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Research Article

New Expansion Formulas for a Family of the λ-Generalized Hurwitz-Lerch Zeta Functions

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We derive several new expansion formulas for a new family of the λ-generalized Hurwitz-Lerch zeta functions which were introduced by Srivastava (2014). These expansion formulas are obtained by making use of some important fractional calculus theorems such as the generalized Leibniz rules, the Taylor-like expansions in terms of different functions, and the generalized chain rule. Several (known or new) special cases are also considered.

1. Introduction

The Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ which is one of the fundamentally important higher transcendental functions is defined by (see, e.g., [1, p. 121 *et seq.*]; see also [2] and [3, p. 194 *et seq.*])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1\right). \tag{1}$$

The Hurwitz-Lerch zeta function contains, as its special cases, the Riemann zeta function $\zeta(s)$, the Hurwitz zeta function $\zeta(s,a)$, and the Lerch zeta function $\ell_s(\xi)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1) = \zeta(s,1) \quad (\Re(s) > 1),$$

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \Phi(1,s,a) \quad \left(\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

$$\ell_s(\xi) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+1)^s} = \Phi\left(e^{2\pi i\xi}, s, 1\right) \quad (\Re(s) > 1;\ \xi \in \mathbb{R}), \tag{2}$$

respectively, but also such other important functions of *Analytic Number Theory* as the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_s(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1) \quad (s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1) \tag{3}$$

and the Lipschitz-Lerch zeta function $\phi(\xi,a,s)$ (see [1, p. 122, Equation 2.5 (11)]):

$$\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\left(e^{2\pi i\xi}, s, a\right) \quad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}\right). \tag{4}$$

Indeed, the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined in (5) can be continued *meromorphically* to the whole complex s-plane, except for a simple pole at $s=1$ with its residue 1. It is also well known that

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-at}}{1-ze^{-t}}dt$$
$$(\Re(a)>0;\Re(s)>0 \text{ when } |z|\leqq 1\ (z\neq 1);\Re(s)>1 \text{ when } z=1). \tag{5}$$

Motivated by the works of Goyal and Laddha [4], Lin and Srivastava [5], Garg et al. [6], and other authors, Srivastava et al. [7] (see also [8]) investigated various properties of a natural multiparameter extension and generalization of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (5) (see also [9]). In particular, they considered the following function:

$$\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a):=\sum_{n=0}^\infty \frac{\prod_{j=1}^p(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^q(\mu_j)_{n\sigma_j}}\frac{z^n}{(n+a)^s}$$
$$\Big(p,q\in\mathbb{N}_0;\ \lambda_j\in\mathbb{C}\ (j=1,\dots,p);\ a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\ (j=1,\dots,q);\ \rho_j,\sigma_k\in\mathbb{R}^+\ (j=1,\dots,p;\ k=1,\dots,q);\ \Delta>-1 \text{ when } s,z\in\mathbb{C};\ \Delta=-1,\ s\in\mathbb{C} \text{ when } |z|<\nabla^*;\ \Delta=-1,\ \Re(\Xi)>\frac{1}{2} \text{ when } |z|=\nabla^*\Big) \tag{6}$$

with

$$\nabla^*:=\left(\prod_{j=1}^p\rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^q\sigma_j^{\sigma_j}\right),$$
$$\Delta:=\sum_{j=1}^q\sigma_j-\sum_{j=1}^p\rho_j, \tag{7}$$
$$\Xi:=s+\sum_{j=1}^q\mu_j-\sum_{j=1}^p\lambda_j+\frac{p-q}{2}.$$

Here, and for the remainder of this paper, $(\lambda)_\kappa$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa=n\in\mathbb{N};\ \lambda\in\mathbb{C})\\ 1 & (\kappa=0;\ \lambda\in\mathbb{C}\setminus\{0\}).\end{cases} \tag{8}$$

It is being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [10, p. 21 *et seq.*]). In terms of the extended Hurwitz-Lerch zeta function defined by (6), the following generalization of several known integral representations arising from (5) was given by Srivastava et al. [7] as follows:

$$\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\cdot{}_p\Psi_q^*\left[\begin{matrix}(\lambda_1,\rho_1),\dots,(\lambda_p,\rho_p);\\(\mu_1,\sigma_1),\dots,(\mu_q,\sigma_q);\end{matrix}\ ze^{-t}\right]dt$$
$$(\min\{\Re(a),\Re(s)\}>0), \tag{9}$$

provided that the integral exists.

Definition 1. The function ${}_l\Psi_m^*$ or ${}_l\Psi_m$ $(l,m\in\mathbb{N}_0)$ involved in the right-hand side of (9) is the well-known Fox-Wright function, which is a generalization of the familiar generalized hypergeometric function ${}_lF_m$ $(l,m\in\mathbb{N}_0)$, with l numerator parameters $a_1,\dots,a_l$ and m denominator parameters $b_1,\dots,b_m$ such that

$$a_j\in\mathbb{C}\quad(j=1,\dots,l),\qquad b_j\in\mathbb{C}\setminus\mathbb{Z}_0^-\quad(j=1,\dots,m), \tag{10}$$

defined by (see, for details, [11, p. 21 *et seq.*] and [10, p. 50 *et seq.*])

$${}_l\Psi_m^*\left[\begin{matrix}(a_1,A_1),\dots,(a_l,A_l);\\(b_1,B_1),\dots,(b_m,B_m);\end{matrix}\ z\right]:=\sum_{n=0}^\infty\frac{(a_1)_{A_1n}\cdots(a_l)_{A_ln}}{(b_1)_{B_1n}\cdots(b_m)_{B_mn}}\frac{z_n}{n!}=\frac{\Gamma(b_1)\cdots\Gamma(b_m)}{\Gamma(a_1)\cdots\Gamma(a_l)}\,{}_l\Psi_m\left[\begin{matrix}(a_1,A_1),\dots,(a_l,A_l);\\(b_1,B_1),\dots,(b_m,B_m);\end{matrix}\ z\right] \tag{11}$$
$$\left(A_j>0\ (j=1,\dots,l);\ B_j>0\ (j=1,\dots,m);\ 1+\sum_{j=1}^mB_j-\sum_{j=1}^lA_j\geqq 0\right),$$

where the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z|<\nabla:=\left(\prod_{j=1}^lA_j^{-A_j}\right)\cdot\left(\prod_{j=1}^mB_j^{B_j}\right). \tag{12}$$

Recently, Srivastava [12] introduced and investigated a significantly more general class of Hurwitz-Lerch zeta type

functions by suitably modifying the integral representation formula (9). Srivastava considered the following function:

$$\begin{aligned}&\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\\&\quad=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp\left(-at-\frac{b}{t^\lambda}\right)\\&\qquad\cdot{}_p\Psi_q^*\left[\begin{matrix}(\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p);\\(\mu_1,\sigma_1),\ldots,(\mu_q,\sigma_q);\end{matrix}\,z\mathrm{e}^{-t}\right]\mathrm{d}t\\&\quad(\min\{\Re(a),\Re(s)\}>0;\ \Re(b)\geqq 0;\ \lambda\geqq 0),\end{aligned}\tag{13}$$

so that, clearly, we have the following relationship:

$$\begin{aligned}&\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;0,\lambda)\\&\quad=\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a)\\&\quad=\mathrm{e}^b\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,0).\end{aligned}\tag{14}$$

In its special case when

$$p-1=q=0,\quad(\lambda_1=\mu;\ \rho_1=1),\tag{15}$$

the definition (13) would reduce to the following form:

$$\begin{aligned}&\Theta_\mu^\lambda(z,s,a;b)\\&\quad:=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\exp\left(-at-\frac{b}{t^\lambda}\right)\left(1-z\mathrm{e}^{-t}\right)^{-\mu}\mathrm{d}t\\&(\min\{\Re(a),\Re(s)\}>0;\ \Re(b)\geqq 0;\ \lambda\geqq 0;\ \mu\in\mathbb{C}),\end{aligned}\tag{16}$$

where we have assumed further that

$$\Re(s)>0\quad\text{when } b=0,\ |z|\leqq 1\ (z\neq 1)\tag{17}$$

or

$$\Re(s-\mu)>0\quad\text{when } b=0,\ z=1,\tag{18}$$

provided that the integral (16) exists. The function $\Theta_\mu^\lambda(z,s,a;b)$ was studied by Raina and Chhajed [13, Eq. (1.6)] and, more recently, by Srivastava et al. [14].

As a particular interesting case of the function $\Theta_\mu^\lambda(z,s,a;b)$, we recall the following function:

$$\begin{aligned}&\Theta_\mu^\lambda(z,s,a;0)=\Phi_\mu^*(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}\mathrm{e}^{-at}}{(1-z\mathrm{e}^{-t})^\mu}\mathrm{d}t\\&(\Re(a)>0;\ \Re(s)>0\text{ when }|z|\leqq 1\ (z\neq 1);\\&\quad\Re(s-\mu)>0\text{ when }z=1).\end{aligned}\tag{19}$$

The function $\Phi_\mu^*(z,s,a)$ was introduced by Goyal and Laddha [4] as follows:

$$\Phi_\mu^*(z,s,a):=\sum_{n=0}^\infty\frac{(\mu)_n}{(a+n)^s}\frac{z^n}{n!}.\tag{20}$$

Another special case of the function $\Theta_\mu^\lambda(z,s,a;b)$ that is worthy to mention occurs when $\lambda=\mu=1$ and $z=1$. We have

$$\begin{aligned}&\Theta_1^1(1,s,a;b)\\&\quad=\zeta_b(s,a)\\&\quad:=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}}{1-\mathrm{e}^{-t}}\exp\left(-at-\frac{b}{t}\right)\mathrm{d}t,\end{aligned}\tag{21}$$

where the function $\zeta_b(s,a)$ is the extended Hurwitz zeta function introduced by Chaudhry and Zubair [15].

In his work, Srivastava [12, p. 1489, Eq. (2.1)] also derived the following series representation of the function $\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$:

$$\begin{aligned}&\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\\&\quad=\frac{1}{\lambda\Gamma(s)}\sum_{n=0}^\infty\frac{\prod_{j=1}^p(\lambda_j)_{\rho_j n}}{(a+n)^s\cdot\prod_{j=1}^q(\mu_j)_{\sigma_j n}}\\&\qquad\cdot H_{0,2}^{2,0}\left[(a+n)b^{1/\lambda}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\frac{z^n}{n!}\quad(\lambda>0),\end{aligned}\tag{22}$$

provided that both sides of (22) exist.

Definition 2. The H-function involved in the right-hand side of (22) is the well-known Fox's H-function defined by [16, Definition 1.1] (see also [10, 17])

$$\begin{aligned}&H_{\mathfrak{p},\mathfrak{q}}^{m,n}(z)\\&\quad=H_{\mathfrak{p},\mathfrak{q}}^{m,n}\left[z\mid\begin{matrix}(a_1,A_1),\ldots,(a_{\mathfrak{p}},A_{\mathfrak{p}})\\(b_1,B_1),\ldots,(b_{\mathfrak{q}},B_{\mathfrak{q}})\end{matrix}\right]\\&\quad=\frac{1}{2\pi i}\int_{\mathscr{L}}\Xi(s)z^{-s}\mathrm{d}s\quad(z\in\mathbb{C}\setminus\{0\};\ |\arg(z)|<\pi),\end{aligned}\tag{23}$$

where

$$\Xi(s)=\frac{\prod_{j=1}^m\Gamma(b_j+B_js)\cdot\prod_{j=1}^n\Gamma(1-a_j-A_js)}{\prod_{j=n+1}^{\mathfrak{p}}\Gamma(a_j+A_js)\cdot\prod_{j=m+1}^{\mathfrak{q}}\Gamma(1-b_j-B_js)}.\tag{24}$$

An empty product is interpreted as 1, $m,n,\mathfrak{p}$, and $\mathfrak{q}$ are integers such that $1\leqq m\leqq\mathfrak{q}$, $0\leqq n\leqq\mathfrak{p}$, $A_j>0$ $(j=1,\ldots,\mathfrak{p})$, $B_j>0$ $(j=1,\ldots,\mathfrak{q})$, $a_j\in\mathbb{C}$ $(j=1,\ldots,\mathfrak{p})$, $b_j\in\mathbb{C}$ $(j=1,\ldots,\mathfrak{q})$, and $\mathscr{L}$ is a suitable Mellin-Barnes type contour separating the poles of the gamma functions

$$\left\{\Gamma(b_j+B_js)\right\}_{j=1}^m\tag{25}$$

from the poles of the gamma functions

$$\left\{\Gamma(1-a_j+A_js)\right\}_{j=1}^n.\tag{26}$$

It is important to recall that Srivastava [12, p. 1490, Eq. (2.10)] presented another series representation for the function $\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$ involving the Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α and degree n in x generated by (see, for details, [10])

$$(1-t)^{-\alpha-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,t^n \quad (|t|<1;\ \alpha\in\mathbb{C}). \tag{27}$$

Explicitly, it was proven by Srivastava [12, p. 1490, Eq. (2.10)] that

$$\begin{aligned}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda) &= \frac{e^{-b}}{\Gamma(s)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot\Gamma(s+\lambda(\alpha+k+1)) \\ &\quad\cdot L_n^{(\alpha)}(b)\,\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s+\lambda(\alpha+j+1),a) \\ &\quad(\Re(a)>0;\ \Re(s+\lambda\alpha)>-\lambda),\end{aligned} \tag{28}$$

provided that each member of (28) exists and

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a) \tag{29}$$

being given by (9).

Motivated by a number of recent works by the present authors [18–20] and also of those of several other authors [4–9, 21, 22], this paper aims to provide many new relationships involving the new family of the λ-generalized Hurwitz-Lerch zeta function $\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$.

2. Pochhammer Contour Integral Representation for Fractional Derivative

The most familiar representation for the fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [23] (see also [24–26]); that is,

$$\mathcal{D}_z^{\alpha}\{z^p f(z)\} = \frac{1}{\Gamma(-\alpha)}\int_0^z f(\xi)\,\xi^p(\xi-z)^{-\alpha-1}d\xi \quad (\Re(\alpha)<0;\ \Re(p)>1), \tag{30}$$

where the integration is carried out along a straight line from 0 to z in the complex ξ-plane. By integrating by part m times, we obtain

$$\mathcal{D}_z^{\alpha}\{z^p f(z)\} = \frac{d^m}{dz^m}\{\mathcal{D}_z^{\alpha-m}\{z^p f(z)\}\}. \tag{31}$$

This allows us to modify the restriction $\Re(\alpha)<0$ to $\Re(\alpha)<m$ (see [26]).

Another representation for the fractional derivative is based on the Cauchy integral formula. This representation, too, has been widely used in many interesting papers (see, e.g., the works of Osler [27–30]).

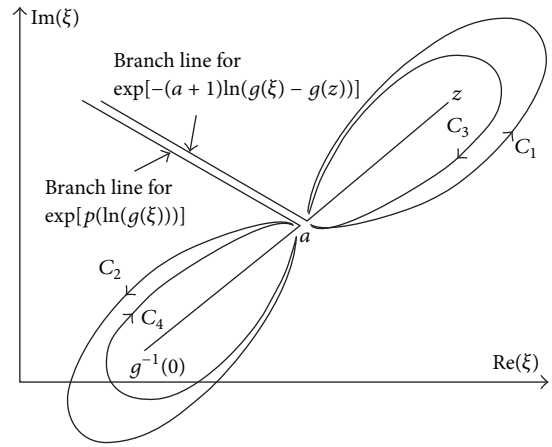

Figure 1: Pochhammer's contour.

The relatively less restrictive representation of the fractional derivative according to parameters appears to be the one based on the Pochhammer's contour integral introduced by Tremblay [31, 32].

Definition 3. Let $f(z)$ be analytic in a simply-connected region $\mathcal{R}$ of the complex z-plane. Let $g(z)$ be regular and univalent on $\mathcal{R}$ and let $g^{-1}(0)$ be an interior point of $\mathcal{R}$. Then, if α is not a negative integer, p is not an integer, and z is in $\mathcal{R}\setminus\{g^{-1}(0)\}$, we define the fractional derivative of order α of $g(z)^p f(z)$ with respect to $g(z)$ by

$$\begin{aligned}D_{g(z)}^{\alpha}\{[g(z)]^p f(z)\} &= \frac{e^{-i\pi p}\Gamma(1+\alpha)}{4\pi\sin(\pi p)} \\ &\quad\times\int_{C(z+,g^{-1}(0)+,z-,g^{-1}(0)-;F(a),F(a))}\cdot\frac{f(\xi)[g(\xi)]^p g'(\xi)}{[g(\xi)-g(z)]^{\alpha+1}}d\xi.\end{aligned} \tag{32}$$

For nonintegers α and p, the functions $g(\xi)^p$ and $[g(\xi)-g(z)]^{-\alpha-1}$ in the integrand have two branch lines which begin, respectively, at $\xi=z$ and $\xi=g^{-1}(0)$, and both branches pass through the point $\xi=a$ without crossing the Pochhammer contour $P(a)=\{C_1\cup C_2\cup C_3\cup C_4\}$ at any other point as shown in Figure 1. Here, $F(a)$ denotes the principal value of the integrand in (32) at the beginning and the ending point of the Pochhammer contour $P(a)$ which is closed on the Riemann surface of the multiple-valued function $F(\xi)$.

Remark 4. In Definition 3, the function $f(z)$ must be analytic at $\xi=g^{-1}(0)$. However, it is interesting to note here that if we could also allow $f(z)$ to have an essential singularity at $\xi=g^{-1}(0)$, then (32) would still be valid.

Remark 5. In case the Pochhammer contour never crosses the singularities at $\xi=g^{-1}(0)$ and $\xi=z$ in (32), then we

know that the integral is analytic for all p and for all α and for z in $\mathscr{R} \setminus \{g^{-1}(0)\}$. Indeed, in this case, the only possible singularities of $D^{\alpha}_{g(z)}\{[g(z)]^p f(z)\}$ are $\alpha = -1, -2, -3, \ldots$ and $p = 0, \pm 1, \pm 2, \ldots$, which can directly be identified from the coefficient of the integral (32). However, by integrating by parts N times the integral in (32) by two different ways, we can show that $\alpha = -1, -2, \ldots$ and $p = 0, 1, 2, \ldots$ are removable singularities (see, for details, [31]).

It is well known that [33, p. 83, Equation (2.4)]

$$D^{\alpha}_{z}\{z^p\} = \frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)} z^{p-\alpha} \quad (\Re(p) > -1). \tag{33}$$

Adopting the Pochhammer based representation for the fractional derivative modifies the restriction to the case when p is not a negative integer.

Now, by using (33) in conjunction with the series representation (22) for $\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)$, we obtain the following important fractional derivative formula that will play an important role in our present investigation:

$$\begin{aligned}
&D^{\alpha}_{z}\left\{z^{\beta-1}\Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a;b,\lambda)\right\} \\
&\quad = \frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}}{(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}} \\
&\qquad \cdot H^{2,0}_{0,2}\left[(a+n)^{b^{1/\lambda}} \mid \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right] D^{\alpha}_{z}\left\{z^{\beta-1}\frac{z^n}{n!}\right\} \\
&\quad = \frac{\Gamma(\beta)}{\lambda\Gamma(s)\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\frac{(\beta)_n}{(\beta-\alpha)_n}\cdot\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}}{(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}} \\
&\qquad \cdot H^{2,0}_{0,2}\left[(a+n)\,b^{1/\lambda} \mid \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\frac{z^{n+\beta-\alpha-1}}{n!} \\
&\quad = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} z^{\beta-\alpha-1}\Phi^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}_{\lambda_1,\ldots,\lambda_p,\beta;\mu_1,\ldots,\mu_q,\beta-\alpha}(z,s,a;b,\lambda) \\
&\qquad (\lambda > 0;\ \beta-1 \text{ not a negative integer}).
\end{aligned} \tag{34}$$

3. Important Results Involving Fractional Calculus

In this section, we recall six fundamental theorems related to fractional calculus that will play central roles in our work. Each of these theorems is a fundamental formula related to the generalized chain rule for fractional derivatives, the Taylor-like expansions in terms of different types of functions, and the generalized Leibniz rules for fractional derivatives.

First of all, Osler [27, p. 290, Theorem 2] discovered a fundamental relation from which he deduced the generalized chain rule for the fractional derivatives. This result is recalled here as Theorem 6 below.

Theorem 6. *Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic in the simply-connected region $\mathscr{R}$ of the complex z-plane and let the origin be an interior or boundary point of $\mathscr{R}$. Suppose also that $g^{-1}(z)$ and $h^{-1}(z)$ are regular univalent functions on $\mathscr{R}$ and that $h^{-1}(0) = g^{-1}(0)$. Let $\oint f(g^{-1}(z))\mathrm{d}z$ vanish over simple closed contour in $\mathscr{R}\cup\{0\}$ through the origin. Then the following relation holds true:*

$$D^{\alpha}_{g(z)}\{f(z)\} = D^{\alpha}_{h(z)}\left\{\frac{f(z)\,g'(z)}{h'(z)}\left(\frac{h(w)-h(z)}{g(w)-g(z)}\right)^{\alpha+1}\right\}\Bigg|_{w=z}. \tag{35}$$

Relation (35) allows us to obtain very easily known and new summation formulas involving special functions of mathematical physics.

By applying the relation (35), Gaboury and Tremblay [34] proved the following corollary which will be useful in the next section.

Corollary 7. *Under the hypotheses of Theorem 6, let p be a positive integer. Then the following relation holds true:*

$$\begin{aligned}
&{}_{z^p}O^{\alpha}_{\beta}\{f(z)\} \\
&= p\left(z^{p-1}\right)^{-\alpha} \\
&\quad \cdot {}_{z}O^{\alpha}_{\beta}\left\{f(z)\left(z^{p-1}\right)^{\alpha}\prod_{s=1}^{p-1}\left(1-\frac{z}{w}\mathrm{e}^{-(2\pi i s)/p}\right)^{\beta-\alpha-1}\right\}\Bigg|_{w=z},
\end{aligned} \tag{36}$$

where

$${}_{g(z)}O^{\alpha}_{\beta}\{\cdots\} := \frac{\Gamma(\beta)}{\Gamma(\alpha)}[g(z)]^{1-\beta}D^{\alpha-\beta}_{g(z)}\left\{[g(z)]^{\alpha-1}\cdots\right\}. \tag{37}$$

Next, in the year 1971, Osler [35] obtained the following generalized Taylor-like series expansion involving fractional derivatives.

Theorem 8. *Let $f(z)$ be an analytic function in a simply-connected region $\mathscr{R}$. Let α and γ be arbitrary complex numbers and let*

$$\theta(z) = (z - z_0)\,q(z) \tag{38}$$

with $q(z)$ a regular and univalent function without any zero in $\mathscr{R}$. Let a be a positive real number and let

$$K = \{0, 1, \ldots, [c]\ \ ([c] \text{ the largest integer not greater than } c)\}. \tag{39}$$

Let b and z_0 be two points in $\mathscr{R}$ such that $b \neq z_0$ and let

$$\omega = \exp\left(\frac{2\pi i}{a}\right). \tag{40}$$

Then the following relationship holds true:

$$\sum_{k\in K} c^{-1}\omega^{-\gamma k} f\left(\theta^{-1}\left(\theta(z)\,\omega^{k}\right)\right)$$
$$= \sum_{n=-\infty}^{\infty} \frac{[\theta(z)]^{cn+\gamma}}{\Gamma(cn+\gamma+1)}$$
$$\cdot D_{z-b}^{cn+\gamma}\left\{f(z)\,\theta'(z)\left(\frac{z-z_0}{\theta(z)}\right)^{cn+\gamma+1}\right\}\Bigg|_{z=z_0} \quad (|z-z_0|=|z_0|). \tag{41}$$

In particular, if $0 < c \leqq 1$ and $\theta(z) = (z - z_0)$, then $k = 0$ and the formula (41) reduces to the following form:

$$f(z) = c\sum_{n=-\infty}^{\infty} \frac{(z-z_0)^{cn+\gamma}}{\Gamma(cn+\gamma+1)} D_{z-b}^{cn+\gamma}\{f(z)\}\Bigg|_{z=z_0}. \tag{42}$$

This last formula (42) is usually referred to as the Taylor-Riemann formula and has been studied in several papers [29, 36–39].

We next recall that Tremblay et al. [40] discovered the power series of an analytic function $f(z)$ in terms of the rational expression $((z-z_1)/(z-z_2))$, where z_1 and z_2 are two arbitrary points inside the region $\mathscr{R}$ of analyticity of $f(z)$. In particular, they obtained the following result.

Theorem 9. *(i) Let c be real and positive and let*

$$\omega = \exp\left(\frac{2\pi i}{a}\right). \tag{43}$$

(ii) Let $f(z)$ be analytic in the simply-connected region $\mathscr{R}$ with z_1 and z_2 being interior points of $\mathscr{R}$. (iii) Let the set of curves

$$\{C(t) : C(t) \subset \mathscr{R},\ 0 < t \leqq r\} \tag{44}$$

be defined by

$$C(t) = C_1(t) \cup C_2(t)$$
$$= \left\{z : |\lambda_t(z_1, z_2; z)| = \left|\lambda_t\left(z_1, z_2; \frac{z_1+z_2}{2}\right)\right|\right\}, \tag{45}$$

where

$$\lambda_t(z_1, z_2; z) = \left[z - \frac{z_1+z_2}{2} + t\left(\frac{z_1-z_2}{2}\right)\right]$$
$$\cdot\left[z - \left(\frac{z_1+z_2}{2}\right) - t\left(\frac{z_1-z_2}{2}\right)\right], \tag{46}$$

which are the Bernoulli type lemniscates (see Figure 2) with center located at $(z_1 + z_2)/2$ and with double-loops in which one loop $C_1(t)$ leads around the focus point

$$\frac{z_1+z_2}{2} + \left(\frac{z_1-z_2}{2}\right)t \tag{47}$$

and the other loop $C_2(t)$ encircles the focus point

$$\frac{z_1+z_2}{2} - \left(\frac{z_1-z_2}{2}\right)t \tag{48}$$

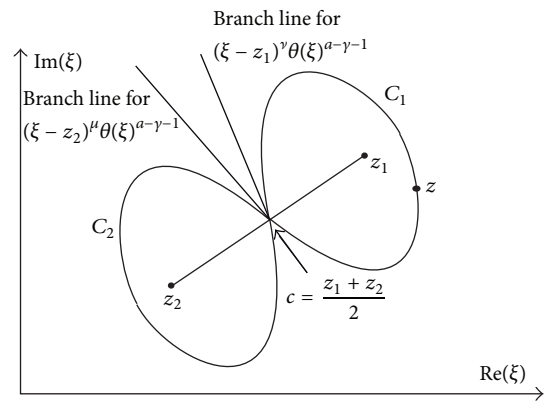

Figure 2: Multiloops contour.

for each t such that $0 < t \leqq r$. (iv) Let

$$[(z-z_1)(z-z_2)]^{\lambda} = \exp(\lambda \ln(\theta((z-z_1)(z-z_2)))) \tag{49}$$

denote the principal branch of that function which is continuous and inside $C(r)$, cut by the respective two branch lines $L_{\pm}$ defined by

$$L_{\pm} = \begin{cases} \left\{z : z = \dfrac{z_1+z_2}{2} \pm t\left(\dfrac{z_1-z_2}{2}\right)\right\} & (0 \leqq t \leqq 1) \\ \left\{z : z = \dfrac{z_1+z_2}{2} \pm it\left(\dfrac{z_1-z_2}{2}\right)\right\} & (t < 0) \end{cases} \tag{50}$$

such that $\ln((z-z_1)(z-z_2))$ is real when $(z-z_1)(z-z_2) > 0$. (v) Let $f(z)$ satisfy the conditions of Definition 3 for the existence of the fractional derivative of $(z - z_2)^p f(z)$ of order α for $z \in \mathscr{R} \setminus \{L_+ \cup L_-\}$, denoted by $D_{z-z_2}^{\alpha}\{(z - z_2)^p f(z)\}$, where α and p are real or complex numbers. (vi) Let

$$K = \left\{k : k \in \mathbb{N},\ \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right)\right.$$
$$< \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right) + \frac{2\pi k}{a} \tag{51}$$
$$\left. < \arg\left(\lambda_t\left(z_1, z_2, \frac{z_1+z_2}{2}\right)\right) + 2\pi\right\}.$$

Then, for arbitrary complex numbers μ,ν,γ and for z on $C_1(1)$ defined by

$$\xi = \frac{z_1+z_2}{2} + \frac{z_1-z_2}{2}\sqrt{1+e^{i\theta}} \quad (-\pi < \theta < \pi),$$

$$\sum_{k\in K} \frac{c^{-1}\omega^{-\gamma k}}{z_1-z_2} f\left(\phi^{-1}\left(\omega^{k}\phi(z)\right)\right)$$
$$\times\left[\phi^{-1}(\omega^{k}\phi(z)) - z_1\right]^{\nu}\left[\phi^{-1}(\omega^{k}\phi(z)) - z_2\right]^{\mu} \tag{52}$$
$$= \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)}\sin[(\mu+cn+\gamma)\pi]}{\sin[(\mu-c+\gamma)\pi]\,\Gamma(1-\nu+cn+\gamma)}$$
$$\cdot D_{z-z_2}^{-\nu+cn+\gamma}\left\{(z-z_2)^{\mu+cn+\gamma-1} f(z)\right\}\Big|_{z=z_1}[\phi(z)]^{cn+\gamma},$$

where

$$\phi(z) = \frac{z - z_1}{z - z_2}. \tag{53}$$

The case $0 < c \leqq 1$ of Theorem 9 reduces to the following form:

$$\begin{aligned}
&\frac{c^{-1} f(z)(z - z_1)^{\nu}(z - z_2)^{\mu}}{z_1 - z_2} \\
&= \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)} \sin[(\mu + cn + \gamma)\pi]}{\sin[(\mu - c + \gamma)\pi]\,\Gamma(1 - \nu + cn + \gamma)} \\
&\quad \cdot D_{z-z_2}^{-\nu+cn+\gamma} \left\{(z - z_2)^{\mu+cn+\gamma-1} f(z)\right\}\Big|_{z=z_1} \left(\frac{z - z_1}{z - z_2}\right)^{cn+\gamma}.
\end{aligned} \tag{54}$$

Tremblay and Fugère [41] developed the power series of an analytic function $f(z)$ in terms of the function $(z - z_1)(z - z_2)$, where z_1 and z_2 are two arbitrary points inside the analyticity region $\mathscr{R}$ of $f(z)$. Explicitly, they gave the following theorem.

Theorem 10. *Under the assumptions of Theorem 9, the following expansion formula holds true*:

$$\begin{aligned}
&\sum_{k\in K} c^{-1}\omega^{-\gamma k}\left[\left(\frac{z_2 - z_1 + \sqrt{\Delta_k}}{2}\right)^{\alpha}\left(\frac{z_1 - z_2 + \sqrt{\Delta_k}}{2}\right)^{\beta}\right. \\
&\qquad \cdot f\left(\frac{z_1 + z_2 + \sqrt{\Delta_k}}{2}\right) \\
&\qquad - e^{i\pi(\alpha-\beta)} \frac{\sin[(\alpha + c - \gamma)\pi]}{\sin[(\beta + c - \gamma)\pi]} \\
&\qquad \cdot \left(\frac{z_2 - z_1 - \sqrt{\Delta_k}}{2}\right)^{\alpha} \\
&\qquad \left.\times\left(\frac{z_1 - z_2 - \sqrt{\Delta_k}}{2}\right)^{\beta} f\left(\frac{z_1 + z_2 - \sqrt{\Delta_k}}{2}\right)\right] \\
&= \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi]}{\sin[(\beta - c - \gamma)\pi]} \frac{e^{-i\pi c(n+1)}[\theta(z)]^{cn+\gamma}}{\Gamma(1 - \alpha + cn + \gamma)} \\
&\quad \cdot D_{z-z_2}^{-\alpha+cn+\gamma} \\
&\quad \times \left\{(z - z_2)^{\beta-cn-\gamma-1}\left(\frac{\theta(z)}{(z - z_2)(z - z_1)}\right)^{-cn-\gamma-1} \theta'(z) f(z)\right\}\Bigg|_{z=z_1},
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
\Delta_k &= (z_1 - z_2)^2 + 4V(\omega^k \theta(z)), \\
V(z) &= \sum_{r=1}^{\infty} D_z^{r-1}\left\{[q(z)]^{-r}\right\}\Big|_{z=0} \frac{z^r}{r!}, \\
\theta(z) &= (z - z_1)(z - z_2)\, q((z - z_1)(z - z_2)).
\end{aligned} \tag{56}$$

As a special case, if we set $0 < c \leqq 1$, $q(z) = 1$ ($\theta(z) = (z - z_1)(z - z_2)$), and $z_2 = 0$ in (55), we obtain

$$\begin{aligned}
f(z) &= c z^{-\beta}(z - z_1)^{-\alpha} \\
&\quad \times \sum_{n=-\infty}^{\infty} \frac{\sin[(\beta - cn - \gamma)\pi]}{\sin[(\beta + c - \gamma)\pi]} \frac{e^{i\pi c(n+1)}[z(z - z_1)]^{cn+\gamma}}{\Gamma(1 - \alpha + cn + \gamma)} \\
&\quad \cdot D_z^{-\alpha+cn+\gamma}\left\{z^{\beta-cn-\gamma-1}(z + w - z_1) f(z)\right\}\Big|_{\substack{z=z_1 \\ (w=z)}}.
\end{aligned} \tag{57}$$

Finally, we give two generalized Leibniz rules for fractional derivatives. Theorem 11 is a slightly modified theorem obtained in 1970 by Osler [28]. Theorem 12 was given, some years ago, by Tremblay et al. [42] with the help of the properties of Pochhammer's contour representation for fractional derivatives.

Theorem 11. (i) *Let $\mathscr{R}$ be a simply-connected region containing the origin.* (ii) *Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 3 for the existence of the fractional derivative. Then, for $\Re(p + q) > -1$ and $\gamma \in \mathbb{C}$, the following Leibniz rule holds true*:

$$\begin{aligned}
&D_z^{\alpha}\left\{z^{p+q} u(z) v(z)\right\} \\
&\quad = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D_z^{\alpha-\gamma-n}\left\{z^p u(z)\right\} D_z^{\gamma+n}\left\{z^q v(z)\right\}.
\end{aligned} \tag{58}$$

Theorem 12. *(i) Let $\mathscr{R}$ be a simply-connected region containing the origin. (ii) Let $u(z)$ and $v(z)$ satisfy the conditions of Definition 3 for the existence of the fractional derivative. (iii) Let $\mathscr{U} \subset \mathscr{R}$ be the region of analyticity of the function $u(z)$ and let $\mathscr{V} \subset \mathscr{R}$ be the region of analyticity of the function $v(z)$. Then, for*

$$z \neq 0, \quad z \in \mathscr{U} \cap \mathscr{V}, \qquad \Re(1 - \beta) > 0, \tag{59}$$

the following product rule holds true:

$$\begin{aligned}
&D_z^{\alpha}\left\{z^{\alpha+\beta-1} u(z) v(z)\right\} \\
&= \frac{z\Gamma(1 + \alpha)\sin(\beta\pi)\sin(\mu\pi)\sin[(\alpha + \beta - \mu)\pi]}{\sin[(\alpha + \beta)\pi]\sin[(\beta - \mu - \nu)\pi]\sin[(\mu + \nu)\pi]} \\
&\quad \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\alpha+\nu+1-n}\left\{z^{\alpha+\beta-\mu-1-n} u(z)\right\} D_z^{-1-\nu+n}\left\{z^{\mu-1+n} v(z)\right\}}{\Gamma(2 + \alpha + \nu - n)\,\Gamma(-\nu + n)}.
\end{aligned} \tag{60}$$

4. Main Expansion Formulas

This section is devoted to the presentation of the new relations involving the new family of the λ-generalized Hurwitz-Lerch zeta function $\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z, s, a; b, \lambda)$.

Theorem 13. *Under the hypotheses of Corollary 7, let k be a positive integer. Then the following relation holds true:*

$$\begin{aligned}
&\Phi_{\lambda_1,\ldots,\lambda_p,\alpha;\mu_1,\ldots,\mu_q,\beta}^{(\rho_1,\ldots,\rho_p,1/k,\sigma_1,\ldots,\sigma_q,1/k)}(z,s,a;b,\lambda)\\
&= \frac{k\Gamma(\beta)\Gamma(k\alpha)}{\lambda\Gamma(s)\Gamma(\alpha)\Gamma(\beta+(k-1)\alpha)}\\
&\quad\cdot\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}(k\alpha)_n H_{0,2}^{2,0}\left[(a+n)b^{1/\lambda}\mid\overline{(s,1),(0,(1/\lambda))}\right]}{(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}(\beta+(k-1)\alpha)_n}\\
&\quad\times\frac{z^n}{n!}\\
&\quad\cdot F_D^{(k-1)}\left[k\alpha+n,1+\alpha-\beta,\ldots,1+\alpha-\beta;\beta+(k-1)\alpha+n;\right.\\
&\qquad\left.\mathrm{e}^{-2\pi i/k},\ldots,\mathrm{e}^{-2(k-1)\pi i/k}\right],
\end{aligned}\tag{61}$$

where $\lambda>0$ and $F_D^{(n)}$ denotes the Lauricella function of n variables defined by [11, p. 60]

$$\begin{aligned}
&F_D^{(n)}[a,b_1,\ldots,b_n;c;x_1,\ldots,x_n]\\
&\quad=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}\\
&\qquad\left(\max\{|x_1|,\ldots,|x_n|\}<1\right),
\end{aligned}\tag{62}$$

provided that both sides of (61) *exist.*

Proof. Putting $p=k$ and letting $f(z)=\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$ in Corollary 7, we get

$$\begin{aligned}
&{}_{z^k}O_\beta^\alpha\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\\
&\quad=\frac{k}{(z^{k-1})^\alpha}\\
&\quad\cdot{}_zO_\beta^\alpha\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\left(z^{k-1}\right)^\alpha\right.\\
&\qquad\left.\cdot\prod_{s=1}^{k-1}\left(1-\frac{z}{w}\mathrm{e}^{-2\pi is/k}\right)^{\beta-\alpha-1}\right\}\Bigg|_{w=z}.
\end{aligned}\tag{63}$$

With the help of the definition of ${}_zO_\beta^\alpha$ given by (37), we find for the left-hand side of (63) that

$$\begin{aligned}
&{}_{z^k}O_\beta^\alpha\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\\
&\quad=\Phi_{\lambda_1,\ldots,\lambda_p,\alpha;\mu_1,\ldots,\mu_q,\beta}^{(\rho_1,\ldots,\rho_p,1/k,\sigma_1,\ldots,\sigma_q,1/k)}(z,s,a;b,\lambda).
\end{aligned}\tag{64}$$

We now expand each factor in the product in (63) in power series and replace the generalized Hurwitz-Lerch zeta function by its H-function series representation. We, thus, find for the right-hand side of (63) that

$$\begin{aligned}
&\frac{k}{(z^{k-1})^\alpha}\,{}_zO_\beta^\alpha\\
&\quad\times\left\{\frac{1}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}z^{n+(k-1)\alpha}}{n!(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}}H_{0,2}^{2,0}\right.\\
&\qquad\times\left[(a+n)b^{1/\lambda}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\\
&\qquad\cdot\sum_{m_1,\ldots,m_{k-1}=0}^{\infty}(1+\alpha-\beta)_{m_1}\cdots(1+\alpha-\beta)_{m_{k-1}}\\
&\qquad\cdot\frac{\left((z/w)\,\mathrm{e}^{-2\pi i/k}\right)^{m_1}}{m_1!}\\
&\qquad\left.\cdots\frac{\left((z/w)\,\mathrm{e}^{-2(k-1)\pi i/k}\right)^{m_{k-1}}}{m_{k-1}!}\right\}\Bigg|_{w=z}\\
&=\frac{k}{\lambda\Gamma(s)(z^{k-1})^\alpha}\\
&\quad\times\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}}{n!(a+n)^s\cdot\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}}H_{0,2}^{2,0}\\
&\quad\times\left[(a+n)b^{1/\lambda}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\\
&\quad\cdot\sum_{m_1,\ldots,m_{k-1}=0}^{\infty}\frac{(1+\alpha-\beta)_{m_1}\cdots(1+\alpha-\beta)_{m_{k-1}}}{z^{m_1+\cdots+m_{k-1}}}\\
&\quad\cdot\frac{\left(\mathrm{e}^{-2\pi i/k}\right)^{m_1}}{m_1!}\cdots\frac{\left(\mathrm{e}^{-2(k-1)\pi i/k}\right)^{m_{k-1}}}{m_{k-1}!}\\
&\quad\times{}_zO_\beta^\alpha\left\{z^{m_1+\cdots+m_{k-1}+n+(k-1)\alpha}\right\}\\
&=\frac{k}{\lambda\Gamma(s)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}}{(a+n)^s\prod_{j=1}^{q}(\mu_j)_{\sigma_j n}}H_{0,2}^{2,0}\\
&\quad\times\left[(a+n)b^{1/\lambda}\mid\overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right]\frac{z^n}{n!}\\
&\quad\cdot\sum_{m_1,\ldots,m_{k-1}=0}^{\infty}(1+\alpha-\beta)_{m_1}\cdots(1+\alpha-\beta)_{m_{k-1}}\\
&\quad\times\frac{\left(\mathrm{e}^{-2\pi i/k}\right)^{m_1}}{m_1!}\cdots\frac{\left(\mathrm{e}^{-2(k-1)\pi i/k}\right)^{m_{k-1}}}{m_{k-1}!}\\
&\quad\cdot\frac{\Gamma(\beta)\Gamma(m_1+\cdots+m_{k-1}+n+k\alpha)}{\Gamma(\alpha)\Gamma(\beta+m_1+\cdots+m_{k-1}+n+k\alpha)}
\end{aligned}$$

$$= \frac{k\Gamma(\beta)\Gamma(k\alpha)}{\lambda\Gamma(s)\Gamma(\alpha)\Gamma(\beta+(k-1)\alpha)}$$
$$\cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{\rho_j n}(k\alpha)_n H_{0,2}^{2,0}\left[(a+n)b^{1/\lambda} \mid \overline{(s,1),(0,1/\lambda)}\right]}{(a+n)^s \cdot \prod_{j=1}^{q}(\mu_j)_{\sigma_j n}(\beta+(k-1)\alpha)_n}$$
$$\times \frac{z^n}{n!}$$
$$\cdot \sum_{m_1,\ldots,m_{k-1}=0}^{\infty} \frac{(n+k\alpha)_{m_1+\cdots+m_{k-1}}(1+\alpha-\beta)_{m_1}\cdots(1+\alpha-\beta)_{m_{k-1}}}{(\beta+(k-1)\alpha)_{m_1+\cdots+m_{k-1}}}$$
$$\cdot \frac{\left(e^{-2\pi i/k}\right)^{m_1}}{m_1!}\cdots\frac{\left(e^{-2(k-1)\pi i/k}\right)^{m_{k-1}}}{m_{k-1}!}. \tag{65}$$

Finally, by combining (64) and (65), we obtain the result (61) asserted by Theorem 13. □

We now shift our focus on the different Taylor-like expansions in terms of different types of functions involving the new family of the λ-generalized Hurwitz-Lerch zeta function $\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$.

Theorem 14. *Under the assumptions of Theorem 8, the following expansion formula holds true*:

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$$
$$= c\sum_{n=-\infty}^{\infty} \frac{z_0^{-cn}(z-z_0)^{cn}}{\Gamma(cn+1)\Gamma(1-cn)} \cdot \Phi_{\lambda_1,\ldots,\lambda_p,1;\mu_1,\ldots,\mu_q,1-cn}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z_0,s,a;b,\lambda) \tag{66}$$
$$\left(|z-z_0| = |z_0|;\ \lambda > 0\right),$$

provided that both members of (66) *exist.*

Proof. Setting $f(z) = \Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$ in Theorem 8 with $b = \gamma = 0$, $0 < c \leqq 1$, and $\theta(z) = z - z_0$, we have

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$$
$$= c\sum_{n=-\infty}^{\infty} \frac{(z-z_0)^{cn}}{\Gamma(1+cn)} \cdot D_z^{cn}\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\Big|_{z=z_0} \tag{67}$$

for $z_0 \neq 0$ and for z such that $|z-z_0| = |z_0|$.

Now, by making use of (34) with $\beta = 1$ and $\alpha = cn$, we find that

$$D_z^{cn}\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\Big|_{z=z_0}$$
$$= \frac{z_0^{-cn}}{\Gamma(1-cn)}\Phi_{\lambda_1,\ldots,\lambda_p,1;\mu_1,\ldots,\mu_q,1-cn}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z_0,s,a;b,\lambda). \tag{68}$$

By combining (67) and (68), we get the result (66) asserted by Theorem 14. □

Theorem 15. *Under the hypotheses of Theorem 9, the following expansion formula holds true:*

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$$
$$= cz^{-\alpha}(z-z_1)^{-\beta}z_1^{\alpha+\beta}$$
$$\cdot \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)}\sin[(\alpha+cn+\gamma)\pi]\Gamma(\alpha+cn+\gamma)}{\sin[(\alpha-c+\gamma)\pi]\Gamma(1-\beta+cn+\gamma)\Gamma(\alpha+\beta)}$$
$$\cdot \Phi_{\lambda_1,\ldots,\lambda_p,\alpha+cn+\gamma;\mu_1,\ldots,\mu_q,\alpha+\beta}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z_1,s,a;b,\lambda)\left(\frac{z-z_1}{z}\right)^{cn+\gamma} \tag{69}$$

for $\lambda > 0$ *and for* z *on* $C_1(1)$ *defined by*

$$z = \frac{z_1}{2} + \frac{z_1}{2}\sqrt{1+e^{i\theta}} \quad (-\pi < \theta < \pi), \tag{70}$$

provided that both sides of (69) *exist.*

Proof. By taking $f(z) = \Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$ in Theorem 9 with $z_2 = 0$, $\mu = \alpha$, $\nu = \beta$, and $0 < c \leqq 1$, we find that

$$\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)$$
$$= c(z-z_1)^{-\beta}z^{-\alpha}z_1$$
$$\cdot \sum_{n=-\infty}^{\infty} \frac{e^{i\pi c(n+1)}\sin[(\alpha+cn+\gamma)\pi]}{\sin[(\alpha-c+\gamma)\pi]\Gamma(1-\beta+cn+\gamma)} \cdot \left(\frac{z-z_1}{z}\right)^{cn+\gamma}$$
$$\cdot D_z^{-\beta+cn+\gamma}\left\{z^{\alpha+cn+\gamma-1}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\Big|_{z=z_1}. \tag{71}$$

Now, with the help of the relation (34) with $\alpha \mapsto -\beta+cn+\gamma$ and $\beta \mapsto \alpha+cn+\gamma-1$, we have

$$D_z^{-\beta+cn+\gamma}\left\{z^{\alpha+cn+\gamma-1}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}\Big|_{z=z_1}$$
$$= z_1^{\alpha+\beta-1}\frac{\Gamma(\alpha+cn+\gamma)}{\Gamma(\alpha+\beta)} \times \Phi_{\lambda_1,\ldots,\lambda_p,\alpha+cn+\gamma;\mu_1,\ldots,\mu_q,\alpha+\beta}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z_1,s,a;b,\lambda). \tag{72}$$

Thus, by combining (71) and (72), we are led to the assertion (69) of Theorem 15. □

Theorem 16. *Under the hypotheses of Theorem 10, the following expansion formula holds true*:

$$\begin{aligned}
&\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\\
&= cz^{-\beta+\gamma}(z-z_1)^{-\alpha+\gamma}z_1^{\beta+\alpha-2\gamma-1}\\
&\quad\cdot\sum_{n=-\infty}^{\infty}\frac{\sin[(\beta-cn-\gamma)\pi]\,e^{i\pi c(n+1)}}{\sin[(\beta+c-\gamma)\pi]\,\Gamma(1-\alpha+cn+\gamma)}\\
&\quad\cdot\left(\frac{z(z-z_1)}{z_1^2}\right)^{cn}\frac{\Gamma(\beta-cn-\gamma)}{\Gamma(\beta+\alpha-2cn-2\gamma)}\\
&\quad\cdot\Big[(z-z_1)\,\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,\beta-cn-\gamma;\mu_1,\dots,\mu_q,\beta+\alpha-2cn-2\gamma}(z_1,s,a;b,\lambda)\\
&\qquad+\left(\frac{\beta-cn-\gamma}{\alpha+\beta-2cn-2\gamma}\right)z_1\\
&\qquad\cdot\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,1+\beta-cn-\gamma;\mu_1,\dots,\mu_q,1+\beta+\alpha-2cn-2\gamma}(z_1,s,a;b,\lambda)\Big]
\end{aligned}\tag{73}$$

for $\lambda>0$ *and for* z *on* $C_1(1)$ *defined by*

$$z=\frac{z_1}{2}+\frac{z_1}{2}\sqrt{1+e^{i\theta}}\quad(-\pi<\theta<\pi),\tag{74}$$

provided that both sides of (73) *exist.*

Proof. Putting $f(z)=\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)$ in Theorem 10 with $z_2=0$, $0<c\leqq 1$, $q(z)=1$, and $\theta(z)=(z-z_1)(z-z_2)$, we find that

$$\begin{aligned}
&\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\\
&= cz^{-\beta}(z-z_1)^{-\alpha}\\
&\quad\cdot\sum_{n=-\infty}^{\infty}\frac{\sin[(\beta-cn-\gamma)\pi]}{\sin[(\beta+c-\gamma)\pi]}\frac{e^{i\pi c(n+1)}[z(z-z_1)]^{cn+\gamma}}{\Gamma(1-\alpha+cn+\gamma)}\\
&\quad\cdot D_z^{-\alpha+cn+\gamma}\\
&\quad\times\left\{z^{\beta-cn-\gamma-1}(z+w-z_1)\,\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\Big|_{\substack{z=z_1\\(w=z)}}.
\end{aligned}\tag{75}$$

With the help of the relations in (34), we have

$$\begin{aligned}
&D_z^{-\alpha+cn+\gamma}\\
&\quad\times\left\{z^{\beta-cn-\gamma-1}(z+w-z_1)\,\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\Big|_{\substack{z=z_1\\(w=z)}}\\
&= D_z^{-\alpha+cn+\gamma}\left\{z^{\beta-cn-\gamma}\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\Big|_{z=z_1}\\
&\quad+(z-z_1)\,D_z^{-\alpha+cn+\gamma}\\
&\quad\times\left\{z^{\beta-cn-\gamma-1}\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\Big|_{z=z_1}\\
&= z_1^{\beta+\alpha-2cn-2\gamma}\\
&\quad\times\Bigg(\frac{\Gamma(1+\beta-cn-\gamma)}{\Gamma(1+\beta+\alpha-2cn-2\gamma)}\\
&\qquad\cdot\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,1+\beta-cn-\gamma;\mu_1,\dots,\mu_q,1+\beta+\alpha-2cn-2\gamma}(z_1,s,a;b,\lambda)\\
&\qquad+\left(\frac{z-z_1}{z_1}\right)\frac{\Gamma(\beta-cn-\gamma)}{\Gamma(\beta+\alpha-2cn-2\gamma)}\\
&\qquad\cdot\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,\beta-cn-\gamma;\mu_1,\dots,\mu_q,\beta+\alpha-2cn-2\gamma}(z_1,s,a;b,\lambda)\Bigg).
\end{aligned}\tag{76}$$

Thus, by combining (75) and (76), we obtain the desired result (73). □

Finally, from the two generalized Leibniz rules for fractional derivatives given in Section 3, we obtain the following two expansion formulas involving the new family of the λ-generalized Hurwitz-Lerch zeta function $\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)$.

Theorem 17. *Under the hypotheses of Theorem 11, the following expansion formula holds true*:

$$\begin{aligned}
&\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,\nu;\mu_1,\dots,\mu_q,\tau}(z,s,a;b,\lambda)\\
&= \frac{\Gamma(\tau)\,\Gamma(1+\nu-\tau)\sin(\gamma\pi)}{\pi}\\
&\quad\cdot\sum_{n=-\infty}^{\infty}\frac{(-1)^n\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,1;\mu_1,\dots,\mu_q,1-\gamma-n}(z,s,a;b,\lambda)}{(\gamma+n)\,\Gamma(1+\nu-\tau-\gamma-n)\,\Gamma(\tau+\gamma+n)},
\end{aligned}\tag{77}$$

provided that both members of (77) *exist.*

Proof. Setting $u(z)=z^{\nu-1}$ and $v(z)=\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)$ in Theorem 11 with $p=q=0$ and $\alpha=\nu-\tau$, we obtain

$$\begin{aligned}
&D_z^{\nu-\tau}\left\{z^{\nu-1}\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\\
&\quad=\sum_{n=-\infty}^{\infty}\binom{\nu-\tau}{\gamma+n}D_z^{\nu-\tau-\gamma-n}\left\{z^{\nu-1}\right\}\\
&\qquad\cdot D_z^{\gamma+n}\left\{\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\},
\end{aligned}\tag{78}$$

which, with the help of (33) and (34), yields

$$\begin{aligned}
&D_z^{\nu-\tau}\left\{z^{\nu-1}\Phi^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}(z,s,a;b,\lambda)\right\}\\
&\quad=\frac{\Gamma(\nu)}{\Gamma(\tau)}z^{\tau-1}\Phi^{(\rho_1,\dots,\rho_p,1,\sigma_1,\dots,\sigma_q,1)}_{\lambda_1,\dots,\lambda_p,\nu;\mu_1,\dots,\mu_q,\tau}(z,s,a;b,\lambda),
\end{aligned}$$

$$D_z^{\nu-\tau-\gamma-n}\left\{z^{\nu-1}\right\} = \frac{\Gamma(\nu)}{\Gamma(\tau+\gamma+n)} z^{\tau+\gamma+n-1},$$
$$D_z^{\gamma+n}\left\{\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}$$
$$= \frac{z^{-\gamma-n}}{\Gamma(1-\gamma-n)} \Phi_{\lambda_1,\ldots,\lambda_p,1;\mu_1,\ldots,\mu_q,1-\gamma-n}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a;b,\lambda). \tag{79}$$

Combining (79) with (78) and making some elementary simplifications, the asserted result (77) follows. □

Theorem 18. *Under the hypotheses of Theorem 12, the following expansion formula holds true*:

$$\begin{aligned}
&\Phi_{\lambda_1,\ldots,\lambda_p,\nu;\mu_1,\ldots,\mu_q,\tau}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a;b,\lambda) \\
&\quad = \left(\Gamma(\tau)\Gamma(1+\nu-\tau)\sin(\beta\pi)\right. \\
&\qquad \left.\times \sin\left[(\nu-\tau+\beta-\theta)\pi\right]\right) \\
&\qquad \times \left(\Gamma(\nu)\Gamma(\tau-\gamma-\theta-1)\Gamma(1+\gamma+\theta)\right. \\
&\qquad\quad \left.\times \sin\left[(\nu-\tau+\beta)\pi\right]\sin\left[(\beta-\theta-\gamma)\pi\right]\right)^{-1} \\
&\qquad \cdot \frac{\sin(\theta\pi)}{\sin\left[(\theta+\gamma)\pi\right]} \sum_{n=-\infty}^{\infty} \frac{\Gamma(\nu-\theta-n)\Gamma(\theta+n)}{\Gamma(2+\nu-\tau+\gamma-n)\Gamma(-\gamma+n)} \\
&\qquad \cdot \Phi_{\lambda_1,\ldots,\lambda_p,\theta+n;\mu_1,\ldots,\mu_q,1+\theta+\gamma}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a;b,\lambda),
\end{aligned} \tag{80}$$

provided that both members of (80) *exist.*

Proof. Upon first substituting $\mu \mapsto \theta$ and $\nu \mapsto \gamma$ in Theorem 12 and then setting

$$\begin{aligned}
\alpha &= \nu-\tau, \qquad u(z) = z^{\tau-\beta}, \\
v(z) &= \Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda),
\end{aligned} \tag{81}$$

in which both $u(z)$ and $v(z)$ satisfy the conditions of Theorem 12, we have

$$\begin{aligned}
&D_z^{\nu-\tau}\left\{z^{\nu-1}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\} \\
&\quad = \frac{z\Gamma(1+\nu-\tau)\sin(\beta\pi)\sin(\theta\pi)\sin\left[(\nu-\tau+\beta-\theta)\pi\right]}{\sin\left[(\nu-\tau+\beta)\pi\right]\sin\left[(\beta-\theta-\gamma)\pi\right]\sin\left[(\theta+\gamma)\pi\right]} \\
&\qquad \cdot \sum_{n=-\infty}^{\infty} \frac{D_z^{\nu-\tau+\gamma+1-n}\left\{z^{\nu-\theta-1-n}\right\}}{\Gamma(2+\nu-\tau+\gamma-n)\Gamma(-\gamma+n)} \\
&\qquad \cdot D_z^{-1-\gamma+n}\left\{z^{\theta-1+n}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\}.
\end{aligned} \tag{82}$$

Now, by using (33) and (34), we find that

$$\begin{aligned}
&D_z^{\nu-\tau}\left\{z^{\nu-1}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\} \\
&\quad = \frac{\Gamma(\nu)}{\Gamma(\tau)} z^{\tau-1}\Phi_{\lambda_1,\ldots,\lambda_p,\nu;\mu_1,\ldots,\mu_q,\tau}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a;b,\lambda), \\
&D_z^{\nu-\tau+\gamma+1-n}\left\{z^{\nu-\theta-1-n}\right\} = \frac{\Gamma(\nu-\theta-n)}{\Gamma(\tau-\gamma-\theta-1)} z^{\tau-\gamma-\theta-2}, \\
&D_z^{-1-\gamma+n}\left\{z^{\theta-1+n}\Phi_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}(z,s,a;b,\lambda)\right\} \\
&\quad = \frac{\Gamma(\theta+n)}{\Gamma(1+\theta+\gamma)} \\
&\qquad \times z^{\theta+\gamma}\Phi_{\lambda_1,\ldots,\lambda_p,\theta+n;\mu_1,\ldots,\mu_q,1+\theta+\gamma}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a;b,\lambda).
\end{aligned} \tag{83}$$

Thus, finally, the result (80) follows by combining (83) and (82). □

5. Corollaries and Consequences

We conclude this paper by presenting some special cases of the main results. These special cases and consequences are given in the form of the following corollaries.

Setting $k = 3$ in Theorem 13 and using the fact that [12, p. 1496, Remark 7]

$$\lim_{b\to 0} H_{0,2}^{2,0}\left[(a+n)b^{1/\lambda} \mid \overline{(s,1),\left(0,\frac{1}{\lambda}\right)}\right] = \lambda\Gamma(s) \qquad (\lambda > 0), \tag{84}$$

we obtain the following corollary given recently by Srivastava et al. [19].

Corollary 19. *Under the hypotheses of Theorem 13, the following expansion formula holds true*:

$$\begin{aligned}
&\Phi_{\lambda_1,\ldots,\lambda_p,\alpha;\mu_1,\ldots,\mu_q,\beta}^{(\rho_1,\ldots,\rho_p,1/3,\sigma_1,\ldots,\sigma_q,1/3)}(z,s,a) \\
&\quad = \frac{3\Gamma(\beta)\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(\beta+\alpha)} \\
&\qquad \cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{(3\alpha)_n}{(\beta+\alpha)_n} \frac{z^n}{(n+a)^s} \\
&\qquad \cdot F_1\left[3\alpha+n, 1+\alpha-\beta, 1+\alpha-\beta; \beta+2\alpha+n; -1, 1\right],
\end{aligned} \tag{85}$$

where F_1 denotes the first Appell function defined by [11, p. 22]

$$\begin{aligned}
&F_1\left[a,b_1,b_2;c;x_1,x_2\right] \\
&\quad = \sum_{m_1,m_2=0}^{\infty} \frac{(a)_{m_1+m_2}(b_1)_{m_1}(b_2)_{m_2}}{(c)_{m_1+m_2}} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!} \\
&\qquad \left(\max\left\{|x_1|,|x_2|\right\} < 1\right),
\end{aligned} \tag{86}$$

provided that both sides of (85) *exist.*

Putting $p-1=q=0$ and setting $\rho_1=1$ and $\lambda_1=\mu$ in Theorem 15 reduces to the following expansion formula given recently by Srivastava et al. [20].

Corollary 20. *Under the hypotheses of Theorem 15, the following expansion formula holds true:*

$$\begin{aligned}&\Theta_{\mu}^{\lambda}(z,s,a;b)\\&=cz^{-\alpha}(z-z_1)^{-\beta}z_1^{\alpha+\beta}\\&\quad\cdot\sum_{n=-\infty}^{\infty}\frac{e^{i\pi c(n+1)}\sin[(\alpha+cn+\gamma)\pi]\,\Gamma(\alpha+cn+\gamma)}{\sin[(\alpha-c+\gamma)\pi]\,\Gamma(1-\beta+cn+\gamma)\,\Gamma(\alpha+\beta)}\\&\quad\cdot\Phi_{\mu,\alpha+cn+\gamma;\alpha+\beta}^{(1,1,1)}(z_1,s,a;b,\lambda)\left(\frac{z-z_1}{z}\right)^{cn+\gamma}\end{aligned}\tag{87}$$

for $\lambda>0$ and for z on $C_1(1)$ defined by

$$z=\frac{z_1}{2}+\frac{z_1}{2}\sqrt{1+e^{i\theta}}\quad(-\pi<\theta<\pi),\tag{88}$$

provided that both sides of (69) *exist.*

Letting $b=0$ in Theorem 18, we deduce the following expansion formula obtained by Srivastava et al. [19].

Corollary 21. *Under the hypotheses of Theorem 18, the following expansion formula holds true*:

$$\begin{aligned}&\Phi_{\lambda_1,\ldots,\lambda_p,\nu;\mu_1,\ldots,\mu_q,\tau}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a)\\&=\frac{\Gamma(\tau)\,\Gamma(1+\nu-\tau)}{\Gamma(\nu)\,\Gamma(\tau-\gamma-\theta-1)}\\&\quad\cdot(\sin\beta\pi\sin[(\nu-\tau+\beta-\theta)\pi]\sin\theta\pi)\\&\quad\times(\Gamma(1+\gamma+\theta)\sin[(\nu-\tau+\beta)\pi]\\&\qquad\times\sin[(\beta-\theta-\gamma)\pi]\sin[(\theta+\gamma)\pi])^{-1}\\&\quad\cdot\sum_{n=-\infty}^{\infty}\frac{\Gamma(\nu-\theta-n)\,\Gamma(\theta+n)\,\Phi_{\lambda_1,\ldots,\lambda_p,\theta+n;\mu_1,\ldots,\mu_q,1+\theta+\lambda}^{(\rho_1,\ldots,\rho_p,1,\sigma_1,\ldots,\sigma_q,1)}(z,s,a)}{\Gamma(2+\nu-\tau+\gamma-n)\,\Gamma(-\gamma+n)}\end{aligned}\tag{89}$$

provided that both members of (89) *exist.*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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