$$\mathcal{L}_{grav}(g_{ab}) \equiv \frac{\sqrt{-g}}{16\pi}(R - 2\Lambda)$$

$$\Psi = \{\varphi(\text{scalar}), A_a(\text{EM}), \rho(\text{fluid}), \psi(\text{Dirac}), \cdots\}$$

Since $\frac{\delta I_{EHM}}{\delta g^{ab}} = 0$

$$\frac{\delta I_{EHM}}{\delta g^{ab}} = \frac{\delta I_{grav}}{\delta g^{ab}} + \frac{\delta I_{mat}}{\delta g^{ab}}$$

And

$$\frac{\delta I_{grav}}{\delta g^{ab}} = \frac{\sqrt{-g}}{16\pi}(G_{ab} + g_{ab}\Lambda) = \frac{\sqrt{-g}}{2} T_{ab}$$

by Einstein's field equation. Thus we get

$$T_{ab} = -\frac{2}{\sqrt{-g}} \frac{\delta I_{mat}}{\delta g^{ab}}$$

or

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta I_{mat}}{\delta g_{ab}}$$

For example, for scalar field $\mathcal{L}_{mat}(g_{ab}, \phi, \phi_{,\mu}) = -\frac{\sqrt{-g}}{2}\partial_\mu\phi\partial^\mu\phi$,

$$\begin{aligned}
\delta I_{mat}[g,\phi] &= \delta\int \mathcal{L}_{mat}d^4x = \delta\int\Big(-\frac{\sqrt{-g}}{2}\Big)\partial_\mu\phi\partial^\mu\phi d^4x \\
&= -\frac{1}{2}\delta\int g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g}d^4x = -\frac{1}{2}\int(\delta g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g} + g^{ab}\phi_{,a}\phi_{,b}\delta\sqrt{-g})d^4x \\
&\quad \text{by } \delta\sqrt{-g} = -\sqrt{-g}\frac{1}{2}g_{cd}\delta g^{cd} \\
&= -\frac{1}{2}\int\{\delta g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g} + g^{ab}\phi_{,a}\phi_{,b}(-\sqrt{-g})\frac{1}{2}g_{cd}\delta g^{cd}\}d^4x \\
&= -\frac{1}{2}\int\{\delta g^{ab}\phi_{,a}\phi_{,b}\sqrt{-g} + \phi_{,c}\phi^{,c}(-\sqrt{-g})\frac{1}{2}g_{ab}\delta g^{ab}\}d^4x \\
&= -\frac{1}{2}\int\delta g^{ab}\sqrt{-g}\{\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c}\}d^4x
\end{aligned}$$

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta I_{mat}}{\delta g^{ab}} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c}$$

With potential $V(\phi)$,

$$\mathcal{L}_{mat}(g_{ab}, \phi, \phi_{,\mu}) = -\frac{\sqrt{-g}}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\sqrt{-g}$$

$$I_{mat}[g,\phi] = \int\mathcal{L}_{mat}d^4x = -\frac{1}{2}\int\partial_\mu\phi\partial^\mu\phi\sqrt{-g}d^4x - \int V(\phi)\sqrt{-g}d^4x$$

$$\begin{aligned}
-\delta\int V(\phi)\sqrt{-g}d^4x &= -\int\underbrace{\delta V(\phi)}_{=0}\sqrt{-g}d^4x - \int V(\phi)\delta\sqrt{-g}d^4x \\
&\quad \delta\sqrt{-g} = -\sqrt{-g}\frac{1}{2}g_{ab}\delta g^{ab} \\
&= \frac{1}{2}\int\delta g^{ab}\sqrt{-g}V(\phi)g_{ab}d^4x
\end{aligned}$$

$$\begin{aligned}
\delta I_{mat}[g,\phi] &= -\frac{1}{2}\int\delta g^{ab}\sqrt{-g}\{\phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c}\}d^4x \\
&\quad +\frac{1}{2}\int\delta g^{ab}\sqrt{-g}V(\phi)g_{ab}d^4x
\end{aligned}$$

$$T_{ab} = \frac{-2}{\sqrt{-g}}\frac{\delta I}{\delta g^{ab}} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi_{,c}\phi^{,c} - V(\phi)g_{ab} \tag{A.11}$$

We can find the field equation for ϕ using $\frac{\delta I}{\delta \phi} = 0$

$$\begin{aligned}
\delta I &= -\frac{1}{2}\delta\int \partial_\mu\phi\partial^\mu\phi\sqrt{-g}d^4x \\
&= 2\cdot\Big(-\frac{1}{2}\Big)\sum_\mu \delta\phi\partial^\mu\phi\sqrt{-g}\Big|_{x_1^\mu}^{x_2^\mu} + 2\cdot\frac{1}{2}\int\delta\phi\partial_\mu(\partial^\mu\phi\sqrt{-g})d^4x \\
&= \int\delta\phi\partial_\mu(\partial^\mu\phi\sqrt{-g})d^4x \\
&\quad \text{using } \sqrt{-g}V^\mu_{|\mu} = (\sqrt{-g}V^\mu)_{,\mu} \\
&= \int\delta\phi\sqrt{-g}(\partial^\mu\phi)_{|\mu}d^4x
\end{aligned}$$

$$\frac{\delta I}{\delta\phi} = \sqrt{-g}(\partial^\mu\phi)_{|\mu} = 0$$

With

$$\mathcal{L}_{mat} = -\frac{\sqrt{-g}}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\sqrt{-g}$$

$$\frac{\delta I}{\delta\phi} = \sqrt{-g}\Big((\partial^\mu\phi)_{|\mu} - \frac{\partial V(\phi)}{\partial\phi}\Big) = 0$$

For $\Gamma = 0$,

$$\partial_\mu\partial^\mu\phi - \frac{\partial V(\phi)}{\partial\phi} = 0$$

The energy momentum tensor of scalar field acquired above with $V(\phi) = 0$

$$T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\phi^{,c}\phi_{,c}$$

satisfies the conservation law. In $\Gamma \neq 0$ frame,

$$T_{ab} = \phi_{|a}\phi_{|b} - \frac{1}{2}g_{ab}\phi^{|c}\phi_{|c}$$

$$\begin{aligned} T_{ab}{}^{|b} &= (\phi_{|a}\phi_{|b})^{|b} - \frac{1}{2}g_{ab}(\phi^{|c}\phi_{|c})^{|b} = (\phi_{|a}{}^{|b}\phi_{|b} + \phi_{|a}\underbrace{\phi_{|b}{}^{|b}}_{=0}) - \frac{1}{2}g_{ab}(\phi^{|c|b}\phi_{|c} + \phi^{|c}\phi_{|c}{}^{|b}) \\ &= \phi_{|a}{}^{|b}\phi_{|b} - \frac{1}{2}\phi^{|c}{}_{|a}\phi_{|c} - \frac{1}{2}\phi^{|c}\phi_{|c|a} = \phi_{|a|b}\phi^{|b} - \frac{1}{2}\phi_{|a|c}\phi^{|c} - \frac{1}{2}\phi_{|a|c}\phi^{|c} = 0 \end{aligned}$$

Let us apply variational principle of general relativity to the perfect fluid and find the energy momentum tensor and the equations of motion. Basic thermodynamics tells us about the change of entropy,

$$TdS = dE + PdV - \mu dN \tag{A.12}$$

$$E = \rho V, \quad N = nV, \quad S = sV$$

$$E, S, V \cdots \text{ of system} \qquad T, P \cdots \text{ of reservoir}$$

$$Td(sV) = d(\rho V) + PdV - \mu d(nV)$$

$$(Ts - \rho - P + \mu n)dV = (d\rho - Tds - \mu dn)V \tag{A.13}$$

Since V and dV are arbitrary, so independent each other.

$$\rightarrow \begin{cases} Tds = d\rho - \mu dn & \times X \quad \rightarrow \rho(n, s) \\ Ts = \rho + P - \mu n & \times dX \end{cases}$$

For an arbitrary variable X,

$$Td(sX) = d(\rho X) + PdX - \mu d(nX) \cdots \text{ always true} \tag{A.14}$$

Or because V and dV are arbitrary in (A.13), we can set V to any variables X in (A.13). Then (A.13) becomes (A.14).

In MCRF(rest frame of fluid): $u^\mu \rightarrow (1, 0, 0, 0)$, $u_a = g_{a\mu}u^\mu \rightarrow (-1, 0, 0, 0)$

$$g_{ab} \overset{*}{=} \eta_{ab} \cdots \text{ local flatness}$$

$$\Delta_{ab} \equiv g_{ab} + u_a u_b = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \Delta^{ab} \cdots \text{ (MCRF)}$$

Local volume is $V \stackrel{*}{=} \sqrt{^3g}d^3x$.
Change V by changing g_{ab} while keeping d^3x fixed

$$\frac{\delta V}{V} \stackrel{*}{=} \frac{\delta\sqrt{^3g}}{\sqrt{^3g}} \stackrel{*}{=} \frac{\frac{1}{2}\sqrt{^3g}g^{ij}\delta g_{ij}}{\sqrt{^3g}}(i,j=1,2,3) \stackrel{*}{=} \frac{1}{2}\Delta^{ab}\delta g_{ab}$$

If no particles are added ($\delta(nV) = 0$) and if the expansion is isentropic ($\delta(sV) = 0$),

$$\delta n V + n\delta V = 0$$

$$\frac{\delta n}{n} = -\frac{\delta V}{V}\Big(=\frac{\delta s}{s}\Big) \stackrel{*}{=} -\frac{1}{2}\Delta^{ab}\delta g_{ab}$$

$$\rightarrow \frac{\delta(s\sqrt{-g})}{s\sqrt{-g}} = \frac{\delta(n\sqrt{-g})}{n\sqrt{-g}} = \frac{\delta n\sqrt{-g} + n\delta\sqrt{-g}}{n\sqrt{-g}} = \frac{\delta n}{n} + \frac{\delta\sqrt{-g}}{\sqrt{-g}} \tag{A.15}$$

$$\stackrel{*}{=} -\frac{1}{2}\Delta^{ab}\delta g_{ab} + \frac{1}{2}g^{ab}\delta g_{ab} \stackrel{*}{=} -\frac{1}{2}u^a u^b \delta g_{ab} \qquad \text{(MCRF)} \tag{A.16}$$

With the following matter action $I[\rho]$ for a perfect fluid, find energy momentum tensor T^{ab} of it.

$$I[\rho] = -\int \rho(n,s)\sqrt{-g}d^4x \tag{A.17}$$

In

$$Td(sX) = d(\rho X) + PdX - \mu d(nX) \tag{A.18}$$

set $X = \sqrt{-g}$,

$$d(\rho\sqrt{-g}) = Td(s\sqrt{-g}) + \mu d(n\sqrt{-g}) - Pd(\sqrt{-g})$$

Variation $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$ will lead to

$$\begin{aligned}
\delta(\rho\sqrt{-g}) &= T\delta(s\sqrt{-g}) + \mu\delta(n\sqrt{-g}) - P\delta(\sqrt{-g}) \\
&\quad \text{In MCRF by } (A.16) \\
&= Ts\sqrt{-g}\Big(-\frac{1}{2}\Big)u^a u^b \delta g_{ab} + \mu n\sqrt{-g}\Big(-\frac{1}{2}\Big)u^a u^b \delta g_{ab} - P\frac{1}{2}\sqrt{-g}g^{ab}\delta g_{ab} \\
&= -\frac{1}{2}\sqrt{-g}\delta g_{ab}[Tsu^a u^b + \mu n u^a u^b + Pg^{ab}] \\
&= -\frac{1}{2}\sqrt{-g}\delta g_{ab}[(\underbrace{Ts + \mu n}_{\rho+P})u^a u^b + Pg^{ab}] \qquad \text{MCRF}
\end{aligned}$$

$$T^{ab} = \frac{2}{\sqrt{-g}}\frac{\delta I_{mat}}{\delta g_{ab}} = (\rho + P)u^a u^b + Pg^{ab} \quad \text{MCRF, thus in all frames} \tag{A.19}$$

We derived T^{ab} for MCRF, thus it is valid for all observers.

With the same procedure for scalar field, we can derive the electro magnetic field equation and T_{ab} by varying the following action of the electro magnetic field.

$$I[A] = -\frac{1}{16\pi}\int \sqrt{-g}F_{ab}F^{ab}d^4x, \quad F_{ab} = \partial_a A_b - \partial_b A_a(\text{tensor}) \tag{A.20}$$

Appendix B

Shell dynamics

This appendix collects and reviews the origins of the shell formulae used in chapter 11.2. More detail can be found in chapter 3 of Poisson's *Toolkit* [8].

B.1 Basic mathematics

We define $e^{\alpha}_{(a)}$ by the differential relation between 4-dimensional bulk $x^{\alpha}(\xi^a)$ and 3-dimensional brane ξ^a coordinates.

$$e^{\alpha}_{(a)} = \frac{\partial x^{\alpha}}{\partial \xi^a}, \qquad dx^{\alpha} = e^{\alpha}_{(a)} d\xi^a$$

where Greek indices running from 1 to 4 are for the bulk coordinates and Latin indices running from 1 to 3 are for the brane coordinates. Thus we can relate brane vectors A_a to bulk vectors A_{α} by

$$A_a = A_{\alpha} \frac{\partial x^{\alpha}}{\partial \xi^a} = A_{\alpha} e^{\alpha}_{(a)} \tag{B.1}$$

The shell history is a timelike 3-space Σ that divides spacetime into two sectors $\mathcal{V}_+$ and $\mathcal{V}_-$ mapped by independent charts x^{α}_+ and x^{α}_- and with metrics

$$ds^2_{\pm} = g_{\alpha\beta} dx^{\alpha} dx^{\beta}|_{\pm}$$

Their common boundary Σ is described by two sets of imbedding relations $x^{\alpha}_{\pm} = x^{\alpha}_{\pm}(\xi^a)$ and intrinsic 3-metric

$$ds^2 = g_{ab}(\xi) d\xi^a d\xi^b$$

in terms of a third independent set of intrinsic coordinates ξ^a. The intrinsic 3-metric g_{ab} is induced compatibly on Σ by each of the two 4-geometries via

$$g_{ab}(\xi) = \Big(g_{\alpha\beta}(x) e^{\alpha}_{(a)} e^{\beta}_{(b)}\Big)_{\pm}, \qquad e^{\alpha}_{(a)}(\xi)|_{\pm} = \frac{\partial x^{\alpha}_{\pm}}{\partial \xi^a} \tag{B.2}$$

where $e_{(a)}$ are three basis vectors tangent to Σ.

The interior and exterior imbeddings $(ds)^2_{\Sigma_+} = (ds)^2_{\Sigma_-}$ thus induce a unique tangential 3-metric g_{ab} on Σ. If we assume isometry $g^+_{ab} = g^-_{ab}$,

$$(ds)^2_{\Sigma_\pm} = g^{\pm}_{\alpha\beta} dx^{\alpha}_{\pm} dx^{\beta}_{\pm} = g^{\pm}_{ab} d\xi^a d\xi^b$$

we get the same distance between points on the inner and outer surface for the same $d\xi^a$. This is called the first junction condition.

We define extrinsic curvature K_{ab} by the variation of normal vector $\underline{n}$ to the brane respect to the change of brane variable ξ^a.

$$\frac{\partial \underline{n}}{\partial \xi^a} = K_{ab} \underline{e}^{(b)} \tag{B.3a}$$

$$\frac{\partial \underline{e}}{\partial \xi^b} = \Gamma^c_{ab} \underline{e}_{(c)} - \epsilon K_{ab} \underline{n} \tag{B.3b}$$

where $\epsilon = n \cdot n = \pm 1$ and $\underline{n}$ is orthogonal to the timelike 3-space Σ. $\underline{n}$ is directed from $\mathcal{V}_-$ to $\mathcal{V}_+$.

The general variation of a bulk vector A^μ along a curve $x^\beta = x^\beta(t)$ in the bulk

$$\frac{d\underline{A}}{dt} = A^{\mu}{}_{|\beta} \frac{dx^\beta}{dt} \underline{e}_{(\mu)} \rightarrow \frac{\delta A^\mu}{\delta t} = \left(\frac{d\underline{A}}{dt}\right)^{\mu} = A^{\mu}{}_{|\beta} \frac{dx^\beta}{dt} \tag{B.4}$$

where the stroke denotes 4-dimensional covariant differentiation; 3-dimensional covariant derivatives with respect to g_{ab} are indicated by a semi-colon.

The Ricci commutation relation tells about the second derivative of A^α.

$$\begin{aligned} \Big(\frac{\delta^2}{\delta\xi^c \delta\xi^b} - \frac{\delta^2}{\delta\xi^b \delta\xi^c}\Big) A^\alpha &= \Big(A^{\alpha}{}_{|\beta\gamma} - A^{\alpha}{}_{|\gamma\beta}\Big) \frac{\partial x^\beta}{\partial \xi^b} \frac{\partial x^\gamma}{\partial \xi^c} \\ &= -A^\mu R^{\alpha}{}_{\mu\beta\gamma} e^{\beta}_{(b)} e^{\gamma}_{(c)} \end{aligned} \tag{B.5}$$

The completeness relation reveals the relation between bulk $g^{\alpha\beta}$ and brane g^{ab}.

$$g^{ab} e^{\alpha}_{(a)} e^{\beta}_{(b)} = g^{\alpha\beta} - \epsilon n^\alpha n^\beta \equiv \Delta^{\alpha\beta} \tag{B.6}$$

Expressing the right hand side of the Ricci commutation relation with $A^\alpha = \underline{e}_{(a)}$ in terms of parallel ($\underline{e}_{(d)}$) and normal ($\underline{n}$) component to the brane results in the Gauss-Codazzi equations

$$R_{\alpha\beta\mu\nu} e^\alpha_{(a)} e^\beta_{(b)} e^\mu_{(c)} e^\nu_{(d)} = R_{abcd} + \epsilon K_{ad} K_{bc} - \epsilon K_{ac} K_{bd} \tag{B.7a}$$

$$R_{\mu\alpha\beta\gamma} n^\mu e^\alpha_{(a)} e^\beta_{(b)} e^\gamma_{(c)} = K_{ab;c} - K_{ac;b} \tag{B.7b}$$

Using the Gauss-Codazzi equations, we can express $G_{\alpha\beta} e^\alpha_{(a)} n^\beta$, $G_{\alpha\beta} n^\alpha n^\beta$ and $G_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)}$. Detailed calculations result in

$$G_{\alpha\beta} e^\alpha_{(a)} n^\beta = K^d{}_{a;d} - K_{,a} \tag{B.8a}$$

$$G_{\alpha\beta} n^\alpha n^\beta = -\frac{\epsilon}{2}({}^{(n)}R + \epsilon K_{cb} K^{bc} - \epsilon K^2) \tag{B.8b}$$

$$\begin{aligned} {}^{(n+1)}G_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)} &= {}^{(n)}G_{ab} + \epsilon\Big(\frac{\partial K}{\partial n} g_{ab} - \frac{\partial K_{ab}}{\partial n}\Big) \\ &\quad -\epsilon K K_{ab} + 2\epsilon K_{ac} K^c{}_b + \frac{\epsilon}{2}(K^2 + K_{cd} K^{cd}) g_{ab} \end{aligned} \tag{B.9}$$

B.2 Application to general relativity.

If we ignore the bounded terms in thin shell in (B.9)

$$\underbrace{G_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)}}_{8\pi T_{\alpha\beta} e^\alpha_{(a)} e^\beta_{(b)}} = \frac{\partial K}{\partial z} g_{ab} - \frac{\partial K_{ab}}{\partial z} = \frac{\partial}{\partial z}(K g_{ab} - K_{ab}) \tag{B.10}$$

Integration by $\int_0^\epsilon dz$ gives

$$8\pi S_{ab} = [K g_{ab} - K_{ab}] \tag{B.11}$$

which describes how the energy momentum tensor of the shell affects the shell geometry where square brakets indicate a jump across the shell.

We can also find an expression for the conservation law of the shell. From (B.8a),

$$G_{\alpha\beta} e^\alpha_{(a)} n^\beta = K^b{}_{a;b} - K_{,a} = (K_a{}^b - \delta^b_a K)_{;b}$$

For both sides of the shell

$$[G_{\alpha\beta} e^{\alpha}_{(a)} n^{\beta}] = [K_a{}^b - \delta^b_a K]_{;b}$$

$$S^b{}_{a;b} = -[e^{\alpha}_{(a)} T^{\beta}_{\alpha} n_{\beta}] \tag{B.12}$$

describing the shell's response to the stresses and energy fluxes in its surroundings.

Variation of the metric in the action

$$I = I_{geom} + I_{mat}$$

where

$$16\pi I_{geom} = \int_{bulk} d^4x \sqrt{-g} R^{\alpha}_{\alpha} - 2\epsilon \int_{\Sigma} [K] d\Sigma_{shell} \tag{B.13}$$

yields, beside Einstein's field equations, also the jump conditions (B.11). Here, the shell is treated simply as an isometric pair of timelike boundaries Σ_+ and Σ_- of the bulk, with a common normal n, directed from $\mathcal{V}_-$ to $\mathcal{V}_+$.

Variation of the material part of the action I_{mat} gives the stress-energy tensor of the shell and its environment according to

$$\delta I_{mat} = \int \frac{1}{2} T^{\alpha\beta} \delta g_{\alpha\beta} \sqrt{-g} d^4x + \int \frac{1}{2} S^{ab} \delta g_{ab} d\Sigma \tag{B.14}$$

$\delta(I_{geom} + I_{mat}) = 0$ gives the Einstein field equations $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$ and the shell equation (B.11).

For a shell of fluid in a fluid environment, I_{mat} takes the form

$$I_{mat} = -\int_{bulk} \rho(n,s) \sqrt{-g} d^4x - \int_{\Sigma} \sigma(n,s) d\Sigma, \quad d\Sigma = 4\pi R^2 d\tau \tag{B.15}$$

where ρ and σ are energy densities per unit volume and unit area respectively, and s, n the corresponding entropy and molecular number densities. (Our use of the same symbols n, s for densities per unit volume and unit area in these different contexts should not cause confusion.)

Let us apply these dynamic laws of shell to the spherical collapsing shell in Schwarzschild geometry. Denote the bulk coordinate and brane coordinate as

$$x^{\mu} = (R, \theta, \phi, t) \tag{B.16a}$$

$$\xi^a = (\quad \theta, \phi, \tau) \tag{B.16b}$$

Diagonal components of the bulk metric $g_{\alpha\beta}$ of the Schwarzschild geometry are

$$g_{\alpha\beta} = (g_{RR}, g_{\theta\theta}, g_{\phi\phi}, g_{tt}) = \Big(\frac{1}{f(R)}, R^2, R^2\sin^2\theta, -f(R)\Big) \tag{B.17}$$

The shell basis vectors are

$$\begin{aligned} e^{\mu}_{(\theta)} = \frac{\partial x^\mu}{\partial \theta} &= \overbrace{(0,1,0,0)}^{(R,\theta,\phi,t)} \\ e^{\mu}_{(\phi)} = \frac{\partial x^\mu}{\partial \phi} &= (0,0,1,0) \\ e^{\mu}_{(\tau)}(= u^\mu) = \frac{\partial x^\mu}{\partial \tau} &= (\dot{R},0,0,\dot{t}) \end{aligned} \tag{B.18}$$

where the dots represent differentiation with respect to the shell proper time τ.

The metric in brane coordinates is

$$\begin{aligned} ds^2 &= \frac{g_{\alpha\beta}dx^\alpha dx^\beta}{d\xi^a d\xi^b} d\xi^a d\xi^b = g_{\alpha\beta} e^{\alpha}_{(a)} e^{\beta}_{(b)} d\xi^a d\xi^b \\ &= R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2 + g_{\alpha\beta} e^{\alpha}_{(\tau)} e^{\beta}_{(\tau)} d\tau^2 \\ &= R^2 d\Omega^2 - d\tau^2 \end{aligned} \tag{B.19}$$

The normal vector to this spherical shell is

$$n_\alpha = (\dot{t}, 0, 0, -\dot{R}) \tag{B.20}$$

The four velocity of the shell satisfies

$$\vec{u}\cdot\vec{u} = \frac{\dot{R}^2}{f(R)} - f(R)\dot{t}^2 = -1$$

$$f\dot{t} = \sqrt{f + \dot{R}^2} \equiv \mathcal{E} \tag{B.21}$$

This $\mathcal{E}$ can be interpreted as the energy of a (fictitious) unit mass attached to the surface of the shell. Detailed calculation shows that

$$K^{\tau}{}_{\tau} = \frac{\dot{\mathcal{E}}}{\dot{R}}, \qquad K^{\theta}{}_{\theta} = K^{\phi}{}_{\phi} = \frac{\mathcal{E}}{R} \tag{B.22}$$

Then the junction condition (B.11) leads to the expression of the shell density σ and shell pressure P,

$$\sigma = -\frac{1}{4\pi}\frac{[\mathcal{E}]}{R} \quad \text{or} \quad [\mathcal{E}] = -\frac{M}{R} \tag{B.23}$$

$$P = \frac{1}{8\pi}\left[\frac{\dot{\mathcal{E}}}{\dot{R}} + \frac{\mathcal{E}}{R}\right] \tag{B.24}$$

where $M = 4\pi R^2\sigma$ is the proper mass of the shell. With an expression for the area of the shell $A = 4\pi R^2$, we get

$$dM + PdA = 0 \tag{B.25}$$

Using the dynamics of a spherical collapsing shell we have developed, prove the backreation effect derived by Kraus and Wilczek. The question is, if we consider the shell proper mass M as a point mass following a geodesic of an effective geometry (even though we know the shell does not follow a geodesic of either the inner or outer actual geometries), then what is the explicit form of the metric driving this point mass?

→From (B.23),

$$\sigma = -\frac{1}{4\pi}\frac{[\mathcal{E}]}{R} \qquad (B.23) \tag{B.26}$$

$$\frac{M}{R} = \mathcal{E}_- - \mathcal{E}_+$$

$$\mathcal{E}_+^2 = \left(\mathcal{E}_- - \frac{M}{R}\right)^2$$

$$f_+ + \dot{R}^2 = f_- + \dot{R}^2 + \left(\frac{M}{R}\right)^2 - \frac{2M}{R}\mathcal{E}_-$$

$$1 - \frac{2m_+}{R} = 1 - \frac{2m_-}{R} + \left(\frac{M}{R}\right)^2 - \frac{2M}{R}\mathcal{E}_-$$

$$\mathcal{E}_- = \frac{1}{M}\underbrace{(m_+ - m_-)}_{\equiv m} + \frac{M}{2R}$$

$$\mathcal{E}_-^2 = \left(\frac{m}{M} + \frac{M}{2R}\right)^2$$

$$1 - \frac{2m_-}{R} + \dot{R}^2 = \left(\frac{m}{M}\right)^2 + \left(\frac{M}{2R}\right)^2 + \frac{m}{R} \tag{B.27}$$

Compare this with the equation of motion of a point mass particle with proper mass

M and energy E,

$$\dot{R}^2 = \Big(\frac{E}{M}\Big)^2 - \Big(1 - \frac{2m_p}{R}\Big) \tag{B.28}$$

Then we get the effective Schwarzschild mass m of spherical collapsing shell.

$$m_p = m_- + \frac{m}{2} + \frac{M^2}{8R} \tag{B.29}$$

This shows that half of the shell energy back-reacts on the back ground geometry. This is the result Kraus and Wilczek derived by quantization of a shell composed of scalar field.

Our specific concern in chapter 11.2 is with a class of spherical shells moving in spherical geometries of the form

$$(ds^2)_\pm = \frac{dr^2}{f(r)} + r^2 d\Omega^2 - f(r)dt^2 \tag{B.30}$$

with different functions f_- and f_+ and time coordinates t_- and t_+ in $\mathcal{V}_-$ and $\mathcal{V}_+$. In all geometries of the form (B.30), the Einstein field equations require $T^r_r = T^t_t$,

$$G^t{}_t = \frac{\frac{\partial f}{\partial r}r - 1 + f(r)}{r^2}, \quad G^r{}_r = \frac{\frac{\partial f}{\partial r}r - 1 + f(r)}{r^2}$$

This leads eventually $S^b_{a;b} = 0$. Thus in this class of spacetimes, the ambient pressures do no work and the shell's internal energy is conserved.

Since the metric inside and outside are different from each other we have two distinct $\mathcal{E}_+$ and $\mathcal{E}_-$ for $\mathcal{E}$. (Physically, $\mathcal{E}_-$ and $\mathcal{E}_+$ could be interpreted as energies of test particles of unit rest mass attached to the inner and outer shell faces.) We will find when we derive the equation of motion of shell that it is very useful to get the expression $\mathcal{E}_\pm$ in terms of the pure metric quantity $f_\pm$ instead of the mechanical quantity $\dot{R}$. Thus let us derive this useful expression of $M\mathcal{E}_+$ and $M\mathcal{E}_-$.

By the junction condition,

$$-8\pi S^\tau{}_\tau = [K^\tau{}_\tau - K] = [-2K^\theta{}_\theta] = -2\Big(\frac{\sqrt{f_+ + \dot{R}^2}}{R} - \frac{\sqrt{f_- + \dot{R}^2}}{R}\Big)$$

where $-S^\tau{}_\tau$ is the surface density σ and we have used the fact that the background geometry is spherical, so $K^{\theta}{}_{\theta} = K^{\phi}{}_{\phi}$. Hence,

$$4\pi\sigma = \frac{1}{R}(\sqrt{f_- + \dot{R}^2} - \sqrt{f_+ + \dot{R}^2})$$

To get the geometrical expression of $\mathcal{E}_-$,

$$\begin{aligned}
\sqrt{f_+ + \dot{R}^2} &= \sqrt{f_- + \dot{R}^2} - \frac{M}{R} \\
f_+ + \dot{R}^2 &= f_- + \dot{R}^2 + \frac{M^2}{R^2} - 2\frac{M}{R}\sqrt{f_- + \dot{R}^2} \\
2\frac{M}{R}\mathcal{E}_- &= \frac{M^2}{R^2} + f_- - f_+
\end{aligned}$$

To get the geometrical expression of $\mathcal{E}_+$,

$$\begin{aligned}
\sqrt{f_- + \dot{R}^2} &= \sqrt{f_+ + \dot{R}^2} + \frac{M}{R} \\
f_- + \dot{R}^2 &= f_+ + \dot{R}^2 + \frac{M^2}{R^2} + 2\frac{M}{R}\sqrt{f_+ + \dot{R}^2} \\
2\frac{M}{R}\mathcal{E}_+ &= -\frac{M^2}{R^2} + f_- - f_+
\end{aligned}$$

$$M\mathcal{E}_- = \frac{R}{2}(f_- - f_+) + \frac{M^2}{2R} \tag{B.31a}$$

$$M\mathcal{E}_+ = \frac{R}{2}(f_- - f_+) - \frac{M^2}{2R} \tag{B.31b}$$

This is a very handy expression of $\mathcal{E}_\pm$ because of their symmetric form in terms of geometrical functions.

Let's continue our calculation to get the equation of motion of a shell $\dot{R}$.

$$M\sqrt{f_- + \dot{R}^2} = \frac{R}{2}(f_- - f_+) + \frac{M^2}{2R}$$

$$M\sqrt{f_+ + \dot{R}^2} = \frac{R}{2}(f_- - f_+) - \frac{M^2}{2R}$$

Squaring both sides of equations gives

$$f_- + \dot{R}^2 = \left(\frac{R}{2M}\right)^2 (f_- - f_+)^2 + \left(\frac{M}{2R}\right)^2 + \frac{M}{2}(f_- - f_+)$$

$$f_+ + \dot{R}^2 = \Big(\frac{R}{2M}\Big)^2 (f_- - f_+)^2 + \Big(\frac{M}{2R}\Big)^2 - \frac{M}{2}(f_- - f_+)$$

Adding both sides of equations results in

$$(f_+ + f_-) + 2\dot{R}^2 = 2\Big(\frac{R}{2M}\Big)^2 (f_- - f_+)^2 + 2\Big(\frac{M}{2R}\Big)^2$$

Finally we get the expression of $\dot{R}$ in symmetric from with respect to f_+ and f_-.

$$\dot{R}^2 = \Big(\frac{R}{2M}\Big)^2 (f_- - f_+)^2 - \frac{1}{2}(f_+ + f_-) + \Big(\frac{M}{2R}\Big)^2 \tag{B.34}$$

For a charged black hole,

$$\dot{R}^2 = \Big(\frac{R}{2M}\Big)^2 \Big(\frac{2m}{R} - \frac{2q\bar{e}}{R^2}\Big)^2 - \frac{1}{2}(f_+ + f_-) + \Big(\frac{M}{2R}\Big)^2 \tag{B.35}$$

(B.34) shows that the equation of motion of a shell of finite mass depends on two different geometries (exterior and interior one) while a test particle moves only in one geometry f (for Schwarzschild geometry). Our goal is to construct a single "effective" geometry in which the shell moves as if it were a test particle, and hence an effective action that describes its tunneling.

(B.34) can be expressed in different ways for our typical classical interpretation.

$$\dot{R}^2 = \overline{\mathcal{E}}^2 - \frac{M}{R}\overline{\mathcal{E}} - f_+ + \Big(\frac{M}{2R}\Big)^2 \tag{B.36a}$$

$$\dot{R}^2 = \overline{\mathcal{E}}^2 + \frac{M}{R}\overline{\mathcal{E}} - f_- + \Big(\frac{M}{2R}\Big)^2 \tag{B.36b}$$

solving for $\overline{\mathcal{E}}$ results in

$$M\overline{\mathcal{E}} = \pm M\sqrt{R_\tau^2 + f_+} + \frac{M^2}{2R} \tag{B.37a}$$

$$M\overline{\mathcal{E}} = \pm M\sqrt{R_\tau^2 + f_-} - \frac{M^2}{2R} \tag{B.37b}$$

(B.37b) gives a kind of classical interpretation of $M\overline{\mathcal{E}}$ for charged shell in empty space ($f_- = 1$). This is a decomposition of the total (conserved) mass-energy m in a form whose Newtonian counterpart is self-evident. Note $M\overline{\mathcal{E}}$ does not contain electric energy for $f_- = 1$; Note that even in the case of $f_- \neq 1$, $M\overline{\mathcal{E}}$ does not contain electric energy. Schwarzschild mass of the shell $m = m_+^U - m_-^U$ contains electric energy where m_+^U is the Schwarzschild mass including shell and m_-^U is the Schwarzschild mass excluding shell.

Later on we will compare the equation of motion derived by Euler-Lagrange equation. Thus let's derive another form of equation of motion of shell in a spherically symmetric geometry. From (B.23) we get the expression

$$M(\mathcal{E}_+ + \mathcal{E}_-) = 2m - \frac{2}{R}q\bar{e}$$

Differentiation both sides results in

$$\dot{M}(\mathcal{E}_+ + \mathcal{E}_-) + M(\dot{\mathcal{E}}_+ + \dot{\mathcal{E}}_-) = 2\dot{m} + \frac{2R_\tau}{R^2}q\bar{e}$$

Since $\dot{m} \equiv \frac{dm}{d\tau} = 0$ for spherical geometries,

$$\frac{dR}{d\tau}\frac{dM}{dR}(\mathcal{E}_+ + \mathcal{E}_-) + M\Big(R_\tau \frac{f'_+(R) + 2R_{\tau\tau}}{2\mathcal{E}_+} + R_\tau \frac{f'_-(R) + 2R_{\tau\tau}}{2\mathcal{E}_-}\Big) = \frac{2R_\tau}{R^2}q\bar{e}$$

Multiply by $\frac{1}{2MR_\tau}$

$$\frac{\overline{\mathcal{E}}}{M}\frac{dM}{dR} + \frac{1}{2}\overline{\Big(\frac{f'(R)}{\mathcal{E}}\Big)} + R_{\tau\tau}\overline{\Big(\frac{1}{\mathcal{E}}\Big)} - \frac{q\bar{e}}{MR^2} = 0 \tag{B.38}$$

where $f'(R) \equiv \frac{df(R)}{dR}$. We are going to use this expression of equation of motion to compare the equation of motion derived from the Euler-Lagrange equation. If they consistent each other, it means the "effective action" we used for that Euler-Lagrange equation is correct.

The action for a charged shell is the action (B.13)+(B.15) with an interaction term

$$I_{int} = \int J^\alpha A_\alpha \sqrt{-g}d^4x \tag{B.39}$$

plus the usual free-field Lagrangian $-\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta}$. Our spherical constraint admits only radial electric fields, for which A_α is gauge-reducible in static (r, t) coordinates to a single component

$$A_t = -\varphi = -\frac{e_\pm}{r} \tag{B.40}$$

in $\mathcal{V}_\pm$ respectively, up to an additive constant which could be adjusted across the shell. The shell's charge is $q = e_+ - e_-$ and (B.39) reduces to

$$I_{int} = \int qA_\alpha dx^\alpha \tag{B.41}$$

Adding (B.41) to the action does not affect the junction conditions or equations of motion in the forms (B.11), (B.12), (B.23), (B.25) and (B.34). (They were obtained by varying the metric in the action; however, (B.41) is independent of the metric.)

The effective Lagrangian of a charged shell emerges if we retain just the mechanical, non-geometrical parts (B.15)+(B.39) of the original action:

$$I_{eff} = -\int M d\tau + \int qA_\alpha dx^\alpha \tag{B.42}$$

To fix the Lagrangian we still need to choose a time-coordinate invariant under Euler-Lagrange variation of the particle's world-line.(Proper time τ(=arc length) is clearly inadmissible) As we shall see, a formally successful(as well as physically desirable) choice is a static observer's time t, split evenly between the alternatives t_+ and t_- by taking the arithmetic mean of the + and - actions. This prescription leads to the following explicit form of (B.42):

$$I_{eff} = \int \overline{Ldt}, \qquad L = -M\frac{f}{\mathcal{E}} - q\varphi \tag{B.43}$$

The upper bar denotes an arithmetic mean, $\overline{X} \equiv \frac{1}{2}(X_+ + X_-)$. With this Lagrangian (B.43), we can drive the same equation of motion. By the definition of $\mathcal{E}$,

$$\mathcal{E}^2 = f + \left(\frac{dt}{d\tau}\right)^2\left(\frac{dR}{dt}\right)^2 = f + \left(\frac{\mathcal{E}}{f}\right)^2 R_t^2$$

$$f^2 - R_t^2 = \frac{f^3}{\mathcal{E}^2} \tag{B.44}$$

Differentiating both sides of (B.44) by R_t and R yields,

$$\frac{\partial}{\partial R_t} \times (B.44) \quad \rightarrow \quad \frac{\partial \mathcal{E}}{\partial R_t} = \left(\frac{\mathcal{E}}{f}\right)^3 R_t = \left(\frac{\mathcal{E}}{f}\right)^2 R_\tau \tag{B.45}$$

$$\frac{\partial}{\partial R} \times (B.44) \quad \rightarrow \quad \frac{\partial \mathcal{E}}{\partial R} = \frac{\mathcal{E}}{2f^2}(3f - 2\mathcal{E}^2)\frac{df}{dR} \tag{B.46}$$

If we couple this with the Lagrangian (B.43)

$$\frac{\partial L}{\partial R_t} = M\frac{f}{\mathcal{E}^2}\frac{\partial \mathcal{E}}{\partial R_t} = M\frac{f}{\mathcal{E}^2}\Big(\frac{\mathcal{E}}{f}\Big)^2 R_\tau = \frac{M}{f}R_\tau \tag{B.47}$$

Now we can calculate the two terms of the Euler-Lagrange equation.

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{\partial L}{\partial R_t}\Big) &= \frac{d\tau}{dt}\frac{d}{d\tau}\Big(\frac{M}{f}R_\tau\Big) = \frac{f}{\mathcal{E}}\frac{\frac{d}{d\tau}(MR_\tau)f - MR_\tau\frac{df}{d\tau}}{f^2} \\
&= \frac{1}{\mathcal{E}f}\Big(\frac{dR}{d\tau}\frac{\partial M}{\partial R}R_\tau f + MR_{\tau\tau}f - MR_\tau\frac{dR}{d\tau}\frac{df}{dR}\Big) \\
&= \frac{f}{\mathcal{E}}\Big(\frac{M}{f}R_{\tau\tau} + \frac{1}{f}\frac{dM}{dR}R_\tau^2 - \frac{M}{f^2}\frac{df}{dR}R_\tau^2\Big) \\
&= \frac{M}{\mathcal{E}}R_{\tau\tau} + \Big(\frac{1}{\mathcal{E}}\frac{dM}{dR} - \frac{M}{\mathcal{E}f}\frac{df}{dR}\Big)R_\tau^2 \\
&= \frac{M}{\mathcal{E}}R_{\tau\tau} + \Big(\frac{dM}{dR} - \frac{M}{f}\frac{df}{dR}\Big)\frac{\mathcal{E}^2 - f}{\mathcal{E}}
\end{aligned}$$

The other term of the Euler-Lagrange equation is

$$\begin{aligned}
-\frac{\partial L}{\partial R} &= \frac{\partial M}{\partial R}\frac{f}{\mathcal{E}} + M\frac{\frac{\partial f}{\partial R}\mathcal{E} - f\frac{\partial \mathcal{E}}{\partial R}}{\mathcal{E}^2} - \frac{qe}{R^2} \\
&= \frac{f}{\mathcal{E}}\frac{dM}{dR} + \frac{M}{\mathcal{E}}\frac{df}{dR} - \frac{Mf}{\mathcal{E}^2}\frac{\mathcal{E}}{2f^2}(3f - 2\mathcal{E}^2)\frac{df}{dR} - \frac{qe}{R^2} \\
&= \frac{f}{\mathcal{E}}\frac{dM}{dR} + \frac{M}{\mathcal{E}}\frac{df}{dR} - \frac{3M}{2\mathcal{E}}\frac{df}{dR} + \frac{M\mathcal{E}}{f}\frac{df}{dR} - \frac{qe}{R^2}
\end{aligned}$$

then Euler-Lagrange equation results in,

$$\begin{aligned}
&\frac{M}{\mathcal{E}}R_{\tau\tau} + \mathcal{E}\frac{dM}{dR} - \frac{\mathcal{E}M}{f}\frac{df}{dR} - \frac{f}{\mathcal{E}}\frac{dM}{dR} + \frac{M}{\mathcal{E}}\frac{df}{dR} \\
&+\frac{f}{\mathcal{E}}\frac{dM}{dR} + \frac{M}{\mathcal{E}}\frac{df}{dR} - \frac{3M}{2\mathcal{E}}\frac{df}{dR} + \frac{M\mathcal{E}}{f}\frac{df}{dR} - \frac{qe}{R^2} = 0 \\
&R_{\tau\tau}\Big(\frac{1}{\mathcal{E}}\Big) + \frac{1}{2\mathcal{E}}\frac{df}{dR} + \frac{\mathcal{E}}{M}\frac{dM}{dR} - \frac{qe}{MR^2} = 0
\end{aligned}$$

Then the resulting equation of motion for our Lagrangian (B.43) is,

$$R_{\tau\tau}\overline{\Big(\frac{1}{\mathcal{E}}\Big)} + \frac{1}{2}\overline{\Big(\frac{f'(R)}{\mathcal{E}}\Big)} + \frac{\overline{\mathcal{E}}}{M}\frac{dM}{dR} - \frac{q}{MR^2}\overline{e} = 0 \tag{B.48}$$

where prime indicates that it is a differentiation with respect to R.

Thus (B.43) is indeed the desired *effective particle action* for the shell.

We checked out that (B.43) is the right form of Lagrangian for charged shell by comparing equation of motion derived from junction condition (B.23). We can check it out in another way by comparing energy derived from junction condition.

The Lagrangian of a charged particle is

$$L = -M\frac{d\tau}{dt} + qA_\mu\frac{dx^\mu}{dt} = -M\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} + qA_\mu\frac{dx^\mu}{dt} \tag{B.49}$$

Corresponding canonical energy momentum p_α is

$$p_\alpha \equiv \frac{\partial L}{\partial \dot{x}^\alpha} = (-M)\frac{-2g_{\alpha\nu}\frac{dx^\nu}{dt}}{2\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}} + qA_\alpha = Mg_{\alpha\nu}\frac{dx^\nu}{d\tau} + qA_\alpha \tag{B.50}$$

Thus the energy of this charged particle $-p_0$ is

$$-p_0 = -Mg_{0\nu}\frac{dx^\nu}{d\tau} - qA_0 \tag{B.51}$$

Note that in pure gravitational system $-p_0 = -Mg_{0\nu}\frac{dx^\nu}{d\tau}$. If there is electric force, this quantity is not conserved so is meaningless.

For our case of a spherical geometry,

$$ds^2 = -f(r,t)dt^2 + \frac{dr^2}{f(r,t)} + r^2 d\Omega^2, \tag{B.52}$$

$$-p_0 = Mf\frac{dt}{d\tau} + q\varphi \tag{B.53}$$

$$-p_0 = Mf\frac{dt}{d\tau} + q\varphi\left(= \frac{fM}{\sqrt{f - \frac{1}{f}\left(\frac{dr}{dt}\right)^2}} + q\varphi\right) \tag{B.54}$$

For the case of a shell, let us assume the effective Lagrangian as the arithmetic mean Lagrangian of inner and outer surfaces,

$$\overline{L} \equiv \frac{1}{2}(L_+ + L_-) = \frac{1}{2}\Big(-M\frac{d\tau}{dt_+} + qA_{\mu+}\frac{dx_+^\mu}{dt_+} - M\frac{d\tau}{dt_-} + qA_{\mu-}\frac{dx_-^\mu}{dt_-}\Big) \tag{B.55}$$

Then the corresponding effective canonical energy momentum is

$$\overline{p_\alpha} = \frac{1}{2}\Big(Mg_{\alpha\nu+}\frac{dx^\nu_+}{d\tau} + qA_{\alpha+} + Mg_{\alpha\nu-}\frac{dx^\nu_-}{d\tau} + qA_{\alpha-}\Big) \tag{B.56}$$

For the spherical geometry (B.52)

$$\begin{aligned} \overline{-p_0} &= \frac{1}{2}\Big(Mf_+\frac{dt_+}{d\tau} + q\varphi_+ + Mf_-\frac{dt_-}{d\tau} + q\varphi_-\Big) && \text{(B.57)} \\ &= \frac{1}{2}(M\mathcal{E}_+ + q\varphi_+ + M\mathcal{E}_- + q\varphi_-) && \text{(B.58)} \\ &= M\overline{\mathcal{E}} + q\overline{\varphi} && \text{(B.59)} \end{aligned}$$

(Or if we define $M\mathcal{E}_\pm = Mf_\pm\frac{dt_\pm}{d\tau} + q\varphi_\pm$ then, $\overline{-p_0} = M\overline{\mathcal{E}}$.) As we have seen for shell dynamics using the junction condition, the shell's internal energy is conserved in the (B.52) geometry. Thus we did check out that (B.55) is the right choice of Lagrangian for a spherical shell.

Using the junction condition, we did not need the detailed expression of the Lagrangian for the shell (B.43) to show that $M\overline{\mathcal{E}} + \frac{q}{R}\overline{e}$ is a conserved quantity.

The conserved energy of a charged shell in Reissner-Nordström fields is

$$E \equiv \overline{-p_0} = m \tag{B.60}$$

where $m = m_+ - m_-$, equal to the shell's Schwarzschild mass.

B.3 Shell junction condition via variational principle

Derive the shell junction condition via variational principle $g_{\alpha\beta} \to g_{\alpha\beta}+\delta g_{\alpha\beta}$. Variation $\delta g_{\alpha\beta}$ can be chosen arbitrarily. Since our interest here is in the shell, we take $\delta g_{\alpha\beta} \neq 0$ only in neighborhood of a shell. The following is the general form of the variation of

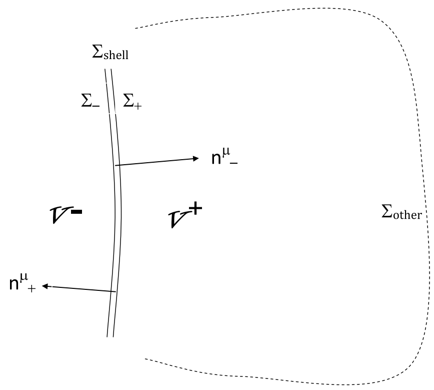

Figure B.1: Inner Σ_- and outer Σ_+ boundaries of the shell. There is also other boundary which is not shell boundary.

the shell action.

$$\begin{aligned}
16\pi\delta I &= -\delta\int_{bulk} d^4x\sqrt{-g}R + \delta\int 2\epsilon K d\Sigma_{other} - \delta\int 2\epsilon[K]d\Sigma_{shell} \\
&\quad +16\pi\int \frac{1}{2}T^{\alpha\beta}\delta g_{\alpha\beta}\sqrt{-g}d^4x \\
&\quad +16\pi\int \frac{1}{2}S^{ab}\delta g_{ab}d\Sigma_{shell} = 0 \qquad \text{(B.61)}
\end{aligned}$$

First, calculate the variation of Einstein Hilbert action part.

$$\begin{aligned}
16\pi\delta I_{bulk} &= \int_{bulk} d^4x\delta(\sqrt{-g}R) \\
&= \int_{bulk} d^4x(\delta\sqrt{-g}R + \sqrt{-g}\delta g^{\alpha\beta}R_{\alpha\beta}) + \int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} \\
&= \int_{\infty} \sqrt{-g}d^4x\delta g^{\alpha\beta}\underbrace{\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)}_{\equiv G_{\alpha\beta}} + \int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}
\end{aligned}$$

Where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is an Einstein tensor. Because of the shell boundary, we cannot ignore second integral. Let's calculate this second integral.

$$\begin{aligned}
\int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} &= \int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\Big((\delta\Gamma^{\mu}_{\alpha\beta})_{|\mu} - (\delta\Gamma^{\mu}_{\alpha\mu})_{|\beta}\Big) \\
&= \int_{bulk} d^4x\sqrt{-g}\Big((g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta})_{|\mu} - (g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\mu})_{|\beta}\Big) \\
&= \int_{bulk} d^4x\sqrt{-g}\Big(\underbrace{g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta}}_{\equiv\sigma^{\mu}}\Big)_{|\mu} \\
&= \int_{bulk} d^4x\sqrt{-g}\sigma^{\mu}{}_{|\mu} = \int_{\mathcal{V}^+} d^4x\sqrt{-g}\sigma^{\mu}{}_{|\mu} + \int_{\mathcal{V}^-} d^4x\sqrt{-g}\sigma^{\mu}{}_{|\mu} \\
&= \int \epsilon\sigma^{\mu}_{+}n_{\mu+}d\Sigma_{+} + \int \epsilon\sigma^{\mu}_{-}n_{\mu-}d\Sigma_{-} + \int \epsilon\sigma^{\mu}n_{\mu}d\Sigma_{other} \\
&= \epsilon\int(\sigma^{\mu}_{+} - \sigma^{\mu}_{-})n_{\mu+}d\Sigma_{shell} + \int \epsilon\sigma^{\mu}n_{\mu}d\Sigma_{other}
\end{aligned}$$

Before we proceed to calculate $\int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}$, let us calculate $\int \sigma^{\mu}n_{\mu}d\Sigma$

in Gaussian coordinate. In Gaussian coordinate ($\stackrel{*}{=}$ means in Gaussian coordinate),

$$ds^2 \stackrel{*}{=} g_{ab}dx^a dx^b + \epsilon dt^2, \quad n^\alpha = \delta^\alpha_0, \quad n_\alpha = g_{\alpha\beta}n^\beta = g_{\alpha 0}n^0 = \epsilon\delta^0_\alpha$$

$${}^{(n+1)}\Gamma^c_{ab} = {}^{(n)}\Gamma^c_{ab}, \qquad \delta^{(n+1)}\Gamma^c_{ab} = \delta^{(n)}\Gamma^c_{ab}$$

$$\Gamma^0_{\alpha 0} = \frac{1}{2}g^{0\gamma}(g_{\alpha\gamma,0} + g_{0\gamma,\alpha} - g_{\alpha 0,\gamma}) \stackrel{*}{=} \frac{1}{2}g^{00}(g_{\alpha 0,0} + g_{00,\alpha} - g_{\alpha 0,0}) \stackrel{*}{=} 0$$

$$\begin{aligned}\Gamma^0_{ab} &= \frac{1}{2}g^{0\gamma}(g_{a\gamma,b} + g_{b\gamma,a} - g_{ab,\gamma}) \stackrel{*}{=} \frac{1}{2}g^{00}(g_{a0,b} + g_{b0,a} - g_{ab,0}) \\ &\stackrel{*}{=} -\frac{1}{2}g^{00}g_{ab,0} \stackrel{*}{=} -g^{00}K_{ab} \stackrel{*}{=} -\epsilon K_{ab}\end{aligned}$$

$$\begin{aligned}\Gamma^a_{0b} &= \frac{1}{2}g^{a\gamma}(g_{0\gamma,b} + g_{b\gamma,0} - g_{0b,\gamma}) \stackrel{*}{=} \frac{1}{2}g^{ac}(g_{0c,b} + g_{bc,0} - g_{0b,c}) \\ &= \frac{1}{2}g^{ac}g_{bc,0} = g^{ac}K_{bc} = K^a_b\end{aligned}$$

Then

$$\delta(d\Sigma) = \delta(\sqrt{{}^3g}d^3x) = -\frac{\sqrt{{}^3g}}{2}g_{ab}\delta g^{ab}d^3x = -\frac{g_{ab}}{2}\delta g^{ab}d\Sigma$$

Hence, under the condition $\delta K_{ab} = 0$,

$$\delta\Gamma^a_{0b} \stackrel{*}{=} \delta g^{ac}K_{bc}, \quad \delta\Gamma^0_{\alpha\beta} \stackrel{*}{=} \delta\Gamma^0_{\alpha b} \stackrel{*}{=} \delta\Gamma^0_{0b} \stackrel{*}{=} 0$$

$$\sigma^0 \equiv g^{\alpha\beta}\delta\Gamma^0_{\alpha\beta} - g^{0\alpha}\delta\Gamma^\beta_{\alpha\beta} \stackrel{*}{=} -g^{00}\delta\Gamma^\beta_{0\beta} = -g^{00}\delta\Gamma^b_{0b} = -g^{00}\delta g^{bc}K_{bc} = -\epsilon\delta g^{bc}K_{bc}$$

Hence

$$\begin{aligned}\int_{bulk} d^4x\sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta} &= \int[\sigma^0]d\Sigma_{shell} + \int \sigma^0 d\Sigma_{other} \\ &= -\epsilon\int[K_{bc}]\delta g^{bc}d\Sigma_{shell} - \epsilon\int K_{bc}\delta g^{bc}d\Sigma_{other}\end{aligned}$$

Thus

$$\begin{aligned}16\pi\delta I_{bulk} &= \int_\infty \sqrt{-g}d^4x\delta g^{\alpha\beta}G_{\alpha\beta} \\ &\quad -\epsilon\int[K_{ab}]\delta g^{ab}d\Sigma_{shell} - \epsilon\int K_{ab}\delta g^{ab}d\Sigma_{other}\end{aligned}$$

$$\begin{aligned}
\delta \int_{boundary} 2\epsilon K d\Sigma &= 2\epsilon \int \underbrace{\delta K_{ab}}_{=0} g^{ab} d\Sigma + 2\epsilon \int K_{ab}\delta g^{ab} d\Sigma + 2\epsilon \int K_{ab} g^{ab} \delta(d\Sigma) \\
&= 2\epsilon \int \delta g^{ab}\Big(K_{ab} - \frac{1}{2} g_{ab} K\Big) d\Sigma
\end{aligned}$$

Thus $\delta I = 0$ in (B.61) yields

$$\begin{aligned}
&-\int_{\infty} \sqrt{-g} d^4 x \delta g^{\alpha\beta} G_{\alpha\beta} + \epsilon \int [K_{ab}]\delta g^{ab} d\Sigma_{shell} + \epsilon \int K_{ab}\delta g^{ab} d\Sigma_{other} \\
&+2\epsilon \int \Big(K_{ab} - \frac{1}{2} K g_{ab}\Big)\delta g^{ab} d\Sigma_{other} - 2\epsilon \int \Big[K_{ab} - \frac{1}{2} K g_{ab}\Big]\delta g^{ab} d\Sigma_{shell} \\
&+16\pi \int \frac{1}{2} T^{\alpha\beta}\delta g_{\alpha\beta}\sqrt{-g} d^4 x + 16\pi \int \frac{1}{2} S^{ab}\delta g_{ab} d\Sigma = 0
\end{aligned}$$

Thus we get $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$ and

$$8\pi S^{ab} = -\epsilon[K_{ab} - K g_{ab}] \tag{B.62}$$

This result is the same as the shell junction condition derived by applying Einstein's field equations to the mathematics of the shell formula.

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