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RESEARCH

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Solution of fractional Volterra–Fredholm integro-differential equations under mixed boundary conditions by using the HOBW method

Mohamed R. Ali^{1*} , Adel R. Hadhoud² and H.M. Srivastava^{3,4}

*Correspondence:

mohamed.reda@bhit.bu.edu.eg;
mohamedreda@bhit@yahoo.com

¹Department of Mathematics,
Benha Faculty of Engineering,
Benha University, Banhā, Egypt
Full list of author information is
available at the end of the article

Abstract

A new approximate technique is introduced to find a solution of FVFIDE with mixed boundary conditions. This paper started from the meaning of Caputo fractional differential operator. The fractional derivatives are replaced by the Caputo operator, and the solution is demonstrated by the hybrid orthonormal Bernstein and block-pulse functions wavelet method (HOBW). We demonstrate the convergence analysis for this technique to emphasize its reliability. The applicability of the HOBW is demonstrated using three examples. The approximate results of this technique are compared with the correct solutions, which shows that this technique has approval with the correct solutions to the problems.

Keywords: Orthonormal Bernstein; Block-pulse functions; Wavelet method; Fractional integro-differential equations; Fractional calculus; Approximate solution

1 Introduction

The applications of fractional calculus can be observed in many fields of physics and engineering such as fluid dynamic traffic [1] and signal processing [2]. Due to the invaluable contribution of fractional calculus in various fields of engineering, the researchers have shown high interest in studying fractional calculus. In this regard as in many cases, it is very difficult to find the correct analytical solutions of fractional differential and integral equations. The approximate methods have gained importance to prevent this difficulty. Initially the authors used different approximate techniques to find the approximate solution of fractional differential and integral equations such as spline collocation method (SCM) [3], fractional transform method (FTM) [4], homotopy perturbation method (HPM) [5], operational Tau method (OTM) [6], rationalized Haar functions method (RHFH) [7], reproducing kernel Hilbert space method (RKHSM) [8], Adomian decomposition method (ADM) [9], and B-spline method [10].

In this paper, we derive the approximate solution of FVFIDE using HOBW. The approximate consequence found by the introduced method is compared with the correct solution of the problem, showing the greatest degree of accuracy.

2 Preliminaries of fractional calculus

In this segment, we first survey some fundamental definitions of the fractional calculus theory which are required for building up our outcomes. The broadly utilized definitions of fractional integral and fractional derivative are the definitions of Riemann–Liouville and Caputo [11–14].

Definition 2.1 A real function $y(x), x > 0$, is said to be in the space $C_\sigma, \sigma \in R$, if there is a genuine number ρ with $\rho > \sigma$ to such an extent that $y(x) = x^\rho y_0(x), y_0(x) \in C[0, \infty)$, and $y(x) \in C_\sigma^n$ if $y^n(x) \in C_\sigma, n \in N$.

Definition 2.2 ([15]) The Riemann fractional integral of order $\alpha > 0$ of a function f is given by

$$(J^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha \in R^+ \tag{2.1}$$

$$(J^0 f)(t) = f(t).$$

This integral operator J has the following properties:

- (a) $J^\alpha J^\beta = J^\beta J^\alpha$;
- (b) $J^\alpha J^\beta = J^{\alpha+\beta}$;
- (c) $J^\alpha (t - a)^\xi = \frac{\Gamma(\xi+1)}{\Gamma(\xi+\alpha+1)} (t - a)^{\alpha+\xi}, \alpha, \beta > 0, \xi > -1$.

Definition 2.3 The Riemann–Liouville fractional derivative is defined by [16]

$$D_*^\alpha f(t) = \begin{cases} D^m \frac{1}{\Gamma(m-\alpha)} \int_0^t (t - \tau)^{m-\alpha-1} f(\tau) d\tau, & m - 1 < \alpha < m, \\ f^{(m)}(t), & \alpha = m. \end{cases} \tag{2.2}$$

In fact $D_*^\alpha J^\alpha f(t) = D^m J^{m-\alpha} J^\alpha f(t) = D^m J^m f(t) = f(t)$. The effect of the operator D^α on the power functions:

$$D_*^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} t^{\gamma-\alpha}, \quad \gamma > -1, t > 0, \tag{2.3}$$

where the fractional derivative $D_*^\alpha f(t)$ is not zero for constant function when $\alpha \notin N$, from (2.3) when $\gamma = 0$, then $D_*^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(-1+\alpha)}$, but $D_*^\alpha t^{\alpha-j} = 0, j : 1(1)m$. Figure 1 shows the effect of Riemann–Liouville fractional derivative (D^α) on t^γ . It is illustrated that when $\gamma = 0$, the Riemann–Liouville derivative is not zero and it is zero when $\gamma = -0.5$. Therefore, this definition does not agree with the principles of integer order calculus.

Definition 2.4 The fractional derivative of $f(t)$ in the Caputo sense is given by [16]

$$D_t^\alpha f(t) = D^m J^{m-\alpha} f(t) = \begin{cases} \frac{d^m f(t)}{dt^m}, & m = \alpha, \\ \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t - s)^{m-\alpha-1} f(s, x) ds, & 0 \leq m - 1 < \alpha < m. \end{cases} \tag{2.4}$$

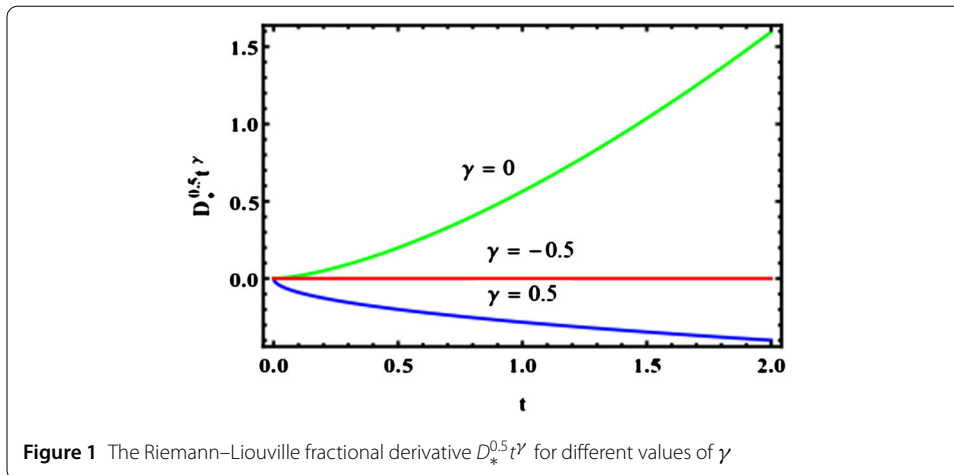


Figure 1 The Riemann–Liouville fractional derivative $D_*^{0.5}t^\gamma$ for different values of γ

We demonstrate the following form of FVFIDE that we will solve by the HOBW technique.

$$(D^\alpha y_i)(x) = g_i(x) + \sum_{j=1}^m \int_0^x k_{1i}(x,t)F_{1i}[t,y(t)] dt + \int_0^1 k_{2i}(x,t)F_{2i}[t,y_i(t)] dt \tag{2.5}$$

with MBC: $\sum_{j=1}^d [a_{ij}y^{(j-1)}(0) + b_{ij}y^{(j-1)}(1)] = r_i, i = 1, 2, \dots, d$, where $y : [0, 1] \rightarrow \mathbb{R}, i = 1, 2$, are continuous functions. $g : [0, 1] \rightarrow \mathbb{R}$ and $k_i : [0, 1] \times [0, 1] \rightarrow \mathbb{R}, i = 1, 2$, are continuous functions. $F_i : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2$, are nonlinear terms and Lipschitz continuous functions. Here D^α is understood as Caputo fractional derivative. Using HOBW this FVFIDE is converted into a system of algebraic equations that can be disbanded by Newton’s method. We applied the Gauss–Legendre quadrature technique for calculating the integration on nonlinear terms. The obtained consequence is compared with that by the Nystrom method.

3 The HOBW method and the operational matrix of the integration

3.1 Wavelets and the HOBW methods

Wavelets constitute a group of functions constructed from dilation and translation of a single function $\psi(x)$ called the mother wavelet, in which the parameter of dilation a and the parameter of translation b vary continuously:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), \quad a, b \in \mathbb{R}, a \neq 0. \tag{3.1}$$

By letting a and b be discrete values such as $a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0$, where n and k are positive integers, we attain the family of discrete wavelets:

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0), \quad n, k \in \mathbb{Z}^+. \tag{3.2}$$

Then $\psi_{k,n}(t)$ shape a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2, b_0 = 1$, then $\psi_{k,n}(t)$ shape an orthonormal basis. Here, $\text{HOBW}_{i,j}(t) = \text{HOBW}(k, i, j, t)$ involves four arguments, $i = 1, \dots, 2^{k-1}, k$ is any positive integer, j is the degree of Bernstein polynomials,

and t is the normalized time. $\text{HOBW}_{ij}(t)$ are defined on $[0, 1)$ as in [17]:

$$\text{HOBW}_{ij}(t) = \begin{cases} 2^{\frac{k-1}{2}} \binom{n}{j} (2^{k-1}x - i + 1)^j (1 - (2^{k-1}x - i + 1))^{n-j} & \frac{i-1}{2^{k-1}} \leq t < \frac{i}{2^{k-1}}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $i = 1, 2, \dots, 2^{k-1}, j = 0, 1, \dots, M - 1$ and k is a positive integer. Thus, we attain our new basis as $\{\text{HOBW}_{1,0}, \text{HOBW}_{1,1}, \dots, \text{HOBW}_{2^{k-1},M-1}\}$ and any function is truncated with them.

The HOBW are orthonormal basis that is given by

$$(\text{HOBW}_{ij}(x), \text{HOBW}_{i'j'}(x)) = \begin{cases} 1, & (i, j) = (i', j'), \\ 0, & (i, j) \neq (i', j'), \end{cases} \tag{3.4}$$

where (\cdot, \cdot) is called the inner product in $L^2[0, 1)$. The HOBW has compact support $[\frac{i-1}{2^{k-1}}, \frac{i}{2^{k-1}}], i = 1, \dots, 2^{k-1}$.

3.2 Function approximation by the HOBW functions

Any function $y(t)$, which is integrable in $[0, 1)$, is truncated by the HOBW method as follows:

$$y(t) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{ij} \text{HOBW}_{ij}(t), \quad i = 1, 2, \dots, \infty, j = 0, 1, 2, \dots, \infty, t \in [0, 1), \tag{3.5}$$

where the HOBW coefficients c_{ij} can be calculated as given below:

$$c_{ij} = \frac{(y(t), \text{HOBW}_{ij}(t))}{(\text{HOBW}_{ij}(t), \text{HOBW}_{ij}(t))}.$$

We approximate $y(t)$ by a truncated series as follows:

$$y(t) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{M-1} c_{ij} \text{HOBW}_{ij}(t) = C^T \text{HOBW}(t), \tag{3.6}$$

where $\text{HOBW}(t)$ and C are $2^{k-1}M \times 1$ vectors given by

$$\text{HOBW}(t) = [\text{HOBW}_{10}, \text{HOBW}_{11}, \dots, \text{HOBW}_{1(M-1)}, \text{HOBW}_{20}, \text{HOBW}_{21}, \dots, \text{HOBW}_{2(M-1)}, \dots, \text{HOBW}_{2^{k-1}0}, \dots, \text{HOBW}_{2^{k-1}(M-1)}]^T$$

and

$$C = [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, c_{21}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^T. \tag{3.7}$$

4 Solution of FVFIDE via the HOBW method

Consider the nonlinear FVFIDE with MBC given in Eq. (2.5), and we approximate the unknown function $y(x) \in [0, 1]$ by the HOBW method as $y(x) = C^T \text{HOBW}(x)$.

We assume

$$F_{1i}[y(x)] = u_i(x), \quad F_{2i}[y(x)] = v_i(x), \tag{4.1}$$

we approximate $u_i(x)$ and $v_i(x)$ as:

$$u_i(x) = A_i^T \text{HOBW}(x_i), \quad v_i(x) = B_i^T \text{HOBW}(x_i),$$

where A and B are like C .

First, applying J to both sides of Eq. (2.5) and using the approximation above, we have

$$\begin{aligned} (J^\alpha D^\alpha y_i)(x) &= J^\alpha [g_i(x)] + J^\alpha \left[\sum_{j=1}^m \int_0^x k_{1i}(x, t) F_{1i}[y(t)] dt \right] \\ &\quad + J^\alpha \left[\int_0^1 k_{2i}(x, t) F_{2i}[y_i(t)] dt \right], \end{aligned} \tag{4.2}$$

$$\begin{aligned} y_i(x) - \sum_{l=0}^{d-1} \frac{x^l}{l!} y_i^{(l)}(0+) &= \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} g_i(\tau) d\tau \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^\tau k_{1i}(\tau, t) u_i(t) dt d\tau \right] \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^1 k_{2i}(\tau, t) v_i(t) dt d\tau \right]. \end{aligned} \tag{4.3}$$

$y_i(x)$ of Eq. (4.3) is replaced with the approximate solution $C_i^T \text{HOBW}(x)$ as follows:

$$\begin{aligned} C_i^T \text{HOBW}(x_i) - \sum_{l=0}^{d-1} \frac{x_i^l}{l!} C_i^T \text{HOBW}_i(0+) &= \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} g_i(\tau) d\tau \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^\tau k_{1i}(\tau, t) A_i^T \text{HOBW}(t) dt d\tau \right] \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^1 k_{2i}(\tau, t) B_i^T \text{HOBW}(t) dt d\tau \right]. \end{aligned} \tag{4.4}$$

We collocate Eq. (4.4) in $2^{k-1}M$ nodal points of Newton–Cotes as $x_i = \frac{2i-1}{2^k M}$. We have

$$\begin{aligned} C_i^T \text{HOBW}(x_i) - \sum_{l=0}^{d-1} \frac{x_i^l}{l!} C_i^T \text{HOBW}_i(0+) &= \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} g_i(\tau) d\tau \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^\tau k_{1i}(\tau, t) A_i^T \text{HOBW}(t) dt d\tau \right] \\ &\quad + \left[\frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} \int_0^1 k_{2i}(\tau, t) B_i^T \text{HOBW}(t) dt d\tau \right]. \end{aligned} \tag{4.5}$$

Applying the Gauss–Legendre quadrature method for evaluating the integrals in Eq. (4.5), we change the domain of integration from $[0, x_i]$ to $[-1, 1]$. Using the transformation $\tau = \frac{x_i}{2}(s + 1)$ and then applying the Gauss–Legendre method yields

$$\begin{aligned}
 & C_i^T \text{HOBW}(x_i) - \sum_{l=0}^{d-1} \frac{x_i^l}{l!} C_i^T \text{HOBW}(0+) \\
 &= \frac{1}{\Gamma(\alpha)} \int_0^{x_i} (x_i - \tau)^{\alpha-1} g_i(\tau) d\tau \\
 &+ \left[\frac{1}{\Gamma(\alpha)} \frac{x_i}{2} \sum_{j=1}^{M_1} w_j \left(\frac{x_i}{2} (1 - s_j)^{\alpha-1} \right) \int_0^{\frac{x_i}{2}(1+s_j)} k_{1i} \left(\frac{x_i}{2} (1 + s_j)^{\alpha-1}, t \right) A_i^T \text{HOBW}(t) dt \right] \\
 &+ \left[\frac{1}{\Gamma(\alpha)} \frac{x_i}{2} \sum_{j=1}^{M_2} w_j \left(\frac{x_i}{2} (1 - s_j)^{\alpha-1} \right) \right. \\
 &\quad \left. \times \int_0^1 k_{2i} \left(\frac{x_i}{2} (1 + s_j), t \right) B_i^T \text{HOBW}(t) dt \right], \tag{4.6}
 \end{aligned}$$

where M_1 and M_2 are the orders of Bernstein polynomial used in the Gauss–Legendre quadrature rule

$$F_{1i} [C_i^T \text{HOBW}(x)] = A_i^T \text{HOBW}(x), \quad F_{2i} [C_i^T \text{HOBW}(x)] = B_i^T \text{HOBW}(x). \tag{4.7}$$

From (4.6) give a system of $2^{k-1}M \times 2^{k-1}M$ nonlinear algebraic equations with the same number of unknowns in the vectors C , A , and B . Numerically disbanding this system by Newton’s technique, we get the solutions for the unknown vectors C , A , and B .

5 Existence and uniqueness

Consider FVIDE (2.5) that can be rewritten in the operator form as follows:

$$(D^\alpha y_i)(x) = g_i(x) + \mathcal{K}_{1i} \mathcal{F}_{1i} y + \mathcal{K}_{2i} \mathcal{F}_{2i} y, \tag{5.1}$$

where

$$\mathcal{K}_{1i} \mathcal{F}_{1i} y = \int_0^x K_{1i}(x, t) F_{1i}[y(t)] dt, \quad \mathcal{K}_{2i} \mathcal{F}_{2i} y = \int_0^1 K_{2i}(x, t) F_{2i}[y(t)] dt. \tag{5.2}$$

Applying J^α to both sides of Eq. (5.1), we have

$$(y_i)(x) = h_i(x) + J^\alpha [g_i(x) + \mathcal{K}_{1i} \mathcal{F}_{1i} y + \mathcal{K}_{2i} \mathcal{F}_{2i} y], \tag{5.3}$$

where $h_i(x) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_i^k(0+)$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$. Equation (5.3) is written in a form of fixed point equation $\mathcal{A}y_i = y_i$, where \mathcal{A} is defined as

$$\mathcal{A}y_i(x) = h_i(x) + J^\alpha [g_i(x) + \mathcal{K}_{1i} \mathcal{F}_{1i} y_i + \mathcal{K}_{2i} \mathcal{F}_{2i} y_i]. \tag{5.4}$$

Let $(C[0, 1], \| \cdot \|_\infty)$ be the Banach space of all continuous functions with the norm $\|f\|_\infty = \max_t |f(t)|$. Also, the operators \mathcal{F}_{1i} and \mathcal{F}_{2i} satisfy the Lipschitz condition on $[0, 1]$

as follows:

$$\begin{aligned} |\mathcal{F}_{1i}\tilde{y}_{im}(x) - \mathcal{F}_{1i}y_i(x)| &\leq L_1|\tilde{y}_{im}(x) - y_i(x)|, \\ |\mathcal{F}_{2i}\tilde{y}_{im}(x) - \mathcal{F}_{2i}y_i(x)| &\leq L_2|\tilde{y}_{im}(x) - y_i(x)|, \end{aligned} \tag{5.5}$$

where L_1 and L_2 are Lipschitz constants. So, we achieve the uniqueness of the solution of Eq. (2.5).

Theorem 5.1 *If $L_1\|K_{1i}\|_\infty + L_2\|K_{2i}\|_\infty < \Gamma(\alpha + 1)$, then problem (2.5) has a unique solution $y \in [0, 1]$.*

Proof Let $A : C[0, 1] \rightarrow C[0, 1]$ such that

$$A_i y_i(x) = h_i(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [g_i(t) + K_{1i}\mathcal{F}_{1i}y_i(t) + K_{2i}\mathcal{F}_{2i}y_i(t)] dt. \tag{5.6}$$

Let $\tilde{y}_i, y_i \in C[0, 1]$ and

$$\begin{aligned} A_i \tilde{y}_i(x) - A_i y_i(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \times [[K_{1i}\mathcal{F}_{1i}\tilde{y}_i(t) - K_{1i}\mathcal{F}_{1i}y_i(t)] \\ &\quad + [K_{2i}\mathcal{F}_{2i}\tilde{y}_i(t) - K_{2i}\mathcal{F}_{2i}y_i(t)]] dt. \end{aligned} \tag{5.7}$$

Then for $x > 0$, we have

$$\begin{aligned} &|A_i \tilde{y}_i(x) - A_i y_i(x)| \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [|K_{1i}| |\mathcal{F}_{1i}\tilde{y}_i(t) - \mathcal{F}_{1i}y_i(t)| \\ &\quad + |K_{2i}| |\mathcal{F}_{2i}\tilde{y}_i(t) - \mathcal{F}_{2i}y_i(t)|] dt \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [|K_{1i}| L_1 |\tilde{y}_i(t) - y_i(t)| + |K_{1i}| L_2 |\tilde{y}_i(t) - y_i(t)|] dt \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} (L_1 \|K_{1i}\|_\infty + L_2 \|K_{2i}\|_\infty) \|\tilde{y}_i - y_i\|_\infty dt \\ &\leq (L_1 \|K_{1i}\|_\infty + L_2 \|K_{2i}\|_\infty) \|\tilde{y}_i - y_i\|_\infty \frac{|x|^\alpha}{\Gamma(\alpha + 1)} \\ &\leq (L_1 \|K_{1i}\|_\infty + L_2 \|K_{2i}\|_\infty) \|\tilde{y}_i - y_i\|_\infty \frac{1}{\Gamma(\alpha + 1)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|A\tilde{y}_i(x) - Ay_i(x)\|_\infty &\leq \Omega_{L_1, L_2, K_1, K_2, \alpha} \|\tilde{y}_i - y_i\|_\infty, \\ \Omega_{L_1, L_2, K_1, K_2, \alpha} &= (L_1 \|K_{1i}\|_\infty + L_2 \|K_{2i}\|_\infty) \frac{1}{\Gamma(\alpha + 1)}. \end{aligned} \tag{5.8}$$

Since $\Omega_{L_1, L_2, K_1, K_2, \alpha} < 1$ by contraction mapping theorem, problem (2.5) has a unique solution in $C[0, 1]$. □

6 Convergence analysis

Theorem 6.1 *Let $y(x)$ be a function defined on $[0, 1]$ and $|y(x)| \leq M_y$, then the sum of absolute values of HOBW coefficients of $y(x)$ defined in Eq. (10) converges absolutely on the interval $[0, 1]$ if $|c_{n,m}| \leq 2^{\frac{1-k}{2}} M_y$.*

Proof Any function $y(x) \in L^2[0, 1]$ can be approximated by HOBW as follows:

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \text{HOBW}_{n,m}(x),$$

where the coefficients $c_{m,n}$ can be determined as follows:

$$c_{m,n} = \langle y(x), \text{HOBW}_{n,m}(x) \rangle.$$

At $m \geq 0$,

$$\begin{aligned} |c_{n,m}| &= |\langle y(x), \text{HOBW}_{n,m}(x) \rangle|, \\ &= \left| \int_0^1 y(x) \text{HOBW}_{n,m}(x) dx \right| \\ &\leq \int_0^1 |y(x)| |\text{HOBW}_{n,m}(x)| dx \\ &\leq M_y \int_0^1 |\text{HOBW}_{n,m}(x)| dx \\ &= M_y \int_{I_{nk}} |\text{HOBW}_{n,m}(x)| dx \\ &= M_y \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} \int_{I_{nk}} |P_m(2^k x - 2n + 1)| dx, \\ I_{nk} &= \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \right). \end{aligned}$$

By putting the variable $2^k x - 2n + 1 = t$, we have

$$|c_{n,m}| = M_y \sqrt{m + \frac{1}{2}} 2^{\frac{k}{2}} \int_{-1}^1 |P_m(t)| dt.$$

Applying Holder’s inequality,

$$\begin{aligned} \left(\int_{-1}^1 |P_m(t)| dt \right)^2 &\leq \left(\int_{-1}^1 1^2 dt \right) \left(\int_{-1}^1 |P_m(t)|^2 dt \right) \\ &= 2 \times \frac{2}{m+1} \\ &= \frac{4}{m+1}. \end{aligned}$$

This proves that

$$\int_{-1}^1 |P_m(t)| dt \leq \frac{2}{\sqrt{2m+1}}.$$

Hence,

$$c_{n,m} \leq 2^{\frac{1-k}{2}} M_y.$$

This means that the series $\sum_{i=1}^M \sum_{j=0}^n c_{ij} \text{HOBW}(x)$ is convergent as $k \rightarrow \infty$. □

Theorem 6.2 *If the sum of absolute values of the HOBW coefficients of a continuous function $y(x)$ shape convergent series, then the HOBW expansion $\sum_{i=1}^M \sum_{j=0}^n c_{ij} \text{HOBW}(x)$ converges with respect to L^2 -norm on $[0, 1]$.*

Proof Let $L^2(R)$ be the Hilbert space and

$$\tilde{y}(x) = \sum_{n=1}^{2^{k-1} M-1} \sum_{m=0}^n c_{n,m} \text{HOBW}_{n,m}(x),$$

where $c_{n,m} = \langle \tilde{y}(x), \text{HOBW}_{n,m}(x) \rangle$ for fixed n .

Let us denote $\text{HOBW}_{n,m}(x) = \chi_l$ and let $\alpha_l = \langle \tilde{y}(x), \chi_l(x) \rangle$.

We define the sequence of partial sums $\{S_n\}$, where

$$S_n(x) = \sum_{l=0}^n \alpha_l \chi_l(x).$$

For every $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$ such that, for every $n > m > N(\varepsilon)$,

$$\begin{aligned} \|S_n(x) - S_m(x)\|_2^2 &= \int_0^1 \sum_{k=m+1}^n |\alpha_k \chi_k(x)|^2 \\ &\leq \sum_{k=m+1}^n |\alpha_k|^2 \int_0^1 |\chi_k(x)|^2 dx \\ &= \sum_{k=m+1}^n |\alpha_k|^2. \end{aligned}$$

From Theorem 5.1, $\sum_{k=0}^\infty |\alpha_k|^2$ is absolutely convergent.

According to the Cauchy criterion, for every $\varepsilon > 0$, there exists a positive number such that

$$\sum_{k=m+1}^n |\alpha_k|^2 < \varepsilon$$

whenever $n > m > N(\varepsilon)$.

Hence $\|S_n(x) - S_m(x)\|_2^2 < \sum_{k=m+1}^n |\alpha_k|^2 < \varepsilon$.

This implies that $\|S_n(x) - S_m(x)\|_2 \leq \sqrt{\varepsilon} < \varepsilon$.

So, the sequence of a partial sum of the series converges with respect to L^2 -norm and hence it completes the proof. □

Table 1 The absolute error for Example 7.1 for different estimations of k, M at $\alpha = 1$

x	$k = 3, M = 4$	$k = 4, M = 5$	$k = 5, M = 6$	$k = 6, M = 7$	$k = 7, M = 8$
0.1	1.6403×10^{-7}	1.4281×10^{-9}	2.0695×10^{-10}	1.574×10^{-12}	4.2461×10^{-14}
0.2	2.5018×10^{-7}	2.7054×10^{-9}	4.1957×10^{-10}	2.1255×10^{-12}	5.0451×10^{-14}
0.3	1.4102×10^{-7}	4.7082×10^{-9}	1.4795×10^{-10}	2.1443×10^{-12}	2.6821×10^{-14}
0.4	2.1512×10^{-7}	6.5014×10^{-9}	3.1759×10^{-10}	3.1682×10^{-12}	3.5171×10^{-14}
0.5	1.9018×10^{-7}	2.7802×10^{-9}	5.6087×10^{-10}	4.6833×10^{-12}	2.6417×10^{-14}
0.6	4.4186×10^{-7}	2.3081×10^{-9}	2.2781×10^{-10}	3.6951×10^{-12}	1.7129×10^{-13}
0.7	3.1042×10^{-6}	4.2721×10^{-8}	1.7309×10^{-9}	1.2071×10^{-11}	2.6357×10^{-13}
0.8	4.3051×10^{-6}	1.3062×10^{-8}	2.9053×10^{-9}	2.2721×10^{-11}	3.6457×10^{-13}
0.9	3.2101×10^{-6}	2.3775×10^{-8}	2.0941×10^{-9}	3.2974×10^{-11}	1.6864×10^{-13}

7 Numerical examples

Example 7.1 Consider the following fractional nonlinear Volterra integro-differential equation:

$$\begin{aligned}
 D^{\alpha+1}y(x) &= \int_0^1 xty^2(t) dt + \int_0^x (e^t - 1)y^2(t) dt + e^x \\
 &\quad - x\left(\frac{e^2}{4} - 2e + \frac{11}{3}\right) - \frac{(e^x - x - 1)^3}{3},
 \end{aligned}
 \tag{7.1}$$

with mixed conditions

$$\begin{aligned}
 y(0) + y'(0) &= 0, \\
 y(1) + y'(1) &= -3 + 2e.
 \end{aligned}
 \tag{7.2}$$

$y(x) = -1 - x - e^x$ is the exact solution at $\alpha = 1$.

Table 1 demonstrates the absolute errors acquired by the present strategy for different estimations of k, M at $\alpha = 1$. The examination of numerical results for $\alpha = 0.75, \alpha = 0.85, \alpha = 0.95, \alpha = 1$ and the exact solution for $\alpha = 1$ is shown in Fig. 2. It is clear from Fig. 2 that as α is near to 1, the related numerical solution converges to the exact solution.

Example 7.2 We consider the nonlinear FVFIDE

$$(D^{\sqrt{3}}y)(x) = \frac{2(2 + \sqrt{3})x^{2-\sqrt{3}}}{\Gamma(2 - \sqrt{3})} - \frac{15x^8}{56} - \frac{x^2}{6} + \int_0^1 x^2ty^2(t) dt + \int_0^x (x + t)y^3(t) dt
 \tag{7.3}$$

with the MBC

$$\begin{aligned}
 y(0) + y'(0) &= 0, \\
 y(1) + y'(1) &= 3,
 \end{aligned}
 \tag{7.4}$$

with the correct solution $y(x) = x$. This problem has been disbanded by HOBW for $M = 4, k = 3$, which reduces the integral equation to a system of algebraic equations that is disbanded by Newton’s method. The consequence obtained by the introduced method is compared with that by the Nystrom method (for $N = 20$). The approximate solutions and absolute errors (Abs. Error) for Example 7.2 are introduced in Table 2.

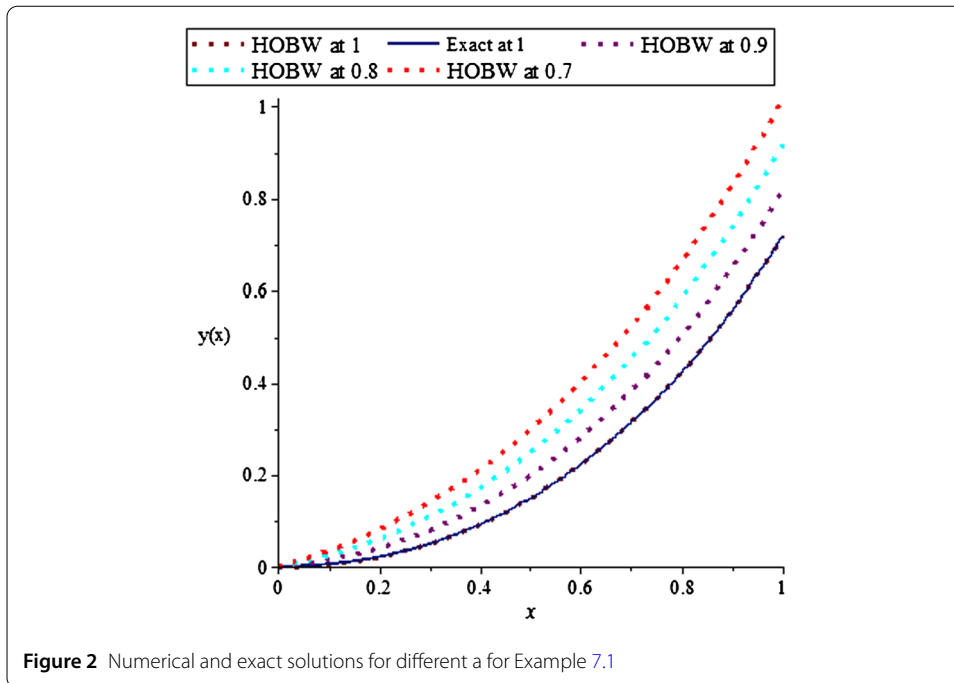


Table 2 Comparison of HOBW results and Abs. Error for Example 7.2

x	Exact	HOBW at $M = 4, k = 3$		Nystrom method ($N = 20$)	
	$y(x)$	$y(x)$	Abs. Error	$y(x)$	Abs. Error
0	0	0.000766339	0.0000766339	0	0
0.1	0.01	0.0106897	0.00006897	0.0100005	0.00000487
0.2	0.04	0.040613	0.0000613	0.0400065	0.0000065
0.3	0.09	0.0905364	0.00005364	0.0900304	0.0000304128
0.4	0.16	0.16046	0.000046	0.160101	0.00010141
0.5	0.25	0.250372	0.0000372	0.250329	0.000329115
0.6	0.36	0.360307	0.0000307	0.361156	0.00115625
0.7	0.49	0.490222	0.0000222	0.494128	0.00412784
0.8	0.64	0.640136	0.0000136	0.653945	0.0139455
0.9	0.81	0.810069	0.000069	0.85402	0.0440196

Example 7.3 We consider the nonlinear FVFIDE

$$\begin{aligned}
 (D^{1.7}y)(x) = & -4.19453 - x - \frac{x^5}{3} + \frac{1}{\Gamma(1.3)} e^x x^{0.3} {}_1F_1[0.3, 1.3; -x] + \int_0^1 x^2 t y^2(t) dt \\
 & + \int_0^x (x+t) y^3(t) dt, \tag{7.5}
 \end{aligned}$$

where ${}_1F_1[0.3, 1.3; -x]$ is the Kummer confluent hypergeometric function defined as

$${}_1F_1[a, b; z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}$$

with $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$ and $(a)_0 = 1$, subject to the MBC $y(0) = 1, y(1) = e$.

The correct solution to this problem is given as $y(x) = e$. This problem is disbanded by HOBW at $M = 4, k = 3$, which reduces the integral equation to a system of algebraic equations that is disbanded by Newton’s method. The consequence obtained by the introduced

Table 3 Comparison of HOBW results and the Nystrom method for Example 7.3

x	Exact	HOBW at $M = 4, k = 3$		Nystrom method	
		$y(x)$	Abs. Error	$y(x)$	Abs. Error
0	1	1	0	1	0
0.1	1.10517	1.105167	0.000003	1.10393	0.00123938
0.2	1.2214	1.22138	0.00002	1.21842	0.00297924
0.3	1.34986	1.349857	0.000003	1.34476	0.00510241
0.4	1.49182	1.491816	0.000004	1.4843	0.00752611
0.5	1.64872	1.648718	0.000002	1.63862	0.0101061
0.6	1.82212	1.822119	0.000001	1.80958	0.0125403
0.7	2.01375	2.013747	0.000003	1.99952	0.0142327
0.8	2.22554	2.225538	0.000002	2.21144	0.0141018
0.9	2.4596	2.45956	0.00004	2.44928	0.0103248

Table 4 Comparison of HOBW results and [18] for Example 7.4

x	Error by HOBW	Error by [18]
	for $M = 8, k = 4$	for $n = 320$
0.1	2.6455×10^{-8}	1.92×10^{-6}
0.3	3.5312×10^{-8}	3.84×10^{-6}
0.5	1.1546×10^{-7}	4.1×10^{-6}
0.7	3.4162×10^{-6}	3.15×10^{-6}
0.9	2.6057×10^{-6}	1.25×10^{-6}

Table 5 Maximum absolute errors at different values of M and k for Example 7.4 via HOBW

M	k			
	4	6	8	10
8	4.59×10^{-7}	8.02×10^{-9}	4.19×10^{-10}	7.38×10^{-11}
12	5.47×10^{-10}	2.49×10^{-12}	3.17×10^{-14}	2.12×10^{-15}
16	3.39×10^{-11}	2.95×10^{-14}	1.21×10^{-15}	4.24×10^{-16}
20	1.02×10^{-14}	4.18×10^{-15}	2.88×10^{-16}	3.05×10^{-17}

method is compared with that by the Nystrom method (for $N = 20$). The numerical solutions and Abs. Errors for Example 7.3 are introduced in Table 3.

Example 7.4 Let us consider the nonlinear FVFIDE

$$\begin{aligned}
 (D^{\frac{\sqrt{7}}{2}} y)(x) &= 1 - e^2 - \log(1 + x + x^2) - \frac{4x^{2+\frac{\sqrt{7}}{2}}}{(\sqrt{7} - 4)\Gamma(2 - \frac{\sqrt{7}}{2})} + \int_0^1 (1 + 2t)e^{y(t)} dt \\
 &+ \int_0^x \frac{(1 + 2t)}{1 + y(t)} dt
 \end{aligned}$$

with boundary conditions $y(0) = 1, y(1) = e$.

The correct solution is $y(x) = x^2 + x$.

This problem is disbanded by HOBW which reduces the integral equation to a system of algebraic equations that is disbanded by Newton’s method. The consequence obtained by the method is compared with that by the Nystrom method [18]. The numerical consequence and Abs. Error for Example 7.4 are introduced in Table 4.

Maximum absolute errors at different values of M and k have been presented in Table 5.

8 Conclusion

In this work, we have fully attempted to find the numerical solution of the fractional system of Volterra integro differential equations by using the HOBW method. The numerical procedure and methodology are done in a very straightforward and effective manner. The numerical accuracy is also a point of interest. Through the numerical calculation, we confirmed that the HOBW method has the highest degree of accuracy. On the basis of this work, the researchers can extend this technique to some other fractional systems of ordinary and partial differential equations.

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Authors' contributions

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Author details

¹Department of Mathematics, Benha Faculty of Engineering, Benha University, Banhā, Egypt. ²Department of Mathematics, Faculty of Science, Menoufia University, Shebeen El-Kom, Egypt. ³Department of Mathematics and Statistics, University of Victoria, Victoria, Canada. ⁴Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, Republic of China.

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