AN APPLICATION OF A CERTAIN FRACTIONAL DERIVATIVE OPERATOR

by

SHIGEYOSHI OWA AND H.M. SRIVASTAVA

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The object of the present paper is to introduce and study a linear operator $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$ which is defined in terms of a certain fractional derivative operator. Various interesting properties of the operator $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$, including its connection with the Carlson–Shaffer operator $\mathcal{L}(a,c)$, are given. It is also shown how these operators can be applied successfully with a view to proving a number of inclusion and connection theorems involving starlike, convex, and prestarlike functions in the open unit disk $\mathcal{U}$.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : |z| < 1\}.$$

A function $f(z) \in \mathcal{A}$ is said to be *starlike of order* α if it satisfies the inequality:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α.

Furthermore, a function $f(z) \in \mathcal{A}$ is said to be *convex of order* α if it satisfies the inequality:

$$\text{(1.3)} \qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions which are convex of order α.

Throughout this paper, it should be understood that functions such as

$$\frac{zf'(z)}{f(z)} \quad \text{and} \quad \frac{zf''(z)}{f'(z)} ,$$

which have *removable singularities* at $z = 0$, have had these singularities removed in statements like (1.2) and (1.3).

It follows readily from (1.2) and (1.3) that (*cf*. Duren [2, p.43, Theorem 2.12] for the special case $\alpha = 0$)

$$\text{(1.4)} \qquad f(z) \in \mathcal{K}(\alpha) \Leftrightarrow z\, f'(z) \in \mathcal{S}^*(\alpha).$$

For the functions $f_j(z)$ defined by

$$\text{(1.5)} \qquad f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1}\, z^{n+1} \qquad (j = 1,2),$$

we denote by $f_1 * f_2(z)$ the Hadamard product or convolution of the functions

$f_1(z)$ and $f_2(z)$, that is,

$$f_1 \star f_2(z) = \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1}\, z^{n+1}. \tag{1.6}$$

With a view to introducing the Carlson–Shaffer operator $\mathscr{L}(a,c)$, we define the function $\varphi(a,c;z)$ by

$$\varphi(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0,-1,-2,\cdots;\ z \in \mathcal{U}), \tag{1.7}$$

where $(\lambda)_n$ is the Pochhammer symbol given by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1,2,3,\cdots. \end{cases} \tag{1.8}$$

Clearly, the function $\varphi(a,c;z)$ is an incomplete Beta function with

$$\varphi(a,c;z) = z\, F(1,a;c;z) \tag{1.9}$$

in terms of the Gaussian hypergeometric function $F(\alpha,\beta;\gamma;z)$ defined by

$$F(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \qquad (c \neq 0,-1,-2,\cdots;\ z \in \mathcal{U}). \tag{1.10}$$

Making use of the function $\varphi(a,c;z)$, Carlson and Shaffer [1] introduced a linear operator $\mathscr{L}(a,c)$ on $\mathcal{A}$ by the convolution:

(1.11) $$\mathscr{L}(a,c)\ f(z) = \varphi(a,c;z)\star f(z) \qquad (f(z) \in \mathcal{A}).$$

We observe that $\mathscr{L}(a,c)$ maps $\mathcal{A}$ onto itself. Moreover, if

$$a \neq 0,-1,-2,\cdots,$$

then $\mathscr{L}(c,a)$ is an inverse of $\mathscr{L}(a,c)$. Note also that (*cf.* [3, p. 1067])

(1.12) $$\mathscr{K}(\alpha) = \mathscr{L}(1,2)\ \mathscr{S}^*(\alpha) \qquad (0 \leqq \alpha < 1)$$

and

(1.13) $$\mathscr{S}^*(\alpha) = \mathscr{L}(2,1)\ \mathscr{K}(\alpha) \qquad (0 \leqq \alpha < 1).$$

Next we introduce the operator $\mathscr{N}_{0,z}^{\alpha,\beta,\eta}$, which is related rather closely to the fractional differential operator $\mathscr{J}_{0,z}^{\alpha,\beta,\eta}$ considered by Sohi [5]. Indeed we have

(1.14) $$\mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ f(z) = \frac{\Gamma(2-\beta)\Gamma(3-\alpha+\eta)}{\Gamma(3-\beta+\eta)}\ z^{\beta}\mathscr{J}_{0,z}^{\alpha,\beta,\eta}\ f(z) \qquad (f(z) \in \mathcal{A}),$$

where the fractional differential operator $\mathscr{J}_{0,z}^{\alpha,\beta,\eta}$ is defined (for real numbers α, β, and η) by (see also [6])

$$(1.15)\qquad \mathcal{J}_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-\beta} \int_0^z (z-\zeta)^{-\alpha} \cdot F\left[\beta-\alpha, -\eta, 1-\alpha; 1 - \frac{\zeta}{z}\right] f(\zeta)\, d\zeta \right\}$$

$$(0 \leqq \alpha < 1;\ \epsilon > \max\{0, \beta-\eta-1\} - 1),$$

$f(z)$ being an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}) \qquad (z \longrightarrow 0),$$

and the multiplicity of $(z-\zeta)^{-\alpha}$ being removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

In this paper we present several interesting properties and characteristics of the linear operator $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$ and apply this operator in conjunction with the Carlson-Shaffer operator $\mathcal{L}(a,c)$ to prove a number of inclusion and connection theorems involving, for example, the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

2. PROPERTIES OF THE LINEAR OPERATOR $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$

We begin by proving an interesting relationship between the operators $\mathcal{N}_{0,z}^{\alpha,\beta,\eta}$ and $\mathcal{L}(a,c)$, which is contained in

Lemma 1. *If* $0 \leqq \alpha < 1$ *and* $\beta - \eta < 3$, *then*

$$(2.1)\qquad \mathcal{N}_{0,z}^{\alpha,\beta,\eta}\, f(z) = \mathcal{L}(2,2-\beta)\ \mathcal{L}(3-\beta+\eta,3-\alpha+\eta)\ f(z) \qquad (f(z)\in\mathcal{A}).$$

Proof. It follows from the definition (1.15) that

$$(2.2)\qquad \mathcal{J}_{0,z}^{\alpha,\beta,\eta}\, z^k = \frac{\Gamma(k+1)\Gamma(k+2-\beta+\eta)}{\Gamma(k+1-\beta)\Gamma(k+2-\alpha+\eta)}\, z^{k-\beta} \qquad (k+2>\beta-\eta),$$

which, in view of (1.1), yields

$$(2.3)\qquad \mathcal{N}_{0,z}^{\alpha,\beta,\eta}\, f(z) = z + \sum_{n=2}^{\infty} \frac{(2)_{n-1}(3-\beta+\eta)_{n-1}}{(2-\beta)_{n-1}(3-\alpha+\eta)_{n-1}}\, z^n$$

$$= \sum_{n=0}^{\infty} \frac{(2)_n(3-\beta+\eta)_n}{(2-\beta)_n(3-\alpha+\eta)_n}\, z^{n+1}$$

$$= \mathcal{L}(2,2-\beta)\ \mathcal{L}(3-\beta+\eta,3-\alpha+\eta)\ f(z),$$

where we have also employed the definition (1.11).

Next we recall the following lemma due essentially to Carlson and Shaffer [1], which will be required in our present investigation (see also Owa and Srivastava [3, p. 1067, Remark 6]).

Lemma 2. *If* $\alpha \leqq \beta < 1$ *and* $0 \leqq \alpha < 1$, *then*

$$(2.4)\qquad \mathcal{L}(2-2\beta,2-2\alpha)\ \mathcal{S}^*(\alpha) \subset \mathcal{S}^*(\beta) \subset \mathcal{S}^*(\alpha).$$

Making use of Lemma 1 and Lemma 2, we now prove an interesting inclusion property of the operators $\mathscr{N}_{0,z}^{\alpha,\beta,\eta}$ and $\mathscr{L}(a,c)$. We first state our result as

Theorem 1. *If* $0 \leqq \alpha < 1$, $\beta-\eta < 3$, *and* $0 \leqq \beta < 1$, *then*

$$\mathscr{L}(3-\alpha+\eta,3-\beta+\eta)\ \mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}\left[\frac{1}{2}\right] \subset \mathscr{S}^*\left[\frac{1}{2}\right]. \tag{2.5}$$

Proof. It is easy to see that, for $0 \leqq \gamma < 1$,

$$\begin{aligned}\mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}(\gamma) &= \mathscr{L}(2,2-\beta)\ \mathscr{L}(3-\beta+\eta,3-\alpha+\eta)\ \mathscr{K}(\gamma)\\ &= \mathscr{L}(2,2-\beta)\ \mathscr{L}(3-\beta+\eta,3-\alpha+\eta)\ \mathscr{L}(1,2)\ \mathscr{S}^*(\gamma)\\ &= \mathscr{L}(1,2-\beta)\ \mathscr{L}(3-\beta+\eta,3-\alpha+\eta)\ \mathscr{S}^*(\gamma).\end{aligned}$$

Therefore, we have

$$\mathscr{L}(3-\alpha+\eta,3-\beta+\eta)\ \mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}(\gamma) = \mathscr{L}(1,2-\beta)\ \mathscr{S}^*(\gamma). \tag{2.6}$$

Since

$$\mathscr{S}^*\left[\frac{1}{2}\right] \subset \mathscr{S}^*\left[\frac{\beta}{2}\right] \qquad (0 \leqq \beta < 1), \tag{2.7}$$

we have

$$\mathscr{L}(1,2-\beta)\ \mathscr{S}^*\left[\frac{1}{2}\right] \subset \mathscr{L}(1,2-\beta)\ \mathscr{S}^*\left[\frac{\beta}{2}\right] \qquad (0 \leqq \beta < 1). \tag{2.8}$$

Thus, by an application of Lemma 2, we obtain

$$\mathscr{L}(1,2-\beta)\ \mathscr{S}^*\left[\frac{\beta}{2}\right] \subset \mathscr{S}^*\left[\frac{1}{2}\right] \subset \mathscr{S}^*\left[\frac{\beta}{2}\right]. \tag{2.9}$$

Finally, on setting $\gamma = \frac{1}{2}$ in (2.6), we complete the proof of Theorem 1.

3. AN APPLICATION INVOLVING PRESTARLIKE FUNCTIONS

A function $f(z) \in \mathcal{A}$ is said to be *prestarlike of order* α $(\alpha \leqq 1)$ if and only if

$$\begin{cases} \dfrac{z}{(1-z)^{2(1-\alpha)}} \star f(z) \in \mathscr{S}^*(\alpha) & (\text{for } \alpha < 1) \\ \operatorname{Re}\left\{\dfrac{f(z)}{z}\right\} > \dfrac{1}{2} & (\text{for } \alpha = 1). \end{cases} \tag{3.1}$$

We denote by $\mathscr{R}(\alpha)$ the subclass of $\mathcal{A}$ consisting of all prestarlike functions of order α. The class $\mathscr{R}(\alpha)$ was first introduced by Ruscheweyh [4].

In view of the definition (3.1) for the class $\mathscr{R}(\alpha)$, we have

$$\mathscr{R}(\alpha) = \mathscr{L}(1,2-2\alpha)\ \mathscr{S}^*(\alpha) \qquad (\alpha < 1) \tag{3.2}$$

and

$$(3.3)\qquad \mathscr{R}(1) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \frac{1}{2} \qquad (z \in \mathcal{U}) \right\}.$$

The following result provides a connection theorem involving the classes $\mathscr{K}(\alpha)$ and $\mathscr{R}(\alpha)$.

Theorem 2. *If* $0 \leqq \alpha < 1$, $\beta - \eta < 3$, *and* $0 \leqq \beta < 2$, *then*

$$(3.4)\qquad \mathscr{L}(3-\alpha+\eta, 3-\beta+\eta)\ \mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}\left[\frac{\beta}{2}\right] = \mathscr{R}\left[\frac{\beta}{2}\right].$$

Proof. Since

$$\mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}\left[\frac{\beta}{2}\right] = \mathscr{L}(2,2-\beta)\ \mathscr{L}(3-\beta+\eta, 3-\alpha+\eta)\ \mathscr{S}^*\left[\frac{1}{2}\right]$$

$$= \mathscr{L}(1,2-\beta)\ \mathscr{L}(3-\beta+\eta, 3-\alpha+\eta)\ \mathscr{S}^*\left[\frac{\beta}{2}\right],$$

we obtain

$$\mathscr{L}(3-\alpha+\eta, 3-\beta+\eta)\ \mathscr{N}_{0,z}^{\alpha,\beta,\eta}\ \mathscr{K}\left[\frac{\beta}{2}\right] = \mathscr{L}(1,2-\beta)\ \mathscr{S}^*\left[\frac{\beta}{2}\right]$$

$$= \mathscr{R}\left[\frac{\beta}{2}\right] \qquad (0 \leqq \beta < 2),$$

which proves the assertion (3.4) of Theorem 2.

Taking $\beta = 0$ in Theorem 2, we have

Corollary 1. *If* $0 \leqq \alpha < 1$ *and* $\eta > -3$, *then*

$$\mathscr{L}(3-\alpha+\eta,3+\eta)\ \mathscr{N}_{0,z}^{\alpha,0,\eta}\ \mathscr{K}(0) = \mathscr{R}(0). \tag{3.5}$$

Finally, setting $\beta = 1$ in Theorem 2, we deduce

Corollary 2. *If* $0 \leqq \alpha < 1$ *and* $\eta > -2$, *then*

$$\mathscr{L}(3-\alpha+\eta,2+\eta)\ \mathscr{N}_{0,z}^{\alpha,1,\eta}\ \mathscr{K}\left(\frac{1}{2}\right) = \mathscr{R}\left(\frac{1}{2}\right). \tag{3.6}$$

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Shigeyoshi Owa:

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

H.M. Srivastava:

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada