STARLIKENESS AND CONVEXITY OF FRACTIONAL CALCULUS OPERATORS

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DMS-707-IR

October 1995

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Abstract

The main object of the present paper is to investigate the starlikeness and convexity of certain general families of operators of fractional calculus (that is, fractional integral and fractional derivative). Relevant connections are also pointed out with various earlier results involving these subclasses of analytic functions.

Key words and phrases. Analytic functions, starlike functions, convex functions, prestarlike functions, fractional calculus, Hadamard product or convolution, Pochhammer symbol, generalized hypergeometric functions, Carlson-Shafter operator.

§1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = \sum_{k=0}^{\infty} a_{k+1}\, z^{k+1} \qquad (a_1 := 1), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{U}$ which are *univalent* in $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *starlike of order* α $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(\frac{z\, f'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U};\ \ 0 \leq \alpha < 1). \tag{1.2}$$

We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α.

1991 *Mathematics Subject Classification.* Primary 30C45, 26A33; Secondary 33C20.

A function $f(z)$ belonging to the class $\mathcal{S}$ is said to be *convex of order* α $(0 \leq \alpha < 1)$ if and only if

$$\Re\left(1+\frac{z\,f''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U};\; 0 \leq \alpha < 1). \tag{1.3}$$

We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order α.

We note that (*cf.* Duren [2]; see also Srivastava and Owa [18])

$$f(z) \in \mathcal{K}(\alpha) \Leftrightarrow z\,f'(z) \in \mathcal{S}^*(\alpha) \qquad (0 \leq \alpha < 1). \tag{1.4}$$

For the functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1}\, z^{n+1} \qquad (a_{j,1} := 1;\; j = 1, 2), \tag{1.5}$$

let $(f_1 * f_2)(z)$ denote the *Hadamard product* or *convolution* of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_{1,n+1}\, a_{2,n+1}\, z^{n+1} \qquad (a_{j,1} := 1;\; j = 1, 2). \tag{1.6}$$

The function

$$s_\alpha(z) := \frac{z}{(1-z)^{2(1-\alpha)}} \qquad (0 \leq \alpha < 1) \tag{1.7}$$

is the well-known extremal function for the class $\mathcal{S}^*(\alpha)$.

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\alpha, \beta)$ if

$$(f * s_\alpha)(z) \in \mathcal{S}^*(\beta) \qquad (0 \leq \alpha < 1;\; 0 \leq \beta < 1). \tag{1.8}$$

Note that $\mathcal{R}(\alpha, \alpha) \equiv \mathcal{R}(\alpha)$ is the subclass of $\mathcal{A}$ consisting of *prestarlike functions of order* α in $\mathcal{U}$. The class $\mathcal{R}(\alpha)$ was introduced by Schild [13].

Let $(\lambda)_\mu$ denote the Pochhammer symbol defined by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} \qquad (\lambda \neq 0, -1, -2, \cdots) \tag{1.9}$$

for real or complex parameters λ and μ, so that

$$(\lambda)_m = \begin{cases} 1 & (m = 0) \\ \lambda(\lambda+1)\cdots(\lambda+m-1) & (m \in \mathbb{N} := \{1, 2, 3, \cdots\}). \end{cases} \tag{1.10}$$

Also let α_j $(j = 1, \cdots, p)$ and β_j $(j = 1, \cdots, q)$ be complex numbers with

$$\beta_j \neq 0, -1, -2, \cdots \qquad (j = 1, \cdots, q).$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots, \beta_q; z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \qquad (p \leq q+1), \end{aligned} \tag{1.11}$$

where $(\lambda)_n$ is given by (1.10).

The ${}_pF_q(z)$ series in (1.11) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathcal{U}$ if $p = q+1$. The condition $p \leq q+1$ stated with the definition (1.11) will be assumed to hold true throughout this paper (see, for details, Srivastava and Karlsson [17]).

Many essentially equivalent definitions of fractional calculus have been given in the literature (*cf.*, *e.g.*, [16, p. 21 *et seq.*] and [19]). We state the following definitions due to Owa and Srivastava [7] which have been used rather frequently in the theory of analytic functions (see also [3] and [6]):

Definition 1. The *fractional integral of order* λ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{-\lambda} f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} \, d\zeta \qquad (\lambda > 0), \tag{1.12}$$

and the *fractional derivative of order* λ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} \, d\zeta \qquad (0 \leq \lambda < 1), \tag{1.13}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ involved in (1.12) (and that of $(z-\zeta)^{-\lambda}$ involved in (1.13)) is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

Definition 2. Under the hypotheses of Definition 1, the *fractional derivative of order* $n + \lambda$ is defined by

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} \mathcal{D}_z^{\lambda} f(z) \qquad (0 \leq \lambda < 1; \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.14}$$

We define the following *generalized* operators of fractional calculus ([12]; see also [8] and [10]):

Definition 3. Let $\alpha, m \in \mathbb{R}_+$, and $\beta, \eta \in \mathbb{R}$. Then the *fractional integral operator* $\mathcal{I}_{0,z;m}^{\alpha,\beta,\eta}$ is defined by

$$\begin{aligned} \mathcal{I}_{0,z;m}^{\alpha,\beta,\eta} f(z) = \frac{z^{-m(\alpha+\beta)}}{\Gamma(\alpha)} \int_0^z (z^m - \zeta^m)^{\alpha-1} \\ \cdot {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{\zeta^m}{z^m}\right) f(\zeta)\, d(\zeta^m), \end{aligned} \tag{1.15}$$

where the function ${}_2F_1$ is Gauss's hypergeometric function defined by (1.11) with

$$p - 1 = q = 1.$$

The corresponding *fractional derivative operator* $\mathcal{D}_{0,z;m}^{\alpha,\beta,\eta}$ is defined by

$$\begin{aligned} \mathcal{D}_{0,z;m}^{\alpha,\beta,\eta}(z) = \frac{d}{dz^m} \left\{ \frac{z^{-m(\beta-\alpha+1)}}{\Gamma(1-\alpha)} \int_0^z (z^m - \zeta^m)^{-\alpha} \right. \\ \left. \cdot {}_2F_1\left(\beta-\alpha+1, -\eta; 1-\alpha; 1 - \frac{\zeta^m}{z^m}\right) f(\zeta)\, d(\zeta^m) \right\} \end{aligned} \tag{1.16}$$

$$\left(0 \leq \alpha < 1;\ \ \beta, \eta \in \mathbb{R};\ \ m \in \mathbb{R}_+\right).$$

In both (1.15) and (1.16), the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^r) \qquad (z \longrightarrow 0),$$

where

$$r > \max\{0, m(\beta - \eta)\} - m,$$

and the multiplicity of $(z^m - \zeta^m)^{\alpha-1}$ in (1.15) (and that of $(z^m - \zeta^m)^{-\alpha}$ in (1.16)) is removed by requiring that $\log(z^m - \zeta^m)$ be real when $z^m - \zeta^m > 0$.

Remark. The definition (1.16) for the generalized fractional derivative operator $\mathcal{D}_{0,z;m}^{\alpha,\beta,\eta}$ $(0 \leq \alpha < 1)$ is based essentially upon the definition (1.13) for the simpler fractional derivative operator $\mathcal{D}_z^\lambda$ $(0 \leq \lambda < 1)$. Indeed, in its special case when $m = 1$, it differs slightly from the fractional derivative operator $\mathcal{J}_{0,z}^{\alpha,\beta,\eta}$ $(0 \leq \alpha < 1)$ used by Sohi [15].

It is easy to observe that

$$\mathcal{I}_{0,z;1}^{\alpha,-\alpha,\eta} f(z) = \mathcal{D}_z^{-\alpha} f(z) \tag{1.17}$$

and

$$\mathcal{D}_{0,z;1}^{\alpha,\alpha-1,\eta} f(z) = \mathcal{D}_z^{\alpha} f(z). \tag{1.18}$$

Following Raina [9], we define the function $\Theta_C^A(a,c)$ by

$$\Theta_C^A(a,c)(z) := \sum_{n=0}^{\infty} \frac{(a)_{An}}{(c)_{Cn}} z^{n+1} \qquad (A>0;\ C>0;\ c \neq 0,-1,-2,\cdots), \tag{1.19}$$

where $C-A>0$ (or $C=A$ and $|z|<A^{-A}C^{C}$).

Corresponding to the function $\Theta_C^A(a,c)$, a linear operator $\mathcal{T}_C^A(a,c)$ is defined by

$$\mathcal{T}_C^A(a,c)\, f := \Theta_C^A(a,c) * f \qquad (f \in \mathcal{A}). \tag{1.20}$$

If $A=C=1$, then

$$\mathcal{T}_1^1(a,c)\, f = \Theta_1^1(a,c) * f = \mathcal{L}(a,c)\, f \qquad (f \in \mathcal{A}), \tag{1.21}$$

where $\mathcal{L}(a,c)$ is the Carlson-Shaffer operator introduced in [1].

By the series expansion, we have

Lemma 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{A}$. Also let $m \in \mathbb{R}_+$ and $\beta, \eta \in \mathbb{R}$.*

(i) *If $0 \leq \alpha < 1$ and $1/m > \beta - \eta - 1$, then*

$$\mathcal{D}_{0,z;m}^{\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma\left(1+\frac{k}{m}\right)\Gamma\left(1-\beta+\eta+\frac{k}{m}\right)}{\Gamma\left(\frac{k}{m}-\beta\right)\Gamma\left(2-\alpha+\eta+\frac{k}{m}\right)} a_k\, z^{k-m(\beta+1)}. \tag{1.22}$$

(ii) *If $\alpha > 0$ and $1/m > \beta - \eta - 1$, then*

$$\mathcal{I}_{0,z;m}^{\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma\left(1+\frac{k}{m}\right)\Gamma\left(1-\beta+\eta+\frac{k}{m}\right)}{\Gamma\left(1-\beta+\frac{k}{m}\right)\Gamma\left(1+\alpha+\eta+\frac{k}{m}\right)} a_k\, z^{k-m\beta}. \tag{1.23}$$

By using Definition 3 of the generalized operators of fractional calculus, we now introduce the linear operator $\Lambda_m(\alpha,\beta,\eta)$ given by

$$\Lambda_m(\alpha,\beta,\eta)\, f(z) := \frac{\Gamma\left(\frac{1}{m}-\beta\right)\Gamma\left(2-\alpha+\eta+\frac{1}{m}\right)}{\Gamma\left(1+\frac{1}{m}\right)\Gamma\left(1-\beta+\eta+\frac{1}{m}\right)} z^{m(\beta+1)}\, \mathcal{D}_{0,z;m}^{\alpha,\beta,\eta} f(z), \tag{1.24}$$

where $f(z) \in \mathcal{A}$ and $1/m > \max\{\beta,\ \alpha-\eta-2,\ \beta-\eta-1\}$.

Let us now define a linear fractional operator $\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta}$ by

$$\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta}\, f(z) := \frac{\Gamma\left(1-\beta+\frac{1}{m}\right)\Gamma\left(1+\alpha+\eta+\frac{1}{m}\right)}{\Gamma\left(1+\frac{1}{m}\right)\Gamma\left(1-\beta+\eta+\frac{1}{m}\right)}\, z^{m\beta}\, \mathcal{I}_{0,z;m}^{\alpha,\beta,\eta}\, f(z), \tag{1.25}$$

where $f(z) \in \mathcal{A}$ and $1/m > \max\{\beta-1,\ -\alpha-\eta-1,\ \beta-\eta-1\}$.

By using the above definitions, we have

Lemma 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and let $0 \leq \alpha < 1$, $m \in \mathbb{R}_+$, $\beta, \eta \in \mathbb{R}$, and $1/m > \max\{\beta,\ \alpha-\eta-2,\ \beta-\eta-1\}$. Then*

$$\begin{aligned}&\Lambda_m(\alpha,\beta,\eta)\, f(z)\\ &= \mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\ \frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},\ 2-\alpha+\eta+\frac{1}{m}\right) f(z).\end{aligned} \tag{1.26}$$

Lemma 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and let $\alpha, m \in \mathbb{R}_+$, $\beta, \eta \in \mathbb{R}$, and $1/m > \max\{\beta-1,\ -\alpha-\eta-1,\ \beta-\eta-1\}$. Then*

$$\begin{aligned}\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta}\, f(z) = {}& \mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\ 1-\beta+\frac{1}{m}\right)\\ &\cdot\ \mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},\ 1+\alpha+\eta+\frac{1}{m}\right) f(z).\end{aligned} \tag{1.27}$$

The following known results will also be required in our present investigation.

Lemma 4 (Ruscheweyh and Sheil-Small [11, p. 126, Lemma 2.4]). *Let $h(z)$ and $g(z)$ be analytic in $\mathcal{U}$ and satisfy*

$$h(0) = g(0) = 0, \qquad h'(0) \neq 0, \quad \text{and} \quad g'(0) \neq 0.$$

Suppose that, for each $\rho(|\rho| = 1)$ and $\sigma(|\sigma| = 1)$, we have

$$h(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) g(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}). \tag{1.28}$$

Then, for each function $F(z)$ analytic in $\mathcal{U}$ and satisfying the inequality $\Re\{F(z)\} > 0$ $(z \in \mathcal{U})$,

$$\Re\left(\frac{(h*G)(z)}{(h*g)(z)}\right) > 0 \qquad (z \in \mathcal{U}), \tag{1.29}$$

where $G(z) = F(z)\, g(z)$.

Lemma 5 (Twomey [21, p. 95, Equation (3)]). *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}^*$. Then*

$$\left|\frac{z\,f'(z)}{f(z)}\right| \leq 1 + \frac{|z|\ell n\left(\frac{(1+|z|)^2|f(z)|}{|z|}\right)}{(1-|z|)\ell n\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \tag{1.30}$$

Equality in (1.30) *holds true for the Koebe function* [*cf.* Equation (1.7)]

$$K(z) := \frac{z}{(1-z)^2} = s_0(z).$$

Lemma 6 (Singh [14, p. 133, Theorem IV]). *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}^*$. Then*

$$\Re\left(\frac{z\,f'(z)}{f(z)}\right) \geq \frac{1-|z|^2}{|z|}\,|f(z)| \qquad (z \in \mathcal{U}) \tag{1.31}$$

and

$$\Re\left(\frac{z\,f'(z)}{f(z)}\right) \leq \frac{1+|z|}{1-|z|} + \frac{2|z|\,\ell n\left(\frac{(1-|z|)^2\,|f(z)|}{|z|}\right)}{(1-|z|^2)\,\ell n\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \tag{1.32}$$

Equality in (1.31) *is attained for a function of the form:*

$$f(z) = \frac{z}{(1-ze^{i\gamma})^{2\delta}\,(1-ze^{-i\gamma})^{2(1-\delta)}} \qquad (0 \leq \delta \leq 1;\ \gamma \in \mathbb{R}), \tag{1.33}$$

and equality in (1.32) *is attained for a function of the form:*

$$f(z) = \frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}} \qquad (0 \leq \delta \leq 1), \tag{1.34}$$

where δ satisfies the equation:

$$2\delta\ \ell n\left(\frac{1+|z|}{1-|z|}\right) = \ell n\left(\frac{(1+|z|)^2\,|f(z)|}{|z|}\right). \tag{1.35}$$

Lemma 7 (Lewis [5, p. 435, Theorem 1]). *Given μ $(-\infty < \mu < \infty)$, let*

$$f_\mu(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^\mu}\ z^{n+1} \qquad (z \in \mathcal{U}). \tag{1.36}$$

Then $f_\mu(z)$ is in the class $\mathcal{K}$ whenever $\mu \geq 0$.

Making use of Lemmas 1 to 7 above, we shall derive several representation, inclusion, and other properties of the aforementioned general families of fractional calculus operators. These properties are associated with the classes $\mathcal{S}^*$, $\mathcal{K}$, and $\mathcal{R}(\alpha,\beta)$. We shall also point out relevant connections of many of the properties obtained in this paper with various earlier results involving these subclasses of analytic functions.

2. Representation and Inclusion Properties

In view of the definition of the class $\mathcal{R}\ (\alpha,\beta)$, we can write

$$\begin{aligned}\mathcal{R}(\alpha,\beta) &= \{f \in \mathcal{A} : (f * s_\alpha)(z) \in \mathcal{S}^*(\beta)\}\\ &= \mathcal{T}_1^1(1, 2-2\alpha)\, \mathcal{S}^*(\beta).\end{aligned} \tag{2.1}$$

The following theorem is a generalization of an earlier result due to Kim *et al.* [4, p. 316, Theorem 2].

Theorem 1. *If* $\alpha, m \in \mathbb{R}_+$, $0 \leq \beta < 2$, $\eta \in \mathbb{R}$, *and*

$$1/m > \max\{\beta - 1,\ -\alpha - \eta - 1,\ \beta - \eta - 1\},$$

then

$$\begin{aligned}\mathcal{R}\left(\frac{\beta}{2}, \alpha\right) = \mathcal{T}_{1/m}^{1/m}\left(1-\beta+\frac{1}{m},\ 1+\frac{1}{m}\right) \mathcal{T}_{1/m}^{1/m}\left(1+\alpha+\eta+\frac{1}{m},\ 1-\beta+\eta+\frac{1}{m}\right)\\ \cdot\, \mathcal{T}_1^1(2, 2-\beta)\ \mathcal{J}_{0,z;m}^{\alpha,\beta,\eta}\ \mathcal{K}(\alpha).\end{aligned}$$

Proof. From the equation (1.27), we readily have

$$\begin{aligned}\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\frac{1}{m},\ 1+\frac{1}{m}\right) \mathcal{T}_{1/m}^{1/m}\left(1+\alpha+\eta+\frac{1}{m},\ 1-\beta+\eta+\frac{1}{m}\right) \mathcal{J}_{0,z,m}^{\alpha,\beta,\eta}\, \mathcal{K}(\alpha)\\ = \mathcal{T}_1^1(1,2)\, \mathcal{S}^*(\alpha)\\ = \mathcal{T}_1^1(2-\beta,2)\, \mathcal{T}_1^1(1,2-\beta)\, \mathcal{S}^*(\alpha),\end{aligned} \tag{2.2}$$

which, in conjunction with the relationship (2.1), yields

$$\begin{aligned}\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\frac{1}{m},\ 1+\frac{1}{m}\right) \mathcal{T}_{1/m}^{1/m}\left(1+\alpha+\eta+\frac{1}{m},\ 1-\beta+\eta+\frac{1}{m}\right)\ \mathcal{J}_{0,z;m}^{\alpha,\beta,\eta}\, \mathcal{K}(\alpha)\\ = \mathcal{T}_1^1(2-\beta,2)\ \mathcal{R}\left(\frac{\beta}{2}, \alpha\right),\end{aligned} \tag{2.3}$$

thus completing the proof of Theorem 1.

Furthermore, we have

Theorem 2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$ *and let, for each* ρ $(|\rho| = 1)$ *and* $\sigma(|\sigma| = 1)$,

$$u(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) f(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}), \tag{2.4}$$

where

$$\begin{aligned} u(z) = \mathcal{T}_{1/m}^{1/m} &\left(1+\frac{1}{m}, 1-\beta+\frac{1}{m}\right) \\ &\cdot \mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m}, 1+\alpha+\eta+\frac{1}{m}\right) \frac{z}{1-z}. \end{aligned} \tag{2.5}$$

Suppose also that $\alpha, m \in \mathbb{R}_+$, $\beta, \eta \in \mathbb{R}$, *and* $1/m > \max\{\beta - 1, -\alpha - \eta - 1\}$. *Then* $\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)$ *belongs to the class* $\mathcal{S}^*$.

Proof. Notice from Lemma 3 and Equation (2.5) that

$$\begin{aligned} \mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z) &= z + \sum_{k=2}^{\infty} \frac{\left(1+\frac{1}{m}\right)_{(k-1)/m}\left(1-\beta+\eta+\frac{1}{m}\right)_{(k-1)/m}}{\left(1-\beta+\frac{1}{m}\right)_{(k-1)/m}\left(1+\alpha+\eta+\frac{1}{m}\right)_{(k-1)/m}} a_k z^k \\ &= (u * f)(z), \end{aligned} \tag{2.6}$$

which readily yields

$$\frac{z\left(\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)\right)'}{\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)} = \frac{z(u*f)'(z)}{(u*f)(z)} = \frac{(u*(zf'))(z)}{(u*f)(z)}. \tag{2.7}$$

Therefore, setting $h(z) = u(z)$, $g(z) = f(z)$, and $F(z) = zf'(z)/f(z)$ in Lemma 4, we find that

$$\Re\left\{\frac{z\left(\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)\right)'}{\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)}\right\} > 0 \qquad (z \in \mathcal{U}), \tag{2.8}$$

which completes the proof of Theorem 2.

From Theorem 2 and Equation (1.4), we easily obtain

Corollary 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{K}$, *and let, for each* ρ $(|\rho| = 1)$ *and* σ $(|\sigma| = 1)$,

$$u(z) * \left(\frac{1+\rho\sigma z}{1-\sigma z}\right) z f'(z) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}), \tag{2.9}$$

where $u(z)$ *is given by* (2.5). *Then* $\mathcal{J}_{0,z;m}^{\alpha,\beta,\eta} f(z)$ *is in the class* $\mathcal{K}$.

Next we prove

Theorem 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{S}^*$ *and let, for each* ρ $(|\rho| = 1)$ *and* σ $(|\sigma| = 1)$,

$$\psi_m(\alpha,\beta,\eta)\left(\frac{1+\rho\sigma z}{1-\sigma z}\, f(z)\right) \neq 0 \qquad (z \in \mathcal{U}\setminus\{0\}), \tag{2.10}$$

where, for convenience,

$$\psi_m(\alpha,\beta,\eta) = \mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m}, \frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m}, 2-\alpha+\eta+\frac{1}{m}\right). \tag{2.11}$$

Suppose also that $0 \leq \alpha < 1$, $m \in \mathbb{R}_+$, $\beta, \eta \in \mathbb{R}$, *and* $1/m > \max\{\beta,\, \alpha-\eta-2\}$. *Then* $\Lambda_m(\alpha,\beta,\eta)\, f(z)$ *is also in the class* $\mathcal{S}^*$.

Proof. The proof of Theorem 3 runs parallel to that of Theorem 2 with

$$u(z) = \psi_m(\alpha,\beta,\eta)\,\frac{z}{1-z}, \tag{2.12}$$

and we omit the details involved.

3. Applications and Further Results

By applying Lemma 5 to Theorem 3, we get

Corollary 2. *Under the hypotheses of Theorem* 3,

$$\left|\frac{\Lambda_m(\alpha,\beta,\eta)\,(z f'(z))}{\Lambda_m(\alpha,\beta,\eta)\, f(z)}\right| \leq 1 + \frac{|z|\,\ell n\left(\frac{(1+|z|)^2\,\Lambda_m(\alpha,\beta,\eta)\, f(z)|}{|z|}\right)}{(1-|z|)\,\ell n\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \tag{3.1}$$

Equality in (3.1) *holds true for the function* $f(z)$ *given by*

$$f(z) = \phi_m(\alpha,\beta,\eta)\left(\frac{z}{(1-z)^2}\right), \tag{3.2}$$

where, for convenience,

$$\begin{aligned}
&\phi_m(\alpha,\beta,\eta)\\
&= \mathcal{T}_{1/m}^{1/m}\left(\frac{1}{m}-\beta,\, 1+\frac{1}{m}\right)\mathcal{T}_{1/m}^{1/m}\left(2-\alpha+\eta+\frac{1}{m},\, 1-\beta+\eta+\frac{1}{m}\right) \qquad (3.3)
\end{aligned}$$

$(0 \leq \alpha < 1;\ m \in \mathbb{R}_+;\ \beta,\eta \in \mathbb{R};\ 1/m > \max\{0,\, \beta-\eta-1\})$.

Next we apply Lemma 6 to Theorem 3 and obtain

Corollary 3. *Under the hypotheses of Theorem* 3,

$$\Re\left\{\frac{\Lambda_m(\alpha,\beta,\eta)\big(z\,f'(z)\big)}{\Lambda_m(\alpha,\beta,\eta)\,f(z)}\right\} \geq \frac{1-|z|^2}{|z|}\,|\Lambda_m(\alpha,\beta,\eta)\,f(z)| \qquad (z \in \mathcal{U}) \qquad (3.4)$$

and

$$\Re\left\{\frac{\Lambda_m(\alpha,\beta,\eta)\big(z\,f'(z)\big)}{\Lambda_m(\alpha,\beta,\eta)\,f(z)}\right\} \leq \frac{1+|z|}{1-|z|} + \frac{2|z|\,\ell n\left(\frac{(1-|z|)^2\,|\Lambda_m(\alpha,\beta,\eta)\,f(z)|}{|z|}\right)}{(1-|z|^2)\,\ell n\left(\frac{1+|z|}{1-|z|}\right)} \qquad (z \in \mathcal{U}). \qquad (3.5)$$

Equality in (3.4) *holds true for the function* $f(z)$ *given by*

$$f(z) = \phi_m(\alpha,\beta,\eta)\left(\frac{z}{(1-ze^{i\lambda})^{2\delta}\,(1-ze^{-i\lambda})^{2(1-\delta)}}\right), \qquad (3.6)$$

and equality in (3.5) *holds true for the function* $f(z)$ *given by*

$$f(z) = \phi_m(\alpha,\beta,\eta)\left(\frac{z}{(1-z)^{2\delta}(1+z)^{2(1-\delta)}}\right), \qquad (3.7)$$

where $\phi_m(\alpha,\beta,\eta)$ *is given by* (3.3), $0 \leq \delta \leq 1$, $\lambda \in \mathbb{R}$, *and* δ *satisfies* (1.35).

From Theorem 3 and Equation (1.4), we obtain

Corollary 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{K}$ *and let, for each* ρ $(|\rho| = 1)$ *and* σ $(|\sigma| = 1)$,

$$\psi_m(\alpha,\beta,\eta)\left(\frac{1+\rho\sigma z}{1-\sigma z}\,z\,f'(z)\right) \neq 0 \qquad (z \in \mathcal{U}\setminus\{0\}), \qquad (3.8)$$

where $\psi_m(\alpha,\beta,\eta)$ *is given by* (2.11). *Then* $\Lambda_m(\alpha,\beta,\eta)\,f(z)$ *is also in the class* $\mathcal{K}$.

We now state and prove

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{A}$ and satisfy, for each ρ ($|\rho| = 1$) and σ ($|\sigma| = 1$),*

$$\psi_m(\alpha,\beta,\eta)\left(\frac{1+\rho\sigma z}{1-\sigma z}\,(f_\mu * f)(z)\right) \neq 0 \qquad (z \in \mathcal{U}\setminus\{0\}), \tag{3.9}$$

where $\psi_m(\alpha,\beta,\eta)$ is given by (2.11), $\mu \geq 0$, and the function $f_\mu(z)$ is given by (1.36). Then $\Lambda_m(\alpha,\beta,\eta)(f_\mu * f)(z)$ is in the class $\mathcal{S}^*$.

Proof. We observe that

$$\begin{aligned}
&\Re\left\{\frac{z(\Lambda_m(\alpha,\beta,\eta)\,(f_\mu * f)(z))')}{\Lambda_m(\alpha,\beta,\eta)(f_\mu * f)(z)}\right\} \\
&= \Re\left\{\frac{\mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},2-\alpha+\eta+\frac{1}{m}\right)(f * zf_\mu')(z)}{\mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},2-\alpha+\eta+\frac{1}{m}\right)(f * f_\mu)(z)}\right\} \\
&= \Re\left\{\frac{\left(\mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},2-\alpha+\eta+\frac{1}{m}\right)f * zf_\mu'\right)(z)}{\left(\mathcal{T}_{1/m}^{1/m}\left(1+\frac{1}{m},\frac{1}{m}-\beta\right)\mathcal{T}_{1/m}^{1/m}\left(1-\beta+\eta+\frac{1}{m},2-\alpha+\eta+\frac{1}{m}\right)f * f_\mu\right)(z)}\right\} \\
&= \Re\left\{\frac{\left(\sum_{k=0}^{\infty}\frac{\left(1+\frac{1}{m}\right)_{k/m}\left(1-\beta+\eta+\frac{1}{m}\right)_{k/m}}{\left(\frac{1}{m}-\beta\right)_{k/m}\left(2-\alpha+\eta+\frac{1}{m}\right)_{k/m}}\,a_{k+1}\,z^{k+1}\right) * \left(z\,f_\mu'(z)\right)}{\left(\sum_{k=0}^{\infty}\frac{\left(1+\frac{1}{m}\right)_{k/m}\left(1-\beta+\eta+\frac{1}{m}\right)_{k/m}}{\left(\frac{1}{m}-\beta\right)_{k/m}\left(2-\alpha+\eta+\frac{1}{m}\right)_{k/m}}\,a_{k+1}\,z^{k+1}\right) * f_\mu(z)}\right\},
\end{aligned} \tag{3.10}$$

which, by virtue of Lemma 7, yields the inclusion relation:

$$f_\mu(z) \in \mathcal{K} \subset \mathcal{S}^* \qquad (\mu \geq 0). \tag{3.11}$$

Setting

$$h(z) = \psi_m(\alpha,\beta,\eta)\,f(z),$$

$$F(z) = \frac{z\,f_\mu'(z)}{f_\mu(z)},$$

and $g(z) = f_\mu(z)$ in Lemma 4, we conclude that

$$\Lambda_m(\alpha,\beta,\eta)(f_\mu * f)(z) \in \mathcal{S}^*,$$

which completes the proof of Theorem 4.

Finally, we have

Corollary 5. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{K}$ and satisfy, for each ρ ($|\rho| = 1$) and σ ($|\sigma| = 1$),*

$$\psi_m(\alpha, \beta, \eta) \left(\frac{1+\rho\sigma z}{1-\sigma z} z(f_\mu * f)'(z) \right) \neq 0 \qquad (z \in \mathcal{U} \setminus \{0\}), \tag{3.12}$$

where $\psi_m(\alpha, \beta, \eta)$ is given by (2.11), $\mu \geq 0$, *and the function $f_\mu(z)$ is given by* (1.36). *Then $\Lambda_m(\alpha, \beta, \eta)(f_\mu * f)(z)$ is in the class $\mathcal{K}$.*

The proof of Corollary 5 is much akin to that of its special case given earlier by Srivastava *et al.* [20, p. 146, Theorem 4], and we omit the details involved.

Acknowledgments

The present investigation was completed during the third-named author's visits to Yeungnam University at Gyongsan and Kyungpook National University at Taegu in August 1995. This work was partially supported by KOSEF (Project No. 94-1400-02-01-3) and TGRC-KOSEF, and by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

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