CERTAIN TRANSFORMATION AND REDUCTION FORMULAS FOR HYPERGEOMETRIC SERIES IN SEVERAL VARIABLES

by

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An (alternative) elementary proof of a certain hypergeometric identity, which was posed recently as a problem, leads naturally first to a multiple–series extension of the problem and then also to its related multivariable hypergeometric transformation and summation formulas. In this paper we aim at presenting a systematic analysis of several general multiple–series identities analogous to these hypergeometric transformations. Many of the general results obtained here are shown to unify and extend numerous transformation and reduction formulas for various classes of double (and multiple) hypergeometric series.

1. INTRODUCTION AND PRELIMINARIES

In terms of the Pochhammer symbol $(\lambda)_n$ given by

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$$(1) \qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall n \in \mathbb{N} = \{1,2,3,\cdots\}, \end{cases}$$

let ${}_pF_q(z)$ denote a generalized hypergeometric series in z with p numerator and q denominator parameters, defined by (*cf.*, *e.g.*, [3, Chapter 2])

$$(2) \qquad {}_pF_q(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z) = {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p; \\ \\ \beta_1,\cdots,\beta_q;\end{matrix}\quad z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!},$$

provided that the series converges (or terminates). Also let

$$F^{p_0:p_1;\cdots;p_r}_{q_0:q_1;\cdots;q_r}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix}$$

denote a generalized multiple hypergeometric series in r variables $z_1,\cdots,z_r$, defined by (*cf.* [17, p. 38, Equation 1.4(38)])

$$(3) \quad F^{p_0:p_1;\cdots;p_r}_{q_0:q_1;\cdots;q_r}\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} \equiv F^{p_0:p_1;\cdots;p_r}_{q_0:q_1;\cdots;q_r}\left[\begin{matrix}(a_{p_0}):(b'_{p_1});\cdots;(b^{(r)}_{p_r}); \\ \\ (c_{q_0}):(d'_{q_1});\cdots;(d^{(r)}_{q_r});\end{matrix}\quad z_1,\cdots,z_r\right]$$

$$= \sum_{m_1,\cdots,m_r=0}^{\infty} \frac{\prod_{j=1}^{p_0} (a_j)_{m_1+\cdots+m_r} \prod_{j=1}^{p_1} (b'_j)_{m_1} \cdots \prod_{j=1}^{p_r} (b^{(r)}_j)_{m_r}}{\prod_{j=1}^{q_0} (c_j)_{m_1+\cdots+m_r} \prod_{j=1}^{q_1} (d'_j)_{m_1} \cdots \prod_{j=1}^{q_r} (d^{(r)}_j)_{m_r}}$$

$$\cdot \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!},$$

where, for convenience, (a_p) abbreviates the array of p parameters

$$a_1, \cdots, a_p,$$

with similar interpretations for (b_q), *et cetera.*

Making use of the case $r = 2$ of Definition (3), we begin by recalling the following interesting problem (posed recently by Grosjean [6]):

$$\text{(4)} \qquad F^{1:2;2}_{1:1;1}\left[\begin{matrix} -2k-1\colon\ 2a, b;\ 2c, d; \\ \\ a+c-k\colon\quad 2b;\quad 2d; \end{matrix}\ \ 1, 1\right] = 0 \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

provided that no zeros appear in the denominator of the double hypergeometric series defining the left–hand side of (4). In his published solution of Grosjean's problem (4), Lossers (*cf.* [6, p. 496]) just set

$$\text{(5)} \qquad \lambda_1 = b, \quad \gamma_1 = 2a, \quad \lambda_2 = d, \quad \text{and} \quad \gamma_2 = 2c$$

in the known result [7, p. 119, Equation (23)]:

$$\text{(6)} \qquad F^{1:2;2}_{1:1;1}\left[\begin{matrix} -2k-1\colon\ \lambda_1, \gamma_1;\ \lambda_2, \gamma_2; \\ \\ \tfrac{1}{2}(\gamma_1+\gamma_2)-k\colon\quad 2\lambda_1;\quad 2\lambda_2; \end{matrix}\ \ 1, 1\right] = 0 \quad (k \in \mathbb{N}_0).$$

It should be remarked in passing that, although Grosjean [6] (and Grosjean and Sharma [7]) stated (4) and (6) only for positive integer values of k, each of these results holds true also when $k = 0$.

The hypergeometric identities (4) and (6), as well as their multidimensional extensions, can indeed by proven *directly* in an elementary way by using certain Eulerian integrals which represent single (and multivariable) Beta functions. The object of the present paper is to prove several general multiple–series identities analogous to the multivariable hypergeometric transformation, reduction, and summation formulas which stem from the aforementioned works of Grosjean *et al.* ([6], [7]).

2. A GENERAL MULTIPLE–SERIES IDENTITY AND ITS CONSEQUENCES

We begin by proving a general multiple–series identity contained in

THEOREM 1. *Let* $\{\omega(m;n_1,\cdots,n_r)\}$ *be a bounded multiple sequence of essentially arbitrary complex numbers. Also let*

$$(7)\qquad \Sigma m_i = m_1 + \cdots + m_r,\quad \Sigma n_i = n_1 + \cdots + n_r,\quad \Sigma j_i = j_1 + \cdots + j_r,$$

et cetera.

Then

$$(8)\qquad \sum_{m_1,\cdots,m_r=0}^{\infty} \omega(\Sigma m_i;\ m_1,\cdots,m_r) \prod_{i=1}^{r} \left\{ \frac{(\lambda_i)_{m_i}}{(2\lambda_i)_{m_i}} \frac{x_i^{m_i}}{m_i!} \right\}$$

$$= \sum_{m_1,j_1,\cdots,m_r,\,j_r=0}^{\infty} \omega(\Sigma m_i + 2\Sigma j_i;\ m_1+2j_1,\cdots,m_r+2j_r)$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{1}{(\lambda_i+\frac{1}{2})_{j_i}} \frac{(\frac{1}{2}x_i)^{m_i}}{m_i!} \frac{(\frac{1}{4}x_i)^{2j_i}}{j_i!} \right\}$$

$$[2\lambda_i \neq 0,-1,-2,\cdots \quad (i = 1,\cdots,r)],$$

provided that each of the series involved converges absolutely.

Proof. The proof of the multiple–series identity (8) is based upon the following known special case of the Gauss summation theorem [13, p. 49]:

$$(9)\qquad {}_2F_1\left(-\tfrac{1}{2}m, -\tfrac{1}{2}m+\tfrac{1}{2}; \lambda+\tfrac{1}{2}; 1\right) = \frac{2^m(\lambda)_m}{(2\lambda)_m} \qquad (m \in \mathbb{N}_0),$$

which may be rewritten in the equivalent form:

$$(10)\qquad \frac{(\lambda)_m}{(2\lambda)_m} = \sum_{j=0}^{[m/2]} \frac{m!}{(m-2j)!\,j!}\,\frac{2^{-m-2j}}{(\lambda+\frac{1}{2})_j} \qquad (m \in \mathbb{N}_0).$$

Denoting, for convenience, the first member of the assertion (8) by $\mathscr{S}$, and making use of (10) to express the product in $\mathscr{S}$ as a multiple sum, we have

$$(11)\qquad \mathscr{S} = \sum_{m_1,\cdots,m_r=0}^{\infty} \omega(\Sigma m_i; m_1,\cdots,m_r) \cdot \prod_{i=1}^{r} \left\{ \sum_{j_i=0}^{[m_i/2]} \frac{x_i^{m_i}}{(m_i-2j_i)!\,j_i!}\,\frac{2^{-m_i-2j_i}}{(\lambda_i+\frac{1}{2})_{j_i}} \right\}.$$

Now invert the order of the multiple m– and j–series in (11), and the second member of (8) will follow readily under the conditions stated with Theorem 1.

Some interesting consequences of Theorem 1 are worthy of note. First of all, if

$$(12)\qquad \omega(m; n_1,\cdots,n_r) = \Omega(m) \qquad (\forall m, n_1,\cdots,n_r \in \mathbb{N}_0),$$

the m–series on the right–hand side of (8) can be simplified considerably by appealing to the elementary identity:

$$(13)\qquad \sum_{m_1,\cdots,m_r=0}^{\infty} f(m_1+\cdots+m_r)\,\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}$$

$$=\sum_{m=0}^{\infty} f(m)\,\frac{(z_1+\cdots+z_r)^m}{m!},$$

and Theorem 1 yields

Theorem 2. *Let* $\{\Omega(n)\}$ *be a bounded sequence of essentially arbitrary complex numbers. Also let* Σm_i *and* Σj_i *be given by Equation* (7).
Then

$$(14)\qquad \sum_{m_1,\cdots,m_r=0}^{\infty} \Omega(\Sigma m_i)\prod_{i=1}^{r}\left\{\frac{(\lambda_i)_{m_i}}{(2\lambda_i)_{m_i}}\,\frac{x_i^{m_i}}{m_i!}\right\}$$

$$=\sum_{m,j_1,\cdots,j_r=0}^{\infty} \Omega(m+2\Sigma j_i)\,\frac{\{\frac{1}{2}(x_1+\cdots+x_r)\}^m}{m!}$$

$$\cdot\prod_{i=1}^{r}\left\{\frac{(\frac{1}{4}x_i)^{2j_i}}{j_i!\,(\lambda_i+\frac{1}{2})_{j_i}}\right\}$$

$$[2\lambda_i\neq 0,-1,-2,\cdots\quad (i=1,\cdots,r)],$$

provided that each of the series involved converges absolutely.

Next we set

$$(15)\qquad \omega(m;n_1,\cdots,n_r)=\Omega(m)(\gamma_1)_{n_1}\cdots(\gamma_r)_{n_r}$$

$$(\forall m,n_1,\cdots,n_r\in\mathbb{N}_0)$$

and

$$(16)\qquad x_i = x \quad (i = 1,\cdots,r)$$

in Theorem 1. Making use of the series identity (*cf.*, *e.g.*, [11, p. 166, Theorem 2]; see also [17, p. 39, Equation 1.4(32)]):

$$(17)\qquad \sum_{m_1,\cdots,m_r=0}^{\infty} f(m_1+\cdots+m_r)\,(\mu_1)_{m_1}\cdots(\mu_r)_{m_r}\,\frac{z^{m_1+\cdots+m_r}}{m_1!\cdots m_r\,!}$$

$$= \sum_{m=0}^{\infty} f(m)\,(\mu_1+\cdots+\mu_r)_m\,\frac{z^m}{m\,!}$$

in order to simplify the resulting multiple m–series on the right–hand side of (8), we shall obtain

Theorem 3. *Under the hypotheses of Theorem* 2,

$$(18)\qquad \sum_{m_1,\cdots,m_r=0}^{\infty} \Omega(\Sigma m_i)\prod_{i=1}^{r}\left\{\frac{(\lambda_i)_{m_i}\,(\gamma_i)_{m_i}}{(2\lambda_i)_{m_i}}\,\frac{x^{m_i}}{m_i!}\right\}$$

$$= \sum_{m,j_1,\cdots,j_r=0}^{\infty} \Omega(m+2\Sigma j_i)\,(\Sigma\gamma_i+2\Sigma j_i)_m\,\frac{(\frac{1}{2}x)^m}{m!}$$

$$\cdot\prod_{i=1}^{r}\left\{\frac{(\gamma_i)_{2j_i}}{(\lambda_i+\frac{1}{2})_{j_i}}\,\frac{(\frac{1}{4}x)^{2j_i}}{j_i!}\right\}$$

$$[2\lambda_i \neq 0,-1,-2,\cdots \quad (i = 1,\cdots,r)],$$

provided that each of the series involved converges absolutely.

For $r=2$, the assertion (14) of Theorem 2 assumes the form:

$$(19)\qquad \sum_{m,n=0}^{\infty} \Omega(m+n)\,\frac{(\lambda)_m(\mu)_n}{(2\lambda)_m(2\mu)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

$$= \sum_{m,p,q=0}^{\infty} \frac{\Omega(m+2p+2q)}{(\lambda+\frac{1}{2})_p(\mu+\frac{1}{2})_q}\,\frac{\{\frac{1}{2}(x+y)\}^m}{m!}\,\frac{(\frac{1}{4}x)^{2p}}{p!}\,\frac{(\frac{1}{4}y)^{2q}}{q!}$$

$$(2\lambda,2\mu \neq 0,-1,-2,\cdots).$$

The (p,q)–series in (19) can be transformed by appealing to the identity [18, p. 11, Equation (9)]

$$(20)\qquad \sum_{m,n=0}^{\infty} \frac{f(m+n)}{(\rho)_m(\sigma)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

$$= \sum_{m,n=0}^{\infty} f(m+n)\,\frac{(\rho+\sigma+m+n-1)_n}{(\rho)_{m+n}(\sigma)_n}\,\frac{(x-y)^m}{m!}\,\frac{y^n}{n!},$$

and we thus have

Corollary 2.1. *Let $\{\Omega(n)\}$ be a bounded sequence of essentially arbitrary complex numbers. Then*

$$(21)\qquad \sum_{m,n=0}^{\infty} \Omega(m+n)\,\frac{(\lambda)_m(\mu)_n}{(2\lambda)_m(2\mu)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

$$= \sum_{m,p,q=0}^{\infty} \Omega(m+2p+2q)\,\frac{(\lambda+\mu+p+q)_q}{(\lambda+\frac{1}{2})_{p+q}(\mu+\frac{1}{2})_q}$$

$$\cdot\frac{\{\tfrac{1}{2}(x+y)\}^m}{m!}\frac{\{\tfrac{1}{16}(x^2-y^2)\}^p}{p!}\frac{(\tfrac{1}{4}y)^{2q}}{q!}$$

$$(2\lambda,2\mu \neq 0,-1,-2,\cdots),$$

provided that each of the series involved converges absolutely.

The p-series in (21) reduces to its first term given by $p = 0$ when $y = \pm x$. In particular, (21) for $y = -x$ yields the identity:

$$(22)\qquad \sum_{m,n=0}^{\infty} (-1)^n\, \Omega(m+n)\, \frac{(\lambda)_m(\mu)_n}{(2\lambda)_m(2\mu)_n}\, \frac{x^{m+n}}{m!\,n!}$$

$$= \sum_{n=0}^{\infty} \Omega(2n)\, \frac{(\lambda+\mu)_{2n}}{(\lambda+\tfrac{1}{2})_n(\mu+\tfrac{1}{2})_n(\lambda+\mu)_n}\, \frac{(\tfrac{1}{4}x)^{2n}}{n!}$$

$$(2\lambda,2\mu \neq 0,-1,-2,\cdots),$$

which was proven by Buschman and Srivastava [4, p. 440, Equation (3.10)] in a markedly different manner using Watson's summation theorem [3, p. 16, Equation 3.3(1)]. On the other hand, (21) for $y = x$ assumes the form:

$$(23)\qquad \sum_{m,n=0}^{\infty} \Omega(m+n)\, \frac{(\lambda)_m(\mu)_n}{(2\lambda)_m(2\mu)_n}\, \frac{x^{m+n}}{m!\,n!}$$

$$= \sum_{m,n=0}^{\infty} \Omega(m+2n)\, \frac{(\lambda+\mu+n)_n}{(\lambda+\tfrac{1}{2})_n(\mu+\tfrac{1}{2})_n}\, \frac{x^m}{m!}\, \frac{(\tfrac{1}{4}x)^{2n}}{n!}$$

$$(2\lambda,2\mu \neq 0,-1,-2,\cdots).$$

The series identity (23) serves as a key formula from which scores of known or new hypergeometric transformation and reduction formulas can be deduced as special cases. First of all, setting

$$\Omega(n)=\frac{(\rho_1)_n\cdots(\rho_u)_n}{(\sigma_1)_n\cdots(\sigma_v)_n}\qquad(n\in\mathbb{N}_0) \tag{24}$$

in the series identity (23), we find that

$$F^{u:1;1}_{v:1;1}\left[\begin{matrix}\rho_1,\cdots,\rho_u\colon\ \lambda;\ \mu;\\ \\ \sigma_1,\cdots,\sigma_v\colon 2\lambda;\ 2\mu;\end{matrix}\ x,\ x\right] \tag{25}$$

$$=\sum_{n=0}^{\infty}\frac{(\rho_1)_{2n}\cdots(\rho_u)_{2n}\ (\lambda+\mu+n)_n}{(\sigma_1)_{2n}\cdots(\sigma_v)_{2n}\ (\lambda+\frac{1}{2})_n(\mu+\frac{1}{2})_n}\ \frac{(\frac{1}{4}x)^{2n}}{n!}$$

$$\cdot\ {}_uF_v\left[\begin{matrix}\rho_1+2n,\cdots,\rho_u+2n;\\ \\ \sigma_1+2n,\cdots,\sigma_v+2n;\end{matrix}\ x\right].$$

In each instance in which the function ${}_uF_v(x)$ in (25) can be put in a simpler form, the identity (25) will yield a hypergeometric transformation or reduction formula. Thus, for $x=\frac{1}{2}$, and

$$u-1=v=1,\quad \rho_1=\alpha,\quad \rho_2=\beta,\quad\text{and}\quad \sigma_1=\tfrac{1}{2}(\alpha+\beta+1),$$

if we apply the summation theorem [3, p. 11, Equation 2.4(2)]

$${}_2F_1\left[\begin{matrix}\alpha,\beta;\\ \\ \frac{1}{2}(\alpha+\beta+1);\end{matrix}\ \frac{1}{2}\right]=\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})}, \tag{26}$$

which is due, in fact, to Kummer [9, p. 134, Formula 2], we shall obtain

$$F^{2:1;1}_{1:1;1}\left[\begin{matrix}\alpha,\beta\colon\ \lambda;\ \mu;\\ \\ \frac{1}{2}(\alpha+\beta+1)\colon 2\lambda;\ 2\mu;\end{matrix}\ \frac{1}{2},\frac{1}{2}\right] \tag{27}$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2})}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})\Gamma(\frac{1}{2}\beta+\frac{1}{2})}\ {}_4F_3\left[\begin{array}{r} \frac{1}{2}\alpha,\ \frac{1}{2}\beta,\ \frac{1}{2}(\lambda+\mu),\ \frac{1}{2}(\lambda+\mu+1); \\ 1 \\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2},\ \lambda+\mu; \end{array}\right],$$

which provides a considerably simplified version of a result due to Grosjean and Sharma [7, p. 112, Equation (18)].

In terms of Appell's double hypergeometric function F_2 (*cf.*, *e.g.*, [1, p. 296]; see also [2, p. 14, Equation (12)]), (27) with $\alpha = \beta - 1$ immediately yields (*cf.* [7, p. 115, Equation (19)])

$$(28)\qquad F_2[\beta-1,\lambda,\mu;\ 2\lambda,2\mu;\ \tfrac{1}{2},\tfrac{1}{2}]$$

$$= 2^{\beta-1}\ {}_4F_3\left[\begin{array}{r} \frac{1}{2}(\beta-1),\ \frac{1}{2}\beta,\ \frac{1}{2}(\lambda+\mu),\ \frac{1}{2}(\lambda+\mu+1); \\ 1 \\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2},\ \lambda+\mu; \end{array}\right].$$

Next we set $x = 1$, and

$$u-1 = v = 1,\quad \rho_1 = -N\ (N \in \mathbb{N}_0),\quad \rho_2 = \beta,\ \text{and}\ \sigma_1 = \delta,$$

in the hypergeometric identity (25), and make use of the Chu–Vandermonde theorem [3, p. 3]. We thus find from (25) that

$$(29)\qquad F^{2:1;1}_{1:1;1}\left[\begin{array}{r} -N,\ \beta\colon\ \lambda;\ \mu; \\ 1,\ 1 \\ \delta\colon\ 2\lambda;\ 2\mu; \end{array}\right]$$

$$= \frac{(\delta-\beta)_N}{(\delta)_N}\ {}_6F_5\left[\begin{array}{r} -\frac{1}{2}N,\ \frac{1}{2}(1-N),\ \frac{1}{2}\beta,\ \frac{1}{2}(\beta+1),\ \frac{1}{2}(\lambda+\mu),\ \frac{1}{2}(\lambda+\mu+1); \\ 1 \\ \frac{1}{2}(\beta-\delta-N+1),\ \frac{1}{2}(\beta-\delta-N+2),\ \lambda+\frac{1}{2},\ \mu+\frac{1}{2},\ \lambda+\mu; \end{array}\right]$$

$$(N \in \mathbb{N}_0),$$

which was given recently by Grosjean and Sharma [7, p. 115, Equation (20);

pp. 116–117, Equation (21)].

Now we turn to a multiple hypergeometric identity which would result from the assertion (18) of Theorem 3 under the special case (24). Indeed we have

$$\text{(30)}\quad F^{u:2;\cdots;2}_{v:1;\cdots;1}\left[\begin{matrix}\rho_1,\cdots,\rho_u\colon \lambda_1,\ \gamma_1;\cdots;\lambda_r,\ \gamma_r;\\ \\ \sigma_1,\cdots,\sigma_v\colon \quad 2\lambda_1;\cdots;\quad 2\lambda_r;\end{matrix}\ x,\cdots,x\right]$$

$$= \sum_{j_1,\cdots,j_r=0}^{\infty} \frac{(\rho_1)_{2\Sigma j_i}\cdots(\rho_u)_{2\Sigma j_i}}{(\sigma_1)_{2\Sigma j_i}\cdots(\sigma_v)_{2\Sigma j_i}} \prod_{i=1}^{r} \left\{\frac{(\gamma_i)_{2j_i}}{(\lambda_i+\frac{1}{2})_{j_i}}\,\frac{(\frac{1}{4}x)^{2j_i}}{j_i!}\right\}$$

$$\cdot\ {}_{u+1}F_v\left[\begin{matrix}\Sigma\gamma_i+2\Sigma j_i,\ \rho_1+2\Sigma j_i,\cdots,\rho_u+2\Sigma j_i;\\ \\ \sigma_1+2\Sigma j_i,\cdots,\sigma_v+2\Sigma j_i;\end{matrix}\ \tfrac{1}{2}x\right].$$

Just as we observed in connection with (25), the multivariable identity (30) will yield a multivariable hypergeometric transformation or reduction formula in each situation in which the function ${}_{u+1}F_v(x)$ can be expressed in a closed form. For example, if in (30) we set $x = 1$, and

$$u = v = 1,\ \rho_1 = \rho,\ \text{and}\ \sigma_1 = \tfrac{1}{2}(\rho+\gamma_1+\cdots+\gamma_r+1),$$

and apply Kummer's summation theorem (26), we shall get the multivariable hypergeometric transformation:

$$\text{(31)}\quad F^{1:2;\cdots;2}_{1:1;\cdots;1}\left[\begin{matrix}\rho\colon \lambda_1,\gamma_1;\cdots;\lambda_r,\gamma_r;\\ \\ \frac{1}{2}(\rho+\gamma_1+\cdots+\gamma_r+1)\colon\quad 2\lambda_1;\cdots;\quad 2\lambda_r;\end{matrix}\ 1,\cdots,1\right]$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(\rho+\gamma_1+\cdots+\gamma_r+1)]}{\Gamma[\frac{1}{2}(\rho+1)]\Gamma[\frac{1}{2}(\gamma_1+\cdots+\gamma_r+1)]}$$

$$\cdot F^{1:2;\cdots;2}_{1:1;\cdots;1}\left[\begin{matrix} \tfrac{1}{2}\rho:\ \tfrac{1}{2}\gamma_1, \tfrac{1}{2}\gamma_1+\tfrac{1}{2};\cdots;\tfrac{1}{2}\gamma_r,\ \tfrac{1}{2}\gamma_r+\tfrac{1}{2}; \\ \\ \tfrac{1}{2}(\gamma_1+\cdots+\gamma_r+1):\quad \lambda_1+\tfrac{1}{2};\cdots;\quad \lambda_r+\tfrac{1}{2}; \end{matrix}\ 1,\cdots,1\right],$$

which provides a multidimensional extension of a result due to Grosjean and Sharma [7, p. 22, Equation (22)]. More importantly, (31) with

$$\rho = -2k-1 \qquad (k \in \mathbb{N}_0)$$

immediately yields the following multidimensional extension of the hypergeometric identity (6), and hence also of Grosjean's problem (4) above:

$$(32)\ F^{1:2;\cdots;2}_{1:1;\cdots;1}\left[\begin{matrix} -2k-1:\ \lambda_1, \gamma_1;\cdots;\lambda_r, \gamma_r; \\ \\ \tfrac{1}{2}(\gamma_1+\cdots+\gamma_r)-k:\quad 2\lambda_1;\cdots;\quad 2\lambda_r; \end{matrix}\ 1,\cdots,1\right] = 0 \quad (k \in \mathbb{N}_0).$$

Several further consequences of the multivariable hypergeometric identities (30) and (31) can be deduced in a similar manner. As an interesting illustration, let us put $\gamma_i = 2\lambda_i + 1$ $(i = 1,\cdots,r)$ in (31), and simplify the second member by applying Lauricella's result (*cf.* [10, 150]; see also [2, p. 117]):

$$(33)\qquad F_D^{(r)}[\alpha,\beta_1,\cdots,\beta_r;\ \gamma;\ 1,\cdots,1]$$

$$= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta_1-\cdots-\beta_r)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta_1-\cdots-\beta_r)}$$

$$(\mathrm{Re}(\gamma-\alpha-\beta_1-\cdots-\beta_r) > 0),$$

which incidentally is an immediate consequence of the Gauss summation theorem [3, p. 49] in view of the hypergeometric reduction formula [17, p. 39, Equation 1.4(32)]. We thus find from (31) that

$$(34)\qquad F^{1:2;\cdots;2}_{1:1;\cdots;1}\left[\begin{array}{l}\rho:\ \lambda_1,\ 2\lambda_1+1;\cdots;\lambda_r,\ 2\lambda_r+1; \\ \lambda_1+\cdots+\lambda_r+\frac{1}{2}(r+\rho+1):\quad 2\lambda_1;\cdots;\quad 2\lambda_r;\end{array}\ 1,\cdots,1\right]$$

$$=\frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(1-r-\rho)]\Gamma[\lambda_1+\cdots+\lambda_r+\frac{1}{2}(r+\rho+1)]}{\Gamma[\frac{1}{2}(\rho+1)]\Gamma[\lambda_1+\cdots+\lambda_r+\frac{1}{2}(r-\rho+1)]\Gamma[\frac{1}{2}(1-r)]}$$

$$(\mathrm{Re}(1-r-\rho)>0).$$

Formula (34) was derived, in a markedly different way, by Karlsson [8, p. 549, Equation (12)]; its special case when $r=2$ was proven earlier by Sharma [14] (and by Grosjean and Sharma [7, p. 120, Equation (25)]). In fact, Karlsson [8] deduced (34) as a special case of his reduction formula [17, p. 39, Equation 1.4(34)].

3. FURTHER TRANSFORMATION AND REDUCTION FORMULAS

By the aforementioned Chu–Vandermonde theorem [3, p. 3], it is easily observed that

$$(35)\qquad {}_2F_1(-m,-n;\alpha;1)=\frac{(\alpha)_{m+n}}{(\alpha)_m(\alpha)_n}\qquad (m,n\in\mathbb{N}_0)$$

or, equivalently,

$$(36)\qquad \frac{(\alpha)_{m+n}}{(\alpha)_m(\alpha)_n}=\sum_{j=0}^{\min(m,n)}\begin{bmatrix}m\\ j\end{bmatrix}\begin{bmatrix}n\\ j\end{bmatrix}\frac{j!}{(\alpha)_j}.$$

Making use of (36) and the series identity [16, p. 139, Theorem 2]:

$$(37)\qquad \sum_{m,n=0}^{\infty} f(m+n)\,(\rho)_m\,(\sigma)_n\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

$$= \sum_{m,n=0}^{\infty} f(m+n)\,(\rho)_m\,(\rho+\sigma+m)_n\,\frac{(x-y)^m}{m!}\,\frac{y^n}{n!},$$

it is not difficult to prove that

$$(38)\qquad \sum_{m,n=0}^{\infty} \Omega(m+n)\,\frac{(\alpha)_{m+n}(\gamma)_m(\delta)_n}{(\alpha)_m(\alpha)_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}$$

$$= \sum_{m,p,\,q=0}^{\infty} \Omega(2m+p+q)\,\frac{(\gamma)_{m+p}(\delta)_m(\gamma+\delta+2m+p)_q}{(\alpha)_m}$$

$$\cdot\,\frac{(xy)^m}{m!}\,\frac{(x-y)^p}{p!}\,\frac{y^q}{q!},$$

provided that each of the series involved converges absolutely.

For $y = x$, (38) immediately yields

$$(39)\qquad \sum_{m,n=0}^{\infty} \Omega(m+n)\,\frac{(\alpha)_{m+n}(\gamma)_m(\delta)_n}{(\alpha)_m(\alpha)_n}\,\frac{x^{m+n}}{m!\,n!}$$

$$= \sum_{m,n=0}^{\infty} \Omega(2m+n)\,\frac{(\gamma)_m(\delta)_m(\gamma+\delta+2m)_n}{(\alpha)_m}\,\frac{x^{2m+n}}{m!\;n!},$$

which, in the special case given by (24), assumes the hypergeometric form:

$$(40)\qquad F^{u+1:1;1}_{v:1;1}\left[\begin{matrix}\alpha,\,\rho_1,\cdots,\rho_u:\ \gamma;\ \delta; & \\ & x,\ x\\ \sigma_1,\cdots,\sigma_v:\ \alpha;\ \alpha; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\rho_1)_{2n}\cdots(\rho_u)_{2n}(\gamma)_n(\delta)_n}{(\sigma_1)_{2n}\cdots(\sigma_v)_{2n}(\alpha)_n}\,\frac{x^{2n}}{n!}$$

$$\cdot\ {}_{u+1}F_{v}\left[\begin{matrix}\gamma+\delta+2n,\ \rho_1+2n,\cdots,\rho_u+2n;\\ \\ \sigma_1+2n,\cdots,\sigma_v+2n;\end{matrix}\ x\right].$$

In particular, for

$$u=v=1,\quad \rho_1=\lambda,\quad \text{and}\quad \sigma_1=\mu,$$

(40) can be rewritten as

$$\text{(41)}\qquad F^{2:1;1}_{1:1;1}\left[\begin{matrix}\lambda,\alpha\colon\ \gamma;\ \delta;\\ \\ \mu\colon\ \alpha;\ \alpha;\end{matrix}\ x,\ x\right]=\sum_{n=0}^{\infty}\frac{(\lambda)_{2n}(\gamma)_n(\delta)_n}{(\mu)_{2n}(\alpha)_n}\,\frac{x^{2n}}{n!}$$

$$\cdot\ {}_2F_1(\lambda+2n,\gamma+\delta+2n;\ \mu+2n;\ x).$$

Let us now recall the following special case of a well–known analytic continuation formula [5, p. 108, Equation 2.10(1)]:

$$\text{(42)}\qquad {}_2F_1(-N,b;c;z)=\frac{(c-b)_N}{(c)_N}\ {}_2F_1(-N,b;\ 1-c+b-N;\ 1-z)$$

$$(N\in\mathbb{N}_0).$$

Applying (42) to the right–hand side of (41) when $x=\frac{1}{2}$, $\gamma=-M$, and $\delta=-N$, we have

$$\text{(43)}\qquad F^{2:1;1}_{1:1;1}\left[\begin{matrix}\lambda,\alpha\colon\ -M;\ -N;\\ \\ \mu\colon\quad \alpha;\quad \alpha;\end{matrix}\ \tfrac{1}{2},\ \tfrac{1}{2}\right]$$

$$=\sum_{n=0}^{\min(M,N)}\frac{(\lambda)_{2n}(-M)_n(-N)_n}{(\mu)_{2n}\ (\alpha)_n}\,\frac{2^{-2n}}{n!}$$

$$\cdot\ {}_2F_1(-M-N+2n,\ \lambda+2n;\ \mu+2n;\ \tfrac{1}{2})$$

$$= \frac{(\mu-\lambda)_{M+N}}{(\mu)_{M+N}} \sum_{n=0}^{\min(M,N)} \frac{(\lambda)_{2n}(-M)_n(-N)_n}{(1-\mu+\lambda-M-N)_{2n}(\alpha)_n} \frac{2^{-2n}}{n!}$$

$$\cdot\ {}_2F_1(-M-N+2n,\ \lambda+2n;\ 1-\mu+\lambda-M-N+2n;\ \tfrac{1}{2}).$$

Upon comparing the two expressions for

$$F^{2:1;1}_{1:1;1}$$

given by (43), we obtain the known transformation formula [7, p. 102, Equation (9)]:

$$(44)\qquad F^{2:1;1}_{1:1;1}\left[\begin{matrix}\lambda,\alpha:\ -M;\ -N; & \\ & \tfrac{1}{2},\tfrac{1}{2}\\ \mu:\quad \alpha;\quad \alpha; & \end{matrix}\right]$$

$$= \frac{(\mu-\lambda)_{M+N}}{(\mu)_{M+N}} F^{2:1;1}_{1:1;1}\left[\begin{matrix}\lambda,\alpha:\ -M;\ -N; & \\ & \tfrac{1}{2},\tfrac{1}{2}\\ 1-\mu+\lambda-M-N:\quad \alpha;\quad \alpha; & \end{matrix}\right]$$

$$(M,N\in \mathbb{N}_0).$$

In precisely the same manner, our result (41) with $x=\frac{1}{2}$ and $\lambda=-N$ will yield another known transformation formula [7, p. 108, Equation (15)]:

$$(45)\qquad F^{2:1;1}_{1:1;1}\left[\begin{matrix}-N,\alpha:\ \gamma;\ \delta; & \\ & \tfrac{1}{2},\tfrac{1}{2}\\ \mu:\ \alpha;\ \alpha; & \end{matrix}\right]$$

$$= \frac{(\mu-\gamma-\delta)_N}{(\mu)_N} F^{2:1;1}_{1:1;1}\left[\begin{matrix} -N, \alpha\colon \gamma;\ \delta; \\ \\ 1-\mu-N+\gamma+\delta\colon \alpha;\ \alpha; \end{matrix}\ \tfrac{1}{2}, \tfrac{1}{2}\right]$$

$$(N \in \mathbb{N}_0).$$

Yet another interesting special case of our result (41) would occur when we set $x = \frac{1}{2}$ and $\mu = \frac{1}{2}(\lambda+\gamma+\delta+1)$, and make use of Kummer's summation theorem (26) once again. We thus obtain the reduction formula:

$$\text{(46)}\quad F^{2:1;1}_{1:1;1}\left[\begin{matrix} \lambda, \alpha\colon \gamma;\ \delta; \\ \\ \frac{1}{2}(\lambda+\gamma+\delta+1)\colon \alpha;\ \alpha; \end{matrix}\ \tfrac{1}{2}, \tfrac{1}{2}\right]$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma[\frac{1}{2}(\lambda+\gamma+\delta+1)]}{\Gamma[\frac{1}{2}(\lambda+1)]\Gamma[\frac{1}{2}(\gamma+\delta+1)]}\ {}_3F_2\left[\begin{matrix} \frac{1}{2}\lambda,\ \gamma,\ \delta; \\ \\ \alpha,\ \frac{1}{2}(\gamma+\delta+1); \end{matrix}\ 1\right],$$

which is presumably new.

Next we record the transformation formula:

$$\text{(47)}\quad F^{4:0;0}_{2:1;1}\left[\begin{matrix} -N,\ \rho,\ \alpha,\ \beta\colon —;\ —; \\ \\ \mu,\ \frac{1}{2}(\alpha+\beta)\colon\ \alpha;\ \beta; \end{matrix}\ \tfrac{1}{4}, \tfrac{1}{4}\right]$$

$$= \frac{(\mu-\rho)_N}{(\mu)_N} F^{4:0;0}_{2:1;1}\left[\begin{matrix} -N,\ \rho,\ \alpha,\ \beta\colon -;\ -; \\ \\ 1-\mu+\rho-N,\ \frac{1}{2}(\alpha+\beta)\colon\ \alpha;\ \beta; \end{matrix}\ \tfrac{1}{4}, \tfrac{1}{4}\right]$$

$$(N \in \mathbb{N}_0),$$

which, in the special case $\beta = \alpha$, was proven by Grosjean and Sharma [7, p. 110, Equation (16); p. 112, Equation (16′)]. As a matter of fact, in view of the hypergeometric reduction formula [17, p. 28, Equation 1.3(33)], the general result (47) is *equivalent* to the ${}_3F_2$ transformation:

$$(48)\qquad {}_3F_2\left[\begin{matrix} -N,\ \rho,\ \tfrac{1}{2}(\alpha+\beta-1); \\ \\ \mu,\ \alpha+\beta-1; \end{matrix}\quad 1\right]$$

$$= \frac{(\mu-\rho)_N}{(\mu)_N}\ {}_3F_2\left[\begin{matrix} -N,\ \rho,\ \tfrac{1}{2}(\alpha+\beta-1); \\ \\ 1-\mu+\rho-N,\ \alpha+\beta-1; \end{matrix}\quad 1\right] \qquad (N \in \mathbb{N}_0),$$

which follows at once from the known result [12, p. 539, Entry 7.4.4(86)]:

$$(49)\qquad {}_3F_2\left[\begin{matrix} -N,\ a,\ b; \\ \\ c,\ d; \end{matrix}\quad 1\right] = \frac{(c-a)_N}{(c)_N}\ {}_3F_2\left[\begin{matrix} -N,\ a,\ d-b; \\ \\ 1-c+a-N,\ d; \end{matrix}\quad 1\right] \qquad (N \in \mathbb{N}_0)$$

upon setting

$$(50)\qquad a = \rho, \quad c = \mu, \ \text{ and } \ d = 2b = \alpha + \beta - 1.$$

Finally, we present a simple proof of the transformation formula [7, p. 112, Equation (17)]:

$$(51)\qquad F^{1:2;2}_{0:2;2}\left[\begin{matrix} \alpha\colon\ -M,\ \mu;\ -N,\ \nu; \\ \\ -\colon\ \ \gamma,\ \alpha;\ \ \delta,\ \alpha; \end{matrix}\quad \tfrac{1}{2},\ \tfrac{1}{2}\right] = \frac{(\gamma-\mu)_M\,(\delta-\nu)_N}{(\gamma)_M\,(\delta)_N}$$

$$\cdot\, F^{1:2;2}_{0:2;2}\left[\begin{matrix} \alpha\colon & -M,\ \mu; & -N,\ \nu; \\ \\ -\colon & 1-\gamma+\mu-M,\ \alpha; & 1-\delta+\nu-N,\ \alpha; \end{matrix}\quad \tfrac{1}{2},\ \tfrac{1}{2}\right]$$

$$(M,N \in \mathbb{N}_0).$$

Indeed, by virtue of a known expansion formula (*cf.* [15, p. 52, Equation (11)′]; see also [17, p. 337, Equation 9.4(243)]), the left–hand side of (51) can be rewritten as

$$(52)\qquad \mathscr{T}=\sum_{n=0}^{\min(M,N)}\frac{(-M)_n(-N)_n(\mu)_n(\nu)_n}{(\gamma)_n(\delta)_n(\alpha)_n}\,\frac{4^{-n}}{n!}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-M+n,\ \mu+n;\\ \gamma+n;\end{matrix}\ \tfrac12\right]\ {}_2F_1\left[\begin{matrix}-N+n,\ \nu+n;\\ \delta+n;\end{matrix}\ \tfrac12\right].$$

Transforming each ${}_2F_1$ function by means of (42) with $z=\frac12$, we have

$$(53)\qquad \mathscr{T}=\sum_{n=0}^{\min(M,N)}\frac{(-M)_n(-N)_n(\mu)_n(\nu)_n}{(\gamma)_n(\delta)_n(\alpha)_n}\,\frac{4^{-n}}{n!}$$

$$\cdot\ \frac{(\gamma-\mu)_{M-n}}{(\gamma+n)_{M-n}}\ {}_2F_1\left[\begin{matrix}-M+n,\ \mu+n;\\ 1-\gamma+\mu-M+n;\end{matrix}\ \tfrac12\right]$$

$$\cdot\ \frac{(\delta-\nu)_{N-n}}{(\delta+n)_{N-n}}\ {}_2F_1\left[\begin{matrix}-N+n,\ \nu+n;\\ 1-\delta+\nu-N+n;\end{matrix}\ \tfrac12\right]$$

or, equivalently,

$$(54)\quad \mathscr{T}=\frac{(\gamma-\mu)_M(\delta-\nu)_N}{(\gamma)_M(\delta)_N}\sum_{n=0}^{\min(M,N)}\frac{(-M)_n(-N)_n(\mu)_n(\nu)_n}{(1-\gamma+\mu-M)_n(1-\delta+\nu-N)_n(\alpha)_n}\,\frac{4^{-n}}{n!}$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-M+n,\ \mu+n;\\ 1-\gamma+\mu-M+n;\end{matrix}\ \tfrac12\right]\ {}_2F_1\left[\begin{matrix}-N+n,\ \nu+n;\\ 1-\delta+\nu-N+n;\end{matrix}\ \tfrac12\right],$$

which, in view of (52), is precisely the same as the right–hand side of the transformation formula (51).

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