A non-local reaction advection-diffusion model for self-interacting species

by

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ABSTRACT

In biological models, advection is inherently a non-local process. In this thesis, we proposed a natural extension of the non-local advection-diffusion model in [7] to include the reaction term (birth and death process). This thesis begins with an investigation of the well-posedness and existence of travelling wave solutions for this non-local reaction-advection-diffusion (RAD) equation. We prove the local-in-time existence and positivity of solutions under $H^3(\mathbb{R})$ initial conditions and provide a continuation criterion of the equation. Subsequently, we explore the existence of travelling wave solutions of this non-local RAD using a combination of perturbation methods, Fredholm operator theory, and Banach's fixed point theorem. Our analysis reveals that such solutions exist when the non-local advection term is small. Finally, we simulate the travelling wave solution to verify our theoretical findings and speculate the solution with a large advection term.

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Chapter 1

Introduction

1.1 Non-local Models in Mathematical Biology

Mathematical Modelling of spatial population dynamics provides a way of establishing connections between observed patterns and the biological mechanism behind them. From predator-prey interactions [20, 23] to inter-species competition [5, 8], these models generally combine a mathematical formulation of the birth and death processes with some spatial consideration, such as dispersal movements. The movements are often assumed to be in the form of diffusion and advection [18]. Therefore, the resulting model is typically a system of reaction-advection-diffusion (RAD) equations. This is our main focus of the thesis, a special class of RAD models in mathematical biology involving non-local advection terms.

Non-locality emerges naturally from individuals gaining information about the neighbouring environment. In ecology, animals sense their surrounding and make decisions to find or keep away from prey or predators and/or aggregate in groups [1, 4, 12]. The process of gaining information about the presence or the absence of others in the environment is fundamentally non-local [1, 14]. This non-local interaction is not restricted to animals and also applies to cells [10]. Mathematically, these non-local interactions of neighbouring individuals can be formulated as non-local advection terms in a continuous model.

From a mathematical perspective, the non-local advection terms appear as integral terms inside a derivative. Many classical theories and methods developed for local models no longer apply because the equation is intrinsically non-local [2, 9]. For instance, single-species aggregation models with non-local advection terms avoid blowup and omit global in-time solutions compared to their local formulation [6, 11]. This shows that the non-locality offers new challenges and opportunities for future studies.

In the next chapter, we will begin by introducing the mathematical formulation of the RAD equations.

1.2 Introduction to the non-local Reaction-Advection-Diffusion models for self-interacting species

Reaction-advection-diffusion equations have long been an area of interest in mathematics. The equation is in the form of PDEs:

$$\frac{\partial c}{\partial t} = \nabla \cdot (D \nabla c) - \nabla \cdot (\vec{V} c) + \rho, \tag{1.1}$$

where each term is given by:

- c: the variable of interest (density concentration for mass transfer, temperature for heat transfer).

- D: the diffusivity constant. For mass transfer, it is the viscosity, and for heat transfer, it is the thermal diffusivity.

- $\vec{V}$: The vector field of quantity c moves in.
- ρ: The source or the sink of the quantity c

Next, we state the physical intuition of each term on the right-hand side of (1.1):

The first term, $\nabla \cdot (D\nabla c)$, describes the diffusion of the quantity. If D is a constant (the case in our paper), this term will be a scalar multiple of the Laplacian Δc. Physically, diffusion is the net movement of the quantity from high-density to low-density regions.

The second term, $\nabla \cdot (\vec{V}c)$, describes the advection of the quantity. In physics, the advection is the process of transporting the substance due to the motion of a fluid. An example is somebody dumping a buck of salt into a river, then the salt concentration will be carried by the stream of the river $\vec{V}$. This advection term models the transportation of the salinity in our example. Compared to diffusion, the salt will also diffuse in the river, but the diffusion will still occur despite no flow in the water. This paper mainly focuses on equations such that $\vec{V}$ is non-local. That is the *non-local advection.*

The last term, ρ, represents the creation or destruction of the quantity. Generally, this term arises when the quantity of interest is created or destroyed due to chemical reactions. The function R could depend on the c and other quantities. In a biological context, this generally corresponds to the birthrate (or death rate) of the species in the model.

Many studies have been conducted on RAD equations and their variants. A common case of these models in the field of mathematical biology is the aggregation equation:

$$\begin{cases} u_t = & D\Delta u - \nabla \cdot (uf) + \rho \\ f = & \nabla(\mathcal{K} * u) \end{cases},$$

where $\mathcal{K}$ is a spatial averaging kernel. For our model, we based on the work of J. Potts and M. Lewis [21], which illustrate the non-local interaction between individuals governed by the indicator function $\mathcal{K} = \chi_{|x|\leq R}$ (See Scenario A), and on top of this, we include a logistic birth and death process:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial}{\partial x}\left(u\frac{\partial A}{\partial x}\right) + ru(1-u/K),$$

where

$$A = \frac{1}{2R}\int_{-R}^{R} u(x+y,t)\,dy.$$

Here $u(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ models the time evolution of the spatial density of a population. The parameters $D, r, K > 0$ are the diffusion coefficient, growth rate and carrying capacity, respectively. The parameter $\beta \in \mathbb{R}\backslash\{0\}$ is the advection coefficient. If $\beta > 0$, then we model the self-attraction between individuals, and $\beta < 0$ shows self-avoidance. Next, consider the non-dimensionalization

$$Ku^* = u \quad t^* = rt \quad x^* = \sqrt{r/D}x \quad R^* = \sqrt{r/D}R \quad \beta^* = \beta\frac{K}{D}.$$

Dropping the asterisk, we have the following dimensionless equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial}{\partial x}\left(u\frac{\partial A}{\partial x}\right) + u(1-u), \quad A = \frac{1}{2R}\int_{-R}^{R} u(x+y,t)\,dy. \tag{1.2}$$

From here, we finished the modelling framework.

1.3 Well-posedness and travelling wave solutions

There are a couple of equations we can ask for our model:

One fundamental question is the well-posedness of the equation. That is, given a reasonable initial condition, will the model yield a reasonable solution? In [7], V. Giunta, T. Hillen, M. Lewis, and J. Potts proved the global existence of a similar equation with a smooth spatial averaging kernel and no reaction term. However, in our case, the indicator function is not a continuous function, and the effect of the logistic growth on the solution is unknown. Therefore, the existence of the solution will be one of the major topics of this thesis.

Another interesting question is the existence of travelling wave solutions. In [25], Y. Zhang demonstrated the existence of travelling wave solutions to a cell-adhesion model. The cell-adhesion model contains a different non-local term compared to our equation (1.2). The impact of different non-local terms on the existence of travelling wave solutions is an unanswered question.

The goal of this thesis is to address the questions listed above. The rest of the thesis is organized as follows: in the second chapter, we proved the well-posedness problem, and then the third chapters show the existence of a travelling wave solution. Finally, we performed numerical approximations to the travelling wave solution in the fourth chapter.

Chapter 2

Well-posedness of the equation

In this chapter, our goal is to study the well-posedness of the reaction-advection-diffusion equations for self-interacting species. Recall the non-dimensionalized equation eq. (1.2)

$$\begin{cases} \frac{\partial u}{\partial t} = & \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial}{\partial x}\left(u \frac{\partial A}{\partial x}\right) + u(1-u) \\ A = & \dfrac{1}{2R} \int_{-R}^{R} u(x+y,t)\,dy \\ u(x,0) = & u_0 \end{cases} \tag{2.1}$$

where $u(x,t) : \mathbb{R} \times [0,\infty)$ describes the spatial density of the population. Naturally, there are questions arising from the equation:

- Under what initial condition u_0 does the unique solution to (2.1) exist?
- Since $u(x,t)$ is a distribution, do we have a positive solution to (2.1). That is $u_0 \geq 0$ gives $u(x,t) \geq 0$

The purpose of this chapter is to answer the above 2 questions.

2.1 Preliminary

Before stating our main results, we will first define the function spaces that we are working with.

Definition 1 (L^p spaces)**.** *Let $\Omega \subseteq \mathbb{R}$ be a measurable set. Let $p \in [1, \infty]$, then we define the spaces $L^p(\Omega)$ to be:*

- *For $p < \infty$, $L^p(\Omega) = \{f : \Omega \to \mathbb{R} : \int_\Omega |f(x)|^p \, dx < \infty\}$. Furthermore, we equip this space with the norm: $\|f\|_{L^p(\Omega)} = (\int_\Omega |f(x)|^p \, dx)^{1/p}$.*

  For $p = \infty$, $L^\infty(\Omega) = \{f : \Omega \to \mathbb{R} : ess\ sup_{x\in\Omega} f(x) < \infty\}$. Furthermore, we quip this space with the norm: $\|f\|_{L^\infty(\Omega)} = ess\ sup_{x\in\Omega} f(x)$

- *For vector-valued functions $\vec{f} \in \mathbb{R}^N$. We equip these functions with the norm: $\left\|\vec{f}\right\|_{L^p(\Omega)} = \sum_{i=1}^N \|f_i\|_{L^p(\Omega)}$*

- *Let $p_1, p_2 \in [1, \infty]$, we equip the space $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$ with the norm: $\|f\|_{L^{p_1}(\Omega)\cap L^{p_2}(\Omega)} = \|f\|_{L^{p_1}(\Omega)} + \|f\|_{L^{p_2}(\Omega)}$*

Definition 2 (C^k spaces)**.** *Let $\Omega \subseteq \mathbb{R}$ be an open set. Let k be a non-negative integer (including ∞), then define the following spaces:*

- *The space of k-th continuously differentiable functions:*

  $$C^k(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is } k\text{-th continuously differentiable}\},$$

  with norm

  $$\|f\|_{C^k(\Omega)} = \sum_{|\alpha|\leq k} \|\partial_\alpha f\|_{L^\infty(\omega)}$$

- *The space of k-th continuously differentiable functions with compact supports:*

  $$C_c^k(\Omega) = \{f \in C^k : f \text{ is compactly supported}\}.$$

Definition 3 (Locally integrable function)**.** *Let $\Omega \in \mathbb{R}$ be an open set. Then, define space:*

- $L^1_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : f \in L^1(K)$ *for all compact subset* $K \subseteq \Omega\}$.

We call the space L^1_{loc} the space of locally integrable functions.

Before defining what a Sobolev space is, we must define the weak derivative.

Definition 4 (Weak derivative)**.** *Let $u, v \in L^1(\Omega)$. Then, we say that v is the k-th weak derivative of u if*

$$\int_\Omega u\partial^k\psi\, dx = (-1)^k \int_\Omega v\psi\, dx \quad \text{for all } \psi(x) \in C^k_c(\Omega).$$

Further, we use $D^k u = v$ to denote the weak derivative.

Definition 5 (Sobolve space)**.** *Let $\Omega \subset \mathbb{R}$ be a measurable set. Let $p \in [1, \infty]$ and m be a non-negative integer. Then, we define the Sobolev space:*

$$W^{m,p}(\Omega) = \{f : \Omega \to \mathbb{R} : D^k f \in L^p(\Omega) \text{ for all } k \leq m\}.$$

We equip this space with the norm:

$$\|f\|_{W^{m,p}} = \sum_{k \leq m} \left\|D^k f\right\|_{L^p(\Omega)},$$

so the space $W^{m,p}(\Omega)$ is a well-defined Banach space.

For the special case $p = 2$, we write:

$$H^m(\Omega) = W^{m,2}(\Omega).$$

One should notice that the $H^m(\Omega)$ is a Hilbert space with the inner product:

$$\langle f, g\rangle_{H^m(\Omega)} = \sum_{k\leq m} \left\langle \partial^k f, \partial^k g\right\rangle_{L^2(\Omega)},$$

for $f, g \in H^m(\Omega)$. Here, $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ denote the L^2 inner product.

Definition 6. *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Then, we equip the space $X \cap Y$ with norm:*

$$\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y,$$

so the space $X \cap Y$ is a well-defined Banach space.

Definition 7 (Convolution)**.** *We define the convolution of two functions $f, g : \mathbb{R}^d \to \mathbb{R}$ to be*

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

Notice that we can rewrite the non-local term in eq. (2.1) as $A = \frac{1}{2R}\chi_{|x|\leq R} * u(x)$, where $\chi_{|x|\leq R}$ is the indicator function.

Then, we can state Young's convolution inequality. For a proof of the lemma, please see [13].

Lemma 2.1.1 (Young's convolution inequality)**.** *Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1,$$

where $1 \leq r, p, q \leq \infty$*. Then, we have*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}. \tag{2.2}$$

Another inequality used in this paper is the Holder's inequality. For the proof, we refer to [13].

Lemma 2.1.2 (Holder's inequality)**.** *Let* $f \in L^p(\mathbb{R})$ *and* $g \in L^p(\mathbb{R})$ *and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r},$$

where $1 \leq r, p, q \leq \infty$*. Then, we have*

$$\|fg\|_{L^r(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \|g\|_{L^q(\mathbb{R})}. \tag{2.3}$$

We will also state the classical Gröwnwall's inequality throughout the paper. We refer to [19] for the proof.

Lemma 2.1.3 (Gröwnwall's inequality)**.** *Let* J *be an interval in the form* $[a, \infty)$, $[a, b]$ *or* $[a, b)$ *with* $a < b$*. Let* $f(t)$ *and* $u(t)$ *be real-valued continuous functions and constant* $C \geq 0$*. If* $f(t) \geq 0$ *and*

$$u(t) \leq C + \int_a^t f(s)u(s)\,ds, \quad \forall t \in J,$$

then

$$u(t) \leq C \exp\left\{\int_a^t f(s)\,ds\right\}, \quad t \in J$$

Definition 8 (Heat equation semi-group)**.** *The heat equation semi-group operator*

$e^{D\Delta t}$ on $\mathbb{R}$ is an operator defined as the following:

$$e^{D\Delta t} f(t,\cdot) = G_D(t,\cdot) * f(\cdot),$$

where G_D is the heat kernel $G_D(t,x) = \dfrac{1}{(4\pi tD)^{1/2}} e^{-\frac{|x|^2}{4Dt}}$ on $\mathbb{R}$.

Remark 1. *The heat equation semi-group provides a way to represent the solution. For example, consider the PDE*

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & t > 0, x \in \mathbb{R} \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases}. \tag{2.4}$$

Then, the solution to (2.4) can be represented as $u(x,t) = G_D(x,t) * f(x) = e^{D\Delta t} f(x)$. That is, the heat equation semi-group maps the initial data to the solution at the corresponding time.

Definition 9. *We define the operator $\partial_x e^{D\Delta t}$ on $\mathbb{R}$ as the following:*

$$\partial_x e^{D\Delta t} f(\cdot) = (\partial_x G_D(t,\cdot)) * f(\cdot).$$

Remark 2. *One should notice that:*

$$e^{D\Delta t} \partial_x f(x) = G_D(x,t) * \partial_x f(x) = (\partial_x G_D(x,t)) * f(x) = \partial_x e^{D\Delta t} f(\cdot).$$

We will use the fact repetitively throughout this chapter.

We also need the following regularizing properties of the heat equation semi-group. The proof can be found in [24].

Lemma 2.1.4. *Let $1 \le p \le q \le \infty$. Then, the operator $e^{D\Delta t}$ and $\partial_x e^{D\Delta t}$ are bounded* *operators from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. In particular, we have*

$$\begin{aligned}\left\|e^{D\Delta t}f\right\|_{L^q} &\lesssim t^{-(1/p-1/q)/2}\|f\|_{L^p}\\ \left\|\partial_x e^{D\Delta t}f\right\|_{L^q} &\lesssim t^{-(1+1/p-1/q)/2}\|f\|_{L^p}\end{aligned}$$

Next, we state the Sobolev Embedding Theorem. The Sobolev Embedding Theorem states that a Sobolev space of a sufficiently high order can be embedded into some space of continuous functions, which can be found in [15]

Lemma 2.1.5 (Sobolev Embedding Theorem)**.** *Let m, k be non-negative integers. If $mp > 1$, then we have*

$$W^{k+m,p}(\mathbb{R}) \hookrightarrow C^k(\mathbb{R}).$$

Here $\hookrightarrow$ denotes continuous embedding. That is, for all $f \in W^{k+m,p}(\mathbb{R})$, we have that $f \in C^k(\mathbb{R})$ and $\|f\|_{C^k(\mathbb{R})} \leq C\|f\|_{W^{k+m,p}(\mathbb{R})}$.

The following Sobolev inequality can be found in [13]:

Lemma 2.1.6. *Let $k > 1/2$, then $H^k(\mathbb{R})$ is an algebra. In particular, for $f, g \in H^k(\mathbb{R})$*

$$\|fg\|_{H^k(\mathbb{R})} \lesssim \|f\|_{H^k(\mathbb{R})}\|g\|_{H^k(\mathbb{R})}$$

Now we can define the mild solution:

Definition 10. *Let $u(x,t) : \mathbb{R} \times [0,\infty)$ be a function satisfies the following integral equality:*

$$u(t) = e^{\Delta t}u_0 - \int_0^t \partial_x e^{\Delta(t-s)}[\beta\overline{u}(s)u(s)] - e^{\Delta(t-s)}[u(s)(1-u(s))]\,ds, \tag{2.5}$$

where $\overline{u}(s) = \dfrac{1}{2R}[u(x+R,s) - u(x-R,s)]$*. Then, we say that* u *is a mild solution to* (2.1)*.*

Remark 3. *To obtain the mild formula, we first use the heat equation semi-group and Duhammel's principle to convert the PDE into an integral equation. Then, we use definition 9 for the advection term. Finally, we use the divergence theorem on the non-local term* $\partial_x A$*:*

$$\frac{1}{2R}\partial_x \int_{-R}^{R} u(x+y,t)\,dt = \frac{1}{2R}[u(x+R,t) - u(x-R,t)] = \overline{u}$$

One should notice that the mild solution defined in (2.5) does not require differentiability. Thus, indeed, the mild solution is a weaker formulation compared to classical solutions.

With the above preliminary information, we can now state the main result.

2.1.1 Statement of the main result

Now we state the main result that we are going to prove in this chapter:

Theorem 2.1.7 (main result)**.** *Let* $u_0 \in H^3(\mathbb{R})$ *and* $u_0 \geq 0$*, then there exists a time* $T > 0$ *and a classical solution* $u(x,t) \geq 0$ *to the system* (2.1) *such that:*

$$u \in C((0,T), C^2(\mathbb{R})), \quad u_t \in C((0,T), H^1(\mathbb{R})).$$

In addition, the solution exists globally if $\|u(t)\|_{L^\infty(\mathbb{R})}$ *is bounded.*

The rest of this chapter aims to prove theorem 2.1.7 and is organized as follows: in the first section, we prove the local-in-time existence; the second section demonstrates the positivity of the solution; and finally, we show the continuation criterion.

2.2 Local in time existence

In this section, our goal is to prove the local in-time existence of a classical solution.

Proposition 2.2.1. *let $k \geq 1$, Let $u_0 \in H^k(\mathbb{R})$, then there exists a time T and a unique mild solution $u(x,t)$ to* (2.5) *with*

$$u \in L^\infty((0,T), H^k(\mathbb{R})).$$

Proof. The core of the proof is to use Banach's fixed point theorem to find a mild solution (2.5). Since the main concern of the theorem is the local-in-time existence, without loss of generality, we shall assume $T_1, T_2 \leq 1$ (T_1, T_2 will be defined later). We define a map $\mathcal{Q}$:

$$\begin{aligned} v(t) \mapsto & \mathcal{Q}v(t) \\ := & e^{\Delta t}u_0 - \int_0^t \beta \partial_x e^{\Delta(t-s)}[\overline{v}v(s)] - e^{\Delta(t-s)}[v(s)(1-v(s))]\, ds, \end{aligned}$$

for $v \in L^\infty((0,T), H^k(\mathbb{R}))$.

Step 1: $\mathcal{Q}$ maps a ball into itself. Define $M := 2\|u_0\|_{H^k(\mathbb{R})}$. Let $B_M(0) \subseteq H^k(\mathbb{R})$ be the ball of radius M in $H^k(\mathbb{R})$ centered at zero. Let $v \in L^\infty((0,T_1), H^k(\mathbb{R}))$. We claim that for any small enough $T_1 > 0$, we have $\mathcal{Q}$ maps $B_M(0)$ to itself. Now, we estimate the $H^k(\mathbb{R})$ norm of $\mathcal{Q}v(t)$:

$$\begin{aligned}
\|\mathcal{Q}v(t)\|_{H^k(\mathbb{R})} \leq& \|u_0\|_{H^k(\mathbb{R})} \\
&+ \frac{\beta}{2R}\int_0^{T_1} \left\|\partial_x e^{\Delta(t-s)}[\overline{v}v(s)]\right\|_{H^k(\mathbb{R})} ds \\
&+ \int_0^{T_1} \left\|e^{\Delta(t-s)}[v(s)(1-v(s))]\right\|_{H^k(\mathbb{R})} ds, \\
=& \frac{M}{2} + A_1 + A_2.
\end{aligned} \tag{2.6}$$

The first part is from the contractive property of the heat equation semi-group. Thus, to show that $\mathcal{Q}$ maps $B_M(0)$ to itself, it suffices to show that $A_1 + A_2 \leq M/2$.

Next, we estimate A_1:

$$\begin{aligned}
A_1 =& \beta \int_0^{T_1} \left\|\partial_x e^{\Delta(t-s)}[\overline{v}v(s)]\right\|_{H^k(\mathbb{R})} ds \\
\lesssim& \beta \int_0^{T_1} (T_1 - s)^{-1/2} \|\overline{v}v(s)\|_{H^k(\mathbb{R})} ds \\
\leq& \beta \sup_{s\in(0,T_1)} \|\overline{v}v(s)\|_{H^k(\mathbb{R})} \int_0^{T_1} (T_1 - s)^{-1/2} ds \\
\lesssim& \beta \sup_{s\in(0,T_1)} \|\overline{v}(s)\|_{H^k(\mathbb{R})} \|v(s)\|_{H^k(\mathbb{R})} \int_0^{T_1} (T_1 - s)^{-1/2} ds \\
\lesssim& \frac{\beta}{R} \sup_{s\in(0,T_1)} \|v(s)\|^2_{H^k(\mathbb{R})} \int_0^{T_1} (T_1 - s)^{-1/2} ds \\
\lesssim& \frac{\beta}{R}\sqrt{T_1} \sup_{s\in(0,T_1)} \|v(s)\|^2_{H^k(\mathbb{R})}.
\end{aligned}$$

The first inequality is from lemma 2.1.4, and the third inequality is from lemma 2.1.6 and $k > 1/2$.

Continue our estimate for A_2:

$$
\begin{aligned}
A_2 = & \int_0^{T_1} \left\| e^{\Delta(t-s)}[v(s)(1-v(s))] \right\|_{H^k(\mathbb{R})} ds \\
\leq & \int_0^{T_1} \|v(s)(1-v(s))\|_{H^k(\mathbb{R})} ds \\
\leq & \int_0^{T_1} \|v(s)(1-v(s))\|_{H^k(\mathbb{R})} ds \\
\leq & T_1 \sup_{0<s<T_1} \|v(s)(1-v(s))\|_{H^k(\mathbb{R})} \\
\lesssim & T_1 \left(\sup_{0<s<T_1} \|v(s)\|_{H^k(\mathbb{R})} + \sup_{0<s<T_1} \|v(s)\|^2_{H^k(\mathbb{R})} \right) \\
\leq & \sqrt{T_1} \left(\sup_{0<s<T_1} \|v(s)\|_{H^k(\mathbb{R})} + \sup_{0<s<T_1} \|v(s)\|^2_{H^k(\mathbb{R})} \right).
\end{aligned}
$$

The fourth inequality is from lemma 2.1.6, and the last one is from the assumption $T_1 \leq 1$.

Combining the estimate of A_1, A_2 and the assumption that $\sup_{0\leq s\leq T_1} \|v(s)\|_{H^k(\mathbb{R})} \leq M$, we obtain:

$$
A_1 + A_2 \lesssim \left(M^2 \frac{\beta}{R} + M + M^2 \right) \sqrt{T_1}.
$$

Thus, indeed, for a sufficiently small T_1, $A_1 + A_2 \leq M/2$. We conclude that $\mathcal{Q}$ maps $B_M(0)$ to itself.

Step 2: $\mathcal{Q}$ is a contraction for small enough T_2. Let $v_1, v_2 \in L^\infty((0,T_2), B_M(0))$, we compute the following:

$$
\begin{aligned}
&\|Qv_1(t) - Qv_2(t)\|_{H^k(\mathbb{R})} \\
=&\left\|\int_0^{T_2} \frac{\beta}{2R}\partial_x e^{\Delta(t-s)}[\overline{v}_1 v_1(s) - \overline{v}_2 v_2(s)]\, ds\right\|_{H^k(\mathbb{R})} \\
&+ \left\|\int_0^{T_2} e^{\Delta(t-s)}[v_1(s)(1-v_1(s)) - v_2(s)(1-v_2(s))]\, ds\right\|_{H^k(\mathbb{R})} \\
&= B_1 + B_2.
\end{aligned}
\tag{2.7}
$$

We first estimate B_1:

$$
\begin{aligned}
B_1 \leq&\left\|\int_0^{T_2} \beta\partial_x e^{\Delta(t-s)}[\overline{v}_1(v_1(s) - v_2(s))]\, ds\right\|_{H^k(\mathbb{R})} + \left\|\int_0^{T_2} \beta\partial_x e^{\Delta(t-s)}[(\overline{v}_1 - \overline{v}_2)v_2(s)]\, ds\right\|_{H^k(\mathbb{R})} \\
\lesssim&\beta\int_0^{T_2}(T-s)^{-1/2}\|\overline{v}_1(v_1(s)-v_2(s))\|_{H^k(\mathbb{R})}\, ds + \beta\int_0^{T_2}(T-s)^{-1/2}\|(\overline{v}_1-\overline{v}_2)v_2(s)\|_{H^k(\mathbb{R})}\, ds \\
\lesssim&\frac{\beta M}{R}\int_0^{T_2}(T-s)^{-1/2}\|v_1(s)-v_2(s)\|_{H^k(\mathbb{R})}\, ds + \frac{\beta M}{R}\int_0^{T_2}(T-s)^{-1/2}\|v_1(s)-v_2(s)\|_{H^k(\mathbb{R})}\, ds \\
\leq&\frac{\beta M}{R}\sup_{0<s<T_2}\|v_1(s)-v_2(s)\|_{H^k(\mathbb{R})}\int_0^{T_2}(T-s)^{-1/2}\, ds \\
\lesssim&\frac{\beta M}{R}\sqrt{T_2}\sup_{0<s<T_2}\|v_1(s)-v_2(s)\|_{H^k(\mathbb{R})}.
\end{aligned}
$$

The second inequality is from lemma 2.1.4. The third inequality is from lemma 2.1.6 and $v_1(s), v_2(s) \in B_M(0)$. For B_2, we follow a similar calculation:

$$
\begin{aligned}
B_2 \leq & \left\| \int_0^{T_2} e^{\Delta(t-s)}[v_1(s) - v_2(s)]\, ds \right\|_{H^k(\mathbb{R})} + \left\| \int_0^{T_2} e^{\Delta(t-s)}[v_1^2(s) - v_2^2(s)]\, ds \right\|_{H^k(\mathbb{R})} \\
\leq & \int_0^{T_2} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}\, ds + \int_0^{T_2} \left\|v_1^2(s) - v_2^2(s)\right\|_{H^k(\mathbb{R})} ds \\
= & \int_0^{T_2} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}\, ds + \int_0^{T_2} \|(v_1(s) - v_2(s))(v_1(s) + v_2(s))\|_{H^k(\mathbb{R})}\, ds \\
\lesssim & \int_0^{T_2} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}\, ds + \int_0^{T_2} M\|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}\, ds \\
\leq & (1+M)T_2 \sup_{s\in(0,T_2)} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})} \\
\leq & (1+M)\sqrt{T_2} \sup_{s\in(0,T_2)} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}.
\end{aligned}
$$

The second inequality is from the contractive property of the heat equation semigroup, the third inequality is due to lemma 2.1.6 and $v_1(s), v_2(s) \in B_M(0)$, and the last one is from the assumption $T_2 \leq 1$. Back to (2.7), we have

$$
\sup_{s\in(0,T_2)} \|Qv_1(t) - Qv_2(t)\|_{H^k(\mathbb{R})} \lesssim \left(\frac{\beta M}{R} + 1 + M \right) \sqrt{T_2} \sup_{s\in(0,T_2)} \|v_1(s) - v_2(s)\|_{H^k(\mathbb{R})}.
$$

Thus, for any small enough T_2, we can conclude that $\mathcal{Q}$ is a contraction on the space $L^\infty((0,T), H^k(\mathbb{R}))$.

Step 3: Existence of the solution. By the Banach's fix point theorem, for $T \leq \min(T_1, T_2)$, the operator $\mathcal{Q}$ has a unique fixed point $u(t)$ in $L^\infty((0,T), H^k(\mathbb{R}))$. Thus, the fixed point is the mild solution (2.5) satisfies:

$$
u(t) \in L^\infty((0,T), H^k(\mathbb{R})).
$$

One should notice that the mild solution satisfies the initial condition automatically:

$$\lim_{t \to 0} u(x,t) = u_0(x).$$

□

Corollary 2.2.2. *Let* $u_0 \in H^3(\mathbb{R})$*, then the mild solution we found in proposition 2.2.1 is, in fact, a classical solution. In particular:*

$$u \in C((0,T), C^2(\mathbb{R})), \quad u_t \in C((0,T), H^1(\mathbb{R})).$$

Proof. Recall the integral form of the solution:

$$u(t) = e^{\Delta t}u_0 - \int_0^t e^{\Delta(t-s)}\partial_x[f(s)u(s)] - e^{\Delta(t-s)}[u(s)(1-u(s))]\, ds.$$

By looking at the integral form, $u(t)$ is continuous in time. Thus, together with proposition 2.2.1, $u(t) \in C((0,T), H^3(\mathbb{R}))$. By Sobolev embedding $H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$, we proved the first part.

In addition, since $u(t) \in H^3(\mathbb{R})$, we have $u_{xx}(t) \in H^1(\mathbb{R})$. Thus, by looking at (2.1), we conclude

$$u_t = C((0,T), H^1(\mathbb{R}))$$

□

Therefore, we prove the local in time existence of a classical solution.

2.3 Positivity

The second section aims to show the positivity of the solution to (2.1). The core of the proof is to show that the solution cannot have negative parts. A similar argument can be found in [6].

We let $u^-(t)$ denote the negative part of the function. Our goal is to show that for a given $u_0(x) \geq 0$, than $\|u^-(t)\|_{L^2(\mathbb{R})} = 0$ for all t. We first write down the rigorous definition.

Definition 11. *Define the negative part of the solution $u^-(x,t) := u(x,t)$ if $u(x,t) < 0$ and $u^-(x,t) := 0$ otherwise. We also define the corresponding domain for $u^-(x,t)$ as $D^-(t) = \{x \in \mathbb{R} : u(x,t) < 0\}$.*

Before proving the positivity, we first state a useful lemma:

Lemma 2.3.1. *If $h(x) \in C^1 \cap L^p(\mathbb{R})$ for $p < \infty$, then*

$$\lim_{|x|\to\infty} h(x) = 0$$

Proof. Claim a contradiction that $\lim_{|x|\to\infty} h(x) \neq 0$. Then, there exists $\delta > 0$ and some sequence $x_n \in \mathbb{R}$ s.t. $|x_n| \to \infty$ and $\inf_n |h(x_n)| > \delta$. Since $h \in C^1(\mathbb{R}^d)$, let $M := \sup_{x\in\mathbb{R}^d} |\partial_x h| < \infty$. Let $\epsilon < \frac{\delta}{2M}$. Since $|x_n| \to \infty$, we pick a subsequence s.t. $|x_{n_i} - c_{n_j}| > 2\epsilon$ for $i \neq j$. Next, we consider the ball with radius ϵ centred at x_{n_i}. By picking the subsequence in this way, we have $B_\epsilon(x_{n_i}) \cap B_\epsilon(x_{n_j}) = \emptyset$ for $i \neq j$. By the mean value theorem, we have $h(y) \geq \delta/2$ for all $y \in B_\epsilon(x_{n_i})$ and $i \in \mathbb{N}$. Then:

$$\begin{aligned}\int_{\mathbb{R}^d} |h(x)|^p \, dx &\geq \sum_{i\in\mathbb{N}} \int_{B_\epsilon(x_{n_i})} |f(x)|^p \, dx \\ &\geq \sum_{i\in\mathbb{N}} \int_{B_\epsilon(x_{n_i})} (\delta/2))^p \, dx \\ &= \infty\end{aligned}$$

This contradicts that $h \in L^p(\mathbb{R}^d)$. Thus, we must have $\lim_{|x|\to\infty} g(x) = 0$. □

Remark 4. *One should notice that by Sobolev embedding, for each $t \in (0, T)$, $u(t) \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$. Thus, by lemma 2.3.1, the solution u we found in corollary 2.2.2 has the property that $u(x, t) \to 0$ as $|x| \to \infty$. We will use this property to cancel boundary terms raised from integration by parts.*

Now, we can prove the positivity of the solution.

Proposition 2.3.2. *Let $u_0(x) \geq 0$, then $u(x, t) \geq 0$. Here, $u(x, t)$ is the solution we found in corollary 2.2.2.*

Proof. We follow a similar method to [6]. The core of the proof is to show that $\|u^-(t)\|_{L^2(\mathbb{R})} = 0$. First, we derive the ODE for $\|u^-(t)\|^2_{L^2(\mathbb{R})}$. Since the L^2 norm of $u(x, t)$ is differentiable in time, we have:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|^2_{L^2(\mathbb{R})} = \int_{D^-(t)} u_t^-(t)u^-(t) \, dx.$$

One should notice that no boundary terms arise due to the time derivative because $u^-(t) = 0$ on the boundary of $D^-(t)$.

Since the derivatives $u, \partial_x u$ and $\partial_{xx} u$ are continuous and differentiable in time, we can substitute u by u^- on $D^-(t)$. Then, by (2.1) we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|_{L^2(\mathbb{R})} &= \int_{D^-(t)} u^-_{xx}(t)u^-(t)\,dx - \beta\int_{D^-(t)}(u^-A^-_x)_x(t)u^-(t)\,dx \\ &\quad + \int_{D^-(t)}(u^-(1-u^-))(t)u^-(t)\,dx \\ &= -\int_{D^-(t)}(u^-_x)^2(t)\,dx + [u^-(t)u^-_x(t)]_{\partial D^-(t)} + \beta\int_{D^-(t)}(u^-A^-_x)(t)u^-_x(t)\,dx \\ &\quad - \beta[(u^-A^-_x)(t)u^-(t)]_{\partial D^-(t)} + \int_{D^-(t)}(u^-(1-u^-))(t)u^-(t)\,dx.\end{aligned}$$

For the second equality, we used integration by parts. Next, we want to show that the boundary term disappears. Indeed, if $D^-(t)$ is bounded, then by definition, $u^- = 0$. If $D^-(t)$ is unbounded, then by lemma 2.3.1 and $u^- = u$ on $D^-(t)$, $u^-(x,t) \to 0$ as $|x| \to \infty$. Thus, we can simplify the equation:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|_{L^2(\mathbb{R})} &= -\int_{D^-(t)}(u^-_x)^2(t)\,dx + \beta\int_{D^-(t)}(u^-A^-_x)(t)u^-_x(t)\,dx \\ &\quad + \int_{D^-(t)}(u^-(1-u^-))(t)u^-(t)\,dx.\end{aligned} \tag{2.8}$$

Here, we obtain the ODE for $\|u^-(t)\|_{L^2}$.

Now, claim a contradiction that the non-negativity ceases to hold at some finite time. Then, we define this time as t_0:

$$t_0 := \sup\{t : u(x,t) \geq 0, \forall x \in \mathbb{R}\}.$$

Thus, we have $u^-(t_0) = 0$. Evaluate the derivative eq. (2.8) at time t_0, we obtain:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u^-(t)\right\|_{L^2(\mathbb{R})} =& -\int_{D^-(t)} (u_x^-)^2(t)\,dx \\
&+ \int_{D^-(t)} (\underbrace{u^-}_{=0} f^-)(t)u_x^-(t)\,dx + \int_{D^-(t)} (\underbrace{u^-}_{=0}(1-u^-))(t)u^-(t)\,dx \\
\leq& 0.
\end{aligned}
$$

Since $u(t) \in C(\mathbb{R})$, this contradicts that non-negativity ceases to hold. $\square$

Hence, we prove the positivity of the solution.

2.4 Continuation Criterion

The goal of this section is to prove the continuation criterion. The strategy is as follows:

Step 1: Assuming that $\|u(t)\|_{L^\infty(\mathbb{R})}$ is bounded in a period of time.

Step 2: We show that $\|u(t)\|_{H^3(\mathbb{R})}$ growth at most exponentially under the assumption in step 1.

We first begin with the proof of the following lemmas.

Lemma 2.4.1. *Let $u_0 \in H^3(\mathbb{R})$, and let $u(t)$ be the solution we found in corollary 2.2.2. Then, we have the following integral equality:*

$$
\partial_x^i u(t) = e^{\Delta t}\partial_x^i u_0 - \int_0^t \beta\partial_x e^{\Delta(t-s)}\partial_x^i[\overline{u}u(s)] + e^{\Delta(t-s)}\partial_x^i[u(1-u)]\,ds,
$$

where $i = 1, 2, 3$.

Proof. Recall the integral form of the solution eq. (2.5). By taking the spatial derivative ∂_x^i for any $i = 1, 2, 3$ on both sides of the equation, we obtain:

$$\partial_x^i u(t) = e^{\Delta t}\partial_x^i u_0 - \partial_x^i \int_0^t \beta\partial_x e^{\Delta(t-s)}\partial_x^i[\overline{u}u(s)] - e^{\Delta(t-s)}[u(1-u)]\,ds$$

Thus, to prove the theorem, it suffices to show that we can commute the integral with the differential operator ∂_x^i:

$$\partial_x^i \int_0^t \beta\partial_x e^{\Delta(t-s)}\overline{u}u(s) - e^{\Delta(t-s)}[u(1-u)]\,ds = \int_0^t \beta\partial_x e^{\Delta(t-s)}\partial_x^i[\overline{u}u(s)] - e^{\Delta(t-s)}\partial_x^i[u(1-u)]\,ds.$$

We can prove this using the dominated convergence theorem. We first estimate the $L^\infty(\mathbb{R})$-norm of the non-local term:

$$\left\|\partial_x e^{\Delta(t-s)}\partial_x^i fu(s)\right\|_{L^\infty} \lesssim (t-s)^{-3/4}\left\|\partial_x^i fu(s)\right\|_{L^2(\mathbb{R})}.$$

The inequality is from lemma 2.1.4. By lemma 2.1.6 and proposition 2.2.1, we have $\overline{u}u(s) \in L^\infty((0,t), H^3(\mathbb{R}))$. Thus, $\partial_x^i\overline{u}u(s) \in L^\infty((0,t), L^2(\mathbb{R}))$. Together with the fact that $(t-s)^{-3/4} \in L_s^1((0,t))$. We can conclude that $\left\|\partial_x e^{\Delta(t-s)}\partial_x^i fu(s)\right\|_{L^\infty} \in L_s^1((0,t))$. Next, we estimate the logistic term:

$$\left\|e^{\Delta(t-s)}\partial_x^i[u(1-u)]\right\|_{L^\infty(\mathbb{R})} \lesssim (t-s)^{-1/4}\left\|\partial_x^i[u(1-u)]\right\|_{L^2(\mathbb{R})}.$$

Again, following the same argument as the non-local term. We have $\left\|e^{\Delta(t-s)}\partial_x^i[u(1-u)]\right\|_{L^\infty(\mathbb{R})} \in L_s^1((0,t))$. Thus, we can indeed use the dominated convergence theorem to commute the integral with the differential operator. □

We also need the following calculus inequality in Sobolev spaces. A similar result can be found in [15].

Lemma 2.4.2. *Let $m \geq 1$, then for any $f, g \in L^\infty \cap H^m(\mathbb{R})$, we have the following inequality:*

$$\|\partial_x^m(fg)\|_{L^2(\mathbb{R})} \lesssim \|f\|_{L^\infty(\mathbb{R})}\|\partial_x^m g\|_{L^2(\mathbb{R})} + \|g\|_{L^\infty(\mathbb{R})}\|\partial_x^m f\|_{L^2(\mathbb{R})}$$

Proof. By Leibniz rule and Holder's inequality, we obtain:

$$\begin{aligned}\|\partial_x^m(fg)\|_{L^2(\mathbb{R})} &\leq \sum_{i=0}^{m} \left\|\partial_x^i f \partial_x^{m-i} g\right\|_{L^2(\mathbb{R})} \\ &\leq \sum_{i=0}^{m} \left\|\partial_x^i f\right\|_{L^{2m/i}(\mathbb{R})}\left\|\partial_x^{m-i} g\right\|_{L^{2m/(m-i)}(\mathbb{R})}.\end{aligned}$$

Recall the Gagliardo–Nirenberg inequality:

$$\left\|\partial_x^j f\right\|_{L^{2r/j}} \lesssim \|f\|_{L^\infty}^{1-j/r}\|\partial_x^r f\|_{L^2}^{j/r}, \quad \text{for all } 0 \leq i \leq r.$$

This implies:

$$\begin{aligned}\|\partial_x^m(fg)\|_{L^2(\mathbb{R})} &\lesssim \sum_{i=0}^{m} \|f\|_{L^\infty}^{1-i/m}\|\partial_x^m f\|_{L^2}^{i/m}\|g\|_{L^\infty(\mathbb{R})}^{1-(m-i)/m}\|\partial_x^m g\|_{L^2(\mathbb{R})}^{(m-i)/m} \\ &\lesssim \sum_{i=0}^{m}[\|\partial_x^m f\|_{L^2}\|g\|_{L^\infty(\mathbb{R})}]^{1-(m-i)/m}[\|f\|_{L^\infty}\|\partial_x^m g\|_{L^2(\mathbb{R})}]^{(m-i)/m} \\ &\lesssim \sum_{i=0}^{m}[\|\partial_x^m f\|_{L^2}\|g\|_{L^\infty(\mathbb{R})}] + [\|f\|_{L^\infty}\|\partial_x^m g\|_{L^2(\mathbb{R})}] \\ &\lesssim \|\partial_x^m f\|_{L^2}\|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty}\|\partial_x^m g\|_{L^2(\mathbb{R})}.\end{aligned}$$

The last inequality is from Young's inequality. Here, we obtain the desired inequality.

□

Now, we prove the following continuation criterion.

Proposition 2.4.3. *Let $u(t)$ be the solution we found in corollary 2.2.2 and propo-*

sition 2.3.2. Let $T^* \geq 0$ *such that* $\sup_{s\in(0,T^*)} \|u(t)\|_{L^\infty(\mathbb{R})} < \infty$. *Then, we have*

$$\sup_{s\in(0,T^*)} \|u(t)\|_{H^3(\mathbb{R})} < \infty.$$

That is, for a period of time that $\|u(t)\|_{L^\infty(\mathbb{R})}$ *is bounded, the solution exists and remains bounded in* $H^3(\mathbb{R})$.

Proof. The goal is to show that $\|\partial^i u(t)\|_{L^2(\mathbb{R})}$ is bounded for $i = 0, 1, 2, 3$. In this proof, we let $M_1 := \sup_{s\in(0,T^*)} \|u(t)\|_{L^\infty(\mathbb{R})}$ for the sake of notation.
Now we estimate $\|u(t)\|_{L^2(\mathbb{R})}$. Since $u(t)$ is continuously differentiable in time. We have:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} u u_t \, dx \\
&= \int_{\mathbb{R}} u\partial_{xx} u \, dx - \beta \int_{\mathbb{R}} u(\partial_x(\overline{u}u)) \, dx + \|u\|^2_{L^2(\mathbb{R})} - \int_{\mathbb{R}} u^3 \, dx \\
&= - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \beta \int_{\mathbb{R}} (\partial_x u)(\overline{u}u) \, dx + \|u\|^2_{L^2(\mathbb{R})} - \int_{\mathbb{R}} u^3 \, dx \\
&\leq - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \beta \int_{\mathbb{R}} (\partial_x u)(\overline{u}u) \, dx + \|u\|^2_{L^2(\mathbb{R})} \\
&\leq - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \beta \|\partial_x u(t)\|_{L^2(\mathbb{R})} \|\overline{u}u(t)\|_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} \\
&\leq - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \frac{\beta}{2R} \|u(t)\|_{L^\infty(\mathbb{R})} \|\partial_x u(t)\|_{L^2(\mathbb{R})} \|u(t)\|_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} \\
&\leq - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \frac{\beta}{2R} M_1 \|\partial_x u(t)\|_{L^2(\mathbb{R})} \|u(t)\|_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} \\
&\leq - \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \left(\epsilon \frac{\beta}{4R} M_1\right)^2 \|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{2\epsilon} \|u(t)\|^2_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

For the third equality, we used integration by parts and lemma 2.3.1; the first inequality is due to the positivity of the solution (proposition 2.3.2); the second inequality is by Cauchy-Schwartz; the third inequality is from Holder; the fourth one is by the definition of M_1, and the last inequality is from Young's inequality.

Now, we let $\epsilon > 0$ and $\left(\epsilon \frac{\beta}{4R} M_1\right)^2 = 1$. Then, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R})} \le (1 + 1/(2\epsilon))\|u(t)\|^2_{L^2(\mathbb{R})}.$$

Applying Gronwall's lemma, we obtain a bound for any $t < T^*$:

$$\|u(t)\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})} e^{(1+1/(2\epsilon))T^*}.$$

Next, we estimate $\|\partial_x^i u(t)\|$ for $i = 1, 2, 3$. Recall lemma 2.4.1, we have:

$$\partial_x^i u(t) = e^{\Delta t}\partial_x^i u_0 - \int_0^t \beta \partial_x e^{\Delta(t-s)}\partial_x^i[\overline{u}u(s)] - e^{\Delta(t-s)}\partial_x^i[u(1-u)]\, ds,$$

Taking L^2 norm on both sides:

$$\begin{aligned}
\left\|\partial_x^i u(t)\right\|_{L^2(\mathbb{R})} - \left\|\partial_x^i u_0\right\| &\le \int_0^t \beta\left\|\partial_x e^{\Delta(t-s)}\partial_x^i[\overline{u}u(s)]\right\|_{L^2(\mathbb{R})} + \left\|e^{\Delta(t-s)}\partial_x^i[u(1-u)]\right\|_{L^2(\mathbb{R})} ds \\
&\lesssim \int_0^t \beta(t-s)^{-1/2}\left\|\partial_x^i[\overline{u}u(s)]\right\|_{L^2(\mathbb{R})} + \left\|\partial_x^i[u(1-u)]\right\|_{L^2(\mathbb{R})} ds \\
&\lesssim \int_0^t \beta(t-s)^{-1/2}\left(\|\overline{u}(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} + \|u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i \overline{u}(s)\right\|_{L^2(\mathbb{R})}\right) \\
&\quad + \left\|\partial_x^i[u(1-u)]\right\|_{L^2(\mathbb{R})} ds \\
&\lesssim \int_0^t \frac{\beta}{R}(t-s)^{-1/2}\|u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} + \left\|\partial_x^i[u(1-u)]\right\|_{L^2(\mathbb{R})} ds \\
&\lesssim \int_0^t \frac{\beta}{R}(t-s)^{-1/2}\|u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} \\
&\quad + \|u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i(1-u)(s)\right\|_{L^2(\mathbb{R})} + \|1-u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} ds \\
&\lesssim \int_0^t \frac{\beta}{R}(t-s)^{-1/2}\|u(s)\|_{L^\infty(\mathbb{R})}\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} \\
&\quad + (\|u(s)\|_{L^\infty(\mathbb{R})} + \|1-u(s)\|_{L^\infty(\mathbb{R})})\left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} ds
\end{aligned}$$

The second inequality is from lemma 2.1.4; the third and the fifth inequality is from lemma 2.4.2. This implies that for $t < T^*$:

$$\left\|\partial_x^i u(t)\right\|_{L^2(\mathbb{R})} - \left\|\partial_x^i u_0\right\| \lesssim \int_0^t \left(\frac{\beta M_1}{R}(t-s)^{-1/2} + 1 + 2M_1\right) \left\|\partial_x^i u(s)\right\|_{L^2(\mathbb{R})} ds.$$

Thus, by Gronwall's inequality, there exists some constant C such that

$$\left\|\partial_x^i u(t)\right\|_{L^2(\mathbb{R})} \leq \left\|\partial_x^i u_0\right\| \exp\left\{C((T^*)^{1/2} + T^*)\right\}.$$

From here, we proved our claim. $\square$

Chapter 3

Existence of travelling wave solutions

The purpose of the chapter is to analyze the existence of travelling wave solutions to our reaction-advection-diffusion equation with non-local advection:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial}{\partial x}\left(u \frac{\partial A}{\partial x}\right) + u(1-u) \quad A = \frac{1}{2R}\int_{-R}^{R} u(x+y,t)\, dy.$$

The travelling wave solutions are the solutions to a PDE that propagates with a constant speed and maintains its shape in space. There are many applications of travelling wave solutions, such as invasive species and tumour invasions [17, 22]. In this chapter, we will be focusing on the existence of travelling wave solutions to (1.2). We structure this chapter as follows: we begin with the rigorous definition of travelling wave solutions and critical wave speed. Then, we introduce the perturbation method. Finally, by applying a combination of Fredholm Operator theory and Banach's fixed point method, we prove the existence of travelling wave solutions.

3.1 Equation for Travelling Wave Solutions and Critical Wave Speed

In this section, we first define the travelling wave solution, present the formula for travelling wave solutions, and then investigate the critical wave speed via standard ODE linear stability analysis.

Travelling wave solutions are solutions of the form $U(z) = U(x + ct)$. Here, $U(z)$ is a non-trivial bounded solution, and $c > 0$ is the travelling wave speed. To construct a travelling wave solution to (1.2), we assume $u(x, t) = U(z)$, $z = x + ct$, and in addition, we set the following boundary condition:

$$U(-\infty) = 0, \quad U(\infty) = 1$$

Substitute this travelling wave ansatz $U(z)$ into (1.2), we obtain the following second order ODE for travelling wave solutions:

$$\begin{aligned} 0 =& U''(z) - cU'(z) - \beta[U(z)\overline{U}(z)]' + U(1 - U), \\ \overline{U}(z) =& \frac{1}{2R}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-R}^{R} U(z + y)\, dz = \frac{1}{2R}(U(z + R) - U(z - R)). \end{aligned} \tag{3.1}$$

Consider a variable change $X = U'$, then we can convertible (3.1) into a 2-dimensional first order system:

$$\begin{cases} U' = X \\ X' = cX + \dfrac{\beta}{2R}[X\overline{U} + U\overline{X}] - U(1 - U) \end{cases}.$$

We can see there exists 2 equilibrium points $(U, X) = (0, 0)$ and $(U, X) = (1, 0)$. This suggests possible travelling wave solutions connect these 2 equilibrium points. Consider linearizing as if they are regular ODEs without non-locality, that is:

$$\frac{\partial[X\overline{U} + U\overline{X}]}{\partial U} = X + \overline{X}, \quad \frac{\partial[X\overline{U} + U\overline{X}]}{\partial X} = U + \overline{U}.$$

Then, we obtain the following linearized system at $(0, 0)$:

$$\begin{bmatrix} U' \\ X' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & c \end{bmatrix} \cdot \begin{bmatrix} U \\ X \end{bmatrix},$$

The eigenvalues of this system are:

$$\frac{c \pm \sqrt{c^2 - 4}}{2}$$

We are only interested in non-negative travelling wave solutions that depart from the origin (saddle point) and connect to $(1, 0)$. To achieve this, we need to exclude spiral behaviour, and we can achieve this by setting the wave speed $c \geq 2$. Then, both eigenvalues of the matrix are real.

The above intuitive connections suggest the possible existence of a travelling wave solution connecting 0 and 1 with wave speed $c \geq 2$. In the subsequent sections, we will tackle this problem with mathematical rigour (i.e. perturbation method).

3.2 Perturbation method

This section aims to introduce the perturbation method. The idea behind the perturbation method is as the following:

Step 1: Finding the existence of the travelling wave solution of a simpler system. In our analysis, we are interested in the case that $\beta = 0$.

Step 2: Consider the difference in the travelling wave solution caused by perturbing the parameter β.

Step 3: Demonstrate the existence of the difference, so we can conclude the existence of the travelling wave solution when $\beta \neq 0$.

Therefore, we will first present some previous results regarding the equation (1.2) with $\beta = 0$

3.2.1 Notations of spaces and norms

Before performing any analysis, we introduce some notations of norms and space. We will use the following notations for the rest of this chapter. For $v \in \mathbb{R}^d$ and $M \in \mathbb{R}^{d\times d}$, we let $|v|$ and $|M|$ denote the vector norm and the matrix norm induced by the vector norm.

Let $C^k(\mathbb{R})$ be the set of functions with up to $k-$th order bounded and continuous derivatives from $\mathbb{R}$ to $\mathbb{R}$. Equip this with supremum norm i.e. $\|\phi\|_{C^k(\mathbb{R})} = \sum_{i\leq k} \sup(|\phi^{(i)}(t)|, t \in \mathbb{R})$ so that C^k is a Banach space.

Similarly, we use $C_0^k(\mathbb{R}) := \{\phi \in C^k(\mathbb{R}) : \lim_{t\to\pm\infty} \phi^{(i)}(t) = 0 \text{ for all } i \leq k\}$ to denote the Banach space of continuous functions with vanishing boundaries using the same supremum norm $\|\phi\|_{C_0(\mathbb{R})} = \|\phi\|_{C(\mathbb{R})}$.

In addition, we let $(C^k(\mathbb{R}))^d$ and $(C_0^k(\mathbb{R}))^d$ denote $\mathbb{R}^d$ vector valued functions such that each component is in $C^k(\mathbb{R})$ and $C_0^k(\mathbb{R})$ respectively. We equip the space $(C^k(\mathbb{R}))^d$

with the norm

$$\|\phi\|_{(C^k(\mathbb{R}))^d} = \left\| \begin{bmatrix} \|\phi_1\|_{C(\mathbb{R})} \\ \|\phi_2\|_{C(\mathbb{R})} \\ \vdots \\ \|\phi_d\|_{C(\mathbb{R})} \end{bmatrix} \right\|.$$

The norm for $(C_0^k(\mathbb{R}))^d$ is similar:

$$\|\phi\|_{(C_0^k(\mathbb{R}))^d} = \left\| \begin{bmatrix} \|\phi_1\|_{C_0(\mathbb{R})} \\ \|\phi_2\|_{C_0(\mathbb{R})} \\ \vdots \\ \|\phi_d\|_{C_0(\mathbb{R})} \end{bmatrix} \right\|.$$

When it is clear from the context, for the sake of notation, we will replace $(C^k(\mathbb{R}))^d$ by $C^k(\mathbb{R})$ and $(C_0^k(\mathbb{R}))^d$ by $C_0^k(\mathbb{R})$.

We will use the above notations throughout the rest of this chapter.

3.2.2 Traveling wave solutions for $\beta = 0$

By setting $\beta = 0$ in (1.2), we obtain the model without the non-local advection term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u). \tag{3.2}$$

This is the well-known Fisher's equation. In [17], Murray showed the existence of travelling wave solutions:

Lemma 3.2.1. *For any $c \geq 2$, there exists a travelling wave solution $N(z) = N(x + ct)$ to (3.2), where $z = x + ct$. In particular,*

1. *$N(\infty) = 1$ and $N(-\infty) = 0$,*

2. *$N(z)$ and $N'(z)$ are uniformly bounded in $\mathbb{R}$,*

3. *$N(z)$ is a monotonically increasing function.*

4. *We also define $N(0) = 1/2$ by translation invariance.*

With the results of Fisher's equation, we are ready to examine our original equation (1.2).

3.2.3 The Perturbed System

To apply the perturbation method, we consider the difference caused by setting $\beta \neq 0$ in the travelling wave solution compared to Fisher's equation. Thus, for a solution $U(z)$ of (3.1), we can represent it as

$$U = N + V,$$

where N denotes the travelling wave solution for Fisher's equation defined in lemma 3.2.1, and V is the perturbation due to the presence of the non-local part ($\beta \neq 0$). Here $V(z)$ is a real-valued function subject to boundary conditions $V(\pm\infty) = 0$.

Then, we can substitute this representation into (3.1) and obtain an equation for V:

$$0 = V'' - cV' - V - \beta[(N + V)\overline{N + V}]' + 2V(1 - N) - V^2. \tag{3.3}$$

We need to transform (3.3) in to an integral equation. Notice that the equation

$$0 = V'' - cV' - V$$

has 2 characteristic roots $\lambda_1 < 0$ and $\lambda_2 > 0$:

$$\lambda_1 = \frac{c - \sqrt{c^2+4}}{2} \quad \lambda_2 = \frac{c + \sqrt{c^2+4}}{2}$$

For the simplicity of the notation, let $\beta[(N+V)\overline{N+V}]' = G'$. Then, the integral form is as the following:

$$V(z) = \frac{1}{\lambda_2 - \lambda_1}\int_{-\infty}^{z} e^{\lambda_1(z-s)}(2V(1-N))\,ds + \frac{1}{\lambda_2 - \lambda_2}\int_{z}^{\infty} e^{\lambda_2(z-s)}(2V(1-N))\,ds + R + Q, \tag{3.4}$$

where

$$R(V) = \frac{1}{\lambda_2 - \lambda_1}\int_{-\infty}^{z} e^{\lambda_1(z-s)}V^2\,ds + \frac{1}{\lambda_2 - \lambda_1}\int_{z}^{\infty} e^{\lambda_2(z-s)}V^2\,ds, \tag{3.5}$$

and

$$Q(V) = \frac{-1}{\lambda_2 - \lambda_1}\left(\int_{-\infty}^{z} e^{\lambda_1(z-s)}G'\,ds + \int_{z}^{\infty} e^{\lambda_2(z-s)}G'\,ds\right).$$

We can further simplify the expression for Q. Rewrite the terms inside the bracket using integration by parts

$$e^{\lambda_1 z}\left([e^{-\lambda_1 s}G]_{-\infty}^{z} + \lambda_1\int_{-\infty}^{z} e^{-\lambda_1 s}G\,ds\right) + e^{\lambda_2 z}\left([e^{-\lambda_2 s}G]_{z}^{\infty} + \lambda_2\int_{z}^{\infty} e^{-\lambda_2 s}G\,ds\right). \tag{3.6}$$

Next, we want to show:

$$\lim_{s\to-\infty} e^{-\lambda_1 s}G(s) = 0, \text{and } \lim_{s\to\infty} e^{-\lambda_2 s}G(s) = 0.$$

Recall the definition of G

$$\begin{aligned}
G(s) =& \beta\left[(N(s)+V(s))\overline{N+V}(s)\right] \\
=& \frac{\beta}{2R}\left[(N(s)+V(s))(N(s+R)-N(s-R)+V(s+R)-V(s-R))\right].
\end{aligned}$$

Since both N and V are bounded, G is bounded as well. In addition, notice that $\lambda_1 < 0$, so $\lim_{s\to-\infty} e^{-\lambda_1 s}G = 0$. Using the same calculation and $\lambda_2 > 0$, we simplify (3.6) to

$$\begin{aligned}
& e^{\lambda_1 z}\left(e^{-\lambda_1 z}G + \lambda_1\int_{-\infty}^{z} e^{-\lambda_1 s}G\,ds\right) + e^{\lambda_2 z}\left(-e^{-\lambda_2 z}G + \lambda_2\int_{z}^{\infty} e^{-\lambda_2 s}G\,ds\right) \\
=& G - G + \lambda_1\int_{-\infty}^{z} e^{\lambda_1(z-s)}G\,ds + \lambda_2\int_{z}^{\infty} e^{\lambda_2(z-s)}G\,ds \\
=& \lambda_1\int_{-\infty}^{z} e^{\lambda_1(z-s)}G\,ds + \lambda_2\int_{z}^{\infty} e^{\lambda_2(z-s)}G\,ds
\end{aligned}$$

plug the above equation back to the definition of G, we obtain:

$$Q = \frac{-\lambda_1}{\lambda_2-\lambda_1}\int_{-\infty}^{z} e^{\lambda_1(z-s)}G\,ds + \frac{-\lambda_2}{\lambda_2-\lambda_1}\int_{z}^{\infty} e^{\lambda_2(z-s)}G\,ds. \tag{3.7}$$

Next, define the operator:

$$L(V) = V - \frac{1}{\lambda_2-\lambda_1}\int_{-\infty}^{z} e^{\lambda_1(z-s)}(2V(1-N))\,ds - \frac{1}{\lambda_2-\lambda_1}\int_{z}^{\infty} e^{\lambda_2(z-s)}(2V(1-N))\,ds. \tag{3.8}$$

One should notice that the operator L is a linear operator. Then, we can simplify the integral equation (3.4) into:

$$L(V) = Q(V) + R(V) \tag{3.9}$$

Remark 5. *In the classical differential equation theory, the standard procedure is to put the linear part on one side and the non-linear part on the other. Then, we can show that the linear part is invertible, and the non-linear part is a contraction. However, in our case, the operator $Q(V)$ contains a linear part, so one might be irritated by the setup of* (3.9). *We chose to retain some linear parts in Q because it is difficult to calculate the range of an operator containing non-local terms. The rest of this chapter will give a detailed discussion of this matter.*

Thus, we obtained an integral equation of the perturbation V. One should notice that from this point, the existence of a travelling wave solution $U(z)$ is equivalent to the existence of $V(z)$. We end the chapter with the following lemma that shows the operators L, Q, R are well-defined:

Lemma 3.2.2. *Let L, Q and R be the operators defined in* (3.8),(3.7) *and* (3.5). *Then, $L(V(z)), Q(V(z)), R(V(z)) \in C_0$ for any $V(z) \in C_0(\mathbb{R})$. That is, the operators L, Q and R are well-defined operators from $C_0(\mathbb{R})$ to itself.*

Proof. We aim to show that $\lim_{z\to\pm\infty} L(V(z)) = 0$ for $V \in C_0(\mathbb{R})$, the following calculation also works for Q and R as well. Recall the definition of $L(V)$ in (3.8), by taking the limit:

$$\lim_{z\to\infty} L(V(z)) = \underbrace{\lim_{z\to\infty} V(z)}_{\text{First part}} - \frac{1}{\lambda_2-\lambda_1} \underbrace{\left(\lim_{z\to\infty} \frac{\int_{-\infty}^{z} e^{-\lambda_1 s}(2V(1-N))\,ds}{e^{-\lambda_1 z}} + \frac{\int_{z}^{\infty} e^{-\lambda_2 s}(2V(1-N))\,ds}{e^{-\lambda_2 z}} \right)}_{\text{Second part}}.$$

The first term disappears because $V \in C_0$. For the second term, an application of L'Hôpital's rule yields

$$\begin{aligned}
&\lim_{z\to\infty} \frac{\int_{-\infty}^{z} e^{-\lambda_1 s}(2V(1-N))\,ds}{e^{-\lambda_1 z}} + \frac{\int_{z}^{\infty} e^{-\lambda_2 s}(2V(1-N))\,ds}{e^{-\lambda_2 z}} \\
=&\lim_{z\to\infty} \frac{e^{-\lambda_1 z}(2V(1-N))}{-\lambda_1 e^{-\lambda_1 z}} + \lim_{z\to\infty} \frac{e^{-\lambda_2 z}(2V(1-N))}{-\lambda_1 e^{-\lambda_2 z}} \\
=&\lim_{z\to\infty} \frac{(2V(1-N))}{-\lambda_1} + \lim_{z\to\infty} \frac{(2V(1-N))}{-\lambda_1} \\
=&0.
\end{aligned}$$

The last equality is from the boundary condition of V and the boundedness of N. Thus, we have $\lim_{z\to\infty} L(V(z)) = 0$ given that $V(z) \in C_0$. Following the same calculation, we can demonstrate that $\lim_{z\to\infty} L(V(z)) = 0$. Therefore, indeed $L(V) \in C_0$ for $V \in C_0$. Following the same calculation in the above, we can make the same conclusion for Q and R. $\square$

In the preceding sections, we will use a combination of Fredholm operator theory and Banach's fixed point theorem to show the existence of V when β is small.

3.3 Surjectivity of the operator $L(V)$

This section aims to prove that the operator L is surjective from $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$. Recall the operator L is from the ODE (3.3):

$$V''(z) - cV'(z) + (1 - 2N(z))V(z) = 0.$$

Consider a substitution $x_1 = V'$ and $x_2 = V$, then we obtain the following ODE:

$$\mathbf{x}' = M(z)\mathbf{x}, \tag{3.10}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad M(z) = \begin{bmatrix} 0 & 1 \\ 2N(z) - 1 & c \end{bmatrix}.$$

To show the surjectivity of the operator L, our first goal is to show that the operator $F_1(\mathbf{x}) : C^1(\mathbb{R}) \to C(\mathbb{R})$:

$$F_1(\mathbf{x}) = \mathbf{x}' - M(z)\mathbf{x}, \tag{3.11}$$

is surjective.

3.3.1 Surjectivity of F_1 on $C(\mathbb{R})$

To prove that the operator $F_1(\mathbf{x}) : C^1(\mathbb{R}) \to C(\mathbb{R})$ is surjective, we first introduce the following definition:

Definition 12 (Exponential Dichotomy)**.** *Let $\mathbf{X}(z)$ be a fundamental matrix for the* *linear ODE:*

$$\mathbf{x}' = M(z)\mathbf{x}, \tag{3.12}$$

where the coefficient matrix $M(z) \in \mathbb{R}^{d\times d}$ *is continuous on an interval* $\mathbb{R}$. *If there exists an interval* J *and a projection matrix* P *that is independent of time and constants* $L_1, L_2, \mu_1, \mu_2 > 0$ *such that*

$$\begin{aligned} |\mathbf{X}(t)P\mathbf{X}^{-1}(s)| &\leq L_1 e^{-\mu_1(t-s)} &\quad \text{for } t \geq s, \\ |\mathbf{X}(t)(I-P)\mathbf{X}^{-1}(s)| &\leq L_2 e^{-\mu_2(s-t)} &\quad \text{for } s \geq t, \end{aligned} \tag{3.13}$$

for all $t, s \in J$, *then the equation* (3.12) *is said to possess an* ***exponential dichotomy*** *on the interval* J.

However, finding the fundamental solution is, in general, difficult, so we introduce the following criterion for an exponential dichotomy (see Proposition 1 in [3]).

Lemma 3.3.1. *Let* $M(z)$ *be a continuous* $d\times d$ *matrix function defined on an interval* J *such that*

1. $M(z)$ *has* k *eigenvalues with real parts* $\leq -a_1 < 0$ *and* $d-k$ *eigenvalues with real part* $\geq a_2 > 0$ *for all* $z \in J$,
2. $|M(z)|$ *is bounded for all* $z \in J$,
3. $|M(z)|$ *has bounded growth on* $z \in J$.

Then, the system

$$\mathbf{x}' = M(z)\mathbf{x},$$

possess an exponential dichotomy on the interval J.

Now we can show that the system (3.10) has an exponential dichotomy on both half line:

Lemma 3.3.2. *Take any $\epsilon > 0$, then the system* (3.10) *has an exponential dichotomy on $(-\infty, -\epsilon)$ and (ϵ, ∞).*

Proof. The core of the proof is to utilize lemma 3.3.1. Consider finding eigenvalues of $M(t)$ in (3.10):

$$\lambda_{\pm}(t) = \frac{c \pm \sqrt{c^2 - 4(1 - 2N(z))}}{2}.$$

By lemma 3.2.1, $N(z)$ increases monotonically from 0 to 1 and $N(0) = 1/2$, so the real part of the eigenvalues of $M(z)$ is uniformly bounded away from 0 on both intervals $(-\infty, -\epsilon)$ and (ϵ, ∞). In addition, bounded $|N(z)|$ and $|N'(z)|$ gives that $|M(z|$ is bounded with bounded growth. Thus, we checked all conditions of lemma 3.3.1. □

Next, we have the following characterization of the range of the operator F_1.

Lemma 3.3.3. *Let $M(z)$ be an $d \times d$ matrix function bounded and continuous on $(-\infty, \infty)$ such that the system:*

$$\mathbf{x}' = M(z)\mathbf{x} \tag{3.14}$$

has an exponential dichotomy on both $(-\infty, -\epsilon)$ and (ϵ, ∞) for any $\epsilon > 0$. Then, for the linear operator $F : C^1(\mathbb{R}) \to C(\mathbb{R})$

$$F_1(\mathbf{x})(z) = \mathbf{x}'(z) - M(z)\mathbf{x}(z),$$

$f \in R(F_1)$ the range of F_1 if and only if

$$\int_{-\infty}^{\infty} \psi^T(z) f(z)\, dz = 0, \tag{3.15}$$

for solution $\psi(z) \in C_0(\mathbb{R})$ of the adjoint system:

$$\mathbf{x}' = -M^T(z)\mathbf{x}. \tag{3.16}$$

Here ^{T} denotes the matrix transpose.

Proof. Let (3.14) have an exponential dichotomy on (ϵ, ∞) with projection P_1 and $(-\infty, -\epsilon)$ with projection P_2, and suppose that the fundamental solution matrix $\mathbf{X}(t)$ such that $\mathbf{X}(0) = I$. We first claim that $(\mathbf{X}(z)^T)^{-1}$ is a fundamental solution matrix for (3.16). Indeed,

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}z} I = (\mathbf{X}(z)^{-1}\mathbf{X}(z))' &= \mathbf{X}^{-1\prime}\mathbf{X} + \mathbf{X}^{-1}\mathbf{X}' \\
\implies \mathbf{X}^{-1\prime} &= -\mathbf{X}^{-1}\mathbf{X}'\mathbf{X}^{-1} \\
\implies \mathbf{X}^{-1\prime} &= -\mathbf{X}^{-1} M(z)\mathbf{X}\mathbf{X}^{-1} \\
\implies \mathbf{X}^{-1\prime} &= -\mathbf{X}^{-1} M(z) \\
\implies (\mathbf{X}^T)^{-1\prime} &= -M^T(z)(\mathbf{X}^T)^{-1}.
\end{aligned}$$

Then, by taking transpose in (3.13), we can conclude that the adjoint system (3.16) have an exponential dichotomy on (ϵ, ∞) with projection $I - P_1^T$ and $(-\infty, -\epsilon)$ with projection $I - P_2^T$.

Next, let $f \in R(F_1)$, let $\psi(t)$ be a solution of (3.16) in $C_0(\mathbb{R})$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \psi^T(z) f(z)\, dt =& \int_{-\infty}^{\infty} \psi^T(z) x'(z) - \psi^T(z) A x(z)\, dt \\ =& \int_{-\infty}^{\infty} \psi^T(z) x'(z) + (\psi^T(z))' x(z)\, dt \\ =& \int_{-\infty}^{\infty} (\psi^T(z) x'(z))'\, dt \\ =& [\psi^T(z) x'(z)]_{-\infty}^{\infty} = 0.\end{aligned}$$

We finished the forward implication.

Conversely, suppose $f \in C_0(\mathbb{R})$ and (3.15) holds for all solution $\phi(z) \in C_0$ of (3.16). Let η be a vector such that:

$$\eta^T [P_1 - (I - P_2)] = 0. \tag{3.17}$$

Then, notice that:

$$\psi(z) = \begin{cases} (X^*(z))^{-1}(I - P_1^T)\eta, & z \geq 0 \\ (X^*(z))^{-1} P_2^T \eta, & z \leq 0 \end{cases}$$

is a solution to (3.16) in $C_0(\mathbb{R})$. By (3.15), it follows that

$$\eta^T \left[\int_{-\infty}^{0} P_2 \mathbf{X}^{-1}(z) f(z)\, dz + \int_{0}^{\infty} (I - P_1) \mathbf{X}^{-1}(z) f(z)\, dz \right] = 0.$$

for all vector η satisfies (3.17). Thus, the vector

$$\int_{-\infty}^{0} P_2 \mathbf{X}^{-1}(z) f(z)\, dt + \int_{0}^{\infty} (I - P_1) \mathbf{X}^{-1}(z) f(z)\, dt,$$

is in the range of $P_1 - (I - P_2)$ (since it is orthogonal to the left null space). Therefore, the following algebraic equation

$$[P_1 - (I - P_2)]\theta = \int_{-\infty}^{0} P_2 \mathbf{X}^{-1}(z) f(z)\, dz + \int_{0}^{\infty} (I - P_1)\mathbf{X}^{-1}(z) f(z)\, dz,$$

has a solution θ. Then,

$$\mathbf{x}(z) := \begin{cases} \mathbf{X}(z)P_1\theta + \int_0^z \mathbf{X}(z)P_1\mathbf{X}^{-1}(s)f(s)\,ds - \int_z^\infty \mathbf{X}(z)(I-P_2)\mathbf{X}^{-1}(s)f(s)\,ds & z \geq 0 \\ \mathbf{X}(I-P_2)\theta + \int_{-\infty}^z \mathbf{X}(z)P_2\mathbf{X}^{-1}(s)f(s)\,ds - \int_z^0 \mathbf{X}(z)(I-P_2)\mathbf{X}^{-1}(s)f(s)\,ds & z \leq 0 \end{cases},$$

is a bounded solution to the system:

$$\mathbf{x}'(z) - M(z)\mathbf{x}(z) = f(z).$$

Therefore, $f \in R(F_1)$. $\square$

Now we can show that F_1 is surjective on $C(\mathbb{R})$:

Proposition 3.3.4. *The operator F_1 defined in* (3.11) *is a surjective operator from $C^1(\mathbb{R})$ to $C(\mathbb{R})$.*

Proof. We first claim that all solutions in $C_0(\mathbb{R})$ of the adjoint system:

$$\mathbf{x}'(z) = -\begin{bmatrix} 0 & 1 \\ 2N(z) - 1 & c \end{bmatrix}^T \mathbf{x}(z) \tag{3.18}$$

are trivial solutions. Notice that we can rewrite the equation into a second order ODE for x_2:

$$x_2'' + cx_2' + (1 - 2N(z))x_2 = 0.$$

Multiply both sides by x_2' and integrate gives:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} x_2''x_2' + cx_2'x_2' + \frac{1}{2}(1-2N(z))\frac{\mathrm{d}}{\mathrm{d}z}(x_2)^2 = 0\\
\implies &\frac{1}{2}[(x_2')^2]_{-\infty}^{\infty} + c\int_{-\infty}^{\infty}(x_2')^2 + \frac{1}{2}[(1-2N)(x_2)^2]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} N'(x_2)^2\,dz = 0\\
\implies &c\int_{-\infty}^{\infty}(x_2')^2 + \int_{-\infty}^{\infty} N'(x_2)^2\,dz = 0.
\end{aligned}
$$

The first implication is due to integration by parts, and the second one is from $x_2, x_2' \in C_0(\mathbb{R})$. Finally, notice that $(x_2')^2 \geq 0$ and $N'(x_2)^2 \geq 0$. Thus, $x_2 = 0$ is the only solution. Substitute this into (3.18), we see that $x_1' = 0$, so with the fact $x_1 \in C_0(\mathbb{R})$, $x_1 = 0$ as well.

Then, by lemma 3.3.3, the operator F_1 in (3.11) is surjective from $C^1(\mathbb{R})$ to $C(\mathbb{R})$ $\square$

The next goal is to prove that F_1 is a surjective operator from $C_0^1(\mathbb{R}) \to C_0(\mathbb{R})$.

3.3.2 Surjectivity of F_1 on $C_0(\mathbb{R})$

We aim to prove that F_1 defined in (3.11) is a surjective operator from $C_0^1(\mathbb{R})$ to $C_0(\mathbb{R})$. One should notice that this is a more fined result compared to proposition 3.3.4. To proceed, we first introduce the following lemma (see Proposition 1.1 and Theorem 1.8 in [16]):

Lemma 3.3.5. *Consider the following non-autonomous and autonomous system:*

$$\mathbf{x}' = g_1(t, \mathbf{x}), \tag{3.19}$$

$$\mathbf{y}' = g_2(t, \mathbf{y}), \tag{3.20}$$

where g_1, g_2 are Lipschitz continuous in $\mathbf{x}$ and continuous in t. Let $(t_0, \mathbf{x}_0)$ be the initial condition. Consider the solution $\mathbf{x}_1(t)$ to (3.19) with the initial condition $(t_0, \mathbf{x}_0)$.

If the following assumptions holds:

1. $g_1(t, \mathbf{x}) \to g_2(\mathbf{x})$ *as* $t \to \infty$ *uniformly in* $\mathbf{x}$,

2. *The orbit* $\mathcal{O}_{g_1}(t_0, \mathbf{x}_0) := \{\mathbf{x}_1(t) : t \geq t_0\}$ *has compact closure.*

Then, the omega ω *limit set of* $\mathbf{x}_1$ *has the following properties:*

1. ω *is nonempty, compact and connected,*

2. ω *is invariant under the autonomous system* (3.20),

3. ω *attracts* $\mathbf{x}_1(t)$*:*

$$|\mathbf{x}_1(t) - \omega| \to 0, \ \textit{as } t \to \infty.$$

Now, we can prove that F_1 is surjective on $C_0(\mathbb{R})$:

Proposition 3.3.6. *The operator* F_1 *defined in* (3.11) *is a surjective operator from* $C_0^1(\mathbb{R})$ *to* $C_0(\mathbb{R})$.

Proof. Take any $f \in C_0(\mathbb{R})$, then by proposition 3.3.4, there exists a solution $\mathbf{x} \in C^1(\mathbb{R})$ s.t.

$$\mathbf{x}'(z) = \begin{bmatrix} 0 & 1 \\ 2N(z) - 1 & c \end{bmatrix} \mathbf{x}(z) + f(z). \tag{3.21}$$

By lemma 3.2.1, $N(z)$ has bounded derivative, and $f(z) \in C_0(\mathbb{R})$. Thus, the left-hand side of (3.3.4) is uniformly continuous in z. Thus, as $z \to \infty$, (3.3.4) converges uniformly to:

$$\mathbf{x}'(z) = \begin{bmatrix} 0 & 1 \\ 1 & c \end{bmatrix} \mathbf{x}(z). \tag{3.22}$$

In addition, $\mathbf{x}(z) \in C^1(\mathbb{R})$ shows that the orbit has compact closure. Thus, by lemma 3.3.5, the omega limit set of (3.21) is non-empty, compact, connected and invariant under (3.22). Notice that the only possible candidate for this kind of set is the equilibrium of (3.22) at the origin. Thus, $\mathbf{x}(z) \to 0$ as $z \to \infty$. The same argument is also true for the alpha limit set. Thus, we can conclude that $\mathbf{x}(z) \in C_0(\mathbb{R})$. Finally, by looking at (3.21) once more, the right-hand side is in $C_0(\mathbb{R})$ as well. Thus, $\mathbf{x}(z) \in C^1(\mathbb{R})$. □

Now, we can prove that $L(V)$ is surjective.

3.3.3 Range of the operator L

Define the operator $F_2 : C_0^2(\mathbb{R}) \to C_0(\mathbb{R})$

$$F_2 V(z) = V''(z) - cV'(z) + (1 - 2N(z))V(z) \tag{3.23}$$

Then, we can state the following corollary

Corollary 3.3.7. *The operator $F_2 : C_0^2(\mathbb{R}) \to C_0(\mathbb{R})$ defined in* (3.23) *is surjective.*

Proof. A direct consequence of proposition 3.3.6. □

Now we prove the main result of the section

Proposition 3.3.8. *The operator L defined in* (3.8) *is a surjective operator from $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$. That is, for any $Z(z) \in C_0(\mathbb{R})$, there exists $V(z) \in C_0(\mathbb{R})$ such that*

$$W(z) = V - \frac{1}{\lambda_2 - \lambda_1} \int_{-\infty}^{z} e^{\lambda_1 (z-s)} (2V(1-N))\, ds - \frac{1}{\lambda_2 - \lambda_1} \int_{z}^{\infty} e^{\lambda_2 (z-s)} (2V(1-N))\, ds. \tag{3.24}$$

Proof. Consider the difference $D(z) = V(z) - W(z)$ and substitute this into (3.24):

$$\begin{aligned} D(z) = &\frac{1}{\lambda_2 - \lambda_1} \int_{-\infty}^{z} e^{\lambda_1(z-s)}(2(D+W)(1-N))\,ds \\ &\frac{1}{\lambda_2 - \lambda_1} \int_{z}^{\infty} e^{\lambda_2(z-s)}(2(D+W)(1-N))\,ds. \end{aligned}$$

Transform this into the differential form:

$$D''(z) - cD'(z) + (1 - 2N(z))D(z) = -(2 - 2N(z))W(z).$$

Notice that $-(2-2N(z))W(z) \in C_0(\mathbb{R})$, thus by corollary 3.3.7, the difference $D(z) \in C_0(\mathbb{R})$ exists. Therefore, $V(z) = D + W \in C_0$ exists as well. □

3.4 Approximation of Q and R

The following two lemmas show that the mappings $R(V)$ and $Q(V)$ are contractions.

Lemma 3.4.1. *For any $k \in [0, 1)$ there exists a constant δ such that*

$$\|R(f) - R(g)\|_{C_0(\mathbb{R})} \leq k\|f - g\|_{C_0(\mathbb{R})},$$

where $f, g \in B(\delta)$ the delta ball in C_0 centered at 0, and R is defined by eq. (3.5).

Proof. Notice that $\|V^2\|_{C_0(\mathbb{R})} = \|V\|^2_{C_0(\mathbb{R})} \leq \delta\|V\|_{C_0(\mathbb{R})}$, thus, by (3.5) and a small enough δ, we have

$$\|R(f) - R(g)\|_{C_0(\mathbb{R})} \leq k\|f - g\|_{C_0(\mathbb{R})},$$

for any $k \in [0, 1)$. □

Lemma 3.4.2. *For any $k \in [0,1)$, there exists δ and ϵ such that for $|\beta| \in (0,\epsilon)$ and $f, g \in B(\delta)$, we have*

$$\|Q(f) - Q(g)\|_{C_0(\mathbb{R})} \leq k\|f - g\|_{C_0(\mathbb{R})}.$$

Proof. Notice that

$$G = \beta(U+V)\frac{\mathrm{d}}{\mathrm{d}z}\int_{-R}^{R} U(z+y) + V(z+y)\,dy.$$

Then, we have

$$\|G(V)\|_{C_0(\mathbb{R})} \leq O(\beta) + O(\beta\|V\|_{C_0(\mathbb{R})}) \text{ as } \|V\|_{C_0(\mathbb{R})} \to 0,$$

for any $V \in B(\delta)$ and $|\beta| \in (0,\sigma)$. Thus, for small enough δ, ϵ and by (3.7), we also have

$$\|Q(f) - Q(g)\| \leq k\|f - g\|,$$

for all $f, g \in B(\delta)$. □

Thus, we can conclude that Q and R defined in (3.7) and (3.5) are contraction mappings.

3.5 Existence of travelling wave solutions for small β

In this section, we give proof of the main results: the existence of the travelling wave solution to (2.1) with a small non-local term.

Theorem 3.5.1. *For any $c \geq 2, R > 0$, there exists $\epsilon > 0$ such that for any $\alpha \in [0, \epsilon)$, the following system:*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial}{\partial x}\left(u \frac{\partial A}{\partial x}\right) + u(1-u) \quad A = \frac{1}{2R}\int_{-R}^{R} u(x+y,t)\,dy$$

admits a unique travelling wave solution $u(x,t) = U(x+ct)$ subject to the boundary condition:

$$U(-\infty) = 0, \quad U(\infty) = 1.$$

Proof. The core of the proof is to show that:

$$L(V) = R(V) + Q(v), \tag{3.25}$$

has a fixed point in $C_0(\mathbb{R})$ using Banach's fixed point theorem. Here L, R and Q are defined in (3.8), (3.5) and (3.7).

First, consider a restriction of the operator L. Let $N(L)$ be the null space of $L : C_0(\mathbb{R}) \to C_0(\mathbb{R})$. Then, the annihilator of the null space of $N^{\perp}(L)$ is a well-defined Banach space, and:

$$N^{\perp}(L) \oplus N(L) = C_0(\mathbb{R}).$$

Then, the restricted operator $P : N^{\perp}(L) \to C_0 : V(z) \mapsto L(V(z))$ is a bijective bounded linear operator. Thus, the inverse is a bounded linear operator as well.

Thus, the equation (3.25) can be written as

$$V(z) = P^{-1}(Q(V(z)) + R(V(z))).$$

We also have the following estimate:

$$\begin{aligned}\left\|P^{-1}(Q(V)+R(V))\right\|_{C_0(\mathbb{R})} \leq & \left\|P^{-1}\right\|(O(\beta)+O(\beta\|V\|_{C_0(\mathbb{R})})+O(\|V\|^2_{C_0(\mathbb{R})})) \\ = & \left\|P^{-1}\right\|(O(\beta)+O(\|V\|_{C_0(\mathbb{R})}))^2,\end{aligned}$$

Thus, there exists δ and ϵ such that for $V, f, g \in B(\delta) \cap N^{\perp}(L)$ and $\beta \in (0, \epsilon)$ we have $\|P^{-1}\|(O(\beta) + O(\|V\|))^2 < \delta$. In addition, by lemma 3.4.1 and lemma 3.4.2

$$\left\|P^{-1}(Q(f + R(f) - P^{-1}(Q(g) + R(g)\right\|_{C_0(\mathbb{R})} \leq k\|\psi - \phi\|_{C_0(\mathbb{R})}.$$

Then, V exists in $B(\delta) \cap N^{\perp}(L)$ using Banach's fixed point theorem. Thus, the solution to the equation (3.1) in the form $U = N + V$ exists. This shows the existence of a travelling wave solution to the equation (1.2) $\square$

Thus, we proved the existence of travelling wave solutions for small β.

Chapter 4

Numerical Simulation

In this chapter, we aim to simulate the travelling wave solution in theorem 3.5.1 using the method of lines and the perturbation method.

4.1 Method of lines with central difference scheme

In this section, our goal is to simulate the PDE using the method of lines(MOL) in time with central difference in space. The idea of the MOL is to turn the PDE into a system of ODE by discretizing the spatial variable only, then solve this huge system of ODE by existing ODE solver packages. Recall that using the divergence theorem on the non-local term, we can rewrite (1.2) as:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial}{\partial x}\left[u \frac{1}{2R}(u(x+R,t) - u(x-R,t))\right] + u(1-u).$$

One should notice that the advantage of this formulation is to avoid computing the numerical integral A. We will first develop the numerical scheme

4.1.1 The scheme

We start by discretizing the space interval $x \in [-l, l]$ for $l > 0$. Let $x_1 = -l, x_2 = -l + h, \dots x_n = l$ be the n discretization in space, where $h = 2l/(n-1)$ is the space step size. In addition, let $m = \lfloor R/h \rfloor$ be the floor of the ratio R/h. Thus, the existing supports $x_{i \pm m} \approx x_i \pm R$ can approximate the non-local terms well.

For ease of notation, denote $u(x_i, t) = u_i(t)$. Then, we can simulate the PDE by solving $u_i(t)$ numerically. This effectively turns the PDE into a n dimensional ODE (This method is so-called "the method of lines"). Next, consider using the central difference scheme to approximate spatial derivatives:

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} =& \frac{\partial^2 u}{\partial x^2}(x_i, t) - \beta \frac{\partial}{\partial x}[u(x_i, t)(u(x_i + R, t) - u(x_i - R, t))] + u(x_i, t)(1 - u(x_i, t)) \\
\approx& \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \\
&- \frac{\beta}{2R} \frac{[u_{i+1}(u_{i+1+m} - u_{i+1-m})] - [u_{i-1}(u_{i-1+m} - u_{i-1-m})]}{2h} \\
&+ u_i(1 - u_i)
\end{aligned}
$$

for $i = 1, 2 \dots n$. Notice that for $i > n$ and $i < 1$, u_i is not defined. We will now fix this problem by specifying the initial condition and the boundary condition.

For the sake of simulating the travelling wave solution. We consider the shifted Heaviside-side initial condition:

$$
u(x_n, 0) = \begin{cases} 0, & x_i \in [-l, h_0) \\ 1, & x_i \in (h_0, l] \\ 0.5, & x_i = h_0 \end{cases}, \tag{4.1}
$$

for $i = 1, 2, \dots n$, where $h_0 \in (-l, l)$ is where the Heaviside function jumps to 1. We also impose the Dirichlet Boundary condition:

$$u_i = \begin{cases} 0, & i < 1 \\ 1, & i > N \end{cases}.$$

In addition, we would like to boost the algorithm efficiency by implementing vectorization. The idea of vectorization is to rewrite the discretized scheme into matrix and vector operations, so we can avoid loop in the simulation. This, in general, will shorten the time to compile the algorithm. For the simplicity of the notation, we define the following $n \times n$ matrix:

$$\mathbf{1}_k = [a_{ij}]_{1 \le i,j \le n} = \begin{cases} a_{ij} = 1, & j - i = k \\ a_{ij} = 0, & j - i \ne k \end{cases},$$

where $0 \le k \le N - 1$. Notice that the matrix $\mathbf{1}_k$ is the off-diagonal matrix. For $k > 0$, we have a super diagonal matrix; for $k < 0$, we have a sub diagonal matrix; and for $k = 0$, we have $\mathbf{1}_0 = I_n$, which is the identity matrix. We also define the following n dimensional vector:

$$\mathbf{b}_k = [a_i]_{1 \le i \le n} = \begin{cases} a_i = 0, & i \le n - k \\ a_i = 1, & i > n - k \end{cases}.$$

That is, the vector $\mathbf{b}_k$ is the vector such that the bottom k entries are 1 and zero everywhere else. Then, we can conveniently denote the approximation schemes into matrix notation. Let $\mathbf{U} = [u_1, u_2, \dots u_n]^T$. In addition, we also assume that $m > 1$

(Naturally, we want $R >> h$). Then,

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} \approx \frac{1}{h^2}[(\mathbf{1}_{+1} - 2I_N + \mathbf{1}_{-1})\mathbf{U} + b_1] \tag{4.2}$$

$$- \frac{\beta}{4Rh}\{[(\mathbf{1}_{+1}\mathbf{U} + \mathbf{b}_{+1}) \otimes (\mathbf{1}_{1+m}\mathbf{U} + \mathbf{b}_{1+m} - \mathbf{1}_{1-m}\mathbf{U})] \tag{4.3}$$

$$- [\mathbf{1}_{-1}\mathbf{U} \otimes (\mathbf{1}_{-1+m}\mathbf{U} + \mathbf{b}_{-1+m} - \mathbf{1}_{-1-m}\mathbf{U})]\} \tag{4.4}$$

$$+ \mathbf{U} \otimes (\mathbf{b}_N - \mathbf{U}), \tag{4.5}$$

is the vector notation for the approximation scheme. Here $\otimes$ denotes Hadamard product (or pointwise multiplication). From this, we can use numerical ODE solvers to solve the above system of equations.

4.1.2 Numerical results of MOL

Using the `scipy.integrate,solve_ivp` in python (for the Python code, please see Appendix A), our numerical scheme (4.2) yields the result with different values of β as demonstrated in fig. 4.1, fig. 4.2, fig. 4.3, fig. 4.4, fig. 4.5 and fig. 4.6. We can observe the formation of travelling wave solution for small $|\beta|$ as evidenced in fig. 4.1 and fig. 4.2. As $|\beta|$ increases, the solution still propagates with speed, but there exists an oscillation in the shape of the solution (fig. 4.5 and fig. 4.6). This possibly shows the non-existence of travelling wave solutions. Overall, the numerical result is consistent with our theoretical fundings (existence of travelling wave solutions for with small $|\beta|$ theorem 3.5.1).

However, one should observe that when $|\beta| = 4$, the numerical results shows the existence of travelling wave solution when $\beta = -4$ but not in the case when $\beta = 4$ (fig. 4.3 and fig. 4.4). Therefore, $|\beta|$ along is not enough to determine the formation of the travelling wave solution; the direction of the advection also plays a role in the travelling wave formation. This will be our future study.

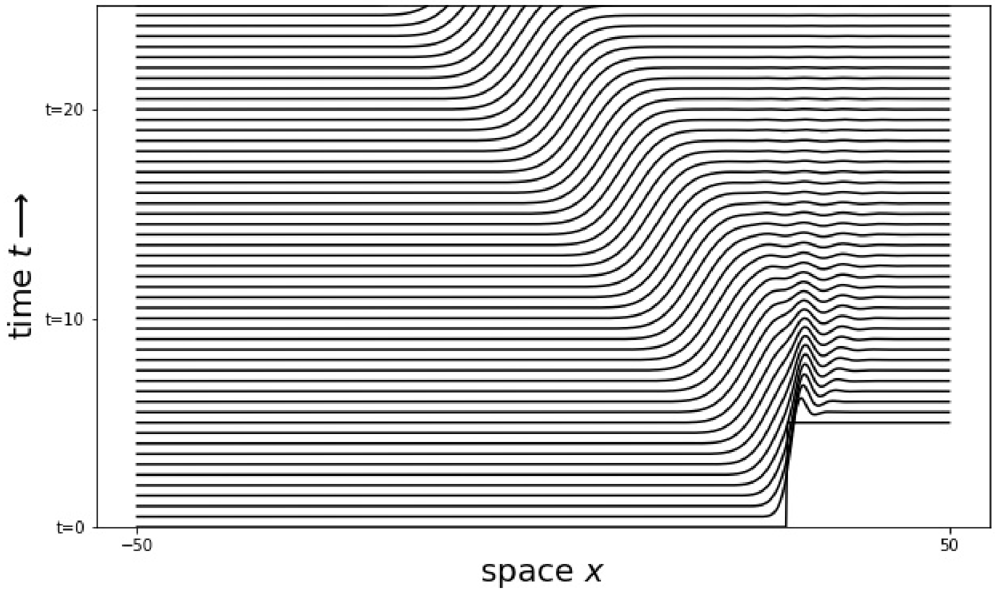

Figure 4.1: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = 2$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution approaching a travelling wave profile.

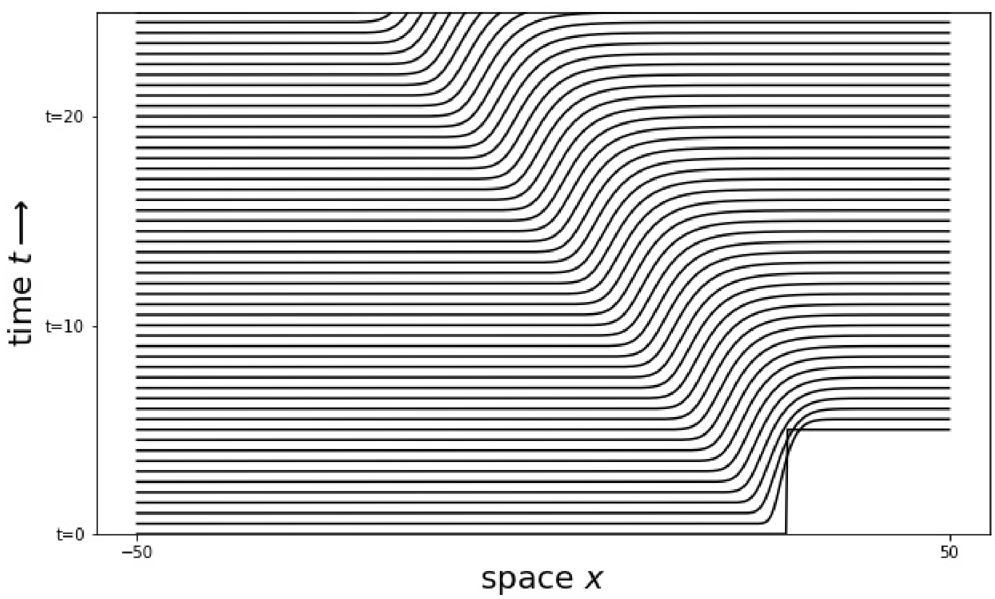

Figure 4.2: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = -2$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution approaching a travelling wave profile.

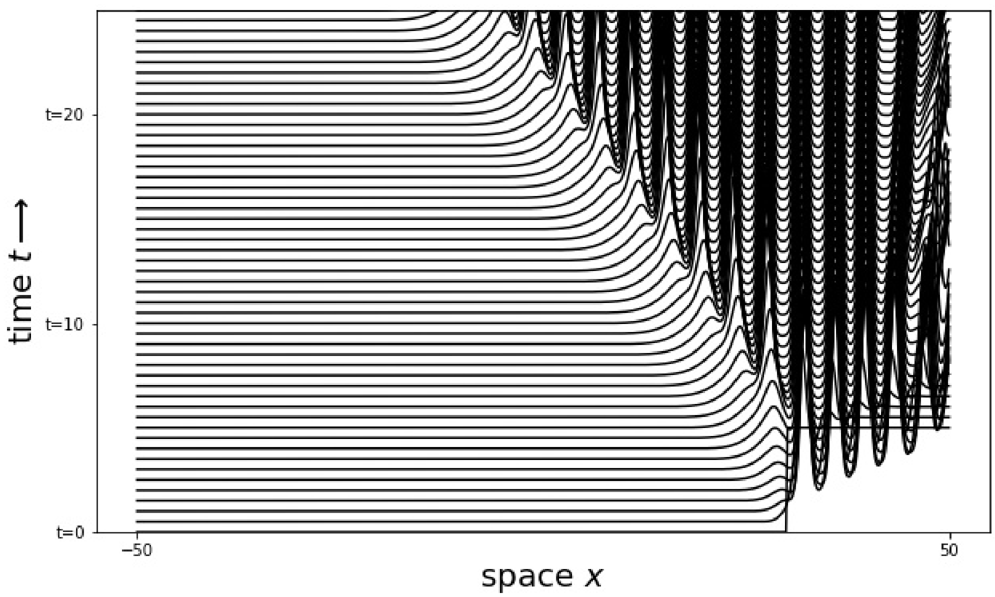

Figure 4.3: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = 4$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution, connecting the zero solution to a homogeneous stationary solution that is oscillatory in space, which approaches an asymptotic speed.

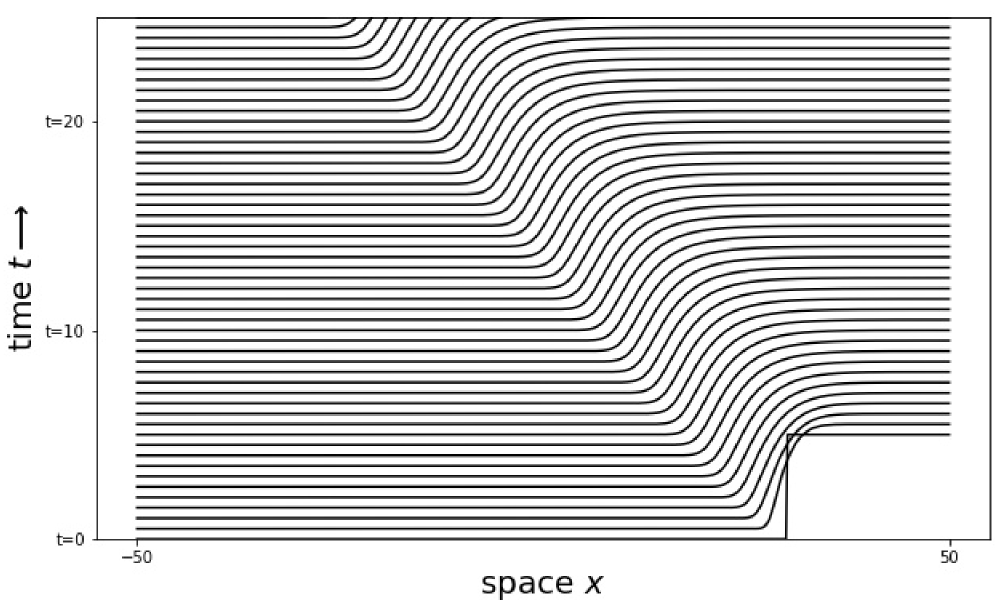

Figure 4.4: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = -4$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution approaching a travelling wave profile.

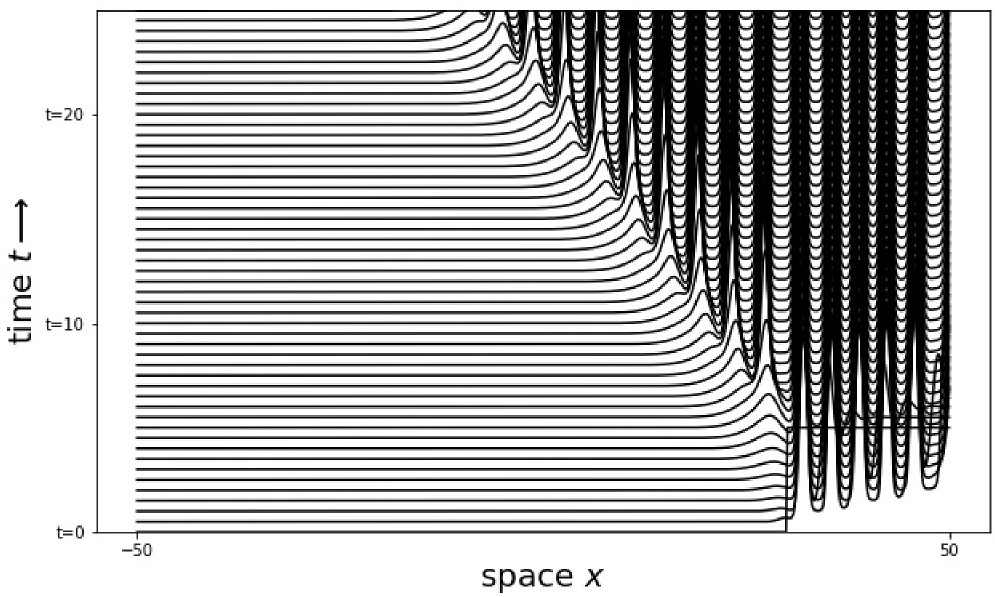

Figure 4.5: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = 6$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution, connecting the zero solution to a homogeneous stationary solution that is oscillatory in space, which approaches an asymptotic speed.

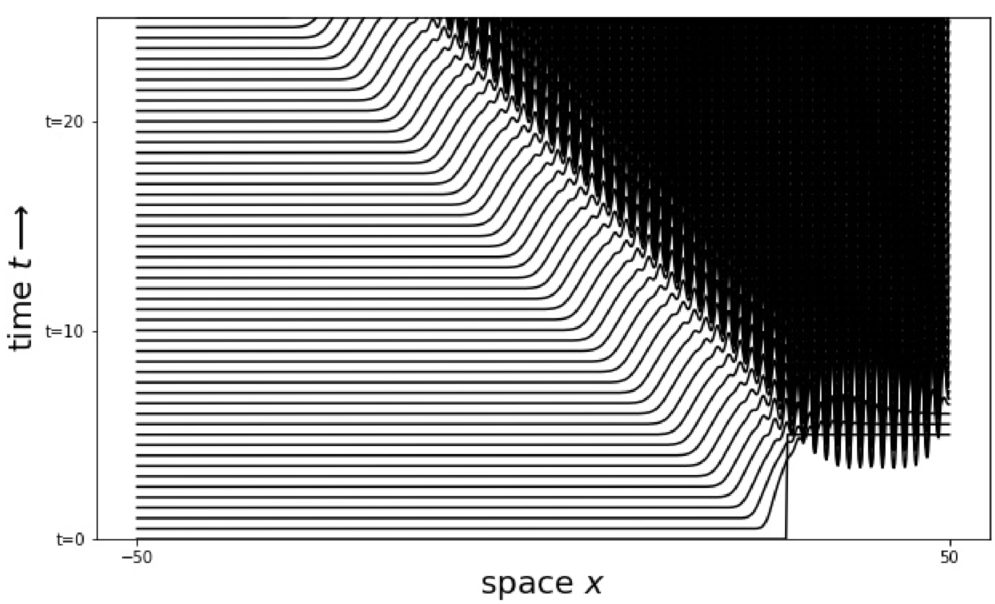

Figure 4.6: Numerical simulation of equations (1.2) using the method of lines with a time domain of $[0, 30]$ and a space domain of $[-50, 50]$. Here $\beta = -6$ and $R = 1$. The initial condition was (4.1) with $h_0 = 30$. Note that the numerical solution forms a wave-type solution, connecting the zero solution to a homogeneous stationary solution that is oscillatory in space, which approaches an asymptotic speed.

4.2 Approximation to the travelling wave profile

This section aims to find the numerical travelling wave profile by a perturbation method. The idea of the perturbation method is to rewrite the solution as a power series expansion, so we can avoid solving a differential equation without involving non-local terms. Recall the ODE for the travelling wave profile (3.1):

$$\begin{aligned} 0 =& U''(z) - cU'(z) - \beta[U(z)\overline{U}(z)]' + U(1-U), \\ \overline{U}(z) =& \frac{1}{2R}\frac{\mathrm{d}}{\mathrm{d}z}\int_{-R}^{R} U(z+y)\,dz = \frac{1}{2R}(U(z+R) - U(z-R)). \end{aligned}$$

Subject to the boundary condition $U(-\infty) = 0$ and $U(\infty) = 1$. Consider looking for solutions of (3.1) as a regular perturbation series in β; that is let

$$U(z) = U_0(z) + \beta U_1(z) + \dots \tag{4.6}$$

From the boundary conditions of U, we gives the following boundary conditions of U_i:

$$\begin{aligned} &U_0(-\infty) = 0, \quad U_0(\infty) = 1, \\ &U_i(-\infty) = 0, \quad U_i(\infty) = 0 \text{ for } i = 1, 2, \dots \end{aligned} \tag{4.7}$$

substitute the ansatz (4.6) into (3.1), we obtain the following equations by grouping terms with the same power of β:

$$O(1) \quad 0 = U_0'' - cU_0' + U_0(1 - U_0) \tag{4.8}$$

$$O(\beta) \quad 0 = U_1'' - cU_1' + U_1 - 2U_1U_0 - U_0\overline{U_0} \tag{4.9}$$

$$O(\beta^i) \quad \vdots \tag{4.10}$$

Notice that (4.8) gives the travelling wave profile of the Fisher's equation subject to the boundary condition (4.7). Again, this is a known solution (lemma 3.2.1). Thus, to find an approximation, all we need is to solve the following boundary value problem:

$$0 = U_1'' - cU_1' + U_1 - 2U_1U_0 - U_0\overline{U_0}, \quad U_1(\infty) = 0. \tag{4.11}$$

One should notice that the non-local term $\overline{U_0}$ is a known function. Thus, the approximate equation is a regular boundary value problem without non-locality. Now, we can solve this problem numerically using existing BVP packages.

4.2.1 Numerical results for the perturbation method

Using the function `scipy.integrate,solve_bvp` in scipy package, we can solve (4.11) numerically. For the Python code, please see Appendix chapter B. The numerical travelling wave profile of (1.2) from different methods can be found in fig. 4.7. We can observe that the numerical results of MOL and the perturbation method are fairly close.

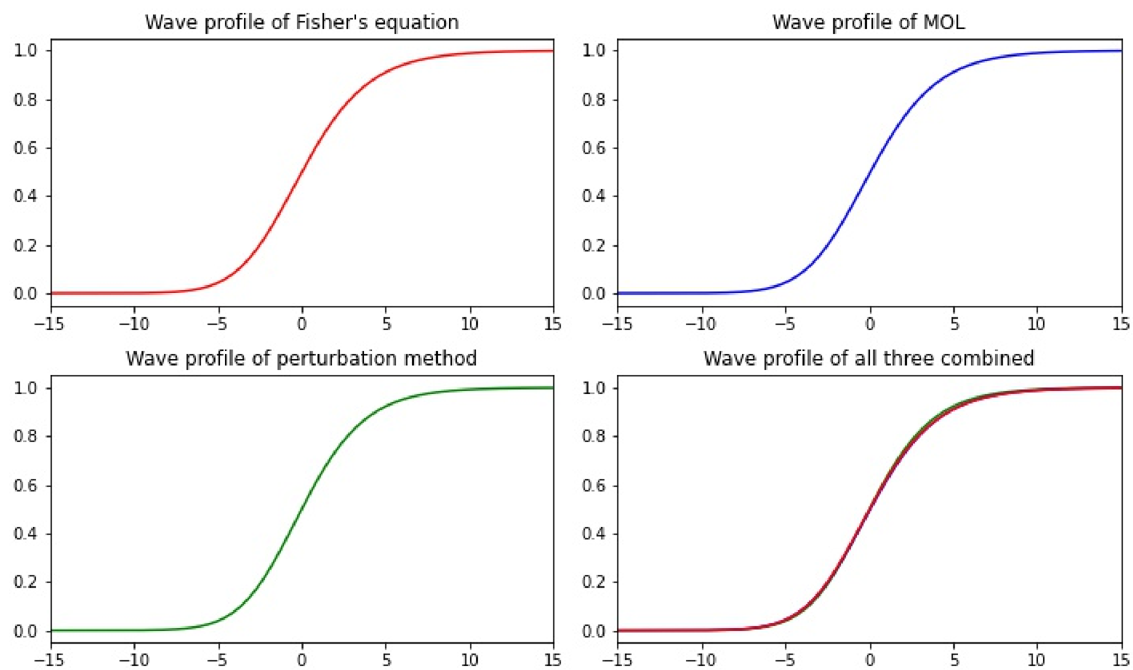

Figure 4.7: Numerical travelling wave profile of (1.2) with different method. The travelling wave speed is $c = 2$. The parameters of the non-local terms are $\beta = 0.2$ and $R = 1$. All solutions are aligned such that the value in the middle is $1/2$.

Chapter 5

Discussions

In this thesis, we have analyzed different aspects of our non-local reaction-advection-diffusion (RAD) equations, including the well-posedness and the travelling wave solution. This discussion will survey the background, summarize our findings, compare them with the existing literature and suggest potential directions for future research.

In [21], J. Potts and M. Lewis proposed a way to model non-local interactions using advection-diffusion equations with non-local advection terms. In our thesis, we considered a natural extension of their work to include the birth and death process (logistic growth). The resulting model is in the form of a non-local reaction-advection-diffusion equation for self-interacting species.

Initially, we demonstrated the local-in-time existence and positivity of the solution given that the initial condition $u_0 \in H^3(\mathbb{R})$. Our analysis revealed that the spatial averaging kernel $\mathcal{K}$ plays a crucial role in this model. For instance, in previous studies [6, 7] of a similar equation without the reaction term, the spatial averaging kernel was assumed to be differentiable with bounded derivatives, allowing the authors to conclude global existence based on the conservation of the L^1 norm. In contrast, our use of a non-smooth indicator function means we can only provide a continuation criterion. As a result, the global existence of the solution to our equation remains an open question, representing a significant area for future research.

Subsequently, we proved the existence of travelling wave solutions using a combination of the perturbation method, Fredholm operator theory, and Banach's fixed point theorem. Our theoretical results indicate that travelling wave solutions exist when the non-local advection term is small. Compared to the methods used in [25], we improved the proof regarding the use of Fredholm operator theory and provided a more refined argument. However, both the work done in [25] and by us cannot predict the existence of the travelling wave solutions when the non-local advection term is large.

Additionally, we employed numerical methods, including the method of lines and perturbation methods, to simulate the travelling wave profile. These numerical results corroborate our theoretical findings, showing the existence of travelling wave solutions with small β. However, we lack precise estimates on how small the non-local term must be to permit the formation of travelling wave solutions. Addressing this will be another promising direction for future research.

In summary, while we have made significant progress in understanding the well-posedness and travelling wave solutions of our non-local RAD equation, several open questions remain. Future studies should focus on:

- Determining the conditions under which the solution exists globally in time.
- Establishing precise estimates for the non-local term that allows the formation of travelling wave solutions.

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Appendix A

Python Code for MOL

```
import numpy as np

from scipy.integrate import solve_ivp

import matplotlib.pyplot as plt


#Specify some basic parameter

l=100


N=1001 #Number of supports is N+1. We recommond to use N odd


beta=-3 #the magnititude of the non-local advection


R=1 #the range of the non-local term


h=2*l/(N-1) #Space step


m=int(R/h//1)
```

```
X=np.linspace(-l,l,N) #Generate supports

#Generate the initial condition
Y0=np.zeros(N)
D=900
for i in range(N):
    if i< D:
        Y0[i]=0
    elif i> D:
        Y0[i]=1
    else:
        Y0[i]=0.5

def offdiagmatrix(k):
    M=np.zeros((N,N))
    for i in range(N):
        for j in range(N):
            if j-i==k:
                M[i,j]=1
    return M

def tailvector(k):
    V=np.zeros(N)
    for i in range(N):
        if i>=N-k:
            V[i]=1
```

```
    return V

offp1=offdiagmatrix(1)

offp0=offdiagmatrix(0)

offm1=offdiagmatrix(-1)

bp1=tailvector(1)

offp1pM=offdiagmatrix(1+m)

bp1pM=tailvector(1+m)

offp1mM=offdiagmatrix(1-m)

offm1pM=offdiagmatrix(-1+m)

bm1pM=tailvector(-1+m)

offm1mM=offdiagmatrix(-1-m)

bN=tailvector(N)

def F(t,Y):
    return (np.matmul(offp1-2*offp0+offm1,Y)+bp1)/h**2\
```

```
        -(beta/(4*R*h))*(((np.matmul(offp1,Y)+bp1)\
        *(np.matmul(offp1pM,Y)+bp1pM-np.matmul(offp1mM,Y)))\
        -(np.matmul(offm1,Y)*(np.matmul(offm1pM,Y)\
        +bm1pM-np.matmul(offm1mM,Y))))\
        +Y*(bN-Y)

Tend=70

tpoints=np.linspace(0,Tend,int(Tend/2+1))

result_solve_ivp = solve_ivp(F, [0,Tend], Y0,method='Radau',t_eval=tpoints)
```

Appendix B

Python Code for Perturbation method

```
import numpy as np
from scipy.integrate import solve_ivp
from scipy.integrate import solve_bvp
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.ticker import LinearLocator

#Specify some basic parameter
l=50

N=1001 #Number of supports is N+1. We recommond to use N odd

beta=0.2 #the magnititude of the non-local advection

R=1 #the range of the non-local term

h=2*l/(N-1) #Space step
```

```
m=int(R/h//1)

X=np.linspace(-l,l,N) #Generate supports

def offdiagmatrix(k):
    M=np.zeros((N,N))
    for i in range(N):
        for j in range(N):
            if j-i==k:
                M[i,j]=1
    return M

def tailvector(k):
    V=np.zeros(N)
    for i in range(N):
        if i>=N-k:
            V[i]=1
    return V

offp1=offdiagmatrix(1)

offp0=offdiagmatrix(0)

offm1=offdiagmatrix(-1)
```

```
bp1=tailvector(1)

offp1pM=offdiagmatrix(1+m)

bp1pM=tailvector(1+m)

offp1mM=offdiagmatrix(1-m)

offm1pM=offdiagmatrix(-1+m)

bm1pM=tailvector(-1+m)

offm1mM=offdiagmatrix(-1-m)

bN=tailvector(N)

def Fisher(t,Y):
    return (np.matmul(offp1-2*offp0+offm1,Y)+bp1)/h**2 +Y*(bN-Y)

Tend=35

tpoints=np.linspace(0,Tend,71)

result_solve_ivpFisher = solve_ivp(Fisher, [0,Tend], Y0,method='Radau',t_eval=tpoi

for i in range(len(X)):
```

```
    if result_solve_ivpFisher.y[i,36]>0.5:
        Fisher05=np. roll(result_solve_ivpFisher.y[:,36], N//2+1-i)
        if 501-i>0:
            Fisher05[0:N//2+1-i]=0
        if 501-i<0:
            Fisher05[(N//2+1-i):]=1
        break

def barU(U):
    Up=np. roll(U, -m)
    Up[-m:]=1
    Um=np.roll(U,m)
    Um[0:m]=0
    return (Up-Um)/(2*R)

barU0=barU(Fisher05)

U0=Fisher05

def fun(x, y):
    return np.vstack((y[1], 2*y[1]-y[0]+2*y[0]*U0[-y[0].size:]-(U0*barU0)[-y[0].si

def bc(ya, yb):
    return np.array([ya[0], yb[0]])

y_a = np.zeros((2,X.size))
```

```
res_a = solve_bvp(fun, bc, X, y_a)


PerturbSol=res_a.y[0]*beta+Fisher05
for i in range(len(X)):
    if PerturbSol[i]>0.5:
        PerturbSol=np. roll(PerturbSol, N//2+1-i)
        if 501-i>0:
            PerturbSol[0:N//2+1-i]=0
        if 501-i<0:
            PerturbSol[(N//2+1-i):]=1
        break
```