Pairwise Balanced Designs of Dimension Three

by

Joanna Niezen
B.Math, University of Waterloo, 2010

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of

MASTERS OF SCIENCE

in the Department of Mathematics and Statistics

Pairwise Balanced Designs of Dimension Three

by

Joanna Niezen
B.Math, University of Waterloo, 2010

Supervisory Committee

Dr. Peter Dukes, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

Supervisory Committee

Dr. Peter Dukes, Supervisor
(Department of Mathematics and Statistics)

Dr. Gary MacGillivray, Departmental Member
(Department of Mathematics and Statistics)

ABSTRACT

A linear space is a set of points and lines such that any pair of points lie on exactly one line together. This is equivalent to a pairwise balanced design PBD(v, K), where there are v points, lines are regarded as blocks, and $K \subseteq \mathbb{Z}_{\geq 2}$ denotes the set of allowed block sizes. The dimension of a linear space is the maximum integer d such that any set of d points is contained in a proper subspace. Specifically for $K = \{3, 4, 5\}$, we determine which values of v admit PBD(v, K) of dimension at least three for all but a short list of possible exceptions under 50. We also observe that dimension can be reduced via a substitution argument.

Contents

ACKNOWLEDGEMENTS

I would like to thank:

my supervisor, Dr. Peter Dukes for his patience and contagious enthusiasm about our research,

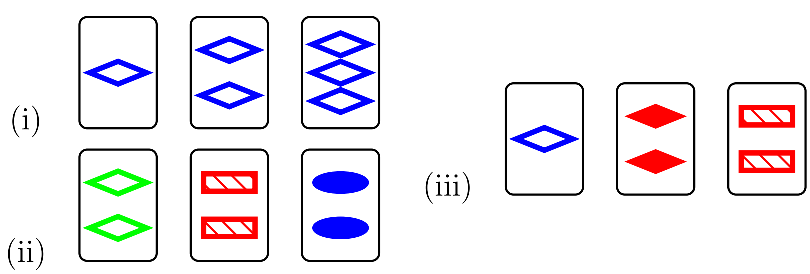
an anonymous party whose programming help was invaluable, and finally

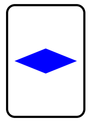
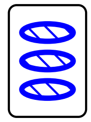
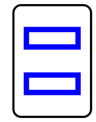
both the Faculty of Graduate Studies and NSERC, for helping to fund my graduate research.

Chapter 1

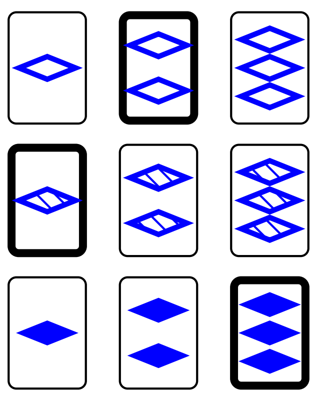
Introduction

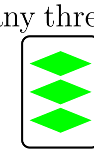
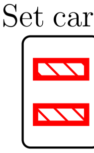
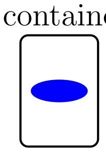
We start with an example. Consider the deck of Set cards, a slight variation of which is used here. There is exactly one card with each unique configuration of the four attributes. Having three possibilities per attribute results in 81 cards. Sets are defined as a collection of three cards such that each of the four attributes are either all shared or all distinct within the collection. For example, (i) and (ii) are both sets, but (iii) is not.

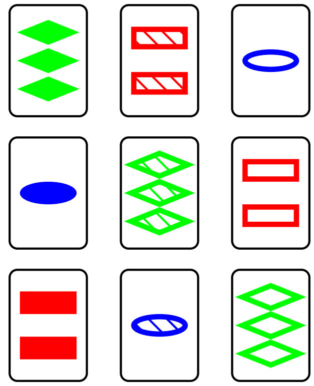


Any two cards chosen either share the attribute shape, or do not. If they share the attribute diamonds for example, then the third card necessary to make a set is also a diamond. If they do not share the attribute, suppose one card is a diamond and the other an oval, then the third card in that set is required to be the unrepresented shape, a rectangle in this case. As each configuration of four attributes is represented on a single card in the deck, every two cards have a unique third card which completes the set. For example, the two cards   are uniquely completed by  to make a set. In this way it is clear that every two cards are in a unique set together within the deck.

The same is true if we vary the number of attributes. Suppose we only allow cards in the Set deck to have a single colour, thereby restricting our deck to 27 cards. The unique third card to complete any set of two blue cards is blue and so all sets containing only blue cards live within the restricted deck. Therefore, the deck with 27 cards also has the property that every two cards define a unique set. We can similarly restrict other properties, or more than one property, to find smaller and smaller decks within the original deck having this same property. As an example, the deck of cards having the attributes blue and diamonds are listed below. Each row, column and diagonal represents a set, including the 'broken-diagonals', one of which is in boldface below.



Within the original Set deck of 81 cards, smaller decks with the Set deck property do not only arise from restricting attributes. In fact, any three Set cards are contained in a nine card deck. For example, the deck containing , , and  is shown below. As in the above example, every row, column, and diagonal is a set.



We can think of the cards as points, and sets as blocks. In this way the Set deck represents a Steiner triple system.

Definition 1.1 (Steiner Triple System)**.** A Steiner triple system, or STS, consists of a set of objects called points, and a set of subsets of the points, called blocks. Any pair of points in an STS is contained in a block exactly once and every block contains exactly three points. A Steiner triple system with v points is called an STS(v).

The deck restricted to blue and diamonds is an example of an STS(9), also known as the affine plane over $\mathbb{F}_3$.

We would like to investigate when subsystems are contained in larger systems. We will look at what happens when the definition of a block is relaxed to allow different sizes and investigate when any d points are contained inside smaller subsystems. Before delving into this investigation, we formalize these ideas.

Definition 1.2 (Incidence Structure)**.** Let P be a set of points and $\mathcal{L}$ be a set of lines. An incidence structure is a triple $(P, \mathcal{L}, \iota)$, where $\iota \subset P \times \mathcal{L}$. We say p and L are incident when $(p, L) \in \iota$.

Definition 1.3 (Linear Space)**.** A linear space is an incidence structure $(P, \mathcal{L}, \iota)$ such that (i) every pair of points in P is incident to exactly one line in $\mathcal{P}$, and (ii) every line in $\mathcal{L}$ is incident to at least two points. We often write $(P, \mathcal{L})$ to denote a linear space, or simply P when the set $\mathcal{L}$ is clear. In any case, lines can be identified with their sets of incident points. Here, we are only concerned with finite linear spaces.

Linear spaces have another name in the context of designs. Let V be a set of points and $\mathcal{B}$ be a set of subsets of V called blocks. The size of a block is the number of points contained in a subset.

Definition 1.4 (Pairwise Balanced Design)**.** A pairwise balanced design is a system of points and blocks, $(V, \mathcal{B})$, such that every pair of points in V is in exactly one block in $\mathcal{B}$. If $|V| = v$, we say $(V, \mathcal{B})$ has order v. Suppose $K \subseteq \mathbb{Z}_{\geq 2}$ includes the set of all block sizes in $\mathcal{B}$. A pairwise balanced design with order v and block sizes in K is called a PBD(v, K). We sometimes refer to a pairwise balanced design as a K-PBD or simply PBD. Note that not all block sizes need to occur.

In what follows, the terminology for linear spaces and pairwise balanced designs is used interchangeably. That is, both the notion of blocks and lines are used interchangeably, as well as 'on' or 'in' and incidence.

Example 1.5 (A PBD of Order 11)**.** We describe a construction for a PBD$(11, \{3, 5\})$ consisting of a single block B of size 5, and a 1-factorization on the complete graph, K_6. The blocks besides B consist of the matched points in a 1-factor of K_6 together with one point from B. More specifically, let $B = \{0', 1', 2', 3', 4'\}$ and the points in K_6 be $0, 1, 2, 3, 4, \infty$. We consider the 1-factor with edges $\{0, \infty\}$, $\{1, 4\}$, and $\{2, 3\}$. The corresponding blocks in the PBD are $\{0', 0, \infty\}$, $\{0', 1, 4\}$, and $\{0', 2, 3\}$. Since the 1-factor spans all points, this set of blocks cover all pairs that include $0'$ with a point outside of B. The remaining blocks are determined by additive shifts modulo 5 to each point of the blocks of size 3, pairing every 1-factor in K_6 with a different point of B. In Section 4.1, we discuss why this is the only PBD of order 11 up to isomorphism.

There is a relationship between the order of a PBD and its permitted block sizes. The following lemma outlines what orders are admissible for a K-PBD, for a given set K.

Lemma 1.6 (Necessary Conditions of Existence)**.** The following are necessary conditions for the existence of a PBD(v, K):

- $v - 1 \equiv 0 \pmod{\alpha(K)}$ where $\alpha(K) = \gcd\{k - 1 \mid k \in K\}$, and
- $v(v - 1) \equiv 0 \pmod{\beta(K)}$ where $\beta(K) = \gcd\{k(k - 1) \mid k \in K\}$.

From Lemma 1.6, Steiner triple systems are admissible (and exist [4]) for orders $v \equiv 1, 3 \pmod 6$. For $K = \{3, 4, 5\}$, the necessary conditions relax to all natural numbers. In fact, it is known that PBD$(v, \{3, 4, 5\})$ exist for every $v \in \mathbb{N}$ except 2, 6, and 8. In general for arbitrary K, Wilson's theory [15] says that the conditions of Lemma 1.6 are sufficient for large v. For these results regarding PBD existence, and many well-known results to come, we refer to [4] in which the reader can find a survey of results and original references.

Definition 1.7 (Subspace)**.** A subspace of $(V, \mathcal{B})$ is a linear space $(W, \mathcal{C})$ that is contained in $(V, \mathcal{B})$. That is, $W \subseteq V$ and $\mathcal{C} \subseteq \mathcal{B}$ such that $(W, \mathcal{C})$ is a linear space itself. We sometimes write $\mathcal{C}$ as $\mathcal{B}|_W$, the set of blocks 'restricted' to W, or containing only points from W. When $W \neq V$, we say $(W, \mathcal{C})$ is a proper subspace of $(V, \mathcal{B})$.

Notice that the points on a single block in any linear space always form a subspace. We further discuss when proper subspaces are guaranteed to exist.

Definition 1.8 (Dimension)**.** The dimension of a linear space V is the maximum integer d such that every set of d points is contained in a proper subspace of V.

Since a block is a trivial subspace of its larger space, by definition a pairwise balanced design with more than one block has dimension at least 2. We therefore are interested in linear spaces of dimension at least 3. We call a Steiner triple system of dimension at least 3 a Steiner space. Teirlinck [14], proved the following theorem on the existence of Steiner spaces.

Theorem 1.9 (Existence of Steiner Spaces)**.** Steiner spaces of order v exist if and only if $v = 15$, 27, 31, 39 and all $v \geq 45$ with $v \equiv 1$ or 3 (mod 6), except possibly $v = 51$, 67, 69, 145.

Further to this result, Dukes and Ling have found an asymptotic existence theory for general pairwise balanced designs with prescribed minimum dimension.

Theorem 1.10 (Asymptotic Existence of PBDs with Prescribed Dimension [8])**.** For any positive integer d and $K \subseteq \mathbb{Z}_{\geq 2}$, there exists a PBD$(v, K)$ of dimension at least d for all sufficiently large v satisfying Lemma 1.6.

In theorem 1.10, sufficiently large v corresponds to a tower of exponents, similarly to Wilson's original asymptotic existence theory [15]. We are interested in exploring the existence of pairwise balanced designs of dimension at least 3 for all orders, especially smaller orders than those treated in Theorem 1.10. After Steiner spaces, the next natural cases to look at are PBDs with block sizes in $\{3, 4\}$ or $\{3, 4, 5\}$. Because the necessary conditions from Lemma 1.6 disappear for $K = \{3, 4, 5\}$, in this thesis we investigate $K = \{3, 4, 5\}$ specifically. Therefore, unless otherwise noted, block sizes $K = \{3, 4, 5\}$ is assumed in what follows. Theorem 1.11 is the main result of this investigation.

Theorem 1.11 (Main Result)**.** Pairwise balanced designs of dimension at least 3 and block sizes in $\{3, 4, 5\}$ exist if and only if $v = 15$ or $v \geq 27$ except $v = 32$ and possibly for $v \in E$ where $E = \{33, 34, 35, 38, 41, 42, 43, 47\}$.

The rest of this thesis is primarily devoted to proving Theorem 1.11. In Chapter 2, we introduce the necessary notions to proceed. In particular, we develop various machinery used in the main construction discussed in Chapter 3. For small orders not covered in this construction, ad-hoc constructions are investigated in Chapter 4. Non-existence results are explored in Chapter 5.

In Chapter 6, we look at forming pairwise balanced designs with smaller dimension from those of higher dimension. Because of Theorem 6.1 to follow, we could alternately state Theorem 1.11 for dimension exactly 3.

Chapter 7 concludes the thesis with a discussion, including implications for the cases $K = \{3, 4\}$ and $K = \{3, 5\}$.

Chapter 2

Background

This chapter introduces the definitions and concepts in the theory of linear spaces and pairwise balanced designs. Following the concepts section we define and explore actions on PBDs that do not disrupt dimension. The final section in the chapter is devoted to confirming the existence of GDDs that will be used in conjunction with these actions in order to uncover PBDs with dimension at least three.

2.1 Fundamentals

Interval notation $[m, n]$ is used to denote the set $\{x \in \mathbb{Z} \mid m \leq x \leq n\}$. Sometimes the notation $[n]$ is used to denote $[1, n]$.

Definition 2.1 (Intersection of Subspaces)**.** Let $(U_1, \mathcal{B}_1)$ and $(U_2, \mathcal{B}_2)$ be any subspaces of a linear space $(V, \mathcal{B})$. Define the intersection $(U_1, \mathcal{B}_1) \cap (U_2, \mathcal{B}_2)$ as the set of all points $W = U_1 \cap U_2$, and the set of blocks from $\mathcal{B}$ containing at least two points in W.

Lemma 2.2 (Linear Spaces Intersect in Subspaces)**.** Let V be a linear space. The intersection of any two (and therefore any collection) of subspaces of V is a linear space. Furthermore, if V is a K-PBD then the intersection of any set of subspaces of V is also a K-PBD.

Proof. Let $(U_1, \mathcal{B}_1)$ and $(U_2, \mathcal{B}_2)$ be subspaces of a linear space $(V, \mathcal{B})$, with intersection $(W, \mathcal{C})$. Suppose that $(W, \mathcal{C})$ is not a linear space. Then W contains two points x and y but there is no block in $\mathcal{C}$ containing the pair $\{x, y\}$. Since $(U_1, \mathcal{B}_1)$ is a linear space containing x and y, there is a block $B \in \mathcal{B}_1$ incident to both x and y. Since $\mathcal{B}$ contains exactly one block incident to both x and y, B is in $\mathcal{B}_2$ also, and thus B is in $\mathcal{C}$.

Suppose V is a K-PBD. We have just shown that for any two points in the intersection $(W, \mathcal{C})$ of V, the block incident to those points is contained in $\mathcal{C}$. Thus $\mathcal{C}$ has the same block sizes $\mathcal{B}$, and is therefore a K-PBD. □

Definition 2.3 (Generated Subspace)**.** For a set of points $S \subset V$, the subspace W generated by S in $(V, \mathcal{B})$ is the intersection of all subspaces in $\mathcal{B}$ containing S. Equivalently, W is the unique minimal subspace containing S. We say S is a basis for W and write $W = \langle S \rangle$. Note that bases need not have the same size.

Definition 2.4 (Non-Degenerate Plane)**.** A non-degenerate plane is a linear space containing more than one line such that any three non-collinear points generate the entire linear space.

Definition 2.5 (Degenerate Plane)**.** A degenerate plane is a linear space containing more than one line such that there are three non-collinear points that generate the entire linear space and three non-collinear points that do not.

From these definitions, we note that both a degenerate and non-degenerate plane have dimension two. We will use the word ‘plane’ to denote any PBD of dimension two, and reserve ‘space’ for PBDs of dimension at least 3.

Definition 2.6 (Projective Plane/Space)**.** Let q be a prime power and $\mathbb{F}_q$ be the finite field of order q. For any positive integer d, consider the vector space $\mathbb{F}_q^{d+1}$. Let the set of all 1-dimensional subspaces of $\mathbb{F}_q^{d+1}$ be the point set. The lattice of all subspaces of $\mathbb{F}_q^{d+1}$ forms the projective space of dimension d over $\mathbb{F}_q$, or $PG_d(q)$. In the terminology of pairwise balanced designs, $PG_d(q)$ is a PBD$(1+q+\cdots+q^d, \{q+1\})$ of dimension d, where blocks are the 2-dimensional subspaces of $\mathbb{F}_q^{d+1}$ and incidence is inclusion.

Example 2.7 (Binary Projective Space)**.** The points of $PG_d(2)$ are the elements of $\mathbb{F}_2^{d+1} \setminus \{0\}$, all non-zero $d+1$-tuples. Lines are the zero-sum triples of these vectors.

Definition 2.8 (Affine Plane/Space)**.** We now start with the vector space $\mathbb{F}_q^d$. Consider the translates of subspaces of $\mathbb{F}_q^d$ as the flats forming the affine space $AG_d(q)$. Thinking of the 1-dimensional flats of $AG_d(q)$ as the set of blocks $\mathcal{B}$, we see that $(\mathbb{F}_q^d, \mathcal{B})$ is a linear space, or equivalently a PBD$(q^d, \{q\})$. From a linear algebra argument, all subspaces generated by d points correspond to proper flats and this linear space has dimension d.

Definition 2.9 (Parallel Class)**.** A parallel class in a linear space is a collection of lines that partition the set of points.

Definition 2.10 (Group Divisible Design)**.** A group divisible design, or GDD, is a triple $(V, G, \mathcal{B})$ where V is a set of points and $\mathcal{B}$ is a set of blocks on V. The set G corresponds to a partition of the points into sets called 'groups'. The defining property of a GDD is that every pair of points is either on exactly one block, or is contained in a single group. For some subset K of the positive integers, a K-GDD is a GDD where all blocks have size in K. If $K = \{k\}$ then we simply write k-GDD.

Let j_i and g_i be natural numbers for each $i = 1, 2, \ldots, m$. We say a GDD is of type $g_1^{j_1} g_2^{j_2} \ldots g_m^{j_m}$ if it has j_i groups each containing g_i points, for each value of i.

Definition 2.11 (Transversal Design)**.** A transversal design, denoted $\mathrm{TD}(k,n)$, is a k-GDD of type n^k.

Definition 2.12 (Mutually Orthogonal Latin Squares)**.** A Latin square of side n is an $n \times n$ array where each cell contains an entry from an n-set, say $[n]$. Every symbol occurs exactly once in each row and column of the array. Two Latin squares are orthogonal if the set of paired elements from corresponding cells exhausts the pairs in $[n]^2$. A set of mutually orthogonal Latin squares of side n, or MOLS(n), is a set of Latin squares that are pairwise orthogonal.

Lemma 2.13 (MOLS equivalence)**.** A $\mathrm{TD}(k,n)$ is equivalent to a set of $k-2$ mutually orthogonal Latin squares of side n.

Definition 2.14 (Resolvable)**.** A PBD or GDD is called resolvable if its blocks can be partitioned into parallel classes.

Definition 2.15 (Subspace of a GDD)**.** A subspace of a GDD $(V, G, \mathcal{B})$ is a GDD $(W, G', \mathcal{C})$ such that $W \subseteq V$ and $G' = W \cap G$ and $\mathcal{C}$ is the set of blocks containing at least two points in W.

2.2 Dimension Preserving Actions

This section discusses actions on PBDs leading to PBDs of new orders, and the impact on dimension.

Definition 2.16 (Deletion)**.** A deletion in a linear space V is the removal of some set of points $S \subseteq V$ and all blocks incident to S.

If a single point x is deleted from V then what remains is a GDD whose groups corresponding to a block incident to x. We call V a 'one point extension' of the GDD that remains when x is deleted.

Construction 2.17 (Filling in Groups)**.** Consider a K-GDD with group sizes G. For each $g \in G$, if a PBD(g, K) exists, replace the groups of the GDD by a PBD(g, K). All pairs within groups are now covered by blocks in these 'filler' PBDs. As pairs between groups are covered in the GDD, it is easy to see that we have created a K-PBD on the same set of points as the GDD.

Notice that if each group is replaced by a single block then the groups of the GDD form a parallel class in the corresponding PBD. In this case, if the original GDD is resolvable, then so is the resulting PBD. In Theorem 2.19 to follow, we additionally discuss how to construct a PBD by adding an additional point to a GDD, and the effect of this construction on dimension.

Definition 2.18 (Strong Dimension of a GDD)**.** A subspace of a GDD is strong if it intersects each group in all or no points.

We say a GDD has strong dimension d if any set of d points generate a strong proper subspace, and there is some set of $d+1$ points that do not.

Notice that strong dimension corresponds to the notion of dimension in linear spaces, when a PBD(v, K) is taken as a K-GDD of type 1^v. We have the following theorem from Dukes and Ling, using a variation of Construction 2.17.

Theorem 2.19 (Strong Dimension and PBD Dimension [8])**.** Suppose there is a K-GDD on v points of strong dimension d. Let $w = 0, 1$. If for every group size g of the GDD, there exists a PBD$(g+w, K)$, then there exists a PBD$(v+w, K)$. Furthermore, PBD$(v + w, K)$ has dimension at least d.

We can similarly let $w > 1$ as long as for every group size g, there is a linear space of order $g + w$ containing a subspace of order w. When adding new points to a linear space, we sometimes call the additional points, 'points at infinity' in order to differentiate them from the original points.

Dukes and Ling extend Wilson's Fundamental Construction [15] to preserve dimension.

Theorem 2.20 (Wilson's Construction and Dimension [7])**.** Let $(V, \mathcal{B})$ be a linear space of dimension d. Let $w : V \to \mathbb{Z}_{\geq 0}$ be a non-degenerate weighting. (That is, an assignment of a non-negative integer to each point such that all points of positive weight generate V.) If for every $B \in \mathcal{B}$ there is a GDD with group sizes $\{w(v) \mid v \in B\}$, then there exists a GDD with group sizes $w(v)$, for any $v \in V$. The associated linear space has strong dimension at least d.

Definition 2.21 (Truncation)**.** A truncation of $(V, \mathcal{B})$ is the removal of some set of points from V. Unlike deletion, blocks that were incident to a truncated point remain in the new configuration with reduced size. We ignore blocks of size 1 in what is left.

Truncating points will be an important tool to find linear spaces with higher dimension. Notice that truncation is the same as Wilson's Construction where each point is assigned a weight of 0 or 1. Dukes and Ling showed that truncating from a linear space preserves dimension, as long as the remaining points are not contained in a subspace of dimension less than d. For dimension at least 3, it is necessary to have points remaining from distinct sub-planes. We make use of truncation from the projective space $PG_d(q)$ whose largest subspace of dimension less than d has order $1 + q + q^2 + \cdots + q^{d-1}$.

Lemma 2.22 (Truncation Preserves Dimension [6])**.** Let W be the linear space resulting from truncating $PG_d(q)$. If the order of W is larger than $1+q+q^2+\cdots+q^{d-1}$, then W has dimension at least d.

The next construction combines Theorems 2.19 and 2.20. We can add points at infinity after inflation using Wilson's construction without affecting the dimension.

Construction 2.23 (Inflation $kv+1$)**.** Start with any PBD($v, \{3,4,5\}$) of dimension at least 3. Inflate each point in V by k, for $k \in \{2,3,4\}$. Replace blocks with $\{3,4,5\}$-GDDs of required group type which exist by Lemma 2.26 to come. This results in a $\{3,4,5\}$-GDD of type k^v and strong dimension at least 3. Add a new point x at infinity and fill in each group with a block of size $k+1$ containing x and each group. The resulting PBD($kv+1, \{3,4,5\}$) has dimension at least 3.

Proof. With x truncated, the configuration is a PBD($kv, \{3,4,5\}$) of dimension at least 3, by Theorem 2.20. Suppose this PBD of order kv has dimension $d \geq 3$ and call it V. As $k+1$ is a valid block size, we only need to show that adding x to each block of inflated points in V preserves the dimension. Because V has dimension d, no set of d points in V generate the entire space. The GDD, where blocks on inflated points in V are taken as groups, has strong dimension d. Thus, by Theorem 2.19, the resulting PBD($kv+1, \{3,4,5\}$) has dimension at least d. □

Definition 2.24 (Contraction)**.** A contraction on a linear space is similar to the notion of contraction in graph theory: merging the points of a block down to a single point. More formally, in a linear space the contraction of a block B is the deletion of each point on B, and the addition of a new point b. Subspaces that contain exactly one point from B are rebuilt using b to have the same set of blocks prior to the contraction. Subspaces containing all of B are replaced with linear spaces on $|B|-1$ fewer points; the set of remaining points of the original subspace together with b.

A contraction is only possible when subspaces of the required reduced order exist. As subspaces are replaced with entirely different linear spaces, the contraction of a K-PBD does not necessarily result in a PBD with blocks sizes in K. Notice, however, that by construction every proper subspace of the original linear space corresponds to a proper subspace after contraction. This property makes contraction useful in terms of finding linear spaces of specified dimension.

Lemma 2.25 (Contraction and Dimension)**.** Let $(V, \mathcal{B})$ be a linear space of dimension $d+1$. Let $(W, \mathcal{C})$ be the linear space created by contracting $B \in \mathcal{B}$. Then W has dimension at least d.

Proof. Take any set S of d points in W. If $b \notin S$, then $\langle S \rangle$ is a proper subspace of W analogous to the subspace $\langle S \rangle$ in V. If $b \in S$, consider the set of $d+1$ points $S' = S \setminus \{b\} \cup \{x, y\}$, for any two points $x, y \in B$. As V has dimension $d+1$, it is clear that $\langle S' \rangle$ is a proper subspace of V containing B. The sister subspace of $\langle S' \rangle$ in W, call it U, is therefore also proper. Moreover, by definition of contraction U contains b. It follows that $\langle S \rangle = U$. Therefore, any set of d points generate a proper subspace in W. □

2.3 GDD Existence

We first look at group divisible designs with constant group sizes 2, 3, or 4. From [4], we know that 3-GDDs of type $2^3, 2^4, 3^3, 3^5, 4^3$, and 4^4 exist. The existence of a 4-GDD of type 3^4, 4^4, and 3^5 as well as a 5-GDD of type 4^5 is also known. All of these GDDs are useful for recursively constructing PBDs.

Lemma 2.26 (GDDs with Constant Group Size)**.** There exists a $\{3, 4, 5\}$-GDD of type k^i for any $k \in \{2, 3, 4\}$ and any $i \in \{3, 4, 5\}$.

The only GDD that is missing for the proof of Lemma 2.26 is a $\{3, 4, 5\}$-GDD of type 2^5 which we construct below.

Construction 2.27 ($\{3, 5\}$-GDD of Type 2^5)**.** Define the groups to be $\{j_a, j_b\}$ for $j \in \mathbb{Z}_5$. The blocks are $\{0_a, 1_a, 2_a, 3_a, 4_a\}$, $\{0_a, 1_b, 4_b\} + j$, and $\{0_a, 2_b, 3_b\} + j$ for all $j \in \mathbb{Z}_5$ where subscripts are preserved. We call this GDD H. No pairs of points $\{j_a, j_b\}$ are covered in this way, so H has the required group type. It is easy to check that the 'ab' and 'bb' differences in blocks of size 3 cover all pairs of points. As pairs of type 'aa' are all covered in the block of size 5, H is therefore a $\{3,5\}$-GDD of type 2^5.

For the main recursion in Chapter 3, it is helpful to obtain all possible five-group GDDs with consecutive group sizes at least two.

Theorem 2.28. There exist $\{3,4,5\}$-GDDs of type $(m-1)^i m^{5-i}$ for all integers $m \geq 3$ and $0 \leq i \leq 5$.

Proof. Lemma 2.29 together with Constructions 2.31, 2.32, and 2.33 complete the proof. □

We use the following lemma from Dukes and Ling.

Lemma 2.29 (Filler GDDs [7])**.** For any integer m, if there is a TD$(5,m)$, then there exists a $\{3,4,5\}$-GDD of type $(m-1)^i m^{5-i}$ for $i = 0,1,\ldots,5$.

By Lemma 2.13, a TD$(5,m)$ is equivalent to three MOLS(m). It is known that the latter exists for every side length $m > 2$ except $m = 3,6$, and possibly 10. The largest set of MOLS(3) is two. There is no set of mutually orthogonal Latin squares of side 6, so we say that the largest set of MOLS(6) is one. It is known that there are at least two MOLS(10) and so far no set of three MOLS(10) has been discovered [4]. To complete the GDD existence when $m = 3,6,10$, alternate constructions are used.

The notion of a set of mutually orthogonal Latin squares having a 'hole' will be useful for finding some of the missing values.

Definition 2.30 (Incomplete Mutually Orthogonal Latin Squares)**.** An incomplete Latin square is an array that is a Latin square with a set of positions that are not assigned with a symbol. A set of incomplete mutually orthogonal Latin squares, or IMOLS, is a set of mutually orthogonal Latin squares each having the same hole.

Construction 2.31 (For $m = 10$)**.** Consider a set of three IMOLS(10) with a 2×2 hole as in [3]. This is equivalent to an 'incomplete' 5-GDD with blocks of size 5. Cover the hole using H, the $\{3, 5\}$-GDD of type 2^5 from Construction 2.27. Now H together with the rest of the blocks from the incomplete MOLS(10) is a $\{3, 5\}$-GDD of type 10^5. Carefully truncating up to five points from this construction gives all the required $\{3, 4, 5\}$-GDD with groups of size $9^i 10^{5-i}$ for all $i = 0, 1, \ldots, 5$.

Proof. We show it is possible to truncate from this GDD while avoiding the blocks of size 3 in H, thus obtaining all possible group types $9^i 10^{5-i}$.

- The outlined construction gives a $\{3, 5\}$-GDD of type 10^5, covering the case of $i = 0$.
- For $i = 1$, truncate any point not in H. Affected blocks are reduced to four points, and blocks in H are not altered. We have a $\{3, 4, 5\}$-GDD of type $10^4 9^1$.
- For $i = 2$, truncate any pair of points $x, y \notin H$. The block covering this pair is reduced to three points. Blocks in H remain unaltered and the rest of the blocks lose at most one point. This gives a $\{3, 4, 5\}$-GDD of type $10^3 9^2$.
- For $i = 3$, truncate three non-collinear points not in H. Similarly, blocks in H are not affected and blocks outside of H lose at most two points, giving a $\{3, 4, 5\}$-GDD of type $10^2 9^3$.
- For $i = 4$, truncate four points from some block B disjoint from H. This is possible as blocks contain at most one point in H, unless they are in H. In what remains, blocks lose at most one point, avoiding H by construction. This gives a $\{3, 4, 5\}$-GDD of type $10^1 9^4$.
- The TD$(5, 9)$ covers the case of $i = 5$.

□

Construction 2.32 (For $m = 6$)**.** This construction provides $\{3, 4, 5\}$-GDDs of type 5^i6^{5-i} for all $i = 0, 1, \ldots, 5$. We work from the known 3-GDD and 4-GDD of type 6^5 (see [4]).

- Either the 3-GDD or 4-GDD of type 6^5 cover the case when $i = 0$.

- For $i = 1$, truncate a single point from the 4-GDD of type 6^5, resulting in a $\{3, 4\}$-GDD of type 5^16^4.

- The case when $i = 2$ requires a different construction as truncating any pair of points from the 3-GDD or 4-GDD of type 6^5 will leave a block of size two. We instead look at three MOLS(7) which is equivalent to a 5-GDD of type 7^5. After truncating an entire block B, a $\{4, 5\}$-GDD of type 6^5 remains. Now additionally truncate any pair of points that remain on a block C of size 5. The block C is reduced to three points and all the other blocks are reduced by at most one, leaving a $\{3, 4, 5\}$-GDD of type 5^26^3.

  It remains to show that a block of size 5 exists after the initial truncation. We will show there are disjoint blocks in a set of three mutually orthogonal Latin squares of order 7. Every point lies in exactly seven blocks, one for each row of the Latin square. Because the truncated points are all incident to B, after truncation there are $6 \times 5 = 30$ distinct blocks of size 4. Since we started with 49 blocks, there are still $49 - 30 - 1 = 18$ blocks of size 5, thus C exists.

- For $i = 3$, truncate three collinear points from a 4-GDD of type 6^5 resulting in a $\{3, 4\}$-GDD of type 5^36^2.

- For $i = 4$, truncate all the points on any block in a 4-GDD of type 6^5. The result is a $\{3, 4\}$-GDD of type 5^46^1 as remaining blocks are reduced by at most one point.

- For $i = 5$, three MOLS(5) and thus a 5-GDD of type 5^5 is known to exist.

Construction 2.33 (For $m = 3$)**.** We construct $\{3, 4, 5\}$-GDDs of type $2^i 3^{5-i}$ for all $i = 0, 1, \ldots, 5$.

- When $i = 0$, it is known that a 3-GDD and 4-GDD of type 3^5 both exist. The former is an STS(15) with a parallel class removed and the latter arises from deleting a point from $AG_2(4)$.

- For $i = 1$, truncate a single point from a 4-GDD of type 3^5, resulting in a $\{3, 4\}$-GDD of type $2^1 3^4$.

- For $i = 2$, work from two intersecting blocks in a 5-GDD of type 4^5. Truncate all but one point on each block, ensuring that the remaining two points lie in distinct groups to create a GDD of type $2^2 3^3$. As only points from two blocks have been truncated, it is clear that this is a $\{3, 4, 5\}$-GDD.

- For $i = 3$, truncate three collinear points from a 4-GDD of type 3^5 to get a $\{3, 4\}$-GDD of type $2^3 3^2$.

- For $i = 4$, truncate an entire block from a 4-GDD of type 3^5. The result is a $\{3, 4\}$-GDD of type $2^4 3^1$.

- For $i = 5$, the GDD H in Construction 2.27 is a $\{3, 5\}$-GDD of type 2^5.

Although the conclusion of Theorem 2.28 fails for $m = 2$, we can instead consider group sizes 1 and 3.

Theorem 2.34 (GDDs of Type $1^i 3^{5-i}$)**.** For $i = 0, 1, \ldots, 5$, there exists a $\{3, 4, 5\}$-GDD of type $1^i 3^{5-i}$.

Proof. Tierlinck [14] showed that a 3-GDD of type $1^i 3^{5-i}$ exists for all $i \neq 2, 5$. A single block of size 5 gives type 1^5. We now construct a $\{3,5\}$-GDD of type $1^2 3^3$. From the PBD$(11, \{3,5\})$, remove three disjoint blocks of size 3. For example, from the PBD of order 11 in Example 1.5, one can remove the disjoint blocks $\{0', 0, \infty\}$, $\{4', 1, 2\}$, and $\{1', 3, 4\}$. These holes become the groups of size 3. The points $2'$ and $3'$ become the two groups of size 1. □

Lemma 2.35 (GDDs of Type $1^i 3^{4-i}$)**.** There exist $\{3,4\}$-GDDs of type 1^4, $1^1 3^3$, and 3^4. There is no $\{3,4\}$-GDD of type $1^2 3^2$ or $1^3 3^1$.

Proof. From [4], it is already known that a 3-GDD of type 3^4 exists. A single block of size 4 covers type 1^4. The PBD$(10, \{3,4\})$ from Lemma 4.5 with three disjoint blocks removed covers the case $1^1 3^3$. A $\{3,4\}$-GDD of type $1^3 3^1$ and $1^2 3^2$ are equivalent to a $\{3,4\}$-PBD of order 6 and 8 respectively, both which fail the necessary conditions of existence (Lemma 1.6). □

Chapter 3

Main Construction

This chapter settles the existence of all but a reasonably small set of orders. We find dimension three PBDs by weighting points in $PG_3(4)$. Increasing the weights gives all orders outlined in the below theorem, which summarizes the chapter.

Theorem 3.1 (Bound from Main Construction)**.** For all orders $v \geq 84$, there exists a PBD$(v, \{3, 4, 5\})$ of dimension at least 3, except possibly for $v = 86$, 88, and 94.

Proof. Theorem 3.2, together with Constructions 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9 complete the proof. The bound itself is discussed in Section 3.4. □

3.1 Main Construction

The strategy of the following construction is to inflate every point of $PG_3(4)$ by either m or $m-1$, for whatever values of $m \in \mathbb{N}$ are possible. Blocks will be replaced with the $\{3, 4, 5\}$-GDDs of Section 2.3 and groups by PBDs of order m and $m-1$. We start with a theorem from Dukes and Ling.

Theorem 3.2 (Base Construction [7])**.** Let $d \geq 3$ and $s = 41(4^d - 1)/3$. Then for $v > s$, there exists a PBD$(v, \{3, 4, 5\})$ of dimension at least d.

Dukes and Ling also discuss a universal bound on the size of the subspaces in terms of dimension, although we do not look at that property here. The proof of Theorem 3.2 is based on the following construction.

Construction 3.3. From a $PG_d(4)$ of dimension $d \geq 3$, give weight $m-1$ or m to every point, for some integer $m \geq 11$. Transversal designs with block size 5 exist for every group size $m \geq 11$. Thus, Lemma 2.29 implies the existence of all required $\{3,4,5\}$-GDDs to use in place of the original blocks of the geometry. Each inflated point becomes a group in this construction. These groups are covered by $\{3,4,5\}$-PBDs of order $m-1$ or m which exist for all values of $m \geq 11$. By Theorem 2.20, the result is a pairwise balanced design with block sizes in $\{3,4,5\}$ and dimension at least d.

It is clear from Construction 3.3 that $v \geq 850$ is settled. We wish to substantially lower this bound. However, another approach must be used in lieu of Lemma 2.29 as a TD$(5,m)$ does not exist when $m = 3, 6, 10$.

Construction 3.4 (Main GDD Construction)**.** We extend Construction 3.3 by inflating each point by m or $m-1$ for all $3 \leq m \leq 10$. On any configuration of inflated points, the GDDs necessary to replace the blocks exist by Theorem 2.28. This replacement results in GDDs of type $(m-1)^j m^{85-j}$, for any $j \in [0, 85]$, and strong dimension at least 3 by Theorem 2.20. Section 3.2 discusses how to fill in the groups of size m and $m-1$.

3.2 Filling in the Groups

The groups outlined in Construction 3.4 can simply be filled in with $\{3,4,5\}$-PBDs of order m and $m-1$ when these PBDs both exist. As in Theorem 3.2, this construction preserves dimension. This completes the cases $(m-1, m) = (3,4)$, $(4,5)$, and $(9,10)$.

We must further investigate the cases $(m-1,m) = (2,3)$, $(5,6)$, $(6,7)$, $(7,8)$ and $(8,9)$ as no PBDs of order 2, 6 or 8 exist.

Construction 3.5 (Add a Point)**.** To fill in the groups of the GDD in Construction 3.4, add a new point x at infinity. When possible, fill in groups with pairwise balanced designs of order m and $m+1$ on each group together with x. By Construction 2.23, the result is a PBD of dimension at least 3. As $\{3,4,5\}$-PBDs of order 3, 4, 9 and 10 exist, this construction takes care of the cases $(m-1,m) = (2,3)$ and $(8,9)$.

Construction 3.6 (Add a Triple)**.** To the GDDs of Construction 3.4, add three points at infinity on a block together. Fill in groups with PBDs of order $m+2$ and $m+3$ on each group with the points at infinity. By extending Construction 2.23 to subspaces we see that the result is still a PBD with dimension at least 3. This takes care of the cases $(m-1,m) = (6,7)$ and $(7,8)$ as orders 9, 10 and 11 are all admissible.

The only remaining case is $(m-1,m) = (5,6)$. A single point or a triple of points cannot be added as this gives holes of order 6 or 8 to fill. Adding a quadruple of points gives admissible PBD orders 9 and 10, however there is no pairwise balanced design on 9 points containing a block of size 4. Similarly adding a block of size 5 gives admissible orders 10 and 11, but as shown in Theorem 4.5 to come, there is only one pairwise balanced design on 10 points which has no block of size 5. Note that it is necessary for points added ‘at infinity’ to lie on a single block.

Construction 3.7 (Repairing the $(5,6)$ Hole)**.** To realize values of $v \in (85 \times 5, 85 \times 6+3) = (425, 513)$, we instead inflate $AG_3(5)$ of order 125. Inflating points by $(m-1,m) = (3,4)$ and $(4,5)$ we achieve all orders in $[375, 625]$. Filler GDDs from Lemma 2.29 are used to replace blocks, as in Construction 3.4. The groups of inflated points are filled with blocks of size 3, 4 or 5.

3.3 A Related Construction

A similar construction instead inflating by 1 and 3 proves the existence of more $\{3,4,5\}$-PBDs.

Construction 3.8 (Odd Orders Greater Than 84)**.** In $PG_3(4)$, inflate any subset of the points by 3. Fill in blocks with the appropriate $\{3,4,5\}$-GDD of type 1^i3^{5-i} from Lemma 2.34. Groups in the resulting GDD, coming from the sets of inflated points, are each placed on a block. By Theorem 2.20, any PBD$(v,\{3,4,5\})$ of dimension at least 3 with odd order $v \in [85, 85 \times 3] = [85, 255]$ is achievable.

Construction 3.9 (Even Orders Greater Than 83)**.** To cover the even values in this range, we first designate a single point at infinity to have weight zero. The rest of the points are carefully inflated by 1 or 3 to avoid inflated blocks with 6 or 8 points. Blocks incident to the point at infinity are replaced by $\{3,4\}$-GDDs of type 1^4, 1^13^3, and 3^4 which exist by Lemma 2.35. The remainder of the blocks are replaced by $\{3,4,5\}$-GDDs from Theorem 2.34 and inflated points are placed on their own block of size 3. This construction covers all even orders in $[84, 84 \times 3] = [84, 252]$ except 86, 88, and 94, where avoiding undesirable inflated blocks sizes is not possible. These orders are discussed in Section 4.3.

3.4 The Bound of Existence

Filling in the groups of the GDD of Construction 3.4 without adding any points takes care of orders in the intervals $[255, 425]$ and $[765, 850]$. Construction 3.5 gives all orders in $[2 \times 85 + 1, 3 \times 85 + 1] = [171, 256]$ and $[8 \times 85 + 1, 9 \times 85 + 1] = [681, 766]$. Construction 3.7 takes care of the orders in $[375, 625]$. Construction 3.6 gives all orders in $[6 \times 85 + 3, 8 \times 85 + 3] = [513, 683]$. Thus we have covered all orders $171 \leq v \leq 850$.

From Constructions 3.8 and 3.9 we know all PBD$(v, \{3, 4, 5\})$ of dimension at least 3 exist for orders $84 \leq v \leq 252$, except possibly 86, 88 and 94. Thus all PBD$(v, \{3, 4, 5\})$ of dimension at least 3 exist for $v \geq 84$ except possibly 86, 88 and 94. In the next chapter, we look at constructions for these orders as well as orders $v < 84$.

Chapter 4

Constructing Small Values

This chapter is devoted to finding PBD$(v, \{3, 4, 5\})$ of dimension at least 3 via ad-hoc constructions, for orders not settled in Chapter 3. The following theorem summarizes the results of this chapter.

Theorem 4.1 (Existence Under 100)**.** PBD$(v, \{3, 4, 5\})$ of dimension at least 3 exist for orders $v = 86$, 88, and 94 and all $27 \leq v \leq 84$ except possibly $v \in E$, for $E = \{32, 33, 34, 35, 38, 41, 42, 43, 47\}$.

Proof. This proof follows from the various constructions in Section 4.3 and 4.2. See Appendix A for a reference table of the orders discussed in this chapter (as well as Chapter 5). For many of these constructions, the dimension of the resulting PBD was verified by computer. Appendix B discusses the program used for verification. □

4.1 Building Blocks

In order to create constructions for smaller values, we first look at some conditions on the configuration of linear spaces of dimension at least 3 and their subspaces.

Lemma 4.2 (Even Order, Odd Pairs)**.** Every point in a PBD($2v, \{3, 4, 5\}$) is incident to an odd number of blocks of size 4. Every point in a PBD($2v+1, \{3, 4, 5\}$) is incident to an even number of blocks of size 4.

Proof. Let x be a point in a PBD($2v, \{3, 4, 5\}$). By definition of a pairwise balanced design, x lies on blocks incident to every other point in the design. Thus, x occurs in an odd numbers of pairs. Any block of size 3 containing x covers two of those pairs. Blocks of size 5 cover four pairs with x, and blocks of size 4 cover 3 pairs with x. In order to cover an odd number of pairs, x therefore lies on an odd number of blocks of size 4.

Similarly, for a PBD of order $2v + 1$, the point x covers $2v$ pairs. Thus, x must be incident to an even number of blocks of size 4. □

Theorem 4.3 (Subspace Bound)**.** Let W be any proper subspace of a linear space V, with orders w and v respectively. If all blocks have size at least three then $2w+1 \leq v$. Furthermore, equality holds if and only if all blocks outside of W have size 3 and contain exactly one point in W.

Proof. We call pairs with one point in W and one point in $V \setminus W$ are called 'crossing pairs'. We look at the number of pairs in $V \setminus W$ necessary to complete blocks with crossing pairs. There are $w(v-w)$ crossing pairs and $\binom{v-w}{2}$ pairs contained in $V \setminus W$. On any block, for every pair contained in $V \setminus W$, there are at most two crossing pairs. Thus, we have $w(v - w) \leq 2\binom{v-w}{2}$ or equivalently $2w + 1 \leq v$.

To show when equality holds, first suppose all crossing pairs are on blocks of size 3 containing one point in W. In this case, there are exactly two crossing pairs covered per block for every pair covered that is contained in $V \setminus W$. The equality is clear. Now suppose there are some pairs not covered in this way. Then there exists a block B of size $b > 3$ with at most one point in W, or with $b = 3$ and no points in W. If the latter is true, B covers all of its pairs in $V \setminus W$. In the case of $b > 3$, $b-1$ crossing pairs are covered, and $\binom{b-1}{2}$ pairs are covered in $V \setminus W$, or equivalently 2 crossing pairs for every $b-2$ pairs in $V \setminus W$. If any blocks exist in either of these cases, the equality is therefore not attainable. □

Corollary 4.4 (Double Subspace Bound)**.** Let V be a linear space with v points and let W_1 and W_2 be a proper subspaces of order w. Then $v - (2w - z) \geq w - z$ where $z = |W_1 \cap W_2|$. Furthermore, equality holds if and only if there is a third subspace W_3 of order w and W_1, W_2, and W_3 all meet in z points.

Lemma 4.5 (PBD 10)**.** There exists a unique PBD of order 10 up to isomorphism.

Proof. By Lemma 4.2 each point is incident to an odd number of blocks of size 4. There cannot be two non-intersecting blocks of size 4. Otherwise the third block of size 4, required to cover the remaining two points, necessarily intersects the first two blocks. This leaves two points on an even number of blocks of size 4 and there cannot be any more blocks of size 4. Therefore, as all blocks of size 4 intersect, they intersect in a single point to satisfy Lemma 4.2. Truncating this point leaves a Steiner triple system of order nine, which is well known to be unique up to isomorphism. □

Lemma 4.6 (PBD 11)**.** There exists a unique PBD$(11, \{3, 4, 5\})$ up to isomorphism.

Proof. Consider the PBD$(11, \{3, 5\})$ constructed in Example 1.5. By Lemma 1.6, a PBD of order 11 contains only blocks of size 3 and 5, and it follows from Corollary 4.4 that only one block of size 5 can be used in such a PBD. The construction in Example 1.5 is unique up to a choice of the point on B and choice of a 1-factorization. □

4.2 Small Values from Truncation

In this section, we determine orders of PBDs of dimension at least 3, relying mainly on Lemma 2.22.

Corollary 4.7 (Truncation from One Block)**.** From Lemma 2.22, we can truncate points from a single block of $PG_d(q)$ without reducing dimension. Truncating any number of points from a block of size b, except $b-2$, avoids reducing the block to two points. Other blocks in the geometry are reduced by at most one point. We create the following PBD$(v, \{3,4,5\})$ of dimension at least 3:

- $v = 36, 37, 39$ from $PG_3(3)$.
- $v = 60$ from $AG_3(4)$, which is itself a truncation of $PG_3(4)$.
- $v = 80, 81, 83, 84$ from $PG_3(4)$.

Because the geometries used above have blocks of size 4 or 5, all the linear spaces created have block sizes in $\{3, 4, 5\}$ when truncated.

Construction 4.8 (Truncating from Two Planes)**.** In this construction, we truncate points from two distinct planes, W_1 and W_2, of $PG_d(q)$ with $q \geq 4$. In order for the result to be a linear space, we truncate points in each plane so that what is left are subspaces $U_1 \subset W_1$ and $U_2 \subset W_2$. This ensures that any pair of points remaining within either plane still lie on a block together. The same is true of pairs in $V \setminus (U_1 \cup U_2)$. Any pair of points between U_1 and U_2 are covered in blocks with points from $V \setminus (U_1 \cup U_2)$ which remain. Blocks not fully contained in W_1 or W_2 lose at most one point per truncated plane. Thus all remaining blocks have size at least 3 so long as $q \geq 4$. (If only one plane is truncated from, we only need $q \geq 3$.) By Lemma 2.22, what is left is a PBD$(v, \{3, 4, 5\})$ of dimension at least 3.

Corollary 4.9 (From Two Planes of $PG_3(4)$)**.** Truncating from two planes of order 21 in $PG_3(4)$ as in Construction 4.8 shows the following PBD$(v, \{3, 4, 5\})$ of dimension at least 3 exist.

- Truncating all points from both planes gives $v = 48$.

- Truncating all but 1 point in each plane gives $v = 50$.
- Truncating all but 1, 3, 4 or 5 points from one of the two planes yields linear spaces of order $v = 51, 52, 53$ and an alternate construction for $v = 49$.
- Leaving 3 or 4 collinear points in each of the two planes gives $v = 54, 56$ and an alternate construction for $v = 55$.
- Leaving 4 points in one plane and 5 in the other plane give an alternate construction for $v = 57$.
- Leaving a Fano plane in one plane, via the subfield $\mathbb{F}_2 \subset \mathbb{F}_4$, and leaving 3 collinear points in another plane results in $v = 58$.
- Leaving an entire plane of order 21 and truncating all but 1, 3, 4 or 5 collinear points in the other plane gives $v = 65, 67, 68, 69$.

Corollary 4.10 (From One Plane of $PG_3(3)$)**.** We truncate from one plane of $PG_3(3)$. From a single plane of order 13, truncate all but one, three or four collinear points. Similarly to Construction 4.8, these truncations result in PBD$(v, \{3, 4\})$ of dimension as least 3 for $v = 28, 30, 31$.

Construction 4.11 (Order 62 and 66)**.** Similarly to Construction 4.8, we find a linear space of dimension at least 3 of order 66 from the PBD$(69, \{4, 5\})$ constructed in Corollary 4.9. Notice that the points left on a plane in Corollary 4.9 are incident to blocks of size 5 only. Points that are incident to blocks of size 4 are disjoint from the plane that was truncated from. We therefore truncate three points from a block of size 4, reducing the remaining blocks to a minimum of size 3.

Similarly, from the PBD$(65, \{4, 5\})$ constructed in Corollary 4.9, truncate three points from a block of size 4. What is left is a PBD$(62, \{3, 4, 5\})$. In both of these cases, the dimension of the resulting linear space is preserved, by Lemma 2.22.

Together with the known Steiner spaces from [14], these constructions complete the existence of PBDs of dimension at least 3 with orders in $[60, 69]$. We now attack orders in the seventies by truncating points from blocks that share a common point.

Construction 4.12 (Truncating from Intersecting Blocks)**.** Again using Lemma 2.22, we truncate from $PG_3(4)$, which recall has blocks of size 5. Find two blocks B and C that intersect in some point x. Truncate x and three additional points from each of these blocks to achieve order 78. This leaves one point per block and so no pairs are lost when B and C are removed.

For a linear space of order 77 and dimension at least 3, truncate x, four points from B and three points from C. In this case, a single point on C is left, so no pairs are lost with the removal of C. For a linear space of order 76 and dimension at least 3, truncate all points on B and C.

In each of these cases, the remaining blocks are reduced to size 3 when they share a point with both B and C, and otherwise are reduced by at most one point.

Construction 4.13 (From Three Intersecting Blocks)**.** We again work with the blocks in Construction 4.12. The two blocks B and C determine a plane W of order 21 in $PG_3(4)$. We wish to find a third block D to truncate that contains their common point x and is not in this plane. Such a D exists outside of W as x lies in more than one plane of $PG_3(4)$. The pairs between B and C are covered in W while the pairs between D and B, or D and C are not, since D is not in W. Therefore any three points $b \in B$, $c \in C$, and $d \in D$, are not collinear. Thus, when B, C, and D are truncated, at most two points are removed for any block. What remains is therefore a PBD$(72, \{3, 4, 5\})$ of dimension at least 3.

Similarly we can obtain a PBD$(74, \{3, 4, 5\})$ of dimension at least 3 by truncating B and all but one point on C and D. As in Construction 4.12, no pairs are lost and Lemma 2.22 ensures the dimension does not decrease.

Construction 4.14 (From Four Intersecting Blocks)**.** We work with the three intersecting blocks in Construction 4.13. In order to create a linear space of order 71, we now look for a fourth block E that is incident to x and in a plane with no more than one of B, C and D at a time. In $PG_3(4)$, the point x appears five times in each of the planes defined by B and C, B and D, and C and D, for a total of 15 occurrences. Because x occurs 21 times in $PG_3(4)$, a block exists incident to x and in none of the mentioned planes, the required E. As in Construction 4.13, we know that no triple from any three of B, C, D, or E is collinear, which prevents creating a block of size 2 after truncation. Truncate all the points on B, C and D and any one point from E. The block E now has size 3 and we have found a linear space of order 71 and dimension at least 3.

To achieve order 70, we truncate all points on B and C, as well as three of the four remaining points from each of D and E.

Another clever way to truncate points is using the specific incidences within a plane. We define an 'oval' of a finite projective plane over $\mathbb{F}_q$ as a set of $q+1$ points, such that no three points are collinear. A line is called 'tangent' to an oval, if it is incident to exactly one point of the oval. It is known that there is a unique point for every oval, called a 'nucleus', such that every line through the nucleus is tangent to the oval. (See [12] for more details.) Tierlinck used the incidence properties of ovals to find linear spaces of dimension 3 from $PG_3(4)$. One such construction is outlined below.

Construction 4.15 (Order 44 [14])**.** Tierlinck's construction of Steiner spaces of order 139 and 141 starts from a truncation of points from $PG_3(4)$. Consider two planes V_1 and V_2 in $PG_3(4)$. In V_1, find an oval $\mathcal{O}$, its nucleus n, and a line L disjoint from $\mathcal{O} \cup \{n\}$. Consider the set of all blocks generated by a point $p \in V_2 \setminus V_1$ together with each point in $\mathcal{O} \cup \{n\} \cup L$. From $PG_3(4)$, we truncate all points not generated by these pairs, leaving 45 points. Finally, we truncate p to find a linear space of order 44. The dimension is preserved after truncation, by Lemma 2.22. We refer to [14] for further details on this construction.

We next construct a PBD of dimension at least 3 by removing points in terms of a contraction.

Construction 4.16 (PBD 29)**.** In $PG_4(2)$, on 31 points, pick a block B and contract it to a single point b. In $PG_4(2)$, $\langle B \cup \{x\} \rangle$ is a Fano plane for all $x \notin B$. After the contraction, replace subspaces containing B by a single block on the four remaining points together with b. Fano planes that originally contained only a single point in B retain the same configuration, now incident to b. The subspaces that do not share a point with B remain unaltered. By Lemma 2.25, what is left is a PBD$(29, \{3, 5\})$ of dimension 3.

4.3 Small Values from Inflation

In this section, we find small values from constructions similar to Theorem 2.20. We first handle cases implied by Construction 2.23.

Corollary 4.17 (PBD Existence from Construction 2.23)**.** From Construction 2.23, we obtain PBD$(v, \{3, 4, 5\})$ of dimension at least three for the following orders:

- $v = 46$ from the Steiner space of order 15 and $k = 3$.
- $v = 82$ from the Steiner space of order 27 and $k = 3$.
- $v = 59$ and 88 from the linear space of order 29 in Construction 4.16, using $k = 2$ and 3 respectively.
- $v = 94$ from $PG_2(4)$ with $k = 3$.

- When $k = 2$, and from the linear spaces of order 36, 37 and 39 in Corollary 4.7, we find alternate constructions for $v = 73, 75, 79$, where Steiner spaces are known to exist.
- When $k = 2$ we can construct alternate examples of PBDs of order $v = 55, 57, 61, 63$ from the spaces of order 27, 28, 30, and 31 respectively.

Similarly to Construction 2.23 and the resulting PBDs in Corollary 4.17, it is sometimes possible to add points at infinity to an existing linear space of dimension at least 3, without inflating the original points. Points are added to existing blocks, however, so the dimension of the resulting linear space depends on how these points are added.

Construction 4.18 (Order 86)**.** Start with the PBD$(82, \{3, 4\})$ from Corollary 4.17. Inflate the point at infinity by five and place these five points on a block called B_∞. In order to cover pairs between B_∞ and the rest of the points, a parallel class of nine parallel STS(9)s are extended to be projective planes of order 13. This covers all pairs with four of the points on B_∞.

The fifth point at infinity is taken care of separately. Within each subspace of order 27, label the three newly created subspaces of order 13 P, Q, and R. The final point on B_∞ is added to the unique parallel class consisting of blocks that contain one point from each of P, Q, and R. In this parallel class, we have created PBDs of order 10 from the existing STS(9)s.

The largest subspace of the PBD of order 82 has order 28, from the subspaces of order 27 together with the point at infinity. Notice that the largest subspace containing B_∞ contains intersecting planes of order 10 and 13. The original points on these planes were contained in PBDs of order 28. All pairs between the 27 points and B_∞ are covered within the PBDs of order 10 and 13. Therefore, any set of points that generated the subspace of order 28, only additionally generates B_∞ in the new linear space. The corresponding subspace in the PBD of order 86 therefore has order 32. Subspaces not containing B_∞ are proper in the usual way. Thus, dimension in the new linear space is preserved and so we have created a PBD$(86, \{3, 4, 5\})$ of dimension at least 3.

Next is an alternate constructions for $v = 75$ where a Steiner space is known to exist.

Construction 4.19 (Alternate 75)**.** We construct an alternate space of dimension at least three containing blocks of size 5. This construction starts with the Steiner space of order 15, inflating every point by five. Blocks are replaced with 3-GDDs of type 5^3, equivalent to a Latin square of side 5. Each group is then placed on a block of size 5. By Theorem 2.20, we have constructed a PBD$(75, \{3, 5\})$ of dimension at least 3.

This same construction, inflating instead by 3, recovers the Steiner space of order 45.

Chapter 5

Non-Existence

In this chapter we explore when $\{3,4,5\}$-PBDs of dimension at least 3 do not exist. The results are summarized in Theorem 5.1 below.

Theorem 5.1 (Main Non-Existence)**.** There is no PBD$(v, \{3,4,5\})$ of dimension at least 3 and order $v = 32$ or $v \leq 26$ except the Steiner space of order 15.

Proof. This follows from a combination of following theorems and corollaries: 5.2, 5.3, 5.5, 5.6, 5.8, 5.9, 5.10, and 5.11. □

Theorem 5.2 (Existence with Order Less Than 22)**.** If there is a linear space of dimension at least 3 and order less than 22 then it is a Steiner space.

Proof. Suppose there is a linear space V of dimension at least 3 that is not a Steiner space. Then it must contain a block B of size $b > 3$. The subspace generated by two points on B and one point not on B is a proper subspace containing B. The smallest non-trivial PBD with a block of size $b > 3$ has order 10. By Theorem 4.3, the order of V is at least $2 \times 10 + 1 = 21$. Furthermore, if V has order exactly 21, there are blocks of size 4 not contained in the subspace of order 10, by Lemma 4.2. Therefore equality cannot met in the subspace bound. Thus, V has order at least 22. □

Corollary 5.3 (Steiner Space of Order 15)**.** From [14] and Theorem 5.2 we know that the only linear space of order less than 22 and dimension at least 3 is the unique Steiner space of order 15, or the 'Fano tetrahedron'.

Theorem 5.4 (Blocks of Size 4 Intersect in One Point)**.** Let V be a linear space of dimension at least 3 and order $v \leq 26$. If V has no subspace of order 12 then all blocks of size 4 intersect in a single point.

Proof. Suppose there are two blocks of size 4 in V that do not intersect. Name these blocks B and C with points b_i, c_i for each $i = 1, 2, 3, 4$. By the subspace bound (4.3) and Lemma 4.6, the only non-trivial subspace of V containing a block of size 4 has order 10.

Let $W_i = \langle b_i, c_1, c_2 \rangle$ be a subspace of order 10 for each $i = 1, 2, 3, 4$. As in Lemma 4.5, the blocks of size 4 in a PBD$(10, \{3, 4\})$ all intersect. Therefore both B and C cannot be contained in any W_i and thus $W_i \neq W_j$. By Lemma 2.2, $W_i \cap W_j = C$ for any i and j. Thus $W_i \setminus C$ are pairwise disjoint sets of size 6 giving 24 unique points. Including C, the linear space V therefore has a minimum of 28 points. This contradicts our assumption that $v \leq 26$. Hence all blocks of size 4 intersect.

Now we show that all the blocks of size 4 intersect in a single point. Suppose instead that there are two points $x, y \in V$ that are incident to more than one block of size 4 (hence, at least three such blocks). Let three of the blocks of size 4 incident to x be called A, B and C. Then choosing one point from each of A, B, and C, not collinear, gives a basis for a PBD of order 10 containing x, call it U_x. If $y \in U_x$, it shares a block of size 4 with x, say A. Let two of the other blocks of size 4 incident to y be D and E. We have shown that D and E must intersect B and C, however y is already incident to each point on B and C in U_x. So D and E cannot have size 4, contradicting our assumption. Thus y is not in U_x. Because D is not incident to x, then $D = \{a, b, c, y\}$ for some $a \in A$, $b \in B$ and $c \in C$, as it must intersect all of A,

B, and C. Three of the pairs on D are then contained in U_x, and so D is generated by any basis generating U_x. However, we know $y \notin U_x$ but $y \in D$. Hence the PBD generated by $\{x, y, d\}$ for any point $d \in D$ is contained in subspace with order at least 11. But a PBD of order 11 has no blocks of size 4 and no PBD of higher order is a subspace of V. Thus all blocks of size 4 in V are incident to a single point. □

Corollary 5.5 (Order 24)**.** There is no PBD$(24, \{3, 4, 5\})$ of dimension at least 3.

Proof. If there were, by Lemma 4.2 and Theorem 5.4, all points lie on blocks of size 4 that intersect in a single point. But $23 \not\equiv 0 \pmod 3$ so this is not possible. □

Corollary 5.6 (Order 22)**.** There is no PBD$(22, \{3, 4, 5\})$ of dimension at least 3.

Proof. By Lemma 4.2 and Theorem 5.4, if there is a PBD$(22, \{3, 4, 5\})$ of dimension at least 3 then all points lie on blocks of size 4 that intersect in a single point. The truncation of the apex point, however, leaves a Steiner space of order 21 which is known not to exist by [14]. □

Construction 5.7 (Necessary Configuration of Small PBDs Containing a Block of Size 5)**.** We determine the necessary configuration for a linear space of dimension at least 3 of order $23 \leq v \leq 26$ containing a block of size 5. By the subspace bound (4.3), subspaces have order at most 12. The only possible proper subspaces containing a block of size 5 have order 5 or 11. Let $B = \{b_1, b_2, \ldots, b_5\}$ be a block of size 5. For any $x \notin B$, the subspace $W_x = \langle x, b_1, b_2 \rangle$ contains B. Similarly define the subspaces W_y and W_z for some $y \notin W_x$ and $z \notin W_x \cup W_y$. By Lemma 2.2, the intersection of these subspaces is a subspace, and is therefore B. The sets $W_x \setminus B$, $W_y \setminus B$, $W_z \setminus B$ are disjoint each containing six points. We have thus considered $3 \times 6 + 5 = 23$ unique points. If there is a point $u \notin W_x \cup W_y \cup W_z$, then to contain W_u similarly requires 29 unique points. Thus, there is no linear space of dimension at least 3 containing a block of size 5 for $v = 24, 25, 26$.

Theorem 5.8 (Order 26)**.** There is no PBD$(26, \{3, 4, 5\})$ of dimension at least 3.

Proof. By the global and local conditions we know that, if such a linear space V exists, then it must contain a block of size 5. By Construction 5.7 we know no linear space of dimension at least 3 exists. □

Theorem 5.9 (Order 25)**.** There is no PBD$(25, \{3, 4, 5\})$ of dimension at least 3.

Proof. Suppose for a contradiction that there is a linear space V of order 25 and dimension at least 3. By Construction 5.7, V does not contain a block of size 5. Because there is no Steiner space of order 25, we know V contains a block of size 4. By the subspace bound, every block of size 4 is contained in a subspace of order 10 or 12. We next show that V does not contain a subspace of order 12.

Suppose W is a subspace of V of order 12 for a contradiction. Because order 12 meets the subspace bound (4.3) with equality, every pair not covered in W must be contained in a block of size 3 with one point in W and two points in $V \setminus W$. So the only blocks of size 4 in V are contained in W. Because 12 is an even order, by Lemma 4.2 every point in W is incident to an odd number of blocks of size 4. But V has odd order, so Lemma 4.2 implies each point in V must be incident to an even number of blocks of size 4. This contradiction implies that there is no subspace of order 12. Thus, the only non-trivial proper subspace of V containing a block of size 4 has order 10.

By Theorem 5.4, all blocks of size 4 in V intersect in a single point. Delete this apex point and what is left is a 3-GDD of type 3^8. Filling in each group with a block constructs a Steiner triple system of order 24. This is absurd as order 24 does not meet the conditions of Lemma 1.6. Thus V has dimension two. □

Theorem 5.10 (Order 23)**.** There is no PBD$(23, \{3, 4, 5\})$ of dimension at least 3.

Proof. Suppose there is a linear space V of order 23 and dimension at least 3. Because of the necessary conditions of existence in Lemma 1.6, it is necessary for to V contain a block of size 5. From Construction 5.7, we know that V contains a single block B of size 5, and three distinct subspaces W_x, W_y, and W_z of order 11 whose intersection is B. This accounts for all 23 points in V.

Consider the subspace generated by any three non-collinear points, $U = \langle x_1, y_1, z_1 \rangle$, with $j_1 \in W_j$ for each $j = x, y, z$. We will show that if U is a proper subspace of V, then U is a Fano plane. Each pair from the basis determines a unique block in U with a third point, say x_2, y_2, and z_2. By Lemma 4.6 on the configuration of PBDs of order 11, we know every block in V intersects B. Thus, the block containing x_1 and x_2 is incident to a point on B, as are the blocks on $\{y_1, y_2\}$, and $\{z_1, z_2\}$. If U contains more than one point on B then it contains B, and therefore all of W_x, W_y, and W_z. Therefore, U contains exactly one point $b \in B$ as it is a proper subspace of V. Thus $\{x_1, x_2, b\}$, $\{y_1, y_2, b\}$, and $\{z_1, z_2, b\}$ are blocks of U. If any other distinct point $j_3 \in W_j$ is in U, then U would contain another point from B to cover the pair $\{j_1, j_3\}$. So no additional point j_3 is in U. The final block of U is therefore $\{x_2, y_2, z_2\}$ and hence U is a Fano plane. Thus, any subspace of V generated by three non-collinear points $\langle x_i, y_i, z_i \rangle$ is a Fano plane.

Now the six points not on B in each W_j form a Latin square of side 6. Call this Latin square L_6. The subspace U with b deleted corresponds to a 3-GDD of type 2^3, or a 2×2 'sub-square' of L_6 up to a permutation of rows and columns. Therefore, every set of three non-collinear points $\{x_i, y_i, z_i\}$, as defined above, are contained in a 2×2 sub-square in some permutation of rows and columns of L_6. It is known, however, that if every pair of entries is contained in a 2×2 sub-square of a Latin square, then the side of the Latin square is a power of 2. So in L_6, there is a set of three non-collinear points $\{x_i, y_i, z_i\}$ with each point $j_i \in W_j$ for $j = x, y, z$, that do not generate a Fano plane. Thus, there is a set of three points who generate the whole space, and so V has dimension 2. □

Theorem 5.11 (Order 32)**.** There is no PBD$(32, \{3, 4, 5\})$ of dimension at least 3.

Proof. Suppose there is a linear space V of order 32 and dimension at least 3. By Lemma 1.6 we know that V contains a block of size 5, call it B. Similarly to Construction 5.7, consider the subspaces W_x, W_y, W_z and possibly W_u, all containing B, where $i \in W_i$ is a point not on B and disjoint from the other W_j. As such, W_i is generated by two points on B with some point $i \in \{x, y, z, u\}$ not on B. By the subspace bound, subspaces of V have size at most 15. Therefore the possible non-trivial proper subspaces containing a block of size 5 have order 11, 13, 14, or 15. We refer to [4] for the existence of these linear spaces, and the non-existence of such a linear space of order 12. If we delete the points on B, what is left is a GDD with possible group sizes 6, 8, 9, or 10. To account for the remaining 27 points, this GDD must have group type $6^3 9^1$, 9^3, or $8^1 9^1 10^1$. The last case is not possible as a unique smallest group cannot cover all pairs in the larger groups.

Suppose when B is deleted we are left with a 3-GDD of type 9^3. The groups are $W_x \setminus B$, $W_y \setminus B$, $W_z \setminus B$. Each W_i is therefore the unique PBD$(14, \{3, 4, 5\})$, as seen in [4]. There exist points $x \in W_x$, $y \in W_y$ and $b \in B$ such that block $\langle x, b\rangle$ has size three and block $\langle y, b\rangle$ has size four. Consider $U = \langle x, y, b\rangle$. This subspace contains a block of size 4 and thus contains at least 10 points. However, at most eight points within the groups can be incident to only one point on B. Therefore, a second point on B must be generated in this case, and thus $U = V$¿ Hence, V has dimension 2.

Now suppose $V \setminus B$ is a $\{3, 4\}$-GDD of type $6^3 9^1$. The groups are $W_x \setminus B$, $W_y \setminus B$, $W_z \setminus B$, and $W_u \setminus B$, letting the latter be the group of size 9. By Lemma 4.2, every point is on an odd number of blocks of size 4. As W_x, W_y, and W_z correspond to PBDs of order 11, having no block of size 4, there must be a block A of size 4 in the

GDD $V \setminus B$. Let $a_i \in W_i \cap A$. For each $b \in B$, consider $\langle b, a_i, a_j \rangle = \langle b, A \rangle$. To make a block with b, at least two points in each group are generated, so $\langle b, A \rangle$ contains at least nine points. To contain the block of size 4, this subspace has at least ten points. Thus $\langle b, a_u \rangle$ is a block of size 4. By the same argument, $\langle b, a_u \rangle$ is a block of size 4 for each $b \in B$. However, $W_u \setminus B$ has only nine points, so a_u cannot be on five blocks of size 4 and thus some $\langle b, a_i, a_j \rangle = V$.

$\square$

Chapter 6

Reducing Dimension

In this chapter, having almost settled the existence of $\{3, 4, 5\}$-PBDs of dimension at least 3, we look at when PBDs with smaller dimension exist. Theorem 6.1 outlines the results of this chapter.

Theorem 6.1 (Lower Dimension Linear Spaces)**.** If there exists a PBD$(v, \{3, 4, 5\})$ of dimension d then there exists a PBD$(v, \{3, 4, 5\})$ of dimension e for all integers $2 \leq e \leq d$.

Proof. This follows from the upcoming Theorems 6.2 and 6.9. □

Tierlinck [14] showed that if there exists a Steiner space of dimension d, then there is a Steiner space of dimension e, for all integers $2 \leq e \leq d$. This theorem relies on the known complete existence of Steiner triple systems of dimension 2. In order to prove Theorem 6.1, we will first prove a result for general block size K, which depends on the existence of planes. Then, specific to $K = \{3, 4, 5\}$, it will be shown that planes exist for every possible order. (That is, for all integers except 2, 6, and 8.) To start, we look at general K.

Theorem 6.2. Suppose there exists a plane with block sizes in K for every order a K-PBD exists. If there exists a PBD(v, K) of dimension d, then there exists a PBD(v, K) of dimension e for all integers $2 \leq e \leq d$.

We break the proof of this theorem into Lemma 6.3, Lemma 6.4 and Theorem 6.6. We first look at the resulting dimension when replacing a maximal proper subspace by a plane, and then look at replacing any non-degenerate plane. These cases are used inductively to determine the dimension when replacing any subspace.

Lemma 6.3. Let $(V, \mathcal{B})$ be a PBD(v, K). If we replace a maximal proper subspace $(X, \mathcal{B}|_X)$ by a space $(X, \mathcal{C}|_X)$ of dimension 2, the resulting pairwise balanced design $(V, \mathcal{C})$ has dimension at most 3.

Proof. As X is a maximal proper subspace of $(V, \mathcal{B})$ it is also a maximal proper subspace in $(V, \mathcal{C})$ as blocks outside of X are left unchanged, so any addition results in the entire space. Pick three points $x_1, x_2, x_3 \in X$ that generate X over $\mathcal{C}$. These points exist since $(X, \mathcal{C}|_X)$ has dimension 2. Pick a point $y \notin X$ and consider the linear space $U = \langle y, x_1, x_2, x_3 \rangle$ in $(V, \mathcal{C})$. As X is maximally proper and contained in U, $U = X$ or $U = V$. Since $y \notin X$ it is the case that $U = V$. We have found a set of four points that generate $(V, \mathcal{C})$ so the dimension of $(V, \mathcal{C})$ is at most 3. □

Lemma 6.4 (Replacing a Non-Degenerate Plane)**.** Suppose $(V, \mathcal{B})$ is a linear space of dimension d, containing a non-degenerate plane $(Z, \mathcal{B}|_Z)$. If we replace $(Z, \mathcal{B}|_Z)$ by any linear space $(Z, \mathcal{C}|_Z)$ then the resulting linear space $(V, \mathcal{C})$ has dimension at least $d - 1$.

Proof. Suppose for a contradiction that there is a linear space $(Z, \mathcal{C}|_Z)$ such that $(V, \mathcal{C})$ has dimension less than $d - 1$. Then there is a set A of $d - 1$ points that generate $(V, \mathcal{C})$. Let $A_B = \langle A \rangle$ and $A_C = \langle A \rangle$ over $\mathcal{B}$ and $\mathcal{C}$ respectively.

- If $A_B \cap Z = \emptyset$ then $A_B \subseteq V \setminus Z$. Since $A_B \setminus Z = A_C \setminus Z$, $A_B = A_C \setminus Z$. So if there is no intersection, then A_C generates the same linear space.

- If $A_B \cap Z = \{z\}$, a single point, then similarly to the last case, no blocks of V are different from A_B.

- Suppose $A_B \cap Z = Z$ in $\mathcal{B}$. Then $A_C \cap Z = Z$ in $\mathcal{C}$ and $A_B \setminus Z = A_C \setminus Z$ so $A_B = A_C$.

- Suppose $A_B \cap Z = L$, for some line L of Z. Let $a_d \in Z \setminus L$ and $W = \langle A, a_d \rangle$ over $(V, \mathcal{B})$. Since W contains L and $a_d \notin L$, and Z is a non-degenerate plane, $Z \subseteq W$. Because nothing besides Z has been altered in $(V, \mathcal{B})$ to create $(V, \mathcal{C})$, W is a subspace of $(V, \mathcal{C})$. But A_B generates a subset of W while $A_C = (V, \mathcal{C})$. So $W = V$ and $\langle A, a_d \rangle = (V, \mathcal{B})$. This completes the contradiction, as $(V, \mathcal{B})$ has dimension d.

□

Lemma 6.4 generalizes to other subspaces which will show that it is possible to reduce the dimension of any linear space. We will need the concept of length for this generalization.

Definition 6.5 (Length)**.** The length of a linear space V is the largest number of subspaces ℓ such that no subspace has the same order, and each smaller subspace is properly contained in the larger ones.

Theorem 6.6 (Replacing General Subspaces)**.** Suppose $(V, \mathcal{B})$ is a linear space of dimension d. Suppose replacing any subspace of length $i \geq 3$ does not reduce the dimension of the resulting space. Let $(Z, \mathcal{B}|_Z)$ be a subspace of V with length $i+1$. If we replace $(Z, \mathcal{B}|_Z)$ by any linear space $(Z, \mathcal{C}|_Z)$ then the resulting linear space $(V, \mathcal{C})$ has dimension at least $d - 1$.

Proof. We proceed by induction on i, the length of the subspace $(Z, \mathcal{B}|_Z)$. Lemma 6.4 takes care of the base case. Suppose that replacing a $(Z, \mathcal{B}|_Z)$ of length i leaves a linear space of dimension at least $d - 1$.

Now consider replacing a subspace $(Z, \mathcal{B}|_Z)$ of length $i+1$ by some space $(Z, \mathcal{C}|_Z)$. Suppose for a contradiction that $(V, \mathcal{C})$ has dimension less than $d-1$. Then there is a set of $d-1$ points A that generate $(V, \mathcal{C})$. Let $A_B = \langle A \rangle$ and $A_C = \langle A \rangle$ over $\mathcal{B}$ and $\mathcal{C}$ respectively. We know from Lemma 2.2 that $A_B \cap Z$ is a subspace with respect to $\mathcal{B}$. If $A_B \cap Z = \emptyset$, a point $\{z\}$, or Z, then as in Lemma 6.4, we get a contradiction. We now consider the case when $A_B \cap Z = M$ where M is a subspace of A_B with length $2 \leq \ell \leq i$. (So $M \neq Z$ over $\mathcal{B}$ by the assumption of length). For some $z \in Z \setminus M$, let $W_B = \langle M, z \rangle$ and $W_C = \langle M, z \rangle$ over $\mathcal{B}$ and $\mathcal{C}$ respectively. Then W_B has length $\ell + 1$. If $\ell < i$, we recursively create a new subspace consisting of the previous space W_B and a point in $Z \setminus W_B$ until we've found a subspace with length $\ell + 1 = i + 1$. Once $\ell = i$, the iterated subspace W_B equals $(Z, \mathcal{B}|_Z)$, since its length is also $i + 1$. This contradicts our assumption. □

This completes the proof of Theorem 6.2. In order to apply this theorem to the case of block size $K = \{3, 4, 5\}$, we show that planes with block size $\{3, 4, 5\}$ exist for all orders except 2, 6, and 8. In order to do so, we first define a special type of Latin square.

Definition 6.7 (Circulant Latin Square)**.** A Latin square is circulant when each row is a cyclic shift of the row before.

In a Latin square, call a point R_i, C_i, or S_i if it represents the i^{th} row, column, or symbol respectively. For example, the blocks $\{R_1, C_1, S_1\}$ and $\{R_1, C_2, S_2\}$ imply the blocks $\{R_2, C_2, S_1\}$ and $\{R_2, C_3, S_2\}$ in a circulant Latin square when rows are shifted cyclically to the right. Call this example of a circulant Latin square L which we will specifically use for this construction. Example 6.8 depicts L with $n = 4$.

Example 6.8. A circulant Latin square with $n = 4$, where each row is the right cyclic shift of the row above.

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

Theorem 6.9 (Dimension Two PBDs)**.** If there exists a PBD($v, \{3, 4, 5\}$), then there is a PBD($v, \{3, 4, 5\}$) of dimension exactly 2.

Proof. Since there exists a PBD($v, \{3, 4, 5\}$) for all orders $v \neq 2, 6, 8$, we wish to find planes for all such orders. From Theorem 5.1, no PBD($v, \{3, 4, 5\}$) of dimension greater than 2 exists with $v \leq 26$, except for $v = 15$. In the case of order 15, the reader is directed to [4] for the complete list of 80 Steiner triple systems of order 15, all but one of which is a plane. Thus, the existence of planes with block sizes $\{3, 4, 5\}$ is only in question for $v \geq 27$.

We start with a 3-GDD of type n^3 which is equivalent to a Latin square of side n, as in Lemma 2.13. Pairwise balanced designs are constructed from this GDD by filling each group with a PBD($n, \{3, 4, 5\}$), which exist for all $n \geq 9$. This settles all $v \geq 27$ with $v \equiv 0, 1 \pmod 3$, the latter case where one point is added at infinity and PBDs of order $n+1$ are used to fill in groups. For orders $v \equiv 2 \pmod 3$, we use a similar construction adding five points at infinity on a block of size 5, however care must be taken to ensure a PBD($n + 5, \{3, 4, 5\}$) containing a block of size 5 exists for all $n \geq 8$. We start from $n = 8$ to obtain $v = 29$ as in the above case, filling in each group with a PBD($13, \{3, 5\}$). Truncating up to four points from two groups of TD($5, n$) covers all $n \geq 46$. For $9 \leq n \leq 46$, we can either truncate or add points to various small designs, including TD($3, n$), TD($4, n$), $PG_2(4)$, and $AG_2(5)$. Note that for $n + 5 \equiv 2 \pmod 3$, the requirement of a block of size 5 is guaranteed.

In order to ensure each of these PBDs has dimension 2, we start from a circulant Latin square L of side n as outlined above. Call the PBD V_L that corresponds to replacing groups of L with $\{3, 4, 5\}$-PBDs.

Consider the subspace $W = \langle R_1, C_2, S_1 \rangle$ of V_L. This subspace clearly contains S_2 from the blocks listed above. Notice that S_1 is on all blocks $\{R_i, C_i, S_1\}$ and S_2 is on all blocks $\{R_i, C_{i+1}, S_2\}$ for each $i \in [g]$. If the block $\{R_i, C_i, S_1\}$ is in W, then $\{R_i, C_{i+1}, S_2\}$ is also in W to cover the pair of R_i and S_2. Since $C_{i+1} \in W$, the block $\{R_{i+1}, C_{i+1}, S_1\}$ is therefore also in the subspace. Continuing in this way, it is clear that W contains all points corresponding to rows and columns of L, and therefore all points corresponding to symbols as well. Thus W is of order at least $3g$. In the case that no additional points are added, we are done. Furthermore, by the subspace bound, there is no subspace of order $3g$ contained in a space of order $3g+1$ or $3g+5$. Thus, when one or five additional points are added we have also generated the entire space. For any order $v \geq 27$, no matter the PBDs used to replace groups of the GDD, we have constructed a PBD$(v, \{3, 4, 5\})$ with dimension 2. $\square$

Chapter 7

Wrap Up

We look at a few related results implied by Theorem 1.11. We then discuss some ideas on the unresolved values, finishing with a brief look at further directions.

7.1 Implications of Existence

We start with a construction from $\{3,4,5\}$-PBDs that implies an existence theory for Steiner spaces.

Construction 7.1 (PBD Existence Implies Steiner Spaces)**.** From a PBD$(v, \{3,4,5\})$ of dimension at least 3, give every point weight 6 and apply Theorem 2.20. Because 3-GDDs of types 6^3, 6^4, and 6^5 exist (see [4]), we can replace inflated blocks with GDDs. Groups are replaced with STS(7) or STS(9), respectively to adding one or three points. By Construction 2.23, this results in Steiner spaces of order $6v+1$ and $6v+3$. We can use any PBD of dimension at least 3 from Theorem 3.1, without circular referencing. From Theorem 2.20, we know this construction preserves dimension and thus have found examples of all Steiner spaces of order $v \geq 505$. As a result of Chapter 4, many smaller Steiner space orders in $187 \leq v \leq 501$ are also constructable this way, independently of Theorem 1.9.

While Tierlinck's treatment [14] of orders is thorough, it is of interest to see that most values can be obtained by PBD existence indirectly. We can similarly use the existence of PBDs with block sizes in $\{3,4,5\}$ to find PBDs with block sizes in $\{3,4\}$ and $\{3,5\}$.

Construction 7.2 ($\{3,4,5\}$ to $\{3,4\}$)**.** Start with any PBD$(v,\{3,4,5\})$ of dimension at least 3. Admissible orders for block sizes $\{3,4\}$ are $0,1 \pmod 3$ from the necessary conditions of Lemma 1.6. To cover both of these congruence classes, we inflate every point of the PBD by 3 and add either one or no additional points at infinity. Blocks of size 3 are replaced by 3-GDDs of type 3^3, blocks of size 4 by 4-GDDs of type 3^4, and blocks of size 5 by 3-GDDs or 4-GDDs of type 3^5. (We refer to [4] for the existence of these GDDs.)

In the case that one point is added, groups with the point at infinity are placed on a block of size 4. Theorem 2.23 implies that the resulting PBD$(3v+1,\{3,4\})$ has dimension at least 3. In the case that no points are added, we replace groups with blocks of size 3. The dimension of these PBD$(3v,\{3,4\})$ follows from Theorem 2.20.

As the largest possible exception of PBD$(v,\{3,4,5\})$ of dimension at least 3 is $v=47$, this construction implies the existence of PBD$(\ell,\{3,4\})$ of dimension at least 3 for all admissible $\ell \geq 144$. Moreover, since there are only 8 possible exceptions for $33 \leq v \leq 47$, there are at most 16 exceptions of admissible order for $99 \leq \ell \leq 142$. Many lower values are known directly as Steiner spaces, or as $\{3,4,5\}$-PBDs with constructions in Chapter 4 having no blocks of size 5.

Construction 7.3 ($\{3,4,5\}$ to $\{3,5\}$)**.** Start with any PBD$(v,\{3,4,5\})$ of dimension at least 3. From Lemma 1.6, any odd order is admissible. We therefore inflate every point by 2 and add one point at infinity. Blocks of size 3 and 4 in the original PBD are replaced by 3-GDDs of type 2^3 or 2^4. Blocks of size 5, once inflated, are replaced by the $\{3,5\}$-GDD of type 2^5 from Construction 2.27. Groups are replaced by blocks of size 3 with the point at infinity. The resulting PBD$(2v+1, \{3,5\})$ has dimension at least 3 by Theorem 2.23. This shows there is a PBD$(\ell, \{3,5\})$ for all $\ell \geq 2 \times 48 + 1 = 97$. Similarly, to Construction 7.2, there are at most 8 exceptions between $67 \leq \ell \leq 95$.

Note that dimension lowering outlined in Chapter 6 will work here as well.

7.2 Notes on Undecided Values

The existence of PBD$(v, \{3,4,5\})$ of dimension at least 3 for some of the possible exceptions $v \in E$ are believed to be difficult to determine. We remind the reader that the set of undetermined orders is $E = \{33, 34, 35, 38, 41, 42, 43, 47\}$. For example, consider $v = 33$ or 34. It is unknown whether or not a PBD$(v, \{3,4\})$ exists of dimension at least 3. Using Construction 2.23 for $k = 2$ with their existence would imply the existence of Steiner spaces of order 67 and 69. This argument is similar to Construction 7.3, and would settle two of the four undetermined orders of Steiner spaces which have been open since 1979 with Tierlinck's publishing of [14]. Furthermore, for $v = 33$ it is known that no Steiner space exists. The same is true of $v = 43$.

7.3 Further Directions

Beyond settling orders within the possible exceptions for PBDs and Steiner spaces, we look at further directions for this line of research.

The Häggkvist number is the minimum length of the longest two-coloured cycle in all edge-colourings of a graph coloured with n colours. In the graphs $K_{n,n}$ and K_{n+1} with n even, the Häggkvist number is bounded by 1720 [6]. This result uses the subspace bound on PBDs outlined in [7], which shows that for all integers $v \geq 861$

(and some smaller as well), there exists a PBD($v, \{3,4,5\}$) of dimension at least d with all subspaces bounded by function of d. Because of the result on Häggkvist numbers being universally bounded, it is believed that perhaps for large enough v, the size of subspaces can be universally bounded as well.

Another problem of interest is the relation of linear spaces to Latin squares. As discussed in Chapter 2, we know that mutually orthogonal Latin squares are equivalent to transversal designs. While the notion of strong dimension in GDDs used here forces any transversal design to have strong dimension 1, perhaps a more relaxed idea of dimension is of interest for Latin squares. For example, one could ask for what orders can any set of d rows, columns or symbols be contained in a proper sub-square of a Latin square.

Appendix A

Small Values Reference

This table lists the known existence of PBDs of dimension at least 3 and block sizes in $\{3, 4, 5\}$. All unlisted orders are taken care of in Theorem 3.1.

Order	**Exists?**	**Reference**
$1 - 14$	No	Corollary 5.3
15	Yes	$PG_3(2)$
$16 - 21$	No	Corollary 5.3
22	No	Corollary 5.6
23	No	Theorem 5.10
24	No	Corollary 5.5
25	No	Theorem 5.9
26	No	Theorem 5.8
27	Yes	Theorem 1.9
28	Yes	Construction 4.10
29	Yes	Construction 4.16

Order	**Exists?**	**Reference**
$30, 31$	Yes	Construction 4.10
32	No	Theorem 5.11
$33 - 35$	Unknown	
$36, 37$	Yes	Corollary 4.7
38	Unknown	
39	Yes	Corollary 4.7
40	Yes	$PG_3(3)$
$41 - 43$	Unknown	
44	Yes	Lemma 4.15
45	Yes	Construction 4.19
46	Yes	Corollary 4.17

Order	**Exists?**	**Reference**
47	Unknown	
$48-58$	Yes	Corollary 4.9
59	Yes	Corollary 4.17
60	Yes	Corollary 4.7
61	Yes	Corollary 4.17
62	Yes	Construction 4.11
63	Yes	Corollary 4.17
64	Yes	$AG_3(4)$
65	Yes	Corollary 4.9
66	Yes	Construction 4.11
$67-69$	Yes	Corollary 4.9
$70, 71$	Yes	Construct 4.14

Order	**Exists?**	**Reference**
72	Yes	Construction 4.13
73	Yes	Corollary 4.17
74	Yes	Construction 4.13
75	Yes	Construction 4.19
$76-78$	Yes	Construction 4.12
79	Yes	Corollary 4.17
$80, 81$	Yes	Corollary 4.7
82	Yes	Corollary 4.17
83	Yes	Corollary 4.7
86	Yes	Construction 4.18
88	Yes	Corollary 4.17
94	Yes	Corollary 4.17

Appendix B

Program to Check Dimension

What follows is the program created to check if a given linear space $(P, \mathcal{B})$ has dimension 3. The program runs in Ruby version 1.9.3. An explanation of each method is given followed by the code itself. The example input is the point-set and blocks of a PBD$(10, \{3, 4\})$.

B.1 Explanation of the Code

Method B.1 (doubles(P): Finding all pairs in P)**.** This method returns the set of a pairs in P. It loops twice over all points in P. It adds a pair to the set of doubles, as long as the pair is valid and does not already exist in the set.

Method B.2 (triples(P): Finding all triples in P)**.** This method returns the set of all triples in P. It loops three times over all points in P. It adds a triple to the set of triples, as long as it is valid and does not already exist in the set.

Method B.3 (checkDesign: Checks if $(P, \mathcal{B})$ is a valid PBD)**.** This method loops over all pairs in doubles(P). For each pair, it counts every block in $\mathcal{B}$ that contains the given pair. The method returns false unless all pairs are counted exactly once.

Method B.4 (Main: Finds the subspace generated by each triple in $(P, \mathcal{B})$)**.** This method finds the subspace generated by each triple $S \in$ triples(P). A point set $Q = S$ and an empty block set $\mathcal{C}$ are created. For every block $B \in \mathcal{B}$, if B contains at least two points from Q then B is added to $\mathcal{C}$. Points in B not already in Q are added to Q. The method loops until no new points or blocks are added. The checkDesign method is used to ensure $(Q, \mathcal{C})$ is a subspace and the size of Q and $\mathcal{C}$ are returned.

B.2 The Code

```
require 'set'
P = Set.new([
        :'1i', :'2i', :'3i',
        :'1j', :'2j', :'3j',
        :'1k', :'2k', :'3k', :'4k'])

B = Set.new([
        Set.new([ :'1i', :'2i', :'3i', :'4k']),
        Set.new([ :'1j', :'2j', :'3j', :'4k']),
        Set.new([ :'1k', :'2k', :'3k', :'4k']),
        Set.new([ :'1i', :'1j', :'1k']),
        Set.new([ :'2i', :'2j', :'2k']),
        Set.new([ :'3i', :'3j', :'3k']),
        Set.new([ :'1i', :'2j', :'3k']),
        Set.new([ :'2i', :'1j', :'3k']),
        Set.new([ :'3i', :'1j', :'2k']),
        Set.new([ :'1i', :'3j', :'2k']),
        Set.new([ :'2i', :'3j', :'1k']),
        Set.new([ :'3i', :'2j', :'1k']),        ])

def doubles(set)
  processed = Set.new
  set.each do |x|
    set.each do |y|
      other = Set.new([x, y])
      next if other.length < 2 or processed.include? other
      processed.add(other)
      yield other
    end
  end
end

def triples(set)
  processed = Set.new
  set.each do |x|
    set.each do |y|
      set.each do |z|
        other = Set.new([x, y, z])
```

```
        next if other.length < 3 or processed.include? other
        processed.add(other)
        yield other
      end
    end
  end
end

def design?(blocks = B, points = P)
  doubles(points) do |double|
    count = 0
    blocks.each do |block|
      count = count.next if double.subset? block
    end
    return false unless count.eql? 1
  end
  return true
end

C = Set.new
Q = Set.new
puts "design?  #{design?}"
triples(P) do |set|
  Q.replace(set)
  C.replace(Set.new)
  puts "PROCESSING: #{set.inspect}"

  loop do
    blocks = Set.new
    B.each do |b|
      count = 0
      b.each do |element|
        count = count.next if Q.include? element
      end
      blocks.add(b) unless count < 2
    end

    added = 0
    blocks.each do |block|
      unless C.include? block
        C.add(block)
        added = added.next
      end
    end

    C.each do |c|
      c.each do |element|
        Q.add(element)
      end
    end

    unless added > 0 then
      puts "design?  #{design?(C, Q)}, C.size: #{C.count}, Q.size: #{Q.count}"
      break
    end
  end
end
```

Bibliography

[1] Lynn Margaret Batten. *Combinatorics of finite geometries*. Cambridge University Press, 1986.

[2] Lynn Margaret Batten and Albrecht Beutelspacher. *The theory of finite linear spaces: combinatorics of points and lines*. Cambridge University Press, 1993.

[3] A. E. Brouwer. "Four MOLS of order 10 with a hole of order 2". In: *Journal of Statistical Planning and Inference* 10 (1984), pp. 203–205.

[4] Charles J. Colbourn and Jeffrey H. Dinitz, eds. *The CRC handbook of combinatorial designs*. CRC Press, Inc., 1996.

[5] A. Delandtsheer. *Handbook of incidence geometry*. North-Holland Publishing Company, 1995. Chap. Dimensional linear spaces, pp. 193–294.

[6] Peter Dukes and Alan C. H. Ling. "Edge-colourings of $K_{n,n}$ with no long two-coloured cycles". In: *Combinatorica* 28.3 (2008), pp. 373–378.

[7] Peter Dukes and Alan C. H. Ling. "Linear spaces with small generated subspaces". In: *Journal of Combinatorial Theory* A.116 (2008), pp. 485–493.

[8] Peter Dukes and Alan C. H. Ling. "Pairwise balanced designs with prescribed minimum dimension". In: *Discrete and Computational Geometry* (in press).

[9] Peter Dukes and Joanna Niezen. "Pairwise balanced designs of dimension 3". preprint.

[10] J. I. Hall. "Steiner triple systems with geometrically minimally generated subsystems". In: *Quart. J. Math Oxford* 2.25 (1974), pp. 41–50.

[11] A. J. W. Hilton. "On the Szamkolowicz-Doyen classification of Steiner triple systems". In: *Proc. London Math Society* 3.34 (1977), pp. 102–116.

[12] F. Kárteszi. *Introduction to finite geometries.* North-Holland Publishing Company, 1976.

[13] Ján Ninčák and Peter J. Owens. "On a problem of R. Hăggkvist concerning edge-colouring of bipartite graphs". In: *Combinatorica* 24.2 (2004), pp. 325–329.

[14] Luc Tierlinck. "On Steiner spaces". In: *Journal of Combinatorial Theory* A.26 (1979), pp. 103–114.

[15] R. M. Wilson. "An existence theory for pairwise balanced designs: II, The structure of PBD-closed sets and the existence conjectures". In: *Journal of Combinatorial Theory.* A 13 (1972), pp. 246–273.