

**Some Multivalent Functions with
Negative Coefficients Defined by Using
a Differential Operator**

M-P. Chen, H. Irmak and H.M. Srivastava

DMS-714-IR

September 1995

**Some Multivalent Functions with Negative Coefficients
Defined by Using a Differential Operator**

MING-PO CHEN

Institute of Mathematics
Academia Sinica
Taipei 11529, Taiwan, Republic of China
E-Mail: MAAPO@CCVAX.SINICA.EDU.TW

HÜSEYİN IRMAK

Department of Mathematics Education
Hacettepe University
06532 Beytepe-Ankara, Turkey
E-Mail: IRMAK@ETI.CC.HUN.EDU.TR

H.M. SRIVASTAVA

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
E-Mail: HMSRI@UVVM.UVIC.CA

1991 *Mathematics Subject Classification*. Primary 30C45.

Key words and phrases. Multivalent functions, differential operator, analytic functions, univalent functions, starlike functions, convex functions, maximum modulus theorem, quasi-Hadamard product (or convolution), Cauchy-Schwarz inequality, extremal functions, growth and distortion theorems, inclusion theorems.

Abstract

Making use of a differential operator, the authors introduce and study rather systematically two new subclasses

$$\mathcal{S}_n(p, q, \alpha) \quad \text{and} \quad \mathcal{C}_n(p, q, \alpha)$$

of p -valently analytic functions with negative coefficients. They also indicate relevant connections of these classes with several other families of univalent and multivalent analytic functions which were considered by many earlier workers on this topic of Geometric Function Theory.

1. Introduction and Definitions

Let us denote by $\mathcal{T}(n, p)$ the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; n, p \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the open unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f(z) \in \mathcal{T}(n, p)$ is said to be p -valently starlike of order α if it also satisfies the inequality:

$$\Re \left(\frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathbb{N}). \quad (1.2)$$

On the other hand, a function $f(z)$ belonging to the class $\mathcal{T}(n, p)$ is said to be p -valently convex of order α if it also satisfies the inequality:

$$\Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathbb{N}). \quad (1.3)$$

It is easily seen that a function $f(z)$ is p -valently convex of order α ($0 \leq \alpha < p$; $p \in \mathbb{N}$) if and only if $z f'(z)$ is p -valently starlike of order α ($0 \leq \alpha < p$; $p \in \mathbb{N}$). (See, for example, Duren [4] and Goodman [5].)

The main purpose of the present paper is to investigate various interesting properties and characteristics of functions belonging to two subclasses

$$\mathcal{S}_n(p, q, \alpha) \quad \text{and} \quad \mathcal{C}_n(p, q, \alpha)$$

of the class $\mathcal{T}(n, p)$, which consist (respectively) of p -valently starlike and p -valently convex functions of order α ($0 \leq \alpha < p$; $p \in \mathbb{N}$). Indeed we have

$$\mathcal{S}_n(p, q, \alpha) := \left\{ f \in \mathcal{T}(n, p) : \Re \left(\frac{z f^{(1+q)}(z)}{f^{(q)}(z)} \right) > \alpha \quad (z \in \mathcal{U}) \right\} \quad (1.4)$$

and

$$\mathcal{C}_n(p, q, \alpha) := \left\{ f \in \mathcal{T}(n, p) : \Re \left(1 + \frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)} \right) > \alpha \quad (z \in \mathcal{U}) \right\}, \quad (1.5)$$

where, for each $f \in \mathcal{T}(n, p)$,

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \quad (j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \quad (1.6)$$

it being understood (*throughout this paper*) that

$$0 \leq \alpha < p - q; \quad n, p \in \mathbb{N}; \quad p > q; \quad q \in \mathbb{N}_0,$$

unless stated otherwise.

Various special cases of the classes defined by (1.4) and (1.5) were considered by many earlier workers on this topic of Geometric Function Theory. For example, we have the following relationships with the classes which were studied in earlier works.

$$\mathcal{S}_n(p, 0, \alpha) = \begin{cases} \mathcal{T}_n^*(p, \alpha) & \text{(Owa [7])} \\ \mathcal{T}_\alpha(p, n) & \text{(Yamakawa [12]);} \end{cases} \quad (1.7)$$

$$\mathcal{C}_n(p, 0, \alpha) = \begin{cases} \mathcal{C}_n(p, \alpha) & \text{(Owa [7])} \\ \mathcal{CT}_\alpha(p, n) & \text{(Yamakawa [12]);} \end{cases} \quad (1.8)$$

$$\mathcal{S}_n(1, 0, \alpha) = \mathcal{T}_n^*(1, \alpha) = \mathcal{T}_\alpha(1, n) = \mathcal{T}_\alpha(n) \quad \text{(Srivastava et al. [11]);} \quad (1.9)$$

$$\mathcal{C}_n(1, 0, \alpha) = \mathcal{C}_n(1, \alpha) = \mathcal{CT}_\alpha(1, n) = \mathcal{C}_\alpha(n) \quad \text{(Srivastava et al. [11]);} \quad (1.10)$$

$$\mathcal{S}_1(1, 0, \alpha) = \mathcal{T}_\alpha(1) = \mathcal{T}^*(\alpha) \quad (\text{Silverman [8]}); \quad (1.11)$$

$$\mathcal{C}_1(1, 0, \alpha) = \mathcal{C}_\alpha(1) = \mathcal{C}(\alpha) \quad (\text{Silverman [8]}). \quad (1.12)$$

In addition, Chen [3], Srivastava and Aouf [9], Murugusundaramoorthy *et al.* [6], Altıntaş *et al.* ([1] and [2]), and many others investigated numerous interesting subclasses of the class $\mathcal{T}(n, p)$, which are *not* included in the aforementioned subclasses (see also Srivastava and Owa [10]).

2. Coefficient Bounds and Inclusion Theorems

A necessary and sufficient condition for a function $f(z) \in \mathcal{T}(n, p)$ to be in the class $\mathcal{S}_n(p, q, \alpha)$ is provided by

Theorem 1. *Let a function $f(z)$ be in the class $\mathcal{T}(n, p)$. Then $f(z)$ belongs to the class $\mathcal{S}_n(p, q, \alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{k!(k-q-\alpha)}{(k-q)!} a_k \leq \frac{p!(p-q-\alpha)}{(p-q)!} \quad (2.1)$$

$$(0 \leq \alpha < p - q; n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

The result is sharp, the extremal function being given by

$$f(z) = z^p - \frac{p!(n+p-q)!(p-q-\alpha)}{(n+p)!(p-q)!(n+p-q-\alpha)} z^{n+p} \quad (2.2)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

Proof. Suppose that the function $f(z)$, given by (1.1), is in the class $\mathcal{S}_n(p, q, \alpha)$. Making use of (1.4) and (1.6), choosing values of z on the real axis, and then letting $z \rightarrow 1-$ through real values, we arrive easily at the assertion (2.1) of Theorem 1.

Conversely, we suppose that the inequality (2.1) holds true and let

$$z \in \partial\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| = 1\}.$$

Then we find from (1.1), (1.4), and (1.6) that

$$\left| \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} - p + q \right| \leq p - q - \alpha \quad (2.3)$$

$$(z \in \partial\mathcal{U}; 0 \leq \alpha < p - q; n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0),$$

where we have also made use of the inequality (2.1). Thus, by the maximum modulus theorem, we conclude from (2.3) that

$$f(z) \in \mathcal{S}_n(p, q, \alpha).$$

Finally, by observing that the function $f(z)$ given by (2.2) is indeed an extremal function for the assertion (2.1), we complete the proof of Theorem 1.

An analogous result for the class $\mathcal{C}_n(p, q, \alpha)$ is contained in

Theorem 2. *Let a function $f(z)$ be in the class $\mathcal{T}(n, p)$. Then $f(z)$ belongs to the class $\mathcal{C}_n(p, q, \alpha)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{k!(k-q-\alpha)}{(k-q-1)!} a_k \leq \frac{p!(p-q-\alpha)}{(p-q-1)!} \quad (2.4)$$

$$(0 \leq \alpha < p - q; n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

The result is sharp, the extremal function being given by

$$f(z) = z^p - \frac{p!(n+p-q-1)!(p-q-\alpha)}{(n+p)!(p-q-1)!(n+p-q-\alpha)} z^{n+p} \quad (2.5)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

Corollary 1 and Corollary 2 below are rather immediate consequences of Theorem 1 and Theorem 2, respectively.

Corollary 1. *If $f(z) \in \mathcal{S}_n(p, q, \alpha)$, then*

$$a_k \leq \frac{p!(k-q)!(p-q-\alpha)}{k!(p-q)!(k-q-\alpha)} \quad (2.6)$$

$$(k \geq n + p; n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

Corollary 2. *If $f(z) \in \mathcal{C}_n(p, q, \alpha)$, then*

$$a_k \leq \frac{p!(k-q-1)!(p-q-\alpha)}{k!(p-q-1)!(k-q-\alpha)} \quad (2.7)$$

$$(k \geq n+p; n, p \in \mathbb{N}; p > q; q \in \mathbb{N}).$$

Next, by appealing to the assertions (2.1) and (2.4) of Theorem 1 and Theorem 2, respectively, it is not difficult to prove the following inclusion properties for the classes $\mathcal{S}_n(p, q, \alpha)$ and $\mathcal{C}_n(p, q, \alpha)$. The details involved may be omitted.

Theorem 3. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geq 0; n, p \in \mathbb{N}) \quad (2.8)$$

be in the same class $\mathcal{S}_n(p, q, \alpha)$. Then the function $h(z)$ defined by

$$h(z) := (1 - \lambda) f(z) + \lambda g(z) = z^p - \sum_{k=n+p}^{\infty} c_k z^k \quad (2.9)$$

$$(c_k := (1 - \lambda) a_k + \lambda b_k \geq 0; 0 \leq \lambda \leq 1; n, p \in \mathbb{N})$$

is also in the class $\mathcal{S}_n(p, q, \alpha)$.

Theorem 4. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.8) be in the same class $\mathcal{C}_n(p, q, \alpha)$. Then the function $h(z)$ defined by (2.9) is also in the class $\mathcal{C}_n(p, q, \alpha)$.*

For the functions $f(z)$ and $g(z)$ given by (1.1) and (2.8), respectively, we define their *quasi-Hadamard product* (or *convolution*) by

$$(f * g)(z) := z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k \quad (2.10)$$

$$(a_k \geq 0; b_k \geq 0; n, p \in \mathbb{N}).$$

Our next results (Theorem 5 and Theorem 6 below) provide inclusion properties of the classes $\mathcal{S}_n(p, q, \alpha)$ and $\mathcal{C}_n(p, q, \alpha)$, respectively, involving the quasi-Hadamard product $(f * g)(z)$ defined by (2.10).

Theorem 5. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.8) be in the same class $\mathcal{S}_n(p, q, \alpha)$. Then their quasi-Hadamard product $(f * g)(z)$ defined by (2.10) is in the class $\mathcal{S}_n(p, q, \beta)$, where

$$\beta \leq p - q - \frac{p!(n+p-q)!(p-q-\alpha)^2}{(n+p)!(p-q)!(n+p-q-\alpha)} \quad (2.11)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z^p - \frac{p!(n+p-q)!(p-q-\alpha)}{(n+p)!(p-q)!(n+p-q-\alpha)} z^{n+p} \quad (2.12)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

Theorem 6. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.8) be in the same class $\mathcal{C}_n(p, q, \alpha)$. Then their quasi-Hadamard product $(f * g)(z)$ defined by (2.10) is in the class $\mathcal{C}_n(p, q, \gamma)$, where

$$\gamma \leq p - q - \frac{p!(n+p-q-1)!(p-q-\alpha)^2}{(n+p)!(p-q-1)!(n+p-q-\alpha)} \quad (2.13)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z^p - \frac{p!(n+p-q-1)!(p-q-\alpha)}{(n+p)!(p-q-1)!(n+p-q-\alpha)} z^{n+p} \quad (2.14)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0).$$

Proof of Theorem 5. In order to prove Theorem 5, it is sufficient to find the largest β such that

$$\sum_{k=n+p}^{\infty} \frac{k!(p-q)!(k-q-\beta)}{p!(k-q)!(p-q-\beta)} a_k b_k \leq 1 \quad (2.15)$$

$$(n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0),$$

and this is accomplished, in view of the Cauchy-Schwarz inequality, by applying Theorem 1 to each of the functions $f(z)$ and $g(z)$. The details involved are fairly straightforward.

Proof of Theorem 6. The proof of Theorem 6 is much akin to that of Theorem 5. Indeed, instead of Theorem 1, we apply Theorem 2 to each of the functions $f(z)$ and $g(z)$.

3. Growth and Distortion Theorems

The growth and distortion properties of functions belonging to the class $\mathcal{S}_n(p, q, \alpha)$ are contained in

Theorem 7. *If $f(z) \in \mathcal{S}_n(p, q, \alpha)$, then*

$$\begin{aligned} & \left(\frac{1}{(p-j)!} - \frac{(n+p-q)!(p-q-\alpha)}{(n+p-j)!(p-q)!(n+p-q-\alpha)} |z|^n \right) p! |z|^{p-j} \leq |f^{(j)}(z)| \\ & \leq \left(\frac{1}{(p-j)!} + \frac{(n+p-q)!(p-q-\alpha)}{(n+p-j)!(p-q)!(n+p-q-\alpha)} |z|^n \right) p! |z|^{p-j} \quad (3.1) \\ & \quad (z \in \mathcal{U}; n, p \in \mathbb{N}; p > \max\{q, j\}; q, j \in \mathbb{N}_0). \end{aligned}$$

The result is sharp for the function $f(z)$ given by (2.2).

Proof. Under the hypothesis of Theorem 7, we find from the assertion (2.1) of Theorem 1 that

$$\begin{aligned} \sum_{k=n+p}^{\infty} k! a_k & \leq \frac{p!(n+p-q)!(p-q-\alpha)}{(p-q)!(n+p-q-\alpha)} \quad (3.2) \\ & \quad (n, p \in \mathbb{N}; p > q; q \in \mathbb{N}_0). \end{aligned}$$

Now the inequalities (3.1) would follow readily when we make use of (3.2) in conjunction with the series expansion for $f^{(j)}(z)$ ($j \in \mathbb{N}_0$) given by (1.6).

In a similar manner, by applying Theorem 2 in place of Theorem 1, we can prove

Theorem 8. *If $f(z) \in \mathcal{C}_n(p, q, \alpha)$, then*

$$\begin{aligned} & \left(\frac{1}{(p-j)!} - \frac{(n+p-q-1)!(p-q-\alpha)}{(n+p-j)!(p-q-1)!(n+p-q-\alpha)} |z|^n \right) p! |z|^{p-j} \leq |f^{(j)}(z)| \\ & \leq \left(\frac{1}{(p-j)!} + \frac{(n+p-q-1)!(p-q-\alpha)}{(n+p-j)!(p-q-1)!(n+p-q-\alpha)} |z|^n \right) p! |z|^{p-j} \quad (3.3) \\ & \quad (z \in \mathcal{U}; n, p \in \mathbb{N}; p > \max\{q, j\}; q, j \in \mathbb{N}_0). \end{aligned}$$

The result is sharp for the function $f(z)$ given by (2.5).

In their special case when $j = 0$, Theorem 7 and Theorem 8 would provide us with the growth properties of functions belonging to the classes $\mathcal{S}_n(p, q, \alpha)$ and $\mathcal{C}_n(p, q, \alpha)$, respectively. For $j \in \mathbb{N}$, these results may be looked upon as the distortion theorems for the classes involved.

Acknowledgments

The present investigation was completed during the third-named author's visits to the Institute of Mathematics (Academia Sinica) at Taipei, the National Chang-Hua University of Education at Chang-Hua, and National Tsing Hua University at Hsin-Chu in July and August 1995. This work was supported, in part, by the *National Science Council of the Republic of China* under Grant NSC-85-2121-M-001-013 and, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

- [1] O. Altıntaş, H. Irmak, and H.M. Srivastava, A subclass of analytic functions defined by using certain operators of fractional calculus, *Comput. Math. Appl.* **30**(1995), no. 1, 1–9.
- [2] O. Altıntaş, H. Irmak, and H.M. Srivastava, Fractional calculus and certain starlike functions with negative coefficients, *Comput. Math. Appl.* **30**(1995), no. 2, 9–15.
- [3] M.-P. Chen, Multivalent functions with negative coefficients in the unit disc, *Tamkang J. Math.* **17**(1986), 127–137.
- [4] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.
- [5] A.W. Goodman, *Univalent Functions*, Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.
- [6] G. Murugusundaramoorthy, P. Balasubramanyam, and K.G. Subramanian, On a generalization of a class of analytic functions with negative coefficients, *Chinese J. Math.* **22**(1994), 11–19.
- [7] S. Owa, On certain classes of p -valent functions with negative coefficients, *Simon Stevin* **59**(1985), 385–402.

- [8] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**(1975), 109–116.
- [9] H.M. Srivastava and M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, *J. Math. Anal. Appl.* **171**(1992), 1–13; *ibid.* **192**(1995), 673–688.
- [10] H.M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.
- [11] H.M. Srivastava, S. Owa, and S.K. Chatterjea, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ. Padova* **77**(1987), 115–124.
- [12] R. Yamakawa, Certain subclasses of p -valently starlike functions with negative coefficients, in *Current Topics in Analytic Function Theory* (H.M. Srivastava and S. Owa, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992, pp. 393–402.