Potential Stability of Sign Pattern Matrices

by

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ABSTRACT

An $n \times n$ sign pattern $\mathcal{A}$ is potentially stable (PS) if there exists a real matrix A having the sign pattern $\mathcal{A}$ and with all its eigenvalues having negative real parts. The identification of non-trivial necessary and sufficient conditions for potential stability remains a long standing open problem. Here we review some of the previous results and give simplified proofs for some of these results. Three techniques are given for the construction of larger order PS sign patterns from given PS sign patterns. These techniques are: construction of a sign pattern that allows a nested sequence of properly signed principal minors (a nest), bordering of a PS sign pattern with additional rows and columns, and use of a similarity transformation of a matrix that is reducible with two diagonal blocks (one of which is a stable matrix and the other a negative scalar). The minimum number of nonzero entries in an irreducible minimally PS sign pattern is determined for $n = 2, \ldots, 6$ and for an arbitrary sign pattern that allows a nest. We also determine lower bounds for the number of nonzero entries in irreducible minimally PS sign patterns having certain sign patterns for their diagonal entries. For irreducible PS sign patterns of order at least four, a bordering construction leads to a new upper bound for the minimum number of nonzero entries.

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DEDICATION

I dedicate this work to my wife, Dana, who has given me her tireless support, understanding and patience. I could not have done this without her. I also dedicate this to our wonderful and patient children, Magdalena, Cohen and Bren, who deserve to have their Dad back again.

Chapter 1

Introduction and Notation

Dynamical systems often are modelled by nonlinear systems that are in general difficult to solve. Let $\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$ be such a system. An *equilibrium point* is a point $(\hat{x}_1, \ldots, \hat{x}_n)$ such that $f_i(\hat{x}_1, \ldots, \hat{x}_n) = 0$ for $i = 1, \ldots, n$. Linearization of the system provides insight into the behaviour of the nonlinear system in the neighbourhood of an equilibrium point. The linearized system can always be written as $\frac{dx_i}{dt} = f_i(\hat{x}_1, \ldots, \hat{x}_n) + \frac{\partial f_i}{\partial x_i}(\hat{x}_1, \ldots, \hat{x}_n)(x_1 - \hat{x}_1) + \cdots + \frac{\partial f_i}{\partial x_n}(\hat{x}_1, \ldots, \hat{x}_n)(x_n - \hat{x}_n)$. Since $(\hat{x}_1, \ldots, \hat{x}_n)$ is an equilibrium point, $f_i(\hat{x}_1, \ldots, \hat{x}_n) = 0$ for $i = 1, \ldots, n$. Using a change of variables, $z_i = x_i - \hat{x}_i$, the linearized system can be written more simply as $\frac{dz_i}{dt} = Jz_i$, where the matrix J is the Jacobian matrix of the system at the equilibrium point $(\hat{x}_1, \ldots, \hat{x}_n)$. The linearized system is defined to be stable if, for arbitrary initial values of perturbations from equilibrium, within a sufficiently small neighbourhood, $\lim_{t \to \infty} z_i = 0$ for $i = 1, \ldots, n$. It is well known that the system is stable in this sense if and only if the real parts of all of the eigenvalues of J are negative [6, 15]. It is common to refer to the stability of both the linearized system and the real $n \times n$ matrix J.

For a matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ denote the multiset of eigenvalues of A. The matrix A is (negative) *stable* if $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$. There exist well known results, such as the Routh-Hurwitz inequalities and Lyapunov's Theorem, that give necessary and sufficient conditions for a matrix to be stable. There are, however, circumstances that arise in areas of economics, ecology and chemistry when the exact value of entries in A may not be known. In fact, in some dynamical systems there are situations in which only the signs may be known [22]. It is still desirable to be able to determine stability properties of such systems by considering only their sign patterns. Samuelson [22], in the mathematical modeling of systems from economics, is usually credited with first raising questions about what qualitative knowledge can be inferred by considering only the signs of the entries of matrix A.

An $n \times n$ *sign pattern* (*matrix*) $\mathcal{A} = [\alpha_{ij}]$ has $\alpha_{ij} \in \{+, 0, -\}$ for $i, j = 1, \ldots, n$. A sign pattern class of real matrices is defined by $Q(\mathcal{A}) = \{B = [b_{ij}] : sign\ b_{ij} = \alpha_{ij}$ for all $i, j\}$. For a matrix $A \in \mathbb{R}^{n \times n}$, $\mathrm{sgn}(A)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of A by $+$ (respectively, $-$, 0), and A is a *realization* of $\mathrm{sgn}(A)$. A sign pattern $\mathcal{A}$ is *sign stable* if for all $A \in Q(\mathcal{A})$, $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$; i.e., $\mathcal{A}$ *requires* stability. Quirk and Ruppert [21] considered the problem of sign stability; such sign patterns were later characterized by Jeffries et al. in [11]. A sign pattern $\mathcal{A}$ is *potentially stable* (PS) if there exists a matrix $A \in Q(\mathcal{A})$ with $\mathrm{Re}(\lambda) < 0$ for all $\lambda \in \sigma(A)$; i.e., $\mathcal{A}$ *allows* stability. A sign pattern that is not PS is called *sign unstable.* The terminology of a potentially stable sign pattern appears to have originated in [20]. Clearly the set of sign stable sign patterns is properly contained in the set of PS sign patterns. However, while there exists a polynomial time algorithm for determining whether or not a sign pattern is sign stable [16], Klee speculated in [15] that "the problem of recognizing potential stability is NP-complete". Bone [3] stated that recognizing PS sign patterns is a decidable problem, although he provided no insight to the computational complexity involved.

Quirk's paper with Ruppert [21] was the first of a number of papers he co-authored in the area of stability of sign pattern matrices [2, 18, 20]. In [18], the statement is made that "the specification of necessary and sufficient conditions [for potential stability] remains an unsolved problem". For the most part, this statement remains true some fifty years later.

If $\gamma \subseteq \{1, \ldots, n\}$, then the *principal submatrix* of A from rows and columns γ is denoted $A[\gamma]$. The *principal minor* of A from rows and columns γ is $\det A[\gamma]$. Sign pattern $\mathcal{A}$ allows a *nested sequence of properly signed principal minors* (abbreviated to a *nest*) if for $1 \le k \le n$, there exists γ_k with $|\gamma_k| = k$ and $\gamma_k \subsetneq \gamma_{k+1}$ such that $sign \det A[\gamma_k] = (-1)^k$. We denote a nest by the sequence of distinct indices $(i_1, \ldots, i_k)$ if $\gamma_k = \{i_1, \ldots, i_k\}$. A *leading* nest has $\gamma_k = \{1, \ldots, k\}$ for $1 \le k \le n$, and is denoted by $(1, \ldots, n)$. If $\mathcal{A}$ allows a nest, then there exists a permutation matrix P such that $P\mathcal{A}P^T$ allows a leading nest. Thus, without loss of generality, if $\mathcal{A}$ allows a nest, then we can assume that it is a leading nest. It has been known at least as far back as [2] that if a sign pattern allows a nest, then the sign pattern is PS. This result, which provides a sufficient condition for potential stability, was exploited by Johnson et al. [12] in considering sufficient conditions for sign patterns to allow a nest.

A sign pattern $\mathcal{A}$ can be represented by a signed digraph $D(\mathcal{A})$ with vertex set $\{1, \ldots, n\}$ and arc set $\{(i, j) : \alpha_{ij} \neq 0\}$ with (i, j) signed $+$ or $-$ as α_{ij}. Note that $D(A) = D(\mathcal{A})$ for all $A \in Q(\mathcal{A})$. A (directed) *cycle* of length $q \geq 2$ (a q-cycle) in $D(\mathcal{A})$ consists of a sequence of arcs $(i_1, i_2), \ldots, (i_{q-1}, i_q), (i_q, i_1)$ such that $i_1, \ldots, i_q$ are distinct vertices. A q-cycle on vertices $i_1, \ldots, i_q$ is denoted by $(i_1 \to \ldots \to i_q \to i_1)$. A cycle of length 1 (a *loop*) is an arc (i_1, i_1). A cycle is signed positive (respectively, negative) if there is an even (respectively odd) number of negative arcs on the cycle. A digraph D is *weakly connected* if the underlying graph G that is obtained by removing the direction on each arc in D is connected. A *weakly connected component* of a digraph is a maximal weakly connected subdigraph.

A *tree sign pattern* describes a sign pattern $\mathcal{A}$ such that the underlying graph G that is obtained by removing the direction on each arc in $D(\mathcal{A})$ is a tree. A *star sign pattern* describes a tree sign pattern $\mathcal{A}$ such that $D(\mathcal{A})$ has one central vertex to which every other vertex is adjacent. Johnson and Summers [13] identified almost all of the $n \times n$ PS tree sign patterns for $n = 2, 3$ and 4. This work was further developed by Gao and Li [7], where they characterized all PS star sign patterns. Jeffries and Johnson [10] presented criteria for tree sign patterns to be sign unstable.

The property of being PS is preserved under transposition, permutation similarity and signature similarity. Two sign patterns are equivalent if one can be obtained from the other by any combination of these three operations. Sign patterns that are PS are usually identified up to equivalence. These operations can be described in terms of their effect on the associated digraph. Transposing a sign pattern corresponds to changing the direction of all the arcs in the corresponding digraph, a permutation similarity of a sign pattern corresponds to changing the numbering of the vertices in the associated digraph, and a signature similarity of a sign pattern corresponds to changing the signs of particular arcs while maintaining the signs of all cycles in the associated digraph.

A matrix A is called *reducible* if there exists a permutation matrix P and square matrices X and Z such that $P^T AP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$. Otherwise, a matrix is called irreducible. A sign pattern $\mathcal{A}$ is called reducible if for all $A \in Q(\mathcal{A})$, A is reducible, otherwise $\mathcal{A}$ is irreducible. If $\mathcal{A}$ is an irreducible PS sign pattern and setting any nonzero α_{ij} to 0 implies that the resulting sign pattern is no longer PS, then $\mathcal{A}$ is a *minimally* PS sign pattern. We often focus on the minimally PS sign patterns when investigating potential stability. Given an $n \times n$ irreducible PS sign pattern $\mathcal{A}$, without loss of generality matrix $A \in Q(\mathcal{A})$ can be normalized to have $n-1$ of its off-diagonal nonzero entries set to ± 1 by using a well known result (see [4, Lemma 2.3]). In addition (by scaling with a positive constant) one diagonal entry can be set to ± 1. This normalization is used in many examples that follow.

An overview of the contents of this thesis, its goals and accomplishments are now described. In Chapter 2, we briefly describe results from the literature related to matrix stability and potential stability. The problem of matrix stability has long been studied, and necessary and sufficient conditions exist to show stability of a matrix by analysis of its characteristic polynomial. In Section 2.1, we review the well known Routh-Hurwitz conditions and state a special case of the Hermite-Biehler theorem. We also state a useful sufficient condition for matrix stability that is due to Fisher and Fuller [5]. In Section 2.2, we review results from the literature on potential stabilty that have been fundamental in the development of our results. Section 2.3 focuses on Miyamichi's work [19], and we highlight some ways in which the proofs from [19] can be simplified and emphasize some conditions that are not clearly articulated there. Miyamichi's work has a direct influence on some of the results in Chapters 3 and 4.

One of the central goals of this thesis is to establish sufficient conditions for potential stability. Some new sufficient conditions for a sign pattern to be PS are developed in Chapter 3 where three techniques for constructing PS sign patterns are presented. In Section 3.1, we describe a construction that can be performed on certain PS sign patterns that allow a nest in order to generate PS sign patterns of higher order that also allow a nest. Section 3.2 focuses on bordering known PS sign patterns with additional rows and columns. We present constructions that involve bordering known PS sign patterns with one additional row and column, with two additional rows and columns and finally with more than two additional rows and columns. The third technique, which is given in Section 3.3, uses a similarity transformation of a matrix that is reducible with two diagonal blocks (one a stable matrix and the other a negative scalar).

Another central goal of this thesis is to determine necessary conditions for potential stability. The number of nonzero entries in a PS sign pattern is investigated in Chapter 4. It is shown in Section 4.1 that the minimum number of nonzero entries in an $n \times n$ sign pattern that allows a nest is $2n-1$. In Section 4.2, the least number of nonzero entries in an $n \times n$ minimally PS sign pattern is determined for orders $n = 2, \ldots, 6$. We also determine lower bounds for the number of nonzero entries in irreducible stable matrices having certain sign patterns for their diagonal entries. Lastly, we determine the number of nonzero entries in certain lower Hessenberg sign patterns produced by constructions presented in Chapter 3, which leads to a new upper bound for the minimum number of nonzero entries in an irreducible PS sign pattern of order at least 4. In Chapter 5 some conclusions are made and suggestions for future research are given.

Chapter 2

Previous Results

In Chapter 2 we are concerned primarily with prior results in the areas of matrix stability and potential stability. Necessary and sufficient conditions for matrix stability are given, followed by the Fisher-Fuller theorem, in Section 2.1. Previous results in the area of potential stability are presented in Section 2.2 along with a discussion of Miyamichi's work [19] in Section 2.3. Alternate proofs of some of the theorems in [19] are also given in Section 2.3.

2.1 Matrix Stability

We begin by stating some well known results for stability of a real $n \times n$ matrix A. If $x^n + k_1 x^{n-1} + \cdots + k_{n-1}x + k_n$ denotes the characteristic polynomial of A, then $k_i = (-1)^i \times$ (the sum of all of the principal minors of order i in A), for $i = 1, \ldots, n$; see, e.g. [9, p. 42]. The following is a well known result for polynomials that provides a method to determine the stability of a matrix by analysis of its characteristic polynomial; see, e.g., [6, p. 220].

Theorem 2.1 (Routh-Hurwitz)**.** *Matrix* $A \in \mathbb{R}^{n\times n}$ *is stable if and only if* $k_i > 0$ *and*

$\Delta_i > 0$ $(1 \leq i \leq n)$, *where*

$$\Delta_i = \begin{vmatrix} k_1 & 1 & 0 & 0 & \cdots & 0 \\ k_3 & k_2 & k_1 & 1 & \cdots & \vdots \\ k_5 & k_4 & k_3 & k_2 & \cdots & 0 \\ k_7 & k_6 & k_5 & k_4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{2i-1} & k_{2i-2} & k_{2i-3} & k_{2i-4} & \cdots & k_i \end{vmatrix}$$

with $k_j = 0$ *if* $j > n$.

The following theorem found in [6, p. 228] is another well known result for polynomials. It is a special case of the Hermite-Biehler theorem. Holtz [8] provides alternate proofs to both of the Routh-Hurwitz and Hermite-Biehler theorems.

Theorem 2.2. *Suppose* $f(x) = h(x^2) + xg(x^2)$ *is a polynomial of degree* $n \geq 3$ *with positive coefficients, where* $h(x^2)$ *is the sum of all even degree terms and* $xg(x^2)$ *is the sum of all odd degree terms. Then* $f(x)$ *is stable if and only if the zeros* $\alpha_1, \ldots, \alpha_{\lfloor \frac{n}{2} \rfloor}$ *of* $h(u)$ *and the zeros* $\beta_1, \ldots, \beta_{\lfloor \frac{n-1}{2} \rfloor}$ *of* $g(u)$ *are all negative and*

$$0 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots .$$

This next theorem is a well known matrix result that is proved by Fisher and Fuller [5]. Ballantine [1] gave alternate proofs to Theorem 2.3 (Fisher-Fuller) and an analogous theorem for $A \in \mathbb{C}^{n \times n}$.

Theorem 2.3 (Fisher-Fuller)**.** *If* $A \in \mathbb{R}^{n \times n}$ *contains a nest, then there exists a positive diagonal matrix* D *such that* DA *is stable.*

2.2 Potential Stability

The following is a collection of results from earlier papers on the topic of potential stability. A necessary condition for a sign pattern to be PS is given in the following proposition by Quirk [20, Proposition 3]. This condition can be seen directly by consideration of Theorem 2.1 (Routh-Hurwitz).

Proposition 2.4. *Suppose* $\mathcal{A}$ *is an* $n \times n$ *PS sign pattern. If* $A \in Q(\mathcal{A})$ *is stable, then* A *has a principal minor of order* i *with* $\operatorname{sign}(-1)^i$ *for every* $i = 1, \ldots, n$.

The following result [12, Theorem 2.1] is a consequence of Theorem 2.3.

Theorem 2.5. *If $\mathcal{A}$ is an $n \times n$ sign pattern that allows a nest, then $\mathcal{A}$ is PS. Moreover, $\mathcal{A}$ contains a nested sequence of PS sign patterns of orders $1, \ldots, n$.*

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ is a *subpattern* of sign pattern $\widehat{\mathcal{A}} = [\hat{\alpha}_{ij}]$ (and $\widehat{\mathcal{A}}$ is a *superpattern* of $\mathcal{A}$) if $\hat{\alpha}_{ij} = \alpha_{ij}$ whenever $\alpha_{ij} \neq 0$. This next theorem is the reason that in investigating potential stability, it is necessary to consider only the minimally PS sign patterns. The proof of the following theorem [13, Theorem 3] uses the fact that the eigenvalues of a real matrix depend continuously on the entries; see also Miyamichi [19, Lemma 4].

Theorem 2.6. *Suppose that $\mathcal{B}$ is an $n \times n$ PS sign pattern and suppose that $\mathcal{B}$ is a subpattern of the $n \times n$ sign pattern $\mathcal{A}$. Then $\mathcal{A}$ is PS.*

A diagonal sign pattern with all entries negative is clearly PS. The next result can be proved using either Theorem 2.5 or Theorem 2.6; see [2, Theorem 1].

Theorem 2.7. *Suppose $\mathcal{A} = [\alpha_{ij}]$ is a sign pattern with $\alpha_{ii} < 0$ for $i = 1, \ldots, n$. Then $\mathcal{A}$ is PS.*

The following theorem is a special case of [13, Theorem 4] with a proof that is similar to [1, Theorem 1].

Theorem 2.8. *If A is an $n \times n$ matrix with a stable $(n-1) \times (n-1)$ principal submatrix and* $\operatorname{sign}\det A = (-1)^n$, *then* $\operatorname{sgn}(A)$ *is PS.*

Proof. Without loss of generality, partition A as $A = \left[\begin{array}{c|c} \hat{A} & a \\ \hline b^T & c \end{array}\right]$, where $\hat{A}$ is stable, a and b are column vectors of length $n-1$ and c is a scalar. Let $D = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & d \end{array}\right]$, where $d \geq 0$, and let $DA = M(d) = \left[\begin{array}{c|c} \hat{A} & a \\ \hline db^T & dc \end{array}\right]$. Then $M(0) = \left[\begin{array}{c|c} \hat{A} & a \\ \hline 0 & 0 \end{array}\right]$ and $\sigma(M(0)) = \sigma(\hat{A}) \cup \{0\}$. The eigenvalues of $M(d)$ are continuous functions of d and for all sufficiently small $d > 0$, the real parts of $n-1$ of the eigenvalues of $M(d)$ are negative since $\hat{A}$ is stable. If, for any such sufficiently small d, μ denotes the remaining eigenvalue, then since $\operatorname{sign}\det DA = \operatorname{sign}\det M(d) = (-1)^n = (-1)^{n-1}\operatorname{sign} \mu$, it follows that μ is negative and $M(d)$ is stable. Since D is a positive diagonal matrix, the matrix product DA has the same sign pattern as A and thus $\operatorname{sgn}(A)$ is PS. $\square$

This next result by Bone [3, Theorem 3] involves bordering a PS sign pattern with two rows and columns. Similar constructions are described in Section 3.2.

Theorem 2.9. *Suppose $\mathcal{A}$ is an $n \times n$ sign pattern with the following properties:*

i. *$D(\mathcal{A})$ contains a positive n-cycle;*

ii. *there exists a permutation matrix P such that $P\mathcal{A}P^T[\{1, \ldots, n-2\}]$ is PS and the digraph of $P\mathcal{A}P^T[\{n-1, n\}]$ contains a negative 2-cycle.*

Then $\mathcal{A}$ is PS.

2.3 Results due to Miyamichi

A number of proofs to theorems of Miyamichi in [19] can be simplified by applying Theorem 2.8. In the following result, we give such an alternate proof to [19, Theorem 2].

Theorem 2.10. *Let $n \geq 2$. Suppose $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ sign pattern with diagonal entries $\alpha_{11} \in \{+, 0\}$ and $\alpha_{ii} = -$ for $i = 2, \ldots, n$. If $D(\mathcal{A})$ has a negative cycle of length ≥ 2 through vertex 1, then $\mathcal{A}$ is PS.*

Proof. Let $A \in Q(\mathcal{A})$ with $\alpha_{11} = 0$ and without loss of generality let there be a negative k-cycle through vertices $1, \ldots, k$ in $D(A)$ for some k, $2 \leq k \leq n$. Let $B = [b_{ij}]$ be the $n \times n$ matrix such that $b_{ij} = a_{ij}$ for all diagonal entries and all entries on the k-cycle, with $b_{ij} = 0$ otherwise, and $\mathcal{B} = \operatorname{sgn}(B)$. The subpattern $\mathcal{B}[\{2, \ldots, k\}]$ is PS by Theorem 2.7 and $\operatorname{sign}\det B[\{1, \ldots, k\}] = (-1)^k$. Thus $\mathcal{B}[\{1, \ldots, k\}]$ is PS by Theorem 2.8. As well $\mathcal{B}[\{k+1, \ldots, n\}]$ is PS by Theorem 2.7. Thus $\mathcal{B}$ is PS as it is the direct sum of two PS sign patterns, and the result follows by Theorem 2.6. □

Analogously, the next theorem gives an alternate proof of potential stability for many of the sign patterns in [19, Theorem 3]. Digraphs (**A**),(**B**),(**C**),(**D**) and (**E**) from Miyamichi [19] are given in Section A.4. Although no alternate proof is provided below for the potential stability of the sign pattern corresponding to the digraph (**D**), in Chapter 3 a generalization of the sign pattern corresponding to digraph (**D**) is given (see Theorem 3.16).

Theorem 2.11. *Let $n \geq 3$. The sign pattern that has digraph* (**A**) *with negative $(n-1)$-cycle and the sign patterns that have digraphs* (**B**), (**C**) *and* (**E**) *are PS.*

Furthermore, the sign patterns that have digraph (**A**) *with a negative* $(n-1)$*-cycle and the sign pattern that has digraph* (**B**) *are minimally PS, whereas the sign pattern that corresponds to digraph* (**C**) *is not minimally PS and it is unknown whether or not the sign pattern that corresponds to digraph* (**E**) *is minimally PS.*

Proof. (**A**)

Suppose $a_{ii} < 0$ for $i = 1, \dots, n-2$ and $a_{n-1,n-1} = a_{nn} = 0$. Assume that digraph $D(A)$ contains a negative 2-cycle through vertices $n-1$ and n and a negative $(n-1)$-cycle that passes through vertices $1, \dots, n-1$. Assume all other entries in A are zero. Let $\mathcal{A} = \text{sgn}(A)$. Then $\mathcal{A}[\{1, \dots, n-1\}]$ is PS by Theorem 2.10. Since $\text{sign} \det A = (-1)^{n-2}(-1)^2 = (-1)^n$, it follows that $\mathcal{A}$ is PS by Theorem 2.8.

Furthermore, zeroing any off-diagonal entry in $\mathcal{A}$ produces a reducible pattern with components that are not both PS, and zeroing any diagonal entry in $\mathcal{A}$ produces a zero determinant for all $A \in Q(\mathcal{A})$. Thus, it follows that the sign pattern corresponding to digraph (**A**) with a negative $(n-1)$-cycle is minimally PS.

(**B**)

Suppose $a_{ii} < 0$ for $i = 1, \dots, n-2$ and $a_{n-1,n-1} = a_{nn} = 0$. Assume that digraph $D(A)$ contains a negative n-cycle and a negative $(n-1)$-cycle that passes through vertices $1, \dots, n-1$. Assume all other entries in A are zero. Let $\mathcal{A} = \text{sgn}(A)$. Then by Theorem 2.10, $\mathcal{A}[\{1, \dots, n-1\}]$ is PS. Since $\mathcal{A}[\{1, \dots, n-1\}]$ is PS and $\text{sign} \det A = (-1)^n$ for all $A \in Q(\mathcal{A})$, by Theorem 2.8, it follows that $\mathcal{A}$ is PS.

Furthermore, for a matrix having digraph (**B**), [19, Lemma 12] implies that zeroing any off-diagonal entry produces an unstable matrix. The zeroing of any diagonal entry in a matrix having digraph (**B**) produces a matrix without a nonzero principal minor of order $n-2$, contradicting Proposition 2.4. Thus, it follows that the sign pattern corresponding to digraph (**B**) is minimally PS.

(**C**)

Suppose $a_{ii} < 0$ for $i = 2, \dots, n-1$ and $a_{11} = a_{nn} = 0$. Assume that digraph $D(A)$ contains a negative 2-cycle through vertices 1 and 2 and a negative n-cycle. Assume all other entries in A are zero. Let $\mathcal{A} = \text{sgn}(A)$. Since sign pattern $\mathcal{A}[\{1, \dots, n-1\}]$ is PS by Theorem 2.10 and $\text{sign} \det A = (-1)^n$ for all $A \in Q(\mathcal{A})$, by Theorem 2.8 it follows that $\mathcal{A}$ is PS. However, the matrix X in Example 3.5 shows that the sign pattern corresponding to digraph (**C**) is not minimally PS.

(**E**)

Suppose $a_{11} > 0, a_{nn} = 0$ and $a_{ii} < 0$ for $i = 2, \dots, n-1$. Assume that $D(A)$ contains a negative 2-cycle through vertices 1 and 2 with $\det A[\{1, 2\}] > 0$, i.e.,

$a_{11}a_{22} > a_{12}a_{21}$, and a positive $(n-1)$-cycle through vertices $2, \ldots, n$. Assume all other entries in A are zero. Let $\mathcal{A} = \text{sgn}(A)$. Since sign pattern $\mathcal{A}[\{1, \ldots, n-1\}]$ is PS by Theorem 2.10 and $\text{sign} \det A = (-1)^n$ for all $A \in Q(\mathcal{A})$, by Theorem 2.8 it follows that $\mathcal{A}$ is PS. Thus, it follows that the sign pattern corresponding to digraph (**E**) is PS.

The sign pattern corresponding to the digraph (**E**) is minimally PS if $n = 3$. Let $n \geq 4$. If any off-diagonal entry in $\mathcal{A} = [\alpha_{ij}]$ is set to zero, then the resulting pattern is reducible with irreducible components that are not all PS. If α_{11} is set to zero, then the resulting pattern is combinatorially singular. If α_{ii} is set to 0 for any one $i \in \{3, \ldots, n-1\}$, then every matrix with this sign pattern has no principal minor of order $n-1$ with sign $(-1)^{n-1}$. However, if α_{22} is set to 0, then it is not clear if the resulting sign pattern is PS. Thus for $n \geq 4$, we do not know if the sign pattern corresponding to the digraph (**E**) is minimally PS.

□

It should be noted that Miyamichi does not explicitly state any restrictions on the order of the higher order PS sign patterns corresponding to the digraphs (**A**) – (**E**) in [19], although the proof for [19, Lemma 11] implies $n \geq 5$ for the digraphs (**A**) – (**D**). However, for all of the digraphs in Theorem 2.11, the above proof requires only $n \geq 3$. We now consider $n = 3$ and 4 and determine the minimum order of a PS sign pattern corresponding to the digraph (**A**) with a positive $(n-1)$-cycle and digraph (**D**).

Although not stated directly in [19], the sign pattern corresponding to digraph (**A**) with a positive $(n-1)$-cycle is not PS for $n = 3$ since a Routh-Hurwitz condition fails, as shown by the normalized matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ b & 0 & 1 \\ 0 & -a & 0 \end{bmatrix},$$

which has characteristic polynomial $\lambda^3 + \lambda^2 + (a-b)\lambda + a$ giving $\Delta_3 = (a-b) - a = -b \not> 0$. Note that the digraph of A is not one that corresponds to any of the 3×3 minimally PS tree sign patterns listed in Section A.2. However, the following example shows that the sign pattern corresponding to digraph (**A**) with a positive $(n-1)$-cycle is PS for $n = 4$.

Example 2.12. Consider

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

with $\sigma(B) \approx \{-0.0433 \pm 1.2272i, -0.9567 \pm 0.6412i\}$. Therefore, B is stable and $\text{sgn}(B)$ is PS.

As shown by sign pattern $\mathcal{A}_{3,4}$ in Section A.2, the sign pattern corresponding to digraph (**D**) in Section A.4 with a positive n-cycle is PS for $n = 3$, despite the lack of a nest. Theorem 2.9 (see Bone [3]) shows the sign pattern to be PS for $n \geq 3$. Although not stated in [19], the sign pattern corresponding to digraph (**D**) with a negative n-cycle is not PS for $n = 3$ or 4, since a Routh-Hurwitz condition fails in these cases as shown by the following two matrices. The normalized matrix

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -b & -a & 0 \end{bmatrix}$$

has characteristic polynomial $\lambda^3 + \lambda^2 + a\lambda + (b+a)$ giving $\Delta_2 = a - (b+a) = -b \not> 0$. Similarly, the normalized matrix

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -b & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -c & 0 & -a & 0 \end{bmatrix}$$

has characteristic polynomial $\lambda^4 + (b+1)\lambda^3 + (a+b)\lambda^2 + a(b+1)\lambda + (ab+c)$ giving $\Delta_3 = a(b+1)[(b+1)(a+b) - a(b+1)] - (ab+c)(b+1)^2 = -c(b+1)^2 \not> 0$.

The following example shows that the sign pattern corresponding to digraph (**D**) with a negative n-cycle is PS for $n = 5$.

Example 2.13. Consider

$$C = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -4 & 0 \end{bmatrix}$$

with $\sigma(C) \approx \{-0.0040 \pm 1.9772i, -1.5395, -0.7263 \pm 0.5507i\}$. Therefore, C is stable and $\text{sgn}(C)$ is PS. Note that matrix X in Example 3.18 shows that this sign pattern associated with digraph ($\mathbf{D}$) is not minimally PS.

Chapter 3

New Potentially Stable Constructions

The problem of potential stability is the determination of whether or not a given sign pattern allows stability. Since every superpattern of a PS sign pattern is also PS, it is sufficient to consider only the minimally PS sign patterns. For 2×2 sign patterns, up to equivalence, there is only one minimally PS sign pattern (see Section A.1). Miyamichi [19] identified the five 3×3 minimally PS sign patterns up to equivalence (see Section A.2). A list of most of the 4×4 PS tree sign patterns was given by Johnson and Summers [13], and the complete list of the 4×4 minimally PS tree sign patterns up to equivalence was presented by Lin et al. [17] (see Section A.3). In other than a few special cases, there has not been an attempt at identifying PS sign patterns of larger order. This chapter focuses on the construction of larger order PS sign patterns from given PS sign patterns, and gives three distinct techniques. The first two techniques are motivated by constructions of Miyamichi [19]. Examples are given to indicate whether or not a construction produces a sign pattern that is minimally PS.

3.1 Identification of a Nest

One sure method of determining the potential stability of sign pattern $\mathcal{A}$ is to find a stable realization $A \in Q(\mathcal{A})$. However, a method for finding a stable realization of a given sign pattern is not always obvious. Determining whether a given sign pattern allows a nest is one approach for showing potential stability that does not require that a stable realization be identified. By Theorem 2.5, if sign pattern $\mathcal{A}$ allows a nest, then $\mathcal{A}$ is PS.

Theorem 3.1, which generalizes a construction used by Miyamichi [19], describes a construction that can be performed on particular sign patterns that allow a nest to generate larger order sign patterns that also allow a nest. For example, this construction could be performed on sign pattern $\mathcal{A}_{3,1}$ in Section A.2 to produce the sign pattern corresponding to a digraph equivalent to (**A**) in Section A.4. Similarly, its application to sign pattern $\mathcal{A}_{3,3}$ in Section A.2 produces the sign pattern corresponding to a digraph equivalent to (**B**) in Section A.4.

Theorem 3.1. *Suppose $\mathcal{A} = [\alpha_{ij}]$ is a sign pattern of order n that allows a leading nest and $\alpha_{12}\alpha_{21} \neq 0$. If the associated 2-cycle $(1 \to 2 \to 1)$ in $D(\mathcal{A})$ is replaced by a k-cycle of the same sign where $k \geq 3$ and all additional vertices have negative loops, then the resulting sign pattern of order $n + k - 2$ allows a nest and consequently is PS.*

Proof. Assume $A \in Q(\mathcal{A})$ has a leading nest. Replace the 2-cycle $(1 \to 2 \to 1)$ in $D(A)$ with a k-cycle (of the same sign) as follows in order to obtain a digraph $\hat{D}$ with $n + k - 2$ vertices:

1. Label the new vertices $n + 1, \ldots, n + k - 2$.

2. Construct $\hat{D}$ from $D(A)$ by adding a negative loop on each new vertex, adding arcs $(1, n+1)$, $(n+1, n+2), \ldots, (n+k-3, n+k-2), (n+k-2, 2)$ and deleting arc $(1, 2)$.

3. Let $\hat{A} = [\hat{a}_{ij}]$ with digraph $D(\hat{A}) = \hat{D}$ be obtained from $A = [a_{ij}]$ by setting $\hat{a}_{ij} = a_{ij}$ if $1 \leq i, j \leq n$ (except that $\hat{a}_{12} = 0$). Let $\hat{a}_{ii} = -1$ for $i = n + 1, \ldots,$ $n + k - 2$ and let

$$\hat{a}_{1,n+1}\hat{a}_{n+1,n+2} \cdots \hat{a}_{n+k-3,n+k-2}\hat{a}_{n+k-2,2} = a_{12}.$$

All other entries in rows and columns $n + 1, \ldots, n + k - 2$ in $\hat{A}$ are zero.

Then the signs of principal minors of $\hat{A}$ are as follows:

$$
\begin{aligned}
\operatorname{sign}\det \hat{A}[\{1\}] &= -1\\
\operatorname{sign}\det \hat{A}[\{1, n+1\}] &= (-1)^2\\
\operatorname{sign}\det \hat{A}[\{1, n+1, n+2\}] &= (-1)^3\\
&\vdots\\
\operatorname{sign}\det \hat{A}[\{1, n+1, \ldots, n+k-2\}] &= (-1)^{k-1}
\end{aligned}
$$

$$
\begin{aligned}
&\operatorname{sign}\det \hat{A}[\{1, 2, n+1, \ldots, n+k-2\}]\\
&\quad = \operatorname{sign}[\hat{a}_{11}\hat{a}_{22}\prod_{i=n+1}^{n+k-2}\hat{a}_{ii} + (-1)^{k+1}\hat{a}_{21}\hat{a}_{1,n+1}\hat{a}_{n+1,n+2}\cdots\hat{a}_{n+k-3,n+k-2}\hat{a}_{n+k-2,2}], \text{ by}\\
&\qquad \text{expansion of the determinant of this submatrix of order } k \text{ about column 1}\\
&\quad = \operatorname{sign}[(-1)^{k-2}a_{11}a_{22} - (-1)^k a_{21}a_{12}]\\
&\quad = \operatorname{sign}[(-1)^k(a_{11}a_{22} - a_{21}a_{12})] = (-1)^k.
\end{aligned}
$$

For $i = 3, \ldots, n$, suppose that $\det A[\{1, \ldots, i\}] = a_{12}p_i(A) + q_i(A)$, where a_{12} does not occur in $q_i(A)$. If $a_{12}p_i(A) \neq 0$, then the arc (1,2) lies on at least one cycle on some subset of the vertices $\{1, \ldots, i\}$ and at least one of the corresponding cycle products occurs in $a_{12}p_i(A)$. By the construction of $\hat{A}$, for each such cycle product from a j-cycle in $D(A)$, there is an associated cycle product from a cycle of length $j + k - 2$ in $D(\hat{A})$. Thus

$$
\begin{aligned}
&\det \hat{A}[\{1, \ldots, i, n+1, \ldots, n+k-2\}]\\
&\quad = (-1)^{k-2}\hat{a}_{1,n+1}\hat{a}_{n+1,n+2}\cdots\hat{a}_{n+k-3,n+k-2}\hat{a}_{n+k-2,2}p_i(A) + \hat{a}_{n+1,n+1}\cdots\\
&\qquad \hat{a}_{n+k-2,n+k-2}q_i(A),
\end{aligned}
$$

where $(-1)^{k-2}$ occurs since a cycle of length j in $D(A)$ has been replaced by a cycle of length $j+k-2$ in $D(\hat{A})$. The second term follows since each term in $q_i(A)$ contains a factor a_{1t}, with $1 \leq t \leq n$, $t \neq 2$, which implies that no other entries of $\hat{A}$ from rows and columns $n+1, \ldots, n+k-2$ can multiply $q_i(A)$. Thus

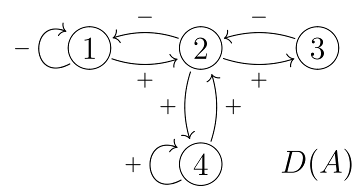
$$
\begin{aligned}
&\operatorname{sign}\det \hat{A}[\{1, \ldots, i, n+1, \ldots, n+k-2\}]\\
&\quad = \operatorname{sign}[(-1)^{k-2}\hat{a}_{1,n+1}\hat{a}_{n+1,n+2}\cdots\hat{a}_{n+k-3,n+k-2}\hat{a}_{n+k-2,2}p_i(A) + \hat{a}_{n+1,n+1}\cdots\\
&\qquad \hat{a}_{n+k-2,n+k-2}q_i(A)]\\
&\quad = \operatorname{sign}[(-1)^{k-2}a_{12}p_i(A) + (-1)^{k-2}q_i(A)]\\
&\quad = (-1)^{k-2}\operatorname{sign}\det A[\{1, \ldots, i\}]\\
&\quad = (-1)^{k-2+i}.
\end{aligned}
$$

Therefore $\hat{A}$ has the nest $(1, n+1, \ldots, n+k-2, 2, \ldots, n)$, and it follows by Theorem 2.5 that $\operatorname{sgn}(\hat{A})$ is PS. □

The construction in Theorem 3.1 is now illustrated for $n = 4$ and $k = 5$.

Example 3.2. Consider the following matrix and its digraph.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -4 & 0 & 1 & 1 \\ 0 & -4 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

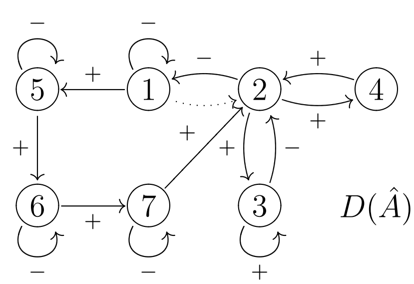


Matrix A has a leading nest with $a_{12}a_{21} = -4 \neq 0$. Note that

$$\sigma(A) = \{-0.1337 \pm 2.5182i, -0.1163 \pm 0.2552i\},$$

and thus A is stable. In fact, $\operatorname{sgn}(A)$ is equal to the minimally PS sign pattern $\mathcal{A}_{4,6}$ in Section A.3. The construction in Theorem 3.1 gives the following matrix and its digraph.

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -4 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$



Since

$$\sigma(\hat{A}) \approx \{-2.0625, 0.4087 \pm 1.7132i, -1.0479 \pm 1.1455i, -0.0796 \pm 0.1615i\},$$

$\hat{A}$ is not stable, although $\hat{A}$ contains the nest $(1, 5, 6, 7, 2, 3, 4)$. However, applying

Theorem 2.3 with the positive diagonal matrix $D = \text{diag}(1, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, 1, 1, 1)$ gives

$$D\hat{A} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{2}{5} & 0 & \frac{1}{10} & \frac{1}{10} & 0 & 0 & 0 \\ 0 & -\frac{2}{5} & \frac{1}{20} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \in \text{sgn}(\hat{A})$$

with

$$\sigma(D\hat{A}) \approx \{-1.6952, -1.1052 \pm 0.7324i, -0.0099 \pm 0.3872i, -0.0123 \pm 0.0311i\}.$$

Example 3.2 shows that the construction in Theorem 3.1 can create a minimally PS sign pattern of larger order from a minimally PS sign pattern of smaller order. The minimality of $\text{sgn}(A)$ can be seen since setting any nonzero diagonal entry in $\text{sgn}(\hat{A})$ to zero gives $\det X = 0$ for all $X \in Q(\text{sgn}(\hat{A}))$ and setting any nonzero off-diagonal entry in $\text{sgn}(\hat{A})$ to zero creates a reducible sign pattern that is not PS. It is not clear whether or not this construction always produces a larger order minimally PS sign pattern from one of smaller order.

3.2 Bordering Potentially Stable Sign Patterns

Another approach to constructing a PS sign pattern involves bordering a sign pattern known to be PS with additional rows and columns. Theorems 3.3, 3.7, 3.9 and 3.12 each involve bordering block upper triangular PS sign patterns with one row and column. The proofs of each of these relies on an application of Theorem 2.8. The constructive nature of these theorems has the added benefit that their proofs provide a method for determining a stable realization of a given construction, thereby confirming potential stability. Of further benefit, it is not necessary to begin with a stable realization of a lower order sign pattern to construct higher order PS sign patterns using these constructions.

A matrix $A = [a_{ij}]$ is a *lower Hessenberg* matrix if $a_{ij} = 0$ for $j - i \geq 2$. Furthermore, A is an *unreduced* lower Hessenberg matrix if in addition $a_{ij} \neq 0$ for $j - i = 1$. A sign pattern $\mathcal{A}$ is unreduced lower Hessenberg if A is a lower Hessenberg matrix for all $A \in Q(\mathcal{A})$. For the next set of results, let $N_i = n_1 + \cdots + n_i$ for $1 \le i \le k$ and let $*$ denote a fixed nonzero entry (either $+$ or $-$). Theorem 3.3 gives a generalization of the sign pattern described in [19, Theorem 2]. This generalization is defined by replacing each negative diagonal entry with a lower Hessenberg PS sign pattern and replacing the negative cycle through vertex 1 with a negative $(N_k + 1)$-cycle.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & \\ & \mathcal{A}_2 & \ddots & & \\ & & \ddots & \mathcal{E}_{k-1} & \\ & & & \mathcal{A}_k & e_k \\ f_1^T & & & & 0 \end{bmatrix},$$

$$\mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Figure 3.1: Sign patterns for Theorem 3.3

Theorem 3.3. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \dots, n_k$, respectively. There exists an order $N_k + 1$ unreduced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.1, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \le i \le k-1)$ and one nonzero entry, e_k and f_1 are the sign pattern column vectors of length n_k and n_1, respectively, each with one nonzero entry, and these $k+1$ nonzero entries are chosen so that the (N_k+1)-cycle $(1 \to \cdots \to N_k + 1 \to 1)$ in $D(\mathcal{B})$ is negative.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \le i \le k$ and $B \in Q(\mathcal{B})$. Then $\sigma(B[\{1, \dots, N_k\}]) = \sigma(A_1 \oplus \cdots \oplus A_k)$ and thus $B[\{1, \dots, N_k\}]$ is stable. Since $\det B = (-1)^{N_k+2} b_{12} \cdots b_{N_k, N_k+1} b_{N_k+1,1}$, it follows that sign $\det B = (-1)^{N_k+1}$. Thus, by Theorem 2.8, $\mathcal{B}$ is PS. □

The following example illustrates the construction in Theorem 3.3 when $k = 2$, $n_1 = 3$, $n_2 = 2$.

Example 3.4. If

$$B = \left[\begin{array}{c|c|c} A_1 & E_1 & 0 \\ \hline 0 & A_2 & e_2 \\ \hline f_1^T & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|cc|c} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-1.7113, -0.1443 \pm 1.8669i\}, \\ \sigma(A_2) &\approx \{-3.4142, -0.5858\} \text{ and} \\ \sigma(B) &\approx \{-0.1521 \pm 1.8643i, -3.4099, -0.1161, -0.4105, -1.7594\}. \end{aligned}$$

Note that if the nonzero entries in E_1 and e_2 are normalized to be 1, then the only entry in B that involves any choice is b_{61}.

Although the construction in Theorem 3.3 produces a PS sign pattern, the sign pattern is not guaranteed to be minimally PS. Example 3.5 illustrates the construction when $k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$ and each $\mathcal{A}_i$ is minimally PS.

Example 3.5. If

$$B = \left[\begin{array}{c|c|c|c} A_1 & E_1 & 0 & 0 \\ \hline 0 & A_2 & E_2 & 0 \\ \hline 0 & 0 & A_3 & e_3 \\ \hline f_1^T & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|c|c|c} -1 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & -1 & 1 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-0.5000 \pm 1.9365i\}, \\ \sigma(A_2) &= \sigma(A_3) = \{-1\} \text{ and} \\ \sigma(B) &\approx \{-1.3965, -0.5321 \pm 1.9300i, -0.2696 \pm 0.3255i\}. \end{aligned}$$

It can be seen that B is not minimally PS by setting the (1,1) entry to zero and obtaining matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$\sigma(X) \approx \{-1.3555, -0.0159 \pm 1.9795i, -0.3063 \pm 0.3073i\}.$$

The following graph construction is used in a number of subsequent theorems.

Construction 3.6. Let A_i be a matrix of order n_i, for $1 \le i \le k$. The digraph $D(B)$ is constructed from the digraphs $D(A_i)$ as follows:

- Vertices $1, \dots, N_1$ are the vertices in $D(A_1)$.
- Vertices $N_j + 1, N_j + 2, \dots, N_{j+1}$ are the vertices in $D(A_{j+1}), 1 \le j \le k-1$.
- $D(B)$ contains all signed arcs from each $D(A_i)$, appropriately relabelled.
- A new vertex $N_k + 1$ is added.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{G}_1 & & & \\ & \mathcal{A}_2 & \ddots & & \\ & & \ddots & \mathcal{G}_{k-1} & \\ & & & \mathcal{A}_k & g_k \\ h_1^T & & & & 0 \end{bmatrix},$$

$$\mathcal{G}_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \dots & 0 & * \end{bmatrix}, g_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } h_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix}$$

Figure 3.2: Sign patterns for Theorem 3.7

Theorem 3.7 is a new result that is similar to Theorem 3.3, but it relaxes the need for $\mathcal{A}_i$ to be lower Hessenberg. In Theorem 3.7, the condition that $\operatorname{sign}\det A_i[\{1, \dots, n_i - 1\}] = (-1)^{n_i - 1}$ can be assumed without loss of generality by Proposition 2.4.

Theorem 3.7. *Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are PS sign patterns of orders $n_1, \ldots, n_k$, respectively, with $n_i \geq 2$ and $A_i \in Q(\mathcal{A}_i)$ is stable $(1 \leq i \leq k)$. Without loss of generality, by permutation similarity suppose that* $\operatorname{sign}\det A_i[\{1, \ldots, n_i - 1\}] = (-1)^{n_i - 1}$. *There exists an order $N_k + 1$ PS sign pattern $\mathcal{B}$ as in Figure 3.2, where $\mathcal{G}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$ and one nonzero entry, g_k and h_1 are the sign pattern column vectors of length n_k and n_1, respectively, each with one nonzero entry, and these $k+1$ nonzero entries are chosen so that the $(k+1)$-cycle in $D(\mathcal{B})$ on which these entries lie is negative.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$ and $B \in Q(\mathcal{B})$. Then $\sigma(B[\{1, \ldots, N_k\}]) = \sigma(A_1 \oplus \cdots \oplus A_k)$ and it follows that $B[\{1, \ldots, N_k\}]$ is stable. Assume digraph $D(B)$ is constructed from the digraphs $D(A_i)$ as in Construction 3.6 along with a negative $(k+1)$-cycle with arcs $(N_1, N_2), \ldots, (N_{k-1}, N_k)$, $(N_k, N_k + 1)$, $(N_k + 1, N_1)$; thus $b_{N_1,N_2} \cdots b_{N_{k-1},N_k} b_{N_k,N_k+1} b_{N_k+1,N_1} < 0$ and

$$\begin{aligned} \operatorname{sign}\det B = & \ (-1)^{n_1 - 1} \cdots (-1)^{n_k - 1} (-1)^{k+1} \\ = & \ (-1)^{N_k + 1}. \end{aligned}$$

Since $B[\{1, \ldots, N_k\}]$ is stable and $\operatorname{sign}\det B = (-1)^{N_k+1}$, it follows by Theorem 2.8 that $\operatorname{sgn}(B)$ is PS. □

The next example illustrates the construction in Theorem 3.7 when $k = 2$ and $n_1 = n_2 = 4$.

Example 3.8. If

$$B = \left[\begin{array}{c|c|c} A_1 & G_1 & 0 \\ \hline 0 & A_2 & g_2 \\ \hline h_1^T & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|cccc|c} -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & \frac{1}{2} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -\frac{1}{1000} & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned}
\sigma(A_1) &\approx \{-0.1704 \pm 2.0694i, -0.3296 \pm 0.5960i\},\\
\sigma(A_2) &\approx \{-0.0716 \pm 0.4333i, -0.2065, -0.0502\} \text{ and}\\
\sigma(B) &\approx \{-0.1705 \pm 2.0694i, -0.3297 \pm 0.5939i, -0.0728 \pm 0.4339i, -0.2159,\\
&\quad -0.0191 \pm 0.0569i\}.
\end{aligned}$$

Suppose that $\mathcal{A}$ is the sign pattern corresponding to the digraph (**E**) (see Section A.4) given by Miyamichi in [19]. The construction in Theorem 3.9 generalizes sign pattern $\mathcal{A}$ in the following way. The 2×2 PS component $\mathcal{A}[\{1, 2\}]$ is replaced with a lower Hessenberg PS component $\mathcal{A}_1$ (as described in Theorem 3.9), all other diagonal entries in $\mathcal{A}$ are replaced with lower Hessenberg PS sign patterns and the positive $(n-1)$-cycle is replaced with a positive $(N_k - n_1 + 2)$-cycle.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & \\ & \mathcal{A}_2 & \ddots & & \\ & & \ddots & \mathcal{E}_{k-1} & \\ & & & \mathcal{A}_k & e_k \\ h_1^T & & & & 0 \end{bmatrix},$$

$$\mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \; e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } h_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix}$$

Figure 3.3: Sign patterns for Theorem 3.9

Theorem 3.9. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \dots, n_k$, respectively, with $n_1 \geq 2$. Let $A_i \in Q(\mathcal{A}_i)$ be stable $(1 \leq i \leq k)$ and suppose that* $\operatorname{sign}\det A_1[\{1, \dots, n_1 - 1\}] = \operatorname{sign}\det A_1$. *There exists an order $N_k + 1$ unreduced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.3, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$ and one nonzero entry, e_k and h_1 are sign pattern column vectors of length n_k and n_1, respectively, each with one nonzero entry, and these $k+1$ nonzero entries are chosen so that the $(N_k - n_1 + 2)$-cycle in $D(\mathcal{B})$ is positive.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$ and $B \in Q(\mathcal{B})$. Then $\sigma(B[\{1, \dots, N_k\}]) = \sigma(A_1 \oplus \cdots \oplus A_k)$ and it follows that $B[\{1, \dots, N_k\}]$ is stable.

Assume digraph $D(B)$ is constructed from the digraphs $D(A_i)$ as in Construction 3.6 along with $k+1$ arcs $(N_1, N_1+1), \ldots, (N_k, N_k+1), (N_k+1, N_1)$ signed so that the $(N_k - n_1 + 2)$-cycle is positive. Then

$$\begin{aligned}
\operatorname{sign}\det B = &\ \operatorname{sign}\det A_1[\{1,\ldots,n_1-1\}](-1)^{N_k-n_1+1}, \text{ where the sign}(-1)^{N_k-n_1+1} \\
&\ \text{is contributed by the positive cycle of length } N_k - n_1 + 2 \\
= &\ \operatorname{sign}\det A_1(-1)^{N_k-n_1+1} \\
= &\ (-1)^{n_1}(-1)^{N_k-n_1+1} = (-1)^{N_k+1}.
\end{aligned}$$

Thus, by Theorem 2.8 it follows that $\operatorname{sgn}(B)$ is PS. □

The construction in Theorem 3.9 is now illustrated for $k=2$, $n_1 = 2$ and $n_2 = 3$.

Example 3.10. If

$$B = \left[\begin{array}{c|c|c} A_1 & E_1 & 0 \\ \hline 0 & A_2 & e_2 \\ \hline h_1^T & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|ccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ -4 & -6 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \\ \hline 0 & \frac{1}{10} & 0 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned}
\sigma(A_1) &\approx \{-0.4384, -4.5616\}, \\
\sigma(A_2) &\approx \{-0.5698, -0.2151 \pm 1.3071i\} \text{ and} \\
\sigma(B) &\approx \{-4.5612, -0.0790, -0.2192 \pm 1.3119i, -0.2254, -0.6960\}.
\end{aligned}$$

The next example shows that bordering a sign pattern that is not minimally PS can produce a minimally PS sign pattern. Example 3.11 illustrates the construction in Theorem 3.9 when $k=1$ and $n_1 = 2$.

Example 3.11. If

$$B = \left[\begin{array}{c|c} A_1 & e_1 \\ \hline h_1^T & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -4 & -6 & 1 \\ \hline 0 & 1 & 0 \end{array}\right],$$

then

$$\begin{aligned}
\sigma(A_1) &\approx \{-0.4384, -4.5616\} \text{ and} \\
\sigma(B) &\approx \{-4.8360, -0.0820 \pm 04473i\}.
\end{aligned}$$

The sign pattern $\text{sgn}(A_1)$ is not minimally PS since replacing the (1,1) entry produces a minimally PS 2×2 sign pattern equivalent to $\mathcal{A}_{2,1}$. However, the sign pattern $\mathcal{B} = \text{sgn}(B)$ obtained by bordering $\text{sgn}(A_1)$ according to the construction in Theorem 3.9 is minimally PS as $\text{sgn}(B)$ is equal to the pattern $\mathcal{A}_{3,2}$ given in Section A.2.

Theorem 3.12 is a new result that is similar to Theorem 3.9, but it relaxes the requirement that $\mathcal{A}_i$ be lower Hessenberg.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{G}_1 & & & \\ & \mathcal{A}_2 & \ddots & & \\ & & \ddots & \mathcal{G}_{k-1} & \\ & & & \mathcal{A}_k & \mathcal{g}_k \\ \mathcal{h}_1^T & & & & 0 \end{bmatrix},$$

$$\mathcal{G}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & 0 & 0 \\ 0 & \dots & 0 & * \end{bmatrix}, \mathcal{g}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } \mathcal{h}_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix}$$

Figure 3.4: Sign patterns for Theorem 3.12

Theorem 3.12. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are PS sign patterns of orders $n_1, \dots, n_k$, respectively, with $n_i \geq 2$, and let $A_i \in Q(\mathcal{A}_i)$ be stable $(1 \leq i \leq k)$. Suppose that* $\text{sign}\det A_1[\{1, \dots, n_1 - 1\}] = (-1)^{n_1}$ *and without loss of generality by permutation similarity suppose sign* $\det A_i[\{1, \dots, n_i - 1\}] = (-1)^{n_i - 1}$ $(2 \leq i \leq k)$. *There exists an order $N_k + 1$ PS sign pattern $\mathcal{B}$ as in Figure 3.4, where $\mathcal{G}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$ and one nonzero entry, $\mathcal{g}_k$ and $\mathcal{h}_1$ are sign pattern column vectors of length n_k and n_1, respectively, each with one nonzero entry, and these $k + 1$ nonzero entries are chosen so that the $(k+1)$-cycle in $D(\mathcal{B})$ on which these entries lie is positive.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$ and $B \in Q(\mathcal{B})$. Assume digraph $D(B)$ is constructed from the digraphs $D(A_i)$ as in Construction 3.6 along with $k+1$ arcs $(N_1, N_2), \dots, (N_{k-1}, N_k), (N_k, N_k + 1), (N_k + 1, N_1)$ signed so that this $(k+1)$-cycle is positive. Thus,

$$\begin{aligned}\operatorname{sign}\det B = &\ \operatorname{sign}\det A_1[\{1,\ldots,n_1-1\}](-1)^{n_2-1}\cdots(-1)^{n_k-1}(-1)^k, \text{ where the} \\ &\ \operatorname{sign}(-1)^k \text{ is contributed by the positive } (k+1)\text{-cycle.} \\ = &\ (-1)^{n_1}(-1)^{n_2-1}\cdots(-1)^{n_k-1}(-1)^k \\ = &\ (-1)^{N_k+1}\end{aligned}$$

By Theorem 2.8, since $\sigma(B[\{1,\ldots,N_k\}]) = \sigma(A_1 \oplus \cdots \oplus A_k)$ and $\operatorname{sign}\det B = (-1)^{N_k+1}$, it follows that $\mathcal{B}$ is PS. □

The construction in Theorem 3.12 is illustrated in the next example for $k = 2$, $n_1 = 2$, and $n_2 = 4$.

Example 3.13. If

$$B = \left[\begin{array}{c|c|c} A_1 & G_1 & 0 \\ \hline 0 & A_2 & g_2 \\ \hline h_1^T & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cccc|c} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ -4 & -6 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & -2 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & \frac{1}{10} & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned}\sigma(A_1) &\approx \{-0.4384, -4.5616\}, \\ \sigma(A_2) &\approx \{-0.3296 \pm 0.5960i, -0.1704 \pm 2.0694i\} \text{ and} \\ \sigma(B) &\approx \{-4.5188, -0.1313 \pm 2.1197i, -1.0205, -0.0726, -0.0628 \pm 0.5107i\}.\end{aligned}$$

The next two theorems involve bordering block upper triangular PS sign patterns with two rows and columns. Similar to the above theorems involving bordering PS sign patterns with one row and column, the theorems are constructive in nature and the proofs provide a method for determining a stable realization of a given sign pattern, although stable realizations are not required to show potential stability. However, the proofs of Theorems 3.14 and 3.16 differ greatly from the previous proofs in that specific knowledge of the characteristic polynomial of a given bordered stable matrix is required in order to construct a higher order stable matrix by bordering. The proofs for both theorems use Theorem 2.2 and some analysis of the characteristic polynomial of the constructed higher order matrix. The construction in Theorem 3.14 generalizes the sign pattern associated with the digraph (**A**) (see Section A.4) given by Miyamichi in [19]. This generalization is defined by replacing each negative diagonal entry with a lower Hessenberg PS sign pattern and replacing the $(n-1)$-cycle with an (N_k+1)-cycle.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & & \\ & \mathcal{A}_2 & \ddots & & & \\ & & \ddots & \mathcal{E}_{k-1} & & \\ & & & \mathcal{A}_k & \mathcal{e}_k & \\ \mathcal{f}_1^T & & & & 0 & * \\ & & & & * & 0 \end{bmatrix},$$

$$\mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \mathcal{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } \mathcal{f}_1 = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Figure 3.5: Sign patterns for Theorem 3.14

Theorem 3.14. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \dots, n_k$, respectively. There exists an order $N_k + 2$ $(N_k \geq 2)$ unreduced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.5, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$, $\mathcal{e}_k$ and $\mathcal{f}_1$ are the sign pattern column vectors of length n_k and n_1, respectively, and for $B \in Q(\mathcal{B}), b_{N_k+1,N_k+2}b_{N_k+2,N_k+1} < 0$.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$, $e_k \in Q(\mathcal{e}_k)$, $f_1 \in Q(\mathcal{f}_1)$, $E_i \in Q(\mathcal{E}_i)$ for $1 \leq i \leq k-1$ and

$$B = \begin{bmatrix} A_1 & E_1 & & & & \\ & A_2 & \ddots & & & \\ & & \ddots & E_{k-1} & & \\ & & & A_k & e_k & \\ f_1^T & & & & 0 & b_{N_k+1,N_k+2} \\ & & & & b_{N_k+2,N_k+1} & 0 \end{bmatrix},$$

where $b_{N_k+1,N_k+2}b_{N_k+2,N_k+1} = C_2 < 0$. Note that $b_{12} \dots b_{N_k,N_k+1}b_{N_k+1,1} = C_{N_k+1} \neq 0$.

The characteristic polynomial of B is given by

$$\det(\lambda I_{N_k+2} - B)$$
$$= \begin{vmatrix} \lambda I_{n_1} - A_1 & -E_1 & & & & & \\ & \lambda I_{n_2} - A_2 & \ddots & & & & \\ & & \ddots & -E_{k-1} & & & \\ & & & \lambda I_{n_k} - A_k & -e_k & & \\ -f_1^T & & & & \lambda & -b_{N_k+1,N_k+2} \\ & & & & -b_{N_k+2,N_k+1} & \lambda \end{vmatrix}.$$

Expanding $\det(\lambda I_{N_k+2} - B)$ about the last row gives

$$\begin{aligned} \det(\lambda I_{N_k+2} - B) = \;& \lambda[\lambda \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) + (-1)^{N_k+2}(-1)^{N_k+1} C_{N_k+1}] \\ & -C_2 \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i), \text{where the sign} (-1)^{N_k+2} \text{ is} \\ & \text{contributed by the } (N_k+1, 1) \text{ position and the sign} \\ & (-1)^{N_k+1} \text{ is contributed by } N_k + 1 \text{ negative signs from} \\ & \text{the cycle in } D(-B) \\ = \;& (\lambda^2 - C_2) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+1}\lambda, \end{aligned}$$

which is of the form (**A**) in [19, Lemma 11]. As in [19, Lemma 11], by using Theorem 2.2, C_2 and C_{N_k+1} can be chosen (dependent on the stable matrices $A_1, \ldots, A_k$) to make B stable. Thus, $\mathcal{B}$ is PS. Note that if $C_{N_k+1} < 0$, then $\mathcal{B}$ is PS also for $k = 1$, $N_1 = 1$ (i.e., $N_k = 1$). This can be seen by the sign pattern $\mathcal{A}_{3,1}$ in Section A.2. $\square$

The following example shows the construction in Theorem 3.14 when $k = 2$, $n_1 = 2$, $n_2 = 3$ and $C_{N_k+1} > 0$.

Example 3.15. If

$$B = \left[\begin{array}{c|c|cc} A_1 & E_1 & 0 & 0 \\ \hline 0 & A_2 & e_2 & 0 \\ \hline f_1^T & 0 & 0 & * \\ 0 & 0 & * & 0 \end{array}\right] = \left[\begin{array}{cc|ccc|cc} -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &= \{-1, -2\}, \\ \sigma(A_2) &\approx \{-0.5698, -0.2151 \pm 1.3071i\} \text{ and} \\ \sigma(B) &\approx \{-2.0500, -0.0934 \pm 1.3236i, -0.1433 \pm 0.8609i, -0.7383 \pm 0.4270i\}. \end{aligned}$$

The following theorem is similar to Theorem 3.14. The subtle difference in Theorem 3.16 is that f_1^T is now in row $N_k + 2$ (rather than in row $N_k + 1$). Theorem 3.16 gives a generalization of the sign pattern corresponding to digraph (**D**) (see Section A.4) given by Miyamichi in [19]. This generalization is defined by replacing each negative diagonal entry with a lower Hessenberg PS sign pattern and replacing the n-cycle with an $(N_k + 2)$-cycle.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & & \\ & \mathcal{A}_2 & \ddots & & & \\ & & \ddots & \mathcal{E}_{k-1} & & \\ & & & \mathcal{A}_k & e_k & \\ & & & & 0 & * \\ f_1^T & & & & * & 0 \end{bmatrix},$$

$$\mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \; e_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Figure 3.6: Sign patterns for Theorem 3.16

Theorem 3.16. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \dots, n_k$, respectively. There exists an order $N_k + 2$ ($N_k \geq 3$) unre-*

duced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.6, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \le i \le k-1)$, $\mathcal{e}_k$ and $\mathcal{f}_1$ are the sign pattern column vectors of length n_k and n_1, respectively, and for $B \in Q(\mathcal{B}), b_{N_k+1,N_k+2}b_{N_k+2,N_k+1} < 0$.

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \le i \le k$, $e_k \in Q(\mathcal{e}_k)$, $f_1 \in Q(\mathcal{f}_1)$, $E_i \in Q(\mathcal{E}_i)$ for $1 \le i \le k-1$ and

$$B = \begin{bmatrix} A_1 & E_1 & & & & & \\ & A_2 & \ddots & & & & \\ & & \ddots & E_{k-1} & & & \\ & & & A_k & & e_k & \\ & & & & & 0 & b_{N_k+1,N_k+2} \\ f_1^T & & & & b_{N_k+2,N_k+1} & & 0 \end{bmatrix},$$

where $b_{N_k+1,N_k+2}b_{N_k+2,N_k+1} = C_2 < 0$, and $b_{12}\dots b_{N_k+1,N_k+2}b_{N_k+2,1} = C_{N_k+2}$ is nonzero. The characteristic polynomial of B is given by

$$\det(\lambda I_{N_k+2} - B) = \begin{vmatrix} \lambda I_{n_1} - A_1 & -E_1 & & & & & \\ & \lambda I_{n_2} - A_2 & \ddots & & & & \\ & & \ddots & -E_{k-1} & & & \\ & & & \lambda I_{n_k} - A_k & & -e_k & \\ & & & & & \lambda & -b_{N_k+1,N_k+2} \\ -f_1^T & & & & -b_{N_k+2,N_k+1} & & \lambda \end{vmatrix}.$$

Expanding $\det(\lambda I_{N_k+2} - B)$ about the last column gives

$$\begin{aligned} \det(\lambda I_{N_k+2} - B) = \;& \lambda[\lambda \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i)] - C_2 \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) \\ & +(-1)^{N_k+3}(-1)^{N_k+2}C_{N_k+2}, \text{ where the sign}(-1)^{N_k+3} \\ & \text{is contributed by the } (N_k+2,1) \text{ position and the sign} \\ & (-1)^{N_k+2} \text{ is contributed by } N_k+2 \text{ negative signs from} \\ & \text{the cycle in } D(-B) \\ = \;& (\lambda^2 - C_2) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+2}, \end{aligned}$$

which is of the form (**D**) in [19, Lemma 11]. As in [19, Lemma 11], by using Theorem 2.2, C_2 and C_{N_k+2} can be chosen (dependent on the stable matrices $A_1, \ldots, A_k$) to make B stable. Thus, $\mathcal{B}$ is PS. Note that if $C_{N_k+2} > 0$, then $\mathcal{B}$ is PS for all N_k (see $\mathcal{A}_{3,4}$ in Section A.2 and Theorem 2.9). □

The following example shows the construction in Theorem 3.16 when $k = 2$, $n_1 = 3$, $n_2 = 2$ and $C_{N_k+2} < 0$.

Example 3.17. If

$$B = \left[\begin{array}{c|c|cc} A_1 & E_1 & 0 & 0 \\ \hline 0 & A_2 & e_2 & 0 \\ \hline 0 & 0 & 0 & * \\ f_1^T & 0 & * & 0 \end{array}\right] = \left[\begin{array}{ccc|cc|cc} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & -3 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-1.7113, -0.1443 \pm 1.8669i\}, \\ \sigma(A_2) &\approx \{-3.4142, -0.5858\} \text{ and} \\ \sigma(B) &\approx \{-0.1373 \pm 1.8855i, -0.0011 \pm 1.7092i, -3.4152, -1.6962, -0.6118\}. \end{aligned}$$

Similar to Example 3.5, the following example shows the construction in Theorem 3.16 need not create a minimally PS sign pattern. Example 3.18 shows the construction when $k = 3$, $n_1 = n_2 = n_3 = 1$, $C_{N_k+2} < 0$ and each $\mathcal{A}_i$ is minimally PS.

Example 3.18. If

$$B = \left[\begin{array}{c|c|c|cc} A_1 & E_1 & 0 & 0 & 0 \\ \hline 0 & A_2 & E_2 & 0 & 0 \\ \hline 0 & 0 & A_3 & e_3 & 0 \\ \hline 0 & 0 & 0 & 0 & * \\ f_1^T & 0 & 0 & * & 0 \end{array}\right] = \left[\begin{array}{c|c|c|cc} -1 & 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -4 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &= \sigma(A_2) = \sigma(A_3) = \{-1\} \\ \sigma(B) &\approx \{-1.5395, -0.0040 \pm 1.9772i, -0.7263 \pm 0.5507i\}. \end{aligned}$$

It can be seen that B is not minimally PS by setting the (1,1) entry to 0 and obtaining matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -4 & 0 \end{bmatrix}$$

with

$$\sigma(X) \approx \{-1.3555, -0.0159 \pm 1.9795i, -0.3063 \pm 0.3073i\}.$$

A polynomial is called *stable* if all of its zeros lie in the open left half plane. The next lemma is used in the proof of Theorem 3.20.

Lemma 3.19. *If $f(x)$ is a monic stable polynomial of degree $n \geq 3$ and $j \geq 1$, then $F(x) = \prod_{i=1}^{j}(x^2 + T_i)f(x) + S$ is stable for appropriate choices of $T_i > 0$ and when*

i. $S > 0$ and $j \leq \lfloor \frac{n-1}{2} \rfloor$; or

ii. $S < 0$ and $j \leq \lfloor \frac{n}{2} \rfloor$.

Proof. If $f(x)$ is as stated, then by Theorem 2.2, $f(x) = h(x^2) + xg(x^2)$ with the zeros of $h(u)$ and $g(u)$ properly interlaced as follows. Let $\alpha_1, \ldots, \alpha_{\lfloor \frac{n}{2} \rfloor}$ be the zeros of $h(u)$ and let $\beta_1, \ldots, \beta_{\lfloor \frac{n-1}{2} \rfloor}$ be the zeros of $g(u)$ such that $0 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \cdots$. The polynomial $F(x)$ can be written

$$\begin{aligned} F(x) = & \prod_{i=1}^{j}(x^2 + T_i)f(x) + S \\ = & \prod_{i=1}^{j}(x^2 + T_i)h(x^2) + S + x\prod_{i=1}^{j}(x^2 + T_i)g(x^2) \\ = & H(x^2) + xG(x^2). \end{aligned}$$

For $S > 0$, if n is odd and $j = \lfloor \frac{n-1}{2} \rfloor$, then let $\alpha_{j+1} = -\infty$. Let $\beta_1 > -T_1 > \alpha_2$, $\beta_2 > -T_2 > \alpha_3$, $\ldots$, $\beta_j > -T_j > \alpha_{j+1}$. The zeros of $G(u)$ are $\beta_1 > -T_1 > \beta_2 > -T_2 > \cdots \beta_j > -T_j > \beta_{j+1} > \cdots > \beta_{\lfloor \frac{n-1}{2} \rfloor}$ where these zeros that are less than $-T_j$ exist only if $\lfloor \frac{n-1}{2} \rfloor \geq j+1$. For small $S > 0$, the zeros of $H(u)$ are $\alpha_1 - \epsilon_1 > -T_1 + \omega_1 > \alpha_2 - \epsilon_2 > -T_2 + \omega_2 > \cdots > \alpha_j - \epsilon_j > -T_j + \omega_j > \alpha_{j+1} - \epsilon_{j+1} > \cdots > \alpha_{\lfloor \frac{n}{2} \rfloor} - \epsilon_{\lfloor \frac{n}{2} \rfloor}$ for some $\epsilon_k > 0$ and $\omega_\ell > 0$ ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, and $1 \leq \ell \leq j$) where these zeros that are less than $-T_j + \omega_j$ exist only if $\lfloor \frac{n}{2} \rfloor \geq j+1$. If $S > 0$ is chosen sufficiently small, then ϵ_k and ω_ℓ are sufficiently small such that the zeros of $H(u)$ and $G(u)$ are interlaced; that is, $0 > \alpha_1 - \epsilon_1 > \beta_1 > -T_1 + \omega_1 > -T_1 > \alpha_2 - \epsilon_2 > \beta_2 > -T_2 + \omega_2 > -T_2 > \cdots > \alpha_j - \epsilon_j > \beta_j > -T_j + \omega_j > -T_j > \alpha_{j+1} - \epsilon_{j+1} > \beta_{j+1} > \cdots$ (as described in Theorem 2.2). Thus $F(x)$ is stable.

Similarly, for $S < 0$, if n is even and $j = \frac{n}{2}$, then let $\beta_j = -\infty$. Let $\alpha_1 > -T_1 > \beta_1, \alpha_2 > -T_2 > \beta_2, \ldots, \alpha_j > -T_j > \beta_j$. The zeros of $G(u)$ are $-T_1 > \beta_1 > -T_2 > \beta_2 \cdots, > -T_j > \beta_j > \cdots > \beta_{\lfloor \frac{n-1}{2} \rfloor}$ where these zeros that are less than $-T_j$ exist only if $\lfloor \frac{n-1}{2} \rfloor \geq j+1$. For small $S < 0$, the zeros of $H(u)$ are $\alpha_1 + \epsilon_1 > -T_1 - \omega_1 > \alpha_2 + \epsilon_2 > -T_2 - \omega_2 > \ldots, \alpha_j + \epsilon_j > -T_j - \omega_j > \alpha_{j+1} + \epsilon_{j+1} > \cdots > \alpha_{\lfloor \frac{n}{2} \rfloor} + \epsilon_{\lfloor \frac{n}{2} \rfloor}$ for some $\epsilon_k > 0$ and $\omega_\ell > 0$ ($1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $1 \leq \ell \leq j$) where these zeros that are less than $-T_j - \omega_j$ exist only if $\lfloor \frac{n}{2} \rfloor \geq j+1$. If $S < 0$ is chosen sufficiently small in magnitude then ϵ_k and ω_ℓ are sufficiently small such that the zeros of $H(u)$ and $G(u)$ are interlaced; that is, $0 > \alpha_1 + \epsilon_1 > -T_1 > -T_1 - \omega_1 > \beta_1 > \alpha_2 + \epsilon_2 > -T_2 > -T_2 - \omega_2 > \beta_2 > \cdots > \alpha_j + \epsilon_j > -T_j > -T_j - \omega_j > \beta_j > \alpha_{j+1} + \epsilon_{j+1} > \beta_{j+1} > \cdots$ (as described in Theorem 2.2). Thus $F(x)$ is stable. $\square$

This next theorem is a generalization of Theorem 3.16 in which the negative 2-cycle in $D(\mathcal{B})$ is replaced by either j weakly connected negative 2-cycles that correspond to $\mathcal{X}_1$ in Figure 3.8, or j negative cycles, one of each length 2, 4,..., $2j$, that correspond to $\mathcal{X}_2$ in Figure 3.8. Note that when $j = 1$, $\mathcal{X}_1 = \mathcal{X}_2$. Figure 3.7 shows an example of the digraph that corresponds to $\mathcal{X}_2$ for $j = 3$.

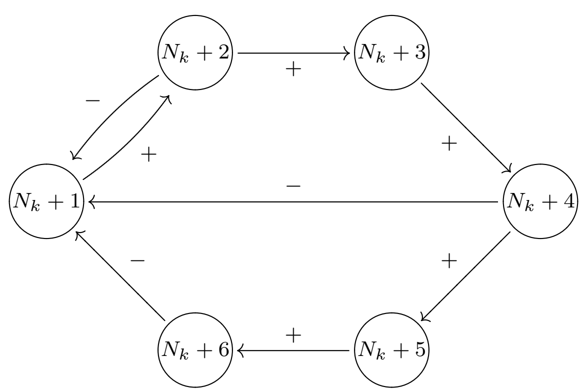


Figure 3.7: Example Digraph of X_2 in Theorem 3.20

Theorem 3.20. *Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \ldots, n_k$, respectively. There exists an order $N_k + 2j$ ($N_k \geq 3$ and $1 \leq$*

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & & \\ & \mathcal{A}_2 & \ddots & & & \\ & & \ddots & \mathcal{E}_{k-1} & & \\ & & & \mathcal{A}_k & \mathcal{F}_k \\ \mathcal{G}_1 & & & & \mathcal{X}_\ell \end{bmatrix}, \mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{F}_k = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \mathcal{G}_1 = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{X}_1 = \begin{bmatrix} 0 & * & 0 & \dots & & & 0 \\ * & 0 & * & \ddots & & & \vdots \\ 0 & 0 & 0 & * & & & \\ \vdots & \ddots & * & 0 & * & & \\ & & & \ddots & \ddots & \ddots & 0 \\ & & & & 0 & 0 & * \\ 0 & \dots & & & 0 & * & 0 \end{bmatrix}, \mathcal{X}_2 = \begin{bmatrix} 0 & * & 0 & \dots & & & 0 \\ * & 0 & * & \ddots & & & \vdots \\ 0 & 0 & 0 & * & & & \\ * & \vdots & \ddots & 0 & * & & \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & & & & & 0 & * \\ * & 0 & \dots & & & 0 & 0 \end{bmatrix}$$

Figure 3.8: Sign patterns for Theorem 3.20

$j \leq \lfloor \frac{N_k-1}{2} \rfloor$) *unreduced lower Hessenberg PS sign pattern* $\mathcal{B}$ *as in Figure 3.8, where* $\mathcal{E}_i$ *has dimensions* $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$*,* $\mathcal{F}_k$ *has dimensions* $n_k \times 2j$*,* $\mathcal{G}_1$ *has dimensions* $2j \times n_1$*, and* $\mathcal{X}_\ell$ *is equal to either* $\mathcal{X}_1$ *or* $\mathcal{X}_2$*. If* $\mathcal{X}_\ell = \mathcal{X}_1$*, then for* $B \in Q(\mathcal{B})$*,* $b_{N_k+2i-1,N_k+2i} b_{N_k+2i,N_k+2i-1} < 0$ $(1 \leq i \leq j)$*; whereas if* $\mathcal{X}_\ell = \mathcal{X}_2$*, then for* $B \in Q(\mathcal{B})$*,* $b_{N_k+1,N_k+2}\ b_{N_k+2,N_k+3} \cdots\ b_{N_k+2r-1,N_k+2r} b_{N_k+2r,N_k+1} < 0$ $(1 \leq r \leq j)$*.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$, $F_k \in Q(\mathcal{F}_k)$, $G_1 \in Q(\mathcal{G}_1)$, $E_i \in Q(\mathcal{E}_i)$ for $1 \leq i \leq k-1$, and $X_\ell = X_1 \in Q(\mathcal{X}_1)$. Construct B as in Figure 3.8, where $b_{N_k+2i-1,N_k+2i} b_{N_k+2i,N_k+2i-1} = C_{2_i} < 0$, for $1 \leq i \leq j$. Note that $b_{12} b_{23} \cdots b_{N_k+2j-1,N_k+2j} b_{N_k+2j,1} = C_{N_k+2j} \neq 0$.

Consideration of the characteristic polynomial of B, $\det(\lambda I_{N_k+2j} - B)$, and expansion about the last row give

$$\begin{aligned}\det(\lambda I_{N_k+2j} - B) = \; & \lambda[\lambda(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_{j-1}}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i)] \\ & -C_{2_j}(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_{j-1}}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) \\ & +(-1)^{N_k+2j+1}(-1)^{N_k+2j} C_{N_k+2j}, \text{ where the sign} \\ & (-1)^{N_k+2j+1} \text{ is contributed by the } (N_k+2j, 1) \text{ position} \\ & \text{and the sign} (-1)^{N_k+2j} \text{ is contributed by } N_k + 2j \\ & \text{negative signs from the cycle in } D(-B) \\ = \; & (\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_j}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+2j},\end{aligned}$$

and by Lemma 3.19, $C_{2_1}, \ldots, C_{2_j}$ and C_{N_k+2j} can be chosen (dependent on the stable matrices $A_1, \ldots, A_k$) so that B is stable. Thus, $\mathcal{B}$ is PS.

When $\mathcal{X}_\ell = \mathcal{X}_2$, construct B as in Figure 3.8, where

$$b_{N_k+1,N_k+2} b_{N_k+2,N_k+3} \cdots b_{N_k+2r-1,N_k+2r} b_{N_k+2r,N_k+1} = C_{2r} < 0 \; (1 \leq r \leq j).$$

Expansion about the last row of $\det(\lambda I_{N_k+2j} - B)$ gives

$$\det(\lambda I_{N_k+2j} - B) = (\lambda^{2j} - C_2\lambda^{2j-2} - \cdots - C_{2j-2}\lambda^2 - C_{2j}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+2j}.$$

Since the magnitudes $\mid C_{2r} \mid$ of the even ordered principal minors in $X_2 \in Q(\mathcal{X}_2)$ can be chosen arbitrarily, the polynomial of degree $2j$ can be factored as $(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_j})$ for some constants $C_{2_i} < 0$ and the proof follows as for the $\mathcal{X}_\ell = \mathcal{X}_1$ case. □

Note, by Theorem 3.19, if $C_{N_k+2j} < 0$ then the above result is true for $j \leq \lfloor \frac{N_k}{2} \rfloor$. The next example illustrates the construction in Theorem 3.20 when $k = 2$, $n_1 = 3$, $n_2 = 2$, $j = 2$, $\mathcal{X}_\ell = \mathcal{X}_1$ and $C_{N_k+4} > 0$.

Example 3.21. If

$$B = \left[\begin{array}{c|c|c} A_1 & E_1 & 0 \\ \hline 0 & A_2 & F_2 \\ \hline G_1 & 0 & X_1 \end{array}\right] = \left[\begin{array}{ccc|cc|cccc} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 15 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-1.7113, -0.1443 \pm 1.8669i\}, \\ \sigma(A_2) &\approx \{-3.4142, -0.5858\} \text{ and} \\ \sigma(B) &\approx \{-0.0142 \pm 2.2553i, -0.0646 \pm 1.5107i, -0.0997 \pm 1.7631i, -3.4132, \\ &\qquad -1.7430, -0.4866\}. \end{aligned}$$

and thus $\text{sgn}(B)$ is PS.

Lemma 3.22. *If $f(x)$ is a monic stable polynomial of degree $n \geq 3$, then $F(x) = x(x^2+T)f(x) + S$ is stable for appropriate choices of $S > 0$ and $T > 0$.*

Proof. If $f(x)$ is as above, then by Theorem 2.2, $f(x) = h(x^2) + xg(x^2)$ where the zeros of $h(u)$ and $g(u)$ are properly interlaced. Let α_1 be the zero of $h(u)$ closest to 0 and let β_1 be the zero of $g(u)$ closest to 0 with $0 > \alpha_1 > \beta_1$. The polynomial $F(x)$ can be written

$$\begin{aligned} F(x) &= x(x^2+T)f(x) + S \\ &= x(x^2+T)h(x^2) + x^2(x^2+T)g(x^2) + S \\ &= xG(x^2) + H(x^2). \end{aligned}$$

If $\beta_1 < -T < \alpha_1$ and S is sufficiently small, then the zeros of $H(u)$ and $G(u)$ are simple, negative and interlaced (as described in Theorem 2.2). Thus $F(x)$ is stable. □

The next theorem involves bordering block upper triangular PS sign patterns with three rows and columns. Similar to all of the preceding theorems involving bordering, the theorem is constructive in nature with a proof that provides a method for determining a stable realization of a given sign pattern, although stable realizations are not required to show potential stability. However, as in Theorems 3.14 and 3.16, specific knowledge of the characteristic polynomial of a given bordered stable matrix is required in order to construct a higher order stable matrix by this bordering procedure.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & & & & \\ & \mathcal{A}_2 & \ddots & & & & & \\ & & \ddots & \mathcal{E}_{k-1} & & & & \\ & & & \mathcal{A}_k & \mathcal{e}_k & & & \\ & & & & 0 & * & & \\ & & & & & 0 & * & \\ f_1^T & & & & & * & 0 & \end{bmatrix},$$

$$\mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \mathcal{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ and } f_1 = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Figure 3.9: Sign patterns for Theorem 3.23

Theorem 3.23. *Suppose $\mathcal{A}_1, \dots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \dots, n_k$, respectively. There exists an order $N_k + 3$ $(N_k \geq 3)$ unreduced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.9, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ $(1 \leq i \leq k-1)$ and one nonzero entry, $\mathcal{e}_k$ and f_1 are the sign pattern column vectors of length n_k and n_1, respectively, each with one nonzero entry, these $k+1$ entries are chosen so that the (N_k+3)-cycle in $D(\mathcal{B})$ is negative, and if $B \in Q(\mathcal{B})$, then $b_{N_k+2,N_k+3}b_{N_k+3,N_k+2} < 0$.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$, $e_k \in Q(\mathcal{e}_k)$, $f_1 \in Q(f_1)$, $E_i \in Q(\mathcal{E}_i)$

for $1 \le i \le k-1$ and

$$B = \begin{bmatrix} A_1 & E_1 & & & & & & \\ & A_2 & \ddots & & & & & \\ & & \ddots & E_{k-1} & & & & \\ & & & A_k & e_k & & & \\ & & & & 0 & b_{N_k+1,N_k+2} & & \\ & & & & & 0 & b_{N_k+2,N_k+3} & \\ f_1^T & & & & & b_{N_k+3,N_k+2} & 0 & \end{bmatrix},$$

where $b_{N_k+2,N_k+3}b_{N_k+3,N_k+2} = C_2 < 0$ and $b_{12} \ldots b_{N_k+2,N_k+3}b_{N_k+3,1} = C_{N_k+3} < 0$. The characteristic polynomial of B is given by

$$\det(\lambda I_{N_k+3} - B) = \begin{vmatrix} \lambda I_{n_1} - A_1 & -E_1 & & & & & \\ & \lambda I_{n_2} - A_2 & \ddots & & & & \\ & & \ddots & -E_{k-1} & & & \\ & & & \lambda I_{n_k} - A_k & -e_k & & \\ & & & & \lambda & -b_{N_k+1,N_k+2} & \\ & & & & & \lambda & -b_{N_k+2,N_k+3} \\ -f_1^T & & & & & -b_{N_k+3,N_k+2} & \lambda \end{vmatrix}.$$

Expanding $\det(\lambda I_{N_k+3} - B)$ about the last column gives

$$\begin{aligned} \det(\lambda I_{N_k+3} - B) = \;& \lambda[\lambda^2 \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i)] - \lambda C_2 \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) \\ & +(-1)^{N_k+4}(-1)^{N_k+3}C_{N_k+3}, \text{ where the sign} (-1)^{N_k+4} \\ & \text{is contributed by the } (N_k+3, 1) \text{ position and the sign} \\ & (-1)^{N_k+3} \text{ is contributed by } N_k+3 \text{ negative signs from} \\ & \text{the cycle in } D(-B) \\ = \;& \lambda(\lambda^2 - C_2) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+3}, \end{aligned}$$

and by Lemma 3.22, C_2 and C_{N_k+3} can be chosen (dependent on the stable matrices $A_1, \ldots, A_k$) so that B is stable. Thus, $\mathcal{B}$ is PS. □

The following example shows the construction in Theorem 3.23 when $k = 2$,

$n_1 = 3$, $n_2 = 2$.

Example 3.24. If

$$B = \left[\begin{array}{c|c|ccc} A_1 & E_1 & 0 & 0 & 0 \\ \hline 0 & A_2 & e_2 & 0 & 0 \\ \hline 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ f_1^T & 0 & 0 & * & 0 \end{array}\right] = \left[\begin{array}{ccc|cc|ccc} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-1.7113, -0.1443 \pm 1.8669i\}, \\ \sigma(A_2) &\approx \{-3.4142, -0.5858\} \text{ and} \\ \sigma(B) &\approx \{-0.1391 \pm 1.8669i, -0.0106 \pm 1.4066i, -3.4139, -1.7215, -0.5178, \\ &\qquad -0.0474\}. \end{aligned}$$

The next result generalizes Lemma 3.22 and is used in the proof of Theorem 3.26.

Lemma 3.25. *If $f(x)$ is a monic stable polynomial of degree $n \geq 3$ and $j \geq 1$, then $F(x) = x\prod_{i=1}^{j}(x^2+T_i)f(x) + S$ is stable for appropriate choices of $T_i > 0$ and $S > 0$ where $j \leq \lfloor \frac{n}{2} \rfloor$.*

Proof. If $f(x)$ is as stated, then by Theorem 2.2, $f(x) = h(x^2) + xg(x^2)$ with the zeros of $h(u)$ and $g(u)$ properly interlaced as follows. Let $\alpha_1, \ldots \alpha_{\lfloor \frac{n}{2} \rfloor}$ be the zeros of $h(u)$ and let $\beta_1, \ldots \beta_{\lfloor \frac{n-1}{2} \rfloor}$ be the zeros of $g(u)$ such that $0 > \alpha_1 > \beta_1 > \alpha_2 > \beta_2 \cdots$. The polynomial $F(x)$ can be written

$$\begin{aligned} F(x) = &\; x\prod_{i=1}^{j}(x^2+T_i)f(x) + S \\ = &\; x\prod_{i=1}^{j}(x^2+T_i)h(x^2) + x^2\prod_{i=1}^{j}(x^2+T_i)g(x^2) + S \\ = &\; xG(x^2) + H(x^2). \end{aligned}$$

If n is even and $j = \frac{n}{2}$, then let $\beta_j = -\infty$. Let $\alpha_1 > -T_1 > \beta_1, \alpha_2 > -T_2 > \beta_2, \ldots,$

$\alpha_j > -T_j > \beta_j$ and let $S > 0$ be sufficiently small, then similar to the proof of Lemma 3.19, the zeros of $H(u)$ and $G(u)$ are simple, negative and interlaced (as described in Theorem 2.2). Thus $F(x)$ is stable. □

The next theorem is a generalization of Theorem 3.23 in which an unsigned vertex connected to a negative 2-cycle in $D(\mathcal{B})$ is replaced by an unsigned vertex connected to either j weakly connected negative 2-cycles that correspond to $\mathcal{Y}_1$ in Figure 3.11 or j negative cycles of even length of each order 2, 4,..., $2j$ that correspond to $\mathcal{Y}_2$ in Figure 3.11. Note that when $j = 1$, $\mathcal{Y}_1 = \mathcal{Y}_2$. Figure 3.10 shows an example of the digraph that corresponds to $\mathcal{Y}_2$ when $j = 3$.

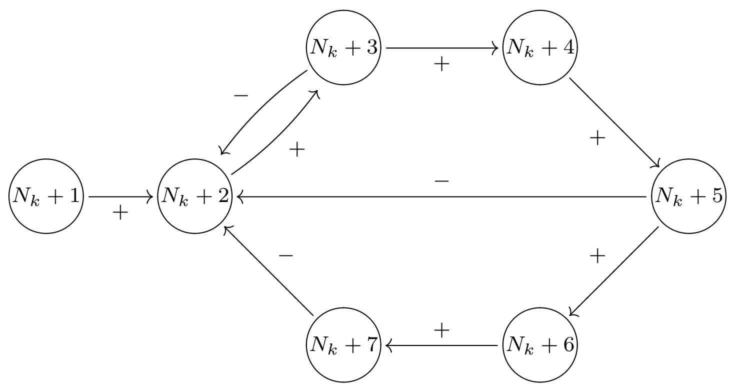


Figure 3.10: Example Digraph of Y_2 in Theorem 3.26

Theorem 3.26. *Suppose $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are unreduced lower Hessenberg PS sign patterns of orders $n_1, \ldots, n_k$, respectively. There exists an order $N_k + 2j + 1$ ($N_k \geq 3$ and $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$) unreduced lower Hessenberg PS sign pattern $\mathcal{B}$ as in Figure 3.11, where $\mathcal{E}_i$ has dimensions $n_i \times n_{i+1}$ ($1 \leq i \leq k-1$), $\mathcal{F}_k$ has dimensions $n_k \times (2j+1)$, $\mathcal{G}_1$ has dimensions $(2j+1) \times n_1$, and the $k+1$ nonzero entries in the patterns $\mathcal{E}_i$, $\mathcal{F}_k$ and $\mathcal{G}_1$ are chosen so that the $(N_k + 2j + 1)$-cycle in $D(\mathcal{B})$ is negative. In addition, $\mathcal{Y}_\ell$ is equal to either $\mathcal{Y}_1$ or $\mathcal{Y}_2$ and for $B \in Q(\mathcal{B})$, $b_{12}b_{23}\cdots b_{N_k+2j,N_k+2j+1}b_{N_k+2j+1,1} < 0$. If $\mathcal{Y}_\ell = \mathcal{Y}_1$ and $B \in Q(\mathcal{B})$, then $b_{N_k+2,N_k+3}b_{N_k+3,N_k+2} < 0$, $b_{N_k+4,N_k+5}b_{N_k+5,N_k+4} < 0$, $\ldots$, $b_{N_k+2j,N_k+2j+1}b_{N_k+2j+1,N_k+2j} < 0$; whereas if $\mathcal{Y}_\ell = \mathcal{Y}_2$ and $B \in Q(\mathcal{B})$, then $b_{N_k+2,N_k+3}b_{N_k+3,N_k+4}\cdots b_{N_k+2r,N_k+2r+1}b_{N_k+2r+1,N_k+2} < 0$ $(1 \leq r \leq j)$.*

Proof. Let $A_i \in Q(\mathcal{A}_i)$ be stable for $1 \leq i \leq k$, $F_k \in Q(\mathcal{F}_k)$, $G_1 \in Q(\mathcal{G}_1)$, $E_i \in Q(\mathcal{E}_i)$ for $1 \leq i \leq k-1$, and $Y_\ell = Y_1 \in Q(\mathcal{Y}_1)$. Construct B as in Figure 3.8, where $b_{N_k+2i,N_k+2i+1}b_{N_k+2i+1,N_k+2i} = C_{2_i} < 0$, for $1 \leq i \leq j$. Note that $b_{12}\ldots b_{N_k+2j,N_k+2j+1}b_{N_k+2j+1,1} = C_{N_k+2j+1} < 0$.

$$\mathcal{B} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{E}_1 & & & & \\ & \ddots & \ddots & & & \\ & & \mathcal{A}_{k-1} & \mathcal{E}_{k-1} & & \\ & & & \mathcal{A}_k & \mathcal{F}_k & \\ \mathcal{G}_1 & & & & \mathcal{Y}_\ell & \end{bmatrix}, \mathcal{E}_i = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{F}_k = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix}, \mathcal{G}_1 = \begin{bmatrix} 0 & \dots & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & & \\ * & 0 & \dots & 0 \end{bmatrix},$$

$$\mathcal{Y}_1 = \begin{bmatrix} 0 & * & 0 & \dots & & & & 0 \\ 0 & 0 & * & \ddots & & & & \vdots \\ 0 & * & 0 & * & & & & \\ \vdots & \ddots & 0 & 0 & * & & & \\ & & & * & 0 & * & & \\ & & & & \ddots & \ddots & \ddots & 0 \\ & & & & & 0 & 0 & * \\ 0 & \dots & & & & 0 & * & 0 \end{bmatrix}, \mathcal{Y}_2 = \begin{bmatrix} 0 & * & 0 & \dots & & & & 0 \\ 0 & 0 & * & \ddots & & & & \vdots \\ 0 & * & 0 & * & & & & \\ 0 & 0 & 0 & 0 & * & & & \\ 0 & * & 0 & 0 & 0 & * & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & & & 0 & 0 & * \\ 0 & * & \dots & & & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.11: Sign patterns for Theorem 3.26

Consideration of the characteristic polynomial of B, $\det(\lambda I_{N_k+2j+1} - B)$, and expansion about the last row give

$$\begin{aligned} \det(\lambda I_{N_k+2j+1} - B) = \; & \lambda[\lambda^2(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_{j-1}}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i)] \\ & - C_{2_j}\lambda(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_{j-1}}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) \\ & + (-1)^{N_k+2j+2}(-1)^{N_k+2j+1} C_{N_k+2j+1}, \text{ where the sign} \\ & (-1)^{N_k+2j+2} \text{ is contributed by the } (N_k+2j+1, 1) \\ & \text{position and the sign} (-1)^{N_k+2j+1} \text{ is contributed} \\ & \text{by } N_k + 2j + 1 \text{ negative signs from the cycle in } D(-B) \\ = \; & \lambda(\lambda^2 - C_{2_1}) \cdots (\lambda^2 - C_{2_j}) \prod_{i=1}^{k} \det(\lambda I_{n_i} - A_i) - C_{N_k+2j+1}, \end{aligned}$$

and by Lemma 3.25, $C_{2_1}, \dots, C_{2_j}$ and C_{N_k+2j+1} can be chosen (dependent on the stable matrices $A_1, \dots, A_k$) so that B is stable. Thus, $\mathcal{B}$ is PS.

When $\mathcal{Y}_\ell = \mathcal{Y}_2$, by an argument similar to that in the proof of Theorem 3.20, the characteristic polynomial of B can similarly be written as above and the result follows. □

The construction in Theorem 3.26 is now illustrated when $k = 2$, $n_1 = 3$, $n_2 = 2$, $j = 2$, $\mathcal{Y}_\ell = \mathcal{Y}_2$ and $C_{N_k+5} < 0$.

Example 3.27. If

$$B = \left[\begin{array}{c|c|c} A_1 & E_1 & 0 \\ \hline 0 & A_2 & F_2 \\ \hline G_1 & 0 & Y_2 \end{array}\right] = \left[\begin{array}{ccc|cc|ccccc} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-1.7113, -0.1443 \pm 1.8669i\}, \\ \sigma(A_2) &\approx \{-3.4142, -0.5858\} \text{ and} \\ \sigma(B) &\approx \{-0.0061 \pm 2.2308i, -0.0344 \pm 1.3869i, -0.1146 \pm 1.8789i, -3.4140, \\ &\qquad -1.7241, -0.4374, -0.1144\}. \end{aligned}$$

By comparing the constructions in Theorems 3.20 and 3.26, it can be seen that Theorem 3.20 might not produce a minimally PS sign pattern. In Theorem 3.20, if $k \geq 2$ and $\mathcal{A}_k$ is a 1×1 PS sign pattern (a negative entry), then setting it to zero produces a sign pattern that can be constructed as in Theorem 3.26. The next example illustrates that the construction in Theorem 3.26 can also produce a sign pattern that is not minimally PS.

Example 3.28. If

$$B = \left[\begin{array}{c|c|c} A_1 & E_1 & 0 \\ \hline 0 & A_2 & F_2 \\ \hline G_1 & 0 & Y_1 \end{array}\right] = \left[\begin{array}{cccc|cc|ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right],$$

then

$$\begin{aligned} \sigma(A_1) &\approx \{-0.1579 \pm 2.0042i, -0.3421 \pm 0.7907i\}, \\ \sigma(A_2) &\approx \{-0.5000 \pm 1.3229i\} \text{ and} \\ \sigma(B) &\approx \{-0.0056 \pm 2.4339i, -0.1270 \pm 2.0585i, -0.6323 \pm 1.3558i, \\ &\qquad -0.0535 \pm 0.6968i, -0.3631\}. \end{aligned}$$

However, setting the (5,5) entry in B produces a stable matrix

$$C = \left[\begin{array}{cccc|cc|ccc} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \end{array}\right]$$

with

$$\begin{aligned} \sigma(C) &\approx \{-0.0157 \pm 2.4384i, -0.0783 \pm 2.0289i, -0.2187 \pm 1.3585i, \\ &\qquad -0.0326 \pm 0.8339i, -0.3094\}. \end{aligned}$$

Thus $\text{sgn}(C)$ is PS and it follows that $\text{sgn}(B)$ is not minimally PS.

3.3 Similarity Transformation

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are similar if $A = X^{-1}BX$ for some invertible $X \in \mathbb{R}^{n\times n}$, and in this case A is said to be obtained from B by a similarity transformation. It is well known that similar matrices have the same eigenvalues. The proof of the following theorem uses a similarity transformation from Kim et al. [14, Theorem 3.1]. As with the bordering techniques in Section 3.2, this method has the advantage that it produces a stable realization of the generated PS sign pattern.

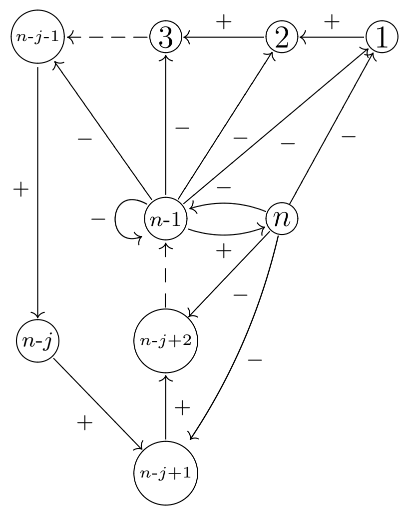


Figure 3.12: Digraph for Theorem 3.29

Theorem 3.29. *Let $n \geq 4$ and $2 \leq j \leq n-2$. Suppose $\mathcal{B}$ is the order n sign pattern with $D(\mathcal{B})$ as in Figure 3.12. Then $\mathcal{B}$ is minimally PS.*

Proof. Let

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix},$$

with $a_i > 0$ for $i = 1, \dots, n-1$ and A a stable matrix of order $n-1$. Take $k = x_{n-1} > 0$ so that $x^T u = k$. Define

$$B = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline x^T & 1 \end{array}\right] \left[\begin{array}{c|c} A & u \\ \hline 0 & -k \end{array}\right] \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline -x^T & 1 \end{array}\right] = \left[\begin{array}{c|c} A - ux^T & u \\ \hline x^T A & 0 \end{array}\right]$$

as in [14, Theorem 3.1] and choose x such that (for any fixed j with $2 \le j \le n-2$)

$$\begin{aligned} b_{n-1,i} &= -a_i - x_i = 0 \Rightarrow x_i = -a_i, \quad \text{for } i = n-j, \ldots, n-2 \\ b_{ni} &= x_{i-1} - a_i x_{n-1} = 0 \Rightarrow x_{i-1} = a_i x_{n-1}, \quad \text{for } i = 2, \ldots, n-j. \end{aligned}$$

By construction, $D(B)$ is the digraph in Figure 3.12 and $\sigma(B) = \sigma(A) \cup \{-k\}$. Thus B is stable and $\text{sgn}(B)$ is PS. Furthermore, since B has only one nonzero principal minor of every order, it follows that $\text{sgn}(B)$ is minimally PS. □

The next example shows the construction in Theorem 3.29 when $n = 7$ and $j = 3$.

Example 3.30. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -6 & -15 & -20 & -15 & -6 \end{bmatrix}, u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} 6 \\ 15 \\ 20 \\ -20 \\ -15 \\ 1 \end{bmatrix}, k = 1.$$

Then A is stable since $\sigma(A) = \{-1, -1, -1, -1, -1, -1\}$. By the similarity transformation in the above proof,

$$B = \left[\begin{array}{c|c} A - ux^T & u \\ \hline x^T A & 0 \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -7 & -21 & -35 & 0 & 0 & -7 & 1 \\ \hline -1 & 0 & 0 & 0 & -35 & -21 & 0 \end{array}\right]$$

and B is stable since $\sigma(B) = \sigma(A) \cup \{-1\}$. Note that B does not contain a nest since there is no nonzero principal minor of order 4 that contains a nonzero principal minor of order 3.

The sign pattern associated with the digraph in Figure 3.12 can be generalized to describe similar classes of PS sign patterns. Note that in the digraph, vertex $n-1$ is included in every cycle (including the loop) and there is exactly one cycle (each negative) of every length $1, \ldots, n$. Any stable realization of a sign pattern associated with a digraph having these properties has exactly one principal minor of every order $1, \ldots, n$ (each of the proper sign), and the magnitudes of the principal minors can be arbitrarily chosen. Thus the sign pattern associated with any such digraph is minimally PS.

Chapter 4

Number of Nonzero Entries

The number of nonzero entries in a given sign pattern can provide insight into whether or not it is PS. There has been very little study done on the number of nonzero entries in an $n \times n$ PS sign pattern. Clearly, allowing reducible patterns, the least number of nonzero entries in a PS sign pattern is n. However, the least number of nonzero entries in an $n \times n$ irreducible PS sign pattern is not known. In the first section of this chapter, we consider the least number of nonzero entries in an irreducible sign pattern that allows a nest and present a new result that shows this minimum number to be $2n - 1$. In Section 4.2, we consider the number of nonzero entries in an $n \times n$ minimally PS sign pattern, state the minimum number for $n = 2, \ldots, 6$ and provide examples. Section 4.2 also considers the property of minimality for PS sign patterns having particular diagonal entries. Finally, we look at the constructions in Theorems 3.3, 3.7 and 3.23, now considering the number of nonzero entries in the constructed PS sign patterns.

4.1 Sign Patterns that Allow a Nest

Theorem 4.3 is a new result that considers the minimum number of nonzero entries in an irreducible sign pattern that allows a nest. The following lemmas are used to prove Theorem 4.3.

Lemma 4.1. *Assume an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ allows a leading nest of order $k < n$. If there exists a leading nest of order $k + 1$, then either $\alpha_{k+1,k+1} \neq 0$ or there exist nonzero entries $\alpha_{i,k+1}$ and $\alpha_{k+1,j}$ for some $i, j \in \{1, \ldots, k\}$.*

Proof. If $\alpha_{k+1,k+1} = 0$, then $\det A[\{1,\ldots,k+1\}] = 0$ for all $A \in Q(\mathcal{A})$ unless there is a nonzero entry in row $k+1$ and column $k+1$ of $\mathcal{A}$. $\square$

Lemma 4.2. *Suppose $\mathcal{A} = [\alpha_{ij}]$ is an $n \times n$ PS sign pattern with nonzero diagonal entries $\alpha_{11}, \ldots, \alpha_{rr}$ $(r \geq 2)$ such that for $2 \leq i \leq r$, $\alpha_{ij} = 0$ and $\alpha_{ji} = 0$ for all $1 \leq j \leq i-1$. If there exist exactly two nonzero entries in $\{\alpha_{r+k,j}, \alpha_{j,r+k} : 1 \leq j \leq r+k\}$ for $k = 1, \ldots, n-r-1$ so that $A \in Q(\mathcal{A})$ implies that $\det A[\{1,\ldots,r+k\}] \neq 0$, then the digraph $D(A[\{1,\ldots,r+k\}])$ has exactly r weakly connected components* (*and $A[\{1,\ldots,r+k\}]$ is reducible*).

Proof. Assume $A \in Q(\mathcal{A})$ is as above. Note that $D(A[\{1,\ldots,r\}])$ consists of r isolated vertices and thus has r weakly connected components. The proof is by induction on k. If $k = 1$, then by Lemma 4.1 the two nonzero entries must be either $a_{r+1,r+1}$ and some entry $a_{r+1,i}$ or $a_{i,r+1}$ for $1 \leq i \leq r$, or $a_{r+1,j}$ and $a_{j,r+1}$ for $1 \leq j \leq r$. In either case $D(A[\{1,\ldots,r+1\}])$ has r weakly connected components.

Suppose the hypothesis is true for $k = t \leq n-2$. Thus $D(A[\{1,\ldots,r+t\}])$ has r weakly connected components. If $k = t+1$, then the two nonzero entries in $\{a_{r+t+1,j}, a_{j,r+t+1} : 1 \leq j \leq r+t+1\}$ must be either:

1. $a_{r+t+1,r+t+1}$ and some entry $a_{r+t+1,i}$ or $a_{i,r+t+1}$ for $1 \leq i \leq r+t$, in which case there are still r weakly connected components since vertices $r+t+1$ and i are in the same weakly connected component; or

2. $a_{i,r+t+1}$ and $a_{r+t+1,j}$ $(1 \leq i, j \leq r+t$ with possibly $i = j)$. Since $\det A[\{1,\ldots, r+t+1\}] \neq 0$ implies that vertex $r+t+1$ must be on a cycle in $D(A[\{1,\ldots, r+t+1\}])$, it follows that vertex $r+t+1$ and the other vertices on this cycle are in the same weakly connected component. All vertices in this component except for vertex $r+t+1$ are weakly connected in $D(A[\{1,\ldots,r+t\}])$, so there are still exactly r weakly connected components.

Thus the digraph result follows by induction. $\square$

Theorem 4.3. *The minimum number of nonzero entries in an $n \times n$ irreducible PS sign pattern that allows a nest is $2n-1$.*

Proof. Without loss of generality, assume there is an $n \times n$ irreducible PS sign pattern matrix $\mathcal{A}$ that allows a leading nest. Suppose that for $2 \leq i \leq n$ there are no nonzero diagonal entries α_{ii} such that $\alpha_{ji} = 0$ and $\alpha_{ij} = 0$ for all $j < i$. It follows by Lemma 4.1 that there must be at least $2(n-1)+1 = 2n-1$ nonzero entries in pattern $\mathcal{A}$.

Assume $\mathcal{A}$ contains exactly $r \geq 2$ nonzero diagonal entries $\alpha_{i_1 i_1}, \ldots, \alpha_{i_r i_r}$ ($1 = i_1 < i_2 < \ldots < i_r < n$) such that for $2 \leq k \leq r$, $\alpha_{j,i_k} = 0$ and $\alpha_{i_k,j} = 0$ for all $1 \leq j \leq i_k - 1$. Without loss of generality, assume that these r nonzero diagonal entries are $\alpha_{11}, \ldots, \alpha_{rr}$. These clearly contribute r nonzero entries to the sign pattern. By Lemma 4.1, the rows and columns $r+1, \ldots, n-1$ contain at least $2(n-r-1)$ nonzero entries. Since $\mathcal{A}$ allows a leading nest, by Lemma 4.2, there exists a subpattern $\widehat{\mathcal{A}}$ of $\mathcal{A}$ that contains exactly two nonzero entries in $\{\hat{\alpha}_{tj}, \hat{\alpha}_{jt} : 1 \leq j \leq t\}$ for each $t \in \{r+1, \ldots, n-1\}$ such that if $\hat{A} \in Q(\widehat{\mathcal{A}})$ then $\det \hat{A}[\{1, \ldots, t\}] \neq 0$. By Lemma 4.2, $\widehat{\mathcal{A}}[\{1, \ldots, n-1\}]$ is reducible and its digraph has r weakly connected components. Therefore, since $\mathcal{A}$ is irreducible it must contain at least $r+1$ additional nonzero entries. It follows that $\mathcal{A}$ has at least $r + 2n - 2r - 2 + r + 1 = 2n - 1$ nonzero entries. □

Example 4.4 contains an irreducible PS sign pattern with $n = 5$, $r = 2$ that allows a leading nest and has $2n - 1 = 9$ nonzero entries, illustrating that the bound in Theorem 4.3 can be obtained.

Example 4.4. Let

$$\mathcal{A} = \begin{bmatrix} - & 0 & + & 0 & 0 \\ 0 & - & 0 & + & 0 \\ - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & - & + \\ - & 0 & 0 & 0 & 0 \end{bmatrix} \text{ with the realization } A = \begin{bmatrix} -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -0.1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix A has a leading nest and $\sigma(A) \approx \{-1.0626 \pm 0.5348i, -0.1836 \pm 0.1147i, -1.5075\}$, thus $\mathcal{A}$ is PS. Given $\mathcal{A}$, the subpattern $\widehat{\mathcal{A}}[\{1,2,3,4\}]$ identified in the proof of Theorem 4.3 is equal to $\mathcal{A}[\{1,2,3,4\}]$ except for 0 in the (3,2) entry. The + in the (3,2) entry (together with the two nonzero entries in row and column 5) ensures that $\mathcal{A}$ is irreducible and $\det A < 0$ for all $A \in Q(\mathcal{A})$.

The following theorem gives a property of 2×2 PS sign patterns. This property ensures that the least number of nonzero entries in a 2×2 PS sign pattern is $2n - 1 = 3$. This minimum value is proved by different means in Theorem 4.8.

Theorem 4.5. *If a 2×2 sign pattern $\mathcal{A}$ is PS, then there exists $A \in Q(\mathcal{A})$ such that A contains a nest.*

Proof. This can be seen by noting that if A is stable, then tr $A < 0$ and det $A > 0$. □

Note that a 3×3 PS sign pattern with $2n - 1$ nonzero entries need not allow a nest as the following example shows.

Example 4.6. Consider

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}$$

with $\sigma(A) \approx \{-0.5698, -0.2151 \pm 1.3071i\}$. The sign pattern, sgn($A$), does not allow a nest since there is no nonzero principal minor of order 1 that is contained in a nonzero principal minor of order 2.

The next example shows that an $n \times n$ irreducible PS sign pattern that contains a nest and $2n - 1$ nonzero entries need not be minimally PS.

Example 4.7. Consider

$$\mathcal{A} = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & 0 & - & + \\ 0 & - & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

The sign pattern $\mathcal{A}$ has $2n - 1$ nonzero entries and requires a leading nest. The matrix B shows that setting the (3,3) entry in $\mathcal{A}$ to 0 produces a PS sign pattern since $\sigma(B) \approx \{-0.3516 \pm 1.4985i, -0.1484 \pm 0.6325i\}$. Note that sgn($B$) does not allow a nest but is minimally PS.

4.2 Minimally Potentially Stable Sign Patterns

It is sufficient, in characterizing potential stability, to consider only the minimally PS sign patterns. The determination of whether or not a given PS sign pattern $\mathcal{A}$ is minimal is, in general, not trivial. A number of methods are helpful in determining minimality. If setting any nonzero entry in $\mathcal{A}$ to zero produces a sign pattern $\tilde{\mathcal{A}}$ such that for all $\tilde{A} \in Q(\tilde{\mathcal{A}})$, a coefficient k_i of the characteristic polynomial is nonpositive, then clearly $\mathcal{A}$ is minimally PS. Similarly, if setting any nonzero entry in $\mathcal{A}$ to zero produces a sign pattern $\tilde{\mathcal{A}}$ such that for all $\tilde{A} \in Q(\tilde{\mathcal{A}})$, one of the Routh-Hurwitz conditions is not satisfied, then again $\mathcal{A}$ is minimally PS. The latter approach, however, can become prohibitively difficult to check for orders ≥ 5. Another method requires knowledge of PS sign patterns of lower order. If setting any nonzero off-diagonal entry in $\mathcal{A}$ to zero produces a sign pattern $\tilde{\mathcal{A}}$ that is reducible and at least one of the irreducible components in $\tilde{\mathcal{A}}$ is not PS, then $\mathcal{A}$ is minimally PS. The arguments following Example 3.2, for minimality of the PS sign pattern $\text{sgn}(\hat{A})$, illustrate that a combination of these methods may be required to determine that a sign pattern is minimally PS.

Knowledge of the minimum number of nonzero entries in a minimally PS sign pattern of fixed order can be used to determine the instability of a sign pattern. If a sign pattern has fewer than the least number of nonzero entries in a minimally PS sign pattern of the given order, then that sign pattern is sign unstable. Clearly a 1×1 PS sign pattern has only $2n-1=1$ nonzero entry. Theorem 4.8 gives a similar result for the number of nonzero entries for any minimally PS sign pattern when $n=2$ or 3.

Theorem 4.8. *Let $n = 2$ or 3 and suppose $\mathcal{A}$ is an $n \times n$ minimally PS sign pattern. Then the least number of nonzero entries in $\mathcal{A}$ is $2n-1$.*

Proof. Assume $\mathcal{A}$ is an $n \times n$ minimally PS sign pattern with stable matrix $A \in Q(\mathcal{A})$.

Suppose $n = 2$. Since A is stable, it must contain at least one negative diagonal entry. The requirement of irreducibility implies there must be at least two nonzero off-diagonal entries and therefore $\mathcal{A}$ has at least $2n-1=3$ nonzero entries.

Suppose $n = 3$. Clearly A has at least one negative diagonal entry. As well, since A is irreducible, it must have at least three nonzero off-diagonal entries. If A has at least four nonzero off-diagonal entries, then the result is true. Otherwise, if A has only three nonzero off-diagonal entries then $D(A)$ contains a 3-cycle. Since A must have a positive principal minor of order 2, there must be at least one more nonzero entry in A, giving the result. □

Note that for $n = 2$, up to equivalence there is only one minimally PS sign pattern $\mathcal{A}_{2,1}$ (see Section A.1) and it has exactly $2n-1=3$ nonzero entries. However, for $n = 3$, all but one of the 3×3 minimally PS sign patterns in Section A.2 has $2n-1=5$ nonzero entries. The sign pattern $\mathcal{A}_{3,2}$ (see Section A.2) is minimally PS but has $2n$ nonzero entries.

It can be seen that for $n = 2$ or 3, no minimally PS sign pattern has an entirely nonzero diagonal. This is not by chance and the next result shows why this is so.

Theorem 4.9. *An irreducible PS sign pattern with a nonzero diagonal is not minimally PS.*

Proof. An irreducible sign pattern with all diagonal entries negative is obviously not minimally PS since by Theorem 2.7 it is possible to set any nonzero off-diagonal entry to zero and produce a PS sign pattern. Assume $\mathcal{A}$ is an irreducible PS sign pattern with a nonzero diagonal and at least one positive diagonal entry. Let $A \in Q(\mathcal{A})$ be a stable matrix. If s is the magnitude of the least positive diagonal entry in A, then clearly $A - sI$ is stable. Since $\text{sgn}(A - sI)$ is a subpattern of $\text{sgn}(A)$, then clearly $\text{sgn}(A)$ is not minimally PS. □

Miyamichi briefly looks at the number of nonzero entries in a PS sign pattern in [19, Lemma 12], and considers the minimum number of arcs in a digraph corresponding to a PS sign pattern that has $n - 2$ negative diagonal entries. The proof of [19, Lemma 12] can be adapted to show a similar constraint on the minimum number of nonzero entries in a PS sign pattern having $n - 3$ negative diagonal entries.

Theorem 4.10. *Let $n \geq 4$. Suppose that A is an $n \times n$ irreducible, stable matrix with $a_{11} \geq 0, a_{22} = a_{33} = 0$ and $a_{ii} < 0$ for $i = 4, 5, \ldots, n$. Then the number of nonzero off-diagonal entries in A is at least $n + 1$, i.e., the number of nonzero entries in A is at least $2n - 2$.*

Proof. Since $D(A)$ is strongly connected, the number of arcs other than loops must be at least n. If the number is equal to n, then $D(A)$ is a simple cycle, with some loops; in this case, let C_n denote the product of the matrix entries in A corresponding to the n-cycle.

The characteristic polynomial of A is

$$(x - a_{11})x^2(x - a_{44}) \cdots (x - a_{nn}) - C_n.$$

The coefficient of x^2 in the polynomial is $(-a_{11})(-a_{44}) \cdots (-a_{nn}) \leq 0$ and it follows that A is unstable. Therefore, $D(A)$ must have at least $n+1$ arcs other than loops, and thus A must have at least $n+1$ off-diagonal entries, giving $2n-2$ nonzero entries. □

It is important to note that in Theorem 4.10 and [19, Lemma 12], the minimum number of nonzero off-diagonal entries is dependent on the number of nonzero diagonal entries. Matrix B in Example 4.7 and matrix X in Example 3.5 show that the minimum in Theorem 4.10 can be realized when $n = 4$ or 5. In fact, the following theorem shows $2n-2$ to be the minimum number of nonzero entries in a minimally PS sign pattern for $n = 4$ or 5.

Theorem 4.11. *Let $n = 4$ or 5 and suppose $\mathcal{A}$ is an $n \times n$ minimally PS sign pattern. Then $\mathcal{A}$ has at least $2n-2$ nonzero entries.*

Proof. Assume $\mathcal{A}$ is an $n \times n$ minimally PS sign pattern with stable matrix $A \in Q(\mathcal{A})$. Suppose $n = 4$. Clearly A has at least one negative diagonal entry. As well, since A is irreducible, it must have at least four nonzero off-diagonal entries. If A has at least five nonzero off-diagonal entries, then the result follows. Otherwise, if A has only four nonzero off-diagonal entries, then $D(A)$ contains a 4-cycle. In order for A to have a positive principal minor of order 2, there must be at least one more nonzero entry in A, again giving at least $2n-2 = 6$ arcs.

Suppose $n = 5$. Clearly A has at least one negative diagonal entry. As well, since A is irreducible, it must have at least five nonzero off-diagonal entries. If A has at least seven nonzero off-diagonal entries, then the result follows. Otherwise A has either five or six nonzero off-diagonal entries. If A has only five nonzero off-diagonal entries, then $D(A)$ contains a 5-cycle. In order for A to have nonzero principal minors of orders 2, 3 and 4, there must be at least two more nonzero entries in A, giving at least $2n-2 = 8$ nonzero entries. If A contains exactly six nonzero off-diagonal entries, then the following cases must be considered.

- $D(A)$ contains two 4-cycles. If this were the case, A must have at least one more nonzero entry to have nonzero principal minors of orders 2 and 3.

- $D(A)$ contains a 4-cycle and a 3-cycle. If this were the case, A must have at least one more nonzero entry to have a nonzero principal minor of order 2.

- $D(A)$ contains a 4-cycle and a 2-cycle. If this were the case, in order for A to have nonzero principal minors of orders 3 and 5 then it is necessary that the negative loop in $D(A)$ be disjoint from both the 4-cycle and the 2-cycle which contradicts irreducibility.

- $D(A)$ contains two 3-cycles. If this were the case, A must have at least one more nonzero entry to have nonzero principal minors of orders 2 and 5.

In each case, $\mathcal{A}$ has at least $2n-2$ nonzero entries. $\square$

The next example is motivation for Theorem 4.13.

Example 4.12. The 6×6 irreducible stable matrix

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -3 & 0 \end{bmatrix}$$

has

$$\sigma(A) \approx \{ -0.1122 \pm 1.6996i, -0.3336 \pm 1.2374i, -0.0542 \pm 0.4549i\}$$

and $2n - 3 = 9$ nonzero entries.

As shown in the next theorem, the number of nonzero entries in the matrix A in Example 4.12 is in fact the minimum number of entries in a 6×6 irreducible PS sign pattern.

Theorem 4.13. *If $\mathcal{A}$ is a 6×6 minimally PS sign pattern, then the least number of nonzero entries in $\mathcal{A}$ is $2n - 3 = 9$ for $n = 6$.*

Proof. Assume $\mathcal{A}$ is a 6×6 minimally PS sign pattern with stable matrix $A \in Q(\mathcal{A})$. Clearly A must have at least one negative diagonal entry. As well, since A is irreducible, it must have at least six nonzero off-diagonal entries. If A has at least eight nonzero off-diagonal entries, then the result follows. Otherwise, A has either six or seven nonzero off-diagonal entries. If A has only six nonzero off-diagonal entries, then $D(A)$ contains a 6-cycle and there must be at least two more nonzero off-diagonal entries to ensure nonzero principal minors of orders 2, 3, 4 and 5. If A has exactly seven nonzero off-diagonal entries, then the following cases must be considered.

- $D(A)$ contains a j-cycle and a k-cycle, where $3 \leq j \leq 6$ and $4 \leq k \leq 6$. If this were the case, A must have at least one more nonzero entry to have a nonzero principal minor of order 2.

- $D(A)$ contains a 2-cycle and a j-cycle, where $j = 5$ or 6. If this were the case, A must have at least one more nonzero entry to have a nonzero principal minor of order 4.

In each case, $\mathcal{A}$ has at least 9 nonzero entries. □

The next theorem, which can be proved by the technique from Theorem 4.10, gives the minimum number of nonzero off-diagonal entries in a minimal $n \times n$ PS sign pattern with $n-4$ negative diagonal entries.

Theorem 4.14. *Let $n \geq 6$. Suppose that A is an $n \times n$ irreducible, stable matrix with $a_{11} \geq 0, a_{22} = a_{33} = a_{44} = 0$ and $a_{ii} < 0$ for $i = 5, \ldots, n$. Then the number of nonzero off-diagonal entries in A is at least $n+1$, i.e., the number of nonzero entries in A is at least $2n-3$.*

The result from Theorem 4.11 ensures that the minimum value of n to consider in Theorem 4.14 is $n = 6$. The next set of lemmas are used in Theorem 4.18 to show that the number of nonzero entries determined by Theorem 4.14 is not sharp for $n = 6$.

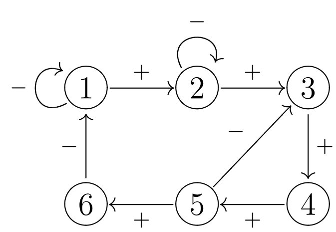


Figure 4.1: Digraph for Lemma 4.15

Lemma 4.15. *The sign pattern corresponding to the digraph in Figure 4.1 is sign unstable. Similarly, if the two negative loops are on any two of the vertices 1,2 and 6, then the corresponding sign pattern is sign unstable.*

Proof. Suppose

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -a & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -b & 0 & 0 & 1 \\ -c & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a normalized realization of the sign pattern corresponding to the digraph in Figure 4.1 with $a, b, c > 0$. The characteristic polynomial of A is $x^6 + (1+a)x^5 + ax^4 + bx^3 + (1+a)bx^2 + abx + c$ with $k_1 = 1+a$, $k_2 = a$, $k_3 = b$, $k_4 = (1+a)b$ and $k_5 = ab$. From the Routh-Hurwitz conditions,

$$\begin{aligned}\Delta_3 = &\; b[a(1+a)-b]-(1+a)^3b+ab(1+a)b \\ = &\; -b(1+a+a^2+a^3+b) \not> 0,\end{aligned}$$

which contradicts stability. Thus, there does not exist such a stable realization A, and it follows that $\text{sgn}(A)$ is sign unstable. Since the characteristic polynomial for any of the other cases is the same as above, the second statement follows. □

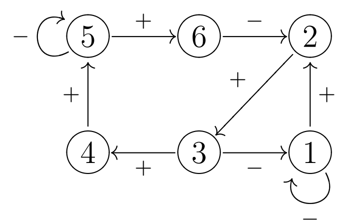


Figure 4.2: Digraph for Lemma 4.16

Lemma 4.16. *The sign pattern corresponding to the digraph in Figure 4.2 is sign unstable. Similarly, if the negative loop in the 5-cycle is on vertex 4 or 6, then the corresponding sign pattern is sign unstable.*

Proof. Suppose

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -a & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -b & 1 \\ 0 & -c & 0 & 0 & 0 & 0 \end{bmatrix}$$

is a normalized stable realization of the sign pattern corresponding to the digraph in Figure 4.2 with $a, b, c > 0$. The characteristic polynomial of A is $f(x) = x^6 + (1 + b)x^5 + bx^4 + ax^3 + abx^2 + cx + c$ with $k_1 = 1 + b$, $k_2 = b$, $k_3 = a$, $k_4 = ab$ and $k_5 = c$. From the Routh-Hurwitz conditions, $\Delta_3 = a[b(1 + b) - a] - ab(1 + b)^2 + c(1 + b) = (c - ab^2)(1 + b) - a^2 > 0$ which implies $(c - ab^2)(1 + b) > a^2$. The characteristic polynomial can also be written $f(x) = h(x^2) + xg(x^2)$ where $h(u) = u^3 + bu^2 + abu + c$ and $g(u) = (1 + b)u^2 + au + c$. Using Theorem 2.2, the zeros of $h(u)$ and $g(u)$ are interlaced, which implies that $g(u)$ has real and distinct zeros. This means that the discriminant in the quadratic formula for the zeros of $g(u)$ is strictly positive, i.e., $a^2-4(1+b)c > 0$ or $a^2 > 4(1+b)c$. However, it is then the case that $(c-ab^2)(1+b) > 4(1+b)c$, which implies that $c-ab^2 > 4c$, giving a contradiction. Thus, there does not exist such a stable realization A, and it follows that $\text{sgn}(A)$ is sign unstable. Since the characteristic polynomial for any of the other cases is the same as above, the second statement follows. □

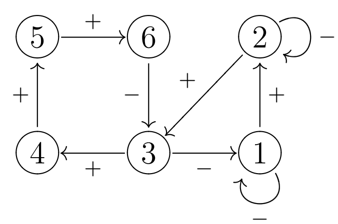


Figure 4.3: Digraph for Lemma 4.17

Lemma 4.17. *The sign pattern corresponding to the digraph in Figure 4.3 is sign unstable.*

Proof. Suppose

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -a & 1 & 0 & 0 & 0 \\ -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -c & 0 & 0 & 0 \end{bmatrix}$$

is a normalized stable realization of the sign pattern corresponding to the digraph in Figure 4.3 with $a, b, c > 0$. The characteristic polynomial of A is $f(x) = x^6 + (1 + a)x^5 + ax^4 + bx^3 + cx^2 + c(1 + a)x + ca$ with $k_1 = 1 + a$, $k_2 = a$, $k_3 = b$, $k_4 = c$ and $k_5 = c(1 + a)$. From the Routh-Hurwitz conditions,

$$\begin{aligned} \Delta_2 = &\ a(1 + a) - b > 0 \Rightarrow a^2 + a - b > 0 \qquad \text{and thus} \\ \Delta_5 = &\ -a^2b^2c - a^3b^2c - b^2c^2 - ab^2c^2 + ab^3c \\ = &\ -b^2c(a^2 + a^3 + c + ac - ab) \\ = &\ -b^2c[a(a + a^2 - b) + c(1 + a)] \not> 0. \end{aligned}$$

Thus, there does not exist such a stable realization A, and it follows that $\text{sgn}(A)$ is sign unstable. □

Theorem 4.18. *Let $n = 6$. Suppose $\mathcal{A}$ is an $n \times n$ minimally PS sign pattern with exactly $n - 4 = 2$ negative diagonal entries. Then the number of nonzero entries in $\mathcal{A}$ is at least $2n - 2$.*

Proof. Let $\mathcal{A}$ be a 6×6 minimally PS sign pattern with 2 negative diagonal entries and let $A \in Q(\mathcal{A})$ be stable. If A has any positive diagonal entries, then the result follows by the irreducibility of A. Otherwise, by Theorem 4.14 there are at least $n+1$ nonzero off-diagonal entries in A. If there are exactly $n+1$ nonzero off-diagonal entries and A is irreducible, then the following cases must be considered.

- $D(A)$ contains a j-cycle and a k-cycle where $4 \leq j \leq 6$ and $4 \leq k \leq 5$. In this case, there must be at least one more nonzero off-diagonal entry to have a nonzero principal minor of order 3 in A.

- $D(A)$ contains a 6-cycle and a 2-cycle. In this case, there must be at least one more nonzero off-diagonal entry to have a nonzero principal minor of order 5 in A.

- $D(A)$ contains a 6-cycle and a 3-cycle. In this case, if either of the loops are contained in the 3-cycle, then there must be at least one more nonzero off-diagonal entry to have a nonzero principal minor of order 5 in A. Otherwise, Lemma 4.15 shows that there must at least one more nonzero off-diagonal entry in A for stability.

- $D(A)$ contains a 5-cycle and a 2-cycle. In this case, in order for A to have a nonzero determinant, then one of the loops has to be disjoint from the 5-cycle and contained in the 2-cycle. If the other loop is also contained in the 2-cycle, then there must be at least one more nonzero off-diagonal entry to have nonzero principal minors of orders 3 and 4. Otherwise, there must at least one more nonzero off-diagonal entry to have a nonzero principal minor of order 4 in A.

- $D(A)$ contains a 5-cycle and a 3-cycle. In this case, in order for A to have a nonzero determinant, then one of the loops has to be disjoint from the 5-cycle and contained in the 3-cycle. If the other loop is contained in the 3-cycle, then there must be at least one more nonzero off-diagonal entry for there to be a nonzero principal minor of order 4 in A. Otherwise, Lemma 4.16 shows that there must be at least one more nonzero off-diagonal entry in A for stability.

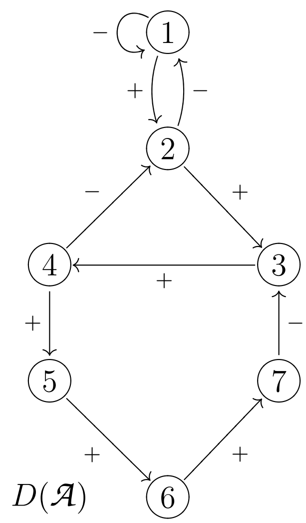
- $D(A)$ contains a 4-cycle and a 3-cycle. In this case, in order for A to have a nonzero determinant, both of the loops have to be disjoint from the 4-cycle. Lemma 4.17 shows that there must be at least one more nonzero off-diagonal entry in A for stability.

In each case, there must be at least $n+2$ nonzero off-diagonal entries in $\mathcal{A}$ and it follows that there must be at least $2n-2$ nonzero entries in $\mathcal{A}$. □

As the order of sign pattern increases, it becomes much more challenging to identify the least number of nonzero entries in a minimally PS sign pattern. Example 4.19 shows that it is possible for a 7×7 irreducible sign pattern to contain $2n-4$ nonzero entries with at least one nonzero principal minor of every order of the correct sign.

Example 4.19. Consider the following matrix and its digraph.

$$\mathcal{A} = \begin{bmatrix} - & + & 0 & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & - & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + \\ 0 & 0 & - & 0 & 0 & 0 & 0 \end{bmatrix}$$



Sign pattern $\mathcal{A}$ allows one nonzero principal minor of order k with sign $(-1)^k$ for $1 \leq k \leq n$. However, it is not clear if it is possible to find a stable realization.

The following theorem considers the total number of nonzero entries possible using the construction in either Theorem 3.3 or Theorem 3.7 and PS sign patterns $\mathcal{A}_i$ with $2n_i - r$ nonzero entries for $1 \leq i \leq k$.

Theorem 4.20. *If either Theorem 3.3 or Theorem 3.7 is used with PS sign patterns $\mathcal{A}_i$ of order n_i $(1 \leq i \leq k)$ to construct a PS sign pattern $\mathcal{B}$ of order $N = N_k + 1$ and if A_i has $2n_i - r$ nonzero entries, then $\mathcal{B}$ has $2N - (k(r-1)+1)$ nonzero entries.*

Proof. The total number of nonzero entries in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is $2n_1 - r + \cdots + 2n_k - r = 2N_k - kr$. There are an additional $k+1$ nonzero entries in $\mathcal{B}$ for a total of $2N_k - kr + k + 1 = 2N - (k(r-1)+1)$ nonzero entries. □

The following example shows the construction of a PS sign pattern of order 13 with $2n-4$ nonzero entries using Theorem 3.7, where $k=3$, $n_1 = n_2 = n_3 = 4$ and $r = 2$.

Example 4.21. Consider matrix

$$C = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{aligned} \sigma(C) \approx \ & \{-0.2798 \pm 1.6082i, -0.4949 \pm 1.5193i, -0.2697 \pm 1.3715i, -0.0103, \\ & -0.2764 \pm 0.6871i, -0.0572 \pm 0.6923i, -0.1169 \pm 0.5120i\}. \end{aligned}$$

The sign pattern $\text{sgn}(C)$ is PS and contains $2n - 4 = 22$ nonzero entries.

The following result is similar to the previous theorem, except that the construction from Theorem 3.23 is used for the PS sign patterns $\mathcal{A}_i$ with $2n_i - r$ nonzero entries for $1 \leq i \leq k$.

Theorem 4.22. *If Theorem 3.23 is used with PS sign patterns $\mathcal{A}_i$ of order n_i ($1 \leq i \leq k$) to construct a PS sign pattern $\mathcal{B}$ of order $N = N_k + 3$ and if A_i has $2n_i - r$ nonzero entries, then $\mathcal{B}$ has $2N - (k(r-1)+2)$ nonzero entries.*

Proof. The total number of nonzero entries in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is $2n_1 - r + \cdots + 2n_k - r = 2N_k - kr$. There are an additional $k+4$ nonzero entries in $\mathcal{B}$ for a total of $2N_k - kr + k + 4 = 2N - (k(r-1)+2)$ nonzero entries. □

Corollary 4.23 is a direct consequence of Theorem 4.22 for $k = 1$.

Corollary 4.23. *For $n \geq 4$, there exist $n \times n$ irreducible PS sign patterns with $2n - (\lfloor \frac{n}{3} \rfloor + 1)$ nonzero entries.*

Proof. The proof is by induction on n. If $n = 4$, then sgn(B) of matrix B in Example 4.7 is a PS sign pattern with $2n - (\lfloor \frac{n}{3} \rfloor + 1) = 6$ nonzero entries. If $n = 5$, then sgn(X) for matrix X in Example 3.18 provides a PS sign pattern with $2n - (\lfloor \frac{n}{3} \rfloor + 1) = 8$ nonzero entries. If $n = 6$, then sgn(A) of matrix A in Example 4.12 shows a PS sign pattern with $2n - (\lfloor \frac{n}{3} \rfloor + 1) = 9$ nonzero entries. Suppose the statement is true for all $n \leq t$. Therefore, there exists a $t \times t$ PS sign pattern $\mathcal{A}$ with $2t - (\lfloor \frac{t}{3} \rfloor + 1)$ nonzero entries. Theorem 4.22 with $k = 1$, $\mathcal{A}_1 = \mathcal{A}$, $N_1 = n_1 = t$ and $r = \lfloor \frac{t}{3} \rfloor + 1$ gives $\mathcal{B}$ having

$$2(t+3) - (\lfloor \tfrac{t}{3} \rfloor + 2) = \quad 2(t+3) - (\lfloor \tfrac{t+3}{3} \rfloor + 1)$$

nonzero entries. The theorem follows by induction. □

In order to see an improvement by Theorem 4.22 over Theorem 4.20 in terms of the least number of nonzero entries in an $N \times N$ PS sign pattern, it is necessary to consider at least $N = 9$. Example 4.24 shows the construction of a PS sign pattern of order 9 with $2n - 4$ nonzero entries using Theorem 4.22, where $k = 1$, $n_1 = 6$ and $r = 3$.

Example 4.24. Consider matrix

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{20} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

with $\sigma(B) \approx \{-0.1133 \pm 1.6994i, -0.3332 \pm 1.2293i, -0.0118 \pm 1.0132, -0.0158 \pm 0.4470, -0.0517\}$. The sign pattern sgn$(B)$ is PS and contains $2n - 4 = 14$ nonzero entries.

Corollary 4.23 shows that there exist $n \times n$ irreducible PS sign patterns with $2n-3$ nonzero entries for $n = 7$ and 8. However, it is not known whether there exist any with fewer than $2n-3$ entries. Example 4.24 confirms that there is a 9×9 irreducible PS sign pattern with $2n - 4 = 14$ nonzero entries, although it is also not known if this is the minimum number for $n = 9$.

Chapter 5

Conclusions

The problem of characterizing potential stability is well known to be a difficult problem. The goals of this thesis are to develop new techniques for constructing PS sign patterns of arbitrary orders and to determine the minimum number of nonzero entries in particular PS sign patterns. The constructions described in Chapter 3 provide sufficient conditions for potential stability and can be used to expand the list of known PS sign patterns. With the exception of Theorem 3.29, the constructions do not require a stable realization of a known lower order PS sign pattern in order to construct a higher order PS sign pattern.

It is a necessary condition that any PS sign pattern of order $n \geq 2$ contains a PS sign pattern of order less than n (since every PS sign pattern must contain a negative diagonal entry). The bordering constructions in Section 3.2 perhaps suggest an approach to characterizing potential stability by considering the PS components in a given sign pattern. Each of the 3×3 minimally PS sign patterns can be constructed by one or more of the constructions in Chapter 3, although as Example 3.11 shows, it is necessary to consider sign patterns that are not minimally PS to construct $\mathcal{A}_{3,2}$ by bordering. Many 4×4 minimally PS sign patterns can be constructed by bordering. However, the 4×4 minimally PS tree sign pattern $\mathcal{A}_{4,11}$ in Section A.3 does not contain a nest, cannot be constructed by any of the bordering techniques described in Section 3.2, and there does not appear to be an obvious similarity transformation as described in Section 3.3 that produces $\mathcal{A}_{4,11}$. This raises the question of finding other constructions that could show potential stability of sign pattern $\mathcal{A}_{4,11}$.

In Chapter 4, the minimum number of nonzero entries in a PS sign pattern is considered. In Section 4.1, it is proved that the minimum number of nonzero entries in an $n \times n$ sign pattern that allows a nest is $2n - 1$. This is helpful in showing that it is impossible to locate a nest in an $n \times n$ sign pattern with $2n-2$ or fewer nonzero entries, and thus another approach to determine the potential stability of such a sign pattern is required. Sign pattern $\mathcal{A}_{3,4}$ in Section A.2 shows that a sign pattern having $2n-1$ nonzero entries is not a sufficient condition for the sign pattern to allow a nest. Furthermore, a sign pattern allowing a nest is not a sufficient condition that the sign pattern is minimally PS as is shown by Example 4.7. Section 4.2 continues with the topic of nonzero entries, this time considering minimally PS sign patterns. It is shown that an $n \times n$ minimally PS sign pattern has at least $2n-1$ nonzero entries for $n = 2$ or 3, at least $2n-2$ nonzero entries for $n = 4$ or 5 and at least $2n-3$ nonzero entries for $n = 6$. Minimally PS sign patterns with certain diagonal patterns are also considered in Section 4.2. It is proved that an $n \times n$ PS sign pattern with $n-3$ negative diagonal entries has at least $2n-2$ nonzero entries and examples are given for $n = 4$ and 5. Pursuing this idea further, it is shown that an $n \times n$ PS sign pattern with $n-4$ negative diagonal entries has at least $2n-3$ nonzero entries. However, Theorem 4.18 shows that when $n = 6$, a sign pattern with $n-4$ negative diagonal entries and $2n-3$ nonzero entries is sign unstable. It is not known, in general, what the minimum number of nonzero entries is in an $n \times n$ minimally PS sign pattern for $n \geq 7$. Constructions are given that can generate $n \times n$ PS sign patterns with $2n - (\lfloor \frac{n}{3} \rfloor + 1)$ nonzero entries. Prior to this work, all of the $n \times n$ PS sign patterns in the literature had at least $2n-2$ nonzero entries.

The results of this thesis suggest the following open problems for future research.

1. Generate 4×4 PS sign patterns from the constructions in Chapter 3 and determine those that are minimally PS.

2. Determine if there exists an $n \times n$ PS sign pattern with $2n-3$ nonzero entries, $n-4$ of which are on the diagonal.

3. Develop constructions that produce all minimally PS sign patterns.

4. Determine the least number of nonzero entries in an $n \times n$ minimally PS sign pattern for $n \geq 7$.

5. Develop other tests for determining minimality of a PS sign pattern. This is important since any superpattern of a PS sign pattern is PS.

Bibliography

[1] C. S. Ballantine, *Stabilization by a diagonal matrix*, Proceedings of the American Mathematical Society **25** (1970), no. 4, 728–734.

[2] L. Bassett, J. Maybee, and J. Quirk, *Qualitative economics and the scope of the correspondence principle*, Econometrica **36** (1968), no. 3/4, 544–563.

[3] T. Bone, *Positive feedback may sometimes promote stability*, Linear Algebra and its Applications **51** (1983), 143 – 151.

[4] T. Britz, J. J. McDonald, D. D. Olesky, and P. van den Driessche, *Minimal spectrally arbitrary sign patterns*, SIAM Journal on Matrix Analysis and Applications **26** (2004), no. 1, 257–271.

[5] M. E. Fisher and A. T. Fuller, *On the stabilization of matrices and the convergence of linear iterative processes*, Proceedings of the Cambridge Philosophical Society **54** (1958), no. 4, 417–425.

[6] F. R. Gantmacher, *The Theory of Matrices*, vol. 2, Chelsea Pub. Co, New York, 1959.

[7] Y. Gao and J. Li, *On the potential stability of star sign pattern matrices*, Linear Algebra and its Applications **327** (2001), no. 1-3, 61–68.

[8] O. Holtz, *Hermite-Biehler, Routh-Hurwitz, and total positivity*, Linear Algebra and its Applications **372** (2003), 105–110.

[9] Roger A. Horn and Charles R. Johnson, *Matrix Analysis*, reprinted with corrections ed., Cambridge University Press, Cambridge; New York, 1990.

[10] C. Jeffries and C. R. Johnson, *Some sign patterns that preclude matrix stability*, SIAM Journal on Matrix Analysis and Applications **9** (1988), no. 1, 19–25.

[11] C. Jeffries, V. Klee, and P. van den Driessche, *Qualitative stability of linear systems*, Linear Algebra and its Applications **87** (1987), 1 – 48.

[12] C. R. Johnson, J. S. Maybee, D. D. Olesky, and P. van den Driessche, *Nested sequences of principal minors and potential stability*, Linear Algebra and its Applications **262** (1997), no. 1-3, 243–257.

[13] C. R. Johnson and T. A. Summers, *The potentially stable tree sign patterns for dimensions less than five*, Linear Algebra and its Applications **126** (1989), 1 – 13.

[14] I.-J. Kim, D. D. Olesky, B. L. Shader, P. van den Driessche, H. van der Holst, and K. N. Vander Meulen, *Generating potentially nilpotent full sign patterns*, Electronic Journal of Linear Algebra **18** (2009), 162–175.

[15] V. Klee, *Sign-patterns and stability. In: F. Roberts (Ed.) Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, IMA Volumes in Mathematics and Applications **17** (1989), 203–219.

[16] V. Klee and P. van den Driessche, *Linear algorithms for testing the sign stability of a matrix and for finding z-maximum matchings in acyclic graphs*, Numerische Mathematik **28** (1977), no. 3, 273–285.

[17] Q. Lin, D. D. Olesky, and P. van den Driessche, *The distance of potentially stable sign patterns to the unstable matrices*, SIAM Journal on Matrix Analysis and Applications **24** (2002), no. 2, 356–367.

[18] J. Maybee and J. Quirk, *Qualitative problems in matrix theory*, SIAM Review **11** (1969), no. 1, 30–51.

[19] J. Miyamichi, *Sign structures of* 3×3 *stable matrices and their generalization to higher-order matrices*, Electronics and Communications in Japan **71** (1988), no. 11, 63–73.

[20] J. Quirk, *The correspondence principle: A macroeconomic application*, International Economic Review **9** (1968), no. 3, 294–306.

[21] J. Quirk and R. Ruppert, *Qualitative economics and the stability of equilibrium*, The Review of Economic Studies **32** (1965), no. 4, 311–326.

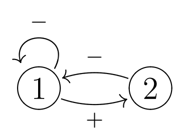
[22] P. A. Samuelson, *Foundations of Economic Analysis*, Harvard University Press, Cambridge, 1955.

Appendix A

The first two sections in the appendix give up to equivalence all of the $n \times n$ minimally PS sign patterns for $n = 2, 3$. Section A.3 gives up to equivalence all of the 4×4 minimally PS tree sign patterns.

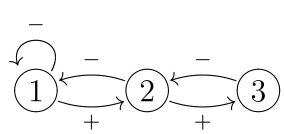
A.1 2×2 minimally PS sign pattern

$$\mathcal{A}_{2,1} = \begin{bmatrix} - & + \\ - & 0 \end{bmatrix}$$

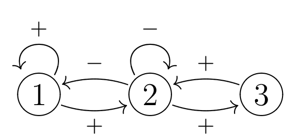


A.2 3×3 minimally PS sign patterns from [19]

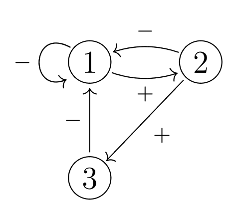
$$\mathcal{A}_{3,1} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & 0 \end{bmatrix}$$



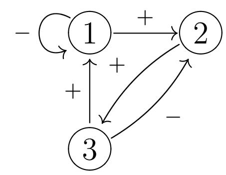
$$\mathcal{A}_{3,2} = \begin{bmatrix} + & + & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix}$$



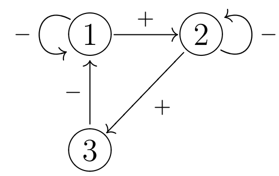
$$\mathcal{A}_{3,3} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & 0 \end{bmatrix}$$



$$\mathcal{A}_{3,4} = \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}$$

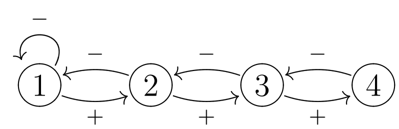


$$\mathcal{A}_{3,5} = \begin{bmatrix} - & + & 0 \\ 0 & - & + \\ - & 0 & 0 \end{bmatrix}$$

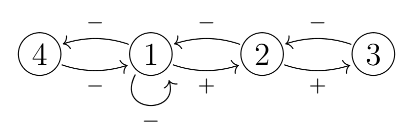


A.3 4×4 minimally PS tree sign patterns from [17]

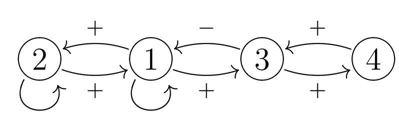
$$\mathcal{A}_{4,1} = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}$$



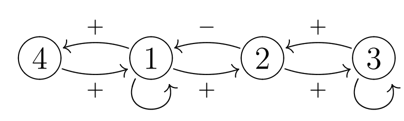
$$\mathcal{A}_{4,2} = \begin{bmatrix} - & + & 0 & + \\ - & 0 & + & 0 \\ 0 & - & 0 & 0 \\ - & 0 & 0 & 0 \end{bmatrix}$$



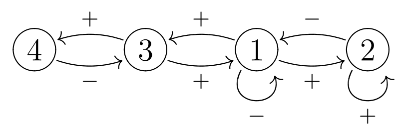
$$\mathcal{A}_{4,3} = \begin{bmatrix} - & + & + & 0 \\ + & - & 0 & 0 \\ - & 0 & 0 & + \\ 0 & 0 & + & 0 \end{bmatrix}$$



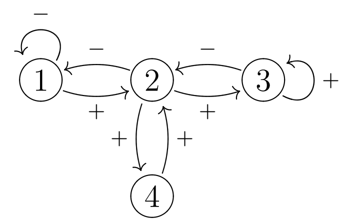
$$\mathcal{A}_{4,4} = \begin{bmatrix} - & + & 0 & + \\ - & 0 & + & 0 \\ 0 & + & - & 0 \\ + & 0 & 0 & 0 \end{bmatrix}$$



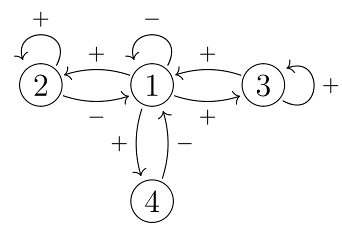
$$\mathcal{A}_{4,5} = \begin{bmatrix} - & + & + & 0 \\ - & + & 0 & 0 \\ + & 0 & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}$$



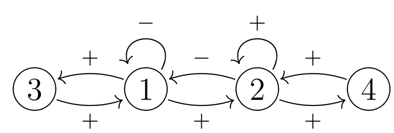
$$\mathcal{A}_{4,6} = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & + \\ 0 & - & + & 0 \\ 0 & + & 0 & 0 \end{bmatrix}$$



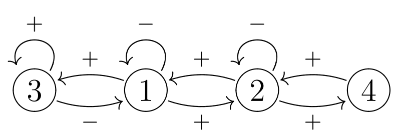
$$\mathcal{A}_{4,7} = \begin{bmatrix} - & + & + & + \\ - & + & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & 0 \end{bmatrix}$$



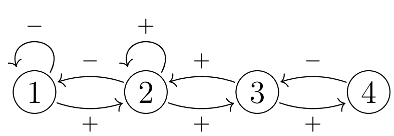
$$\mathcal{A}_{4,8} = \begin{bmatrix} - & + & + & 0 \\ - & + & 0 & + \\ + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \end{bmatrix}$$



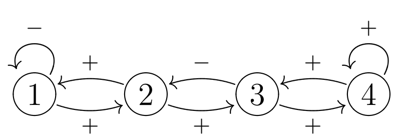
$$\mathcal{A}_{4,9} = \begin{bmatrix} - & + & + & 0 \\ + & - & 0 & + \\ - & 0 & + & 0 \\ 0 & + & 0 & 0 \end{bmatrix}$$



$$\mathcal{A}_{4,10} = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}$$



$$\mathcal{A}_{4,11} = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}$$



A.4 Higher order PS sign patterns from [19]