SOME CLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS†

by

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ABSTRACT

A function

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_1 > 0,\ a_n \geqq 0)$$

is said to be in the class $\mathcal{T}\{b_n\}$ if there exists a sequence $\{b_n\}$ of positive real numbers such that $\sum_{n=2}^{\infty} b_n a_n \leqq a_1$. Two subclasses $\mathcal{T}_0\{b_n,\ z_0\}$ and $\mathcal{T}_1\{b_n,\ z_0\}$ of $\mathcal{T}\{b_n\}$ are introduced, and several interesting results for these classes are proved. Furthermore, the convolution product, radius of convexity, and order of starlikeness of functions belonging to these classes are also considered.

1. INTRODUCTION

A single-valued function $f(z)$ is said to be univalent in a domain $\mathcal{D}$ if it never takes on the same value twice, that is, if $f(z_1) = f(z_2)$ for $z_1,\ z_2 \in \mathcal{D}$ implies that $z_1 = z_2$ (cf., e.g., [1]). Let $\mathcal{T}$ denote the class of functions of the form

$$(1.1) \qquad f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_1 > 0,\ a_n \geqq 0)$$

which are analytic and univalent in the unit disk $\mathcal{U} = \{z\colon |z| < 1\}$. Then a function $f(z)$ of $\mathcal{T}$ is said to be in the class $\mathcal{T}\{b_n\}$ if there exists a sequence $\{b_n\}$ of positive real numbers such that

$$(1.2) \qquad \sum_{n=2}^{\infty} b_n a_n \leqq a_1 .$$

Gupta and Ahmad [2] studied the classes

$$\mathcal{T}\{[(n-1) + \beta(n+1-2\alpha)]/[2\beta(1-\alpha)]\}$$

and

$$\mathcal{T}\{n[(n-1) + \beta(n+1-2\alpha)]/[2\beta(1-\alpha)]\},$$

and the classes

$$\mathcal{T}\{[(1-\beta)n - \alpha\beta]/[\beta(1-\alpha)]\}, \quad \mathcal{T}\{[(n-1) + \beta(1-\alpha n)]/[\beta(1+\alpha)]\},$$

and

$$\mathcal{T}\{n[(n-1) + \beta(1+\alpha n)]/[\beta(1+\alpha)]\}$$

were studied recently by Owa ([5], [6]). In particular, for $a_1 = 1$, Silverman ([8], [9]) introduced the classes

$$\mathcal{T}\{b_n\}, \quad \mathcal{T}\{(n-\alpha)/(1-\alpha)\} \quad \text{and} \quad \mathcal{T}\{n(n-\alpha)/(1-\alpha)\},$$

Gupta and Jain ([3], [4]) introduced the classes

$$\mathcal{T}\{[(n-1) + \beta(n+1-2\alpha)]/[2\beta(1-\alpha)]\}, \quad \mathcal{T}\{n[(n-1) + \beta(n+1-2\alpha)]/[2\beta(1-\alpha)]\},$$

and

$$\mathcal{T}\{n(1+\beta)/[2\beta(1-\alpha)]\},$$

and Owa [7] introduced the class

$$\mathcal{T}\{(n+m-1)!(2n+m-1)/[(n-1)!(m+1)!]\}.$$

Let $\mathcal{T}_0\{b_n, z_0\}$ and $\mathcal{T}_1\{b_n, z_0\}$ denote the subclasses of $\mathcal{T}\{b_n\}$ consisting of functions $f(z)$ satisfying the conditions $f(z_0) = z_0$ and $f'(z_0) = 1$ for $0 < z_0 < 1$, respectively. The object of this paper is to present several interesting results for functions belonging to these classes with $a_1 \geqq 1$.

2. THE CLASS $\mathcal{T}_0\{b_n, z_0\}$

We begin by stating the following result for functions $f(z)$ belonging to $\mathcal{T}_0\{b_n, z_0\}$.

THEOREM 1. <u>If the function</u> $f(z)$ <u>defined by</u> (1.1) <u>is in the class</u> $\mathcal{T}_0\{b_n, z_0\}$ <u>for</u> $0 < z_0 < 1$, <u>then</u>

$$\sum_{n=2}^{\infty} \left(b_n - z_0^{n-1}\right) a_n \leqq 1. \tag{2.1}$$

PROOF. By the definition of $\mathcal{T}_0\{b_n, z_0\}$, $f(z)$ satisfies the inequality (1.2) and $f(z_0) = z_0$. Note that

$$a_1 = 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1}. \tag{2.2}$$

Hence (1.2) becomes

$$\sum_{n=2}^{\infty} b_n a_n \leqq 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1}, \tag{2.3}$$

which implies (2.1). This completes the proof of Theorem 1.

THEOREM 2. <u>Let the function</u> $f(z)$ <u>be defined by</u> (1.1). <u>If there exists a sequence</u> $\{b_n\}$ <u>of positive real numbers which satisfy</u> (2.1) <u>for</u> $0 < z_0 < 1$, <u>then</u> $f(z) \in \mathcal{T}_0\{b_n, z_0\}$.

PROOF. It suffices to prove that (2.1) implies (1.2) and $f(z_0) = z_0$ for $0 < z_0 < 1$. Substituting for a_1 from (2.2) into (2.1), we observe that (2.1) implies (1.2).

Further, for a_1 given by (2.2), we have

$$f(z_0) = a_1 z_0 - \sum_{n=2}^{\infty} a_n z_0^n \tag{2.4}$$

$$= z_0\left(1 + \sum_{n=2}^{\infty} a_n z_0^{n-1}\right) - \sum_{n=2}^{\infty} a_n z_0^n = z_0,$$

and the proof of Theorem 2 is thus completed.

With the aid of Theorems 1 and 2, we now prove the following result.

THEOREM 3. *Let the function* $f(z)$ *defined by* (1.1) *and the function* $g(z)$ *defined by*

$$g(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0,\ c_n \geqq 0) \tag{2.5}$$

be in the same class $\mathcal{T}_0\{b_n, z_0\}$. *Then the function*

$$F(z) = \lambda f(z) + (1-\lambda) g(z) \qquad (0 \leqq \lambda \leqq 1) \tag{2.6}$$

is also in the class $\mathcal{T}_0\{b_n, z_0\}$.

PROOF. Since

$$F(z) = [\lambda a_1 + (1-\lambda) c_1] z - \sum_{n=2}^{\infty} [\lambda a_n + (1-\lambda) c_n] z^n, \tag{2.7}$$

we find that

$$(2.8)\qquad \sum_{n=2}^{\infty}\left(b_n - z_0^{n-1}\right)[\lambda\, a_n + (1-\lambda)c_n]$$

$$= \lambda \sum_{n=2}^{\infty}\left(b_n - z_0^{n-1}\right)a_n + (1-\lambda)\sum_{n=2}^{\infty}\left(b_n - z_0^{n-1}\right)c_n$$

$$\leqq 1,$$

by means of Theorem 1. Consequently, we have

$$F(z) \in \mathcal{T}_0\{b_n, z_0\}$$

in view of Theorem 2.

Theorem 3 shows that the class $\mathcal{T}_0\{b_n, z_0\}$ is convex. Naturally, therefore, we are interested in finding the extreme points of $\mathcal{T}_0\{b_n, z_0\}$.

THEOREM 4. *Let*

$$(2.9)\qquad f_1(z) = z$$

and

$$(2.10)\qquad f_n(z) = \frac{b_n z - z^n}{b_n - z_0^{n-1}} \qquad (n \geqq 2).$$

Then $f(z)$ *is in the class* $\mathcal{T}_0\{b_n, z_0\}$ *if and only if it can be expressed in the form*

$$(2.11)\qquad f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geqq 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = 1$.

PROOF. Assume that $f(z)$ is expressible in the form (2.11). Then we have

$$(2.12)\qquad f(z) = \left(\lambda_1 + \sum_{n=2}^{\infty} \frac{\lambda_n b_n}{b_n - z_0^{n-1}}\right) z - \sum_{n=2}^{\infty} \frac{\lambda_n}{b_n - z_0^{n-1}} z^n$$

and $f(z_0) = z_0$. We also have

$$(2.13)\qquad \sum_{n=2}^{\infty} \left(b_n - z_0^{n-1}\right) \cdot \frac{\lambda_n}{b_n - z_0^{n-1}} = 1 - \lambda_1 \leqq 1.$$

Hence $f(z) \in \mathcal{T}_0\{b_n, z_0\}$.

For the converse, suppose that the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}_0\{b_n, z_0\}$. Then we can set

$$(2.14)\qquad \lambda_n = \left(b_n - z_0^{n-1}\right) a_n \qquad (n \geqq 2)$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. We thus obtain the representation (2.11).

COROLLARY 1. The extreme points of $\mathcal{T}_0\{b_n, z_0\}$ are $f_n(z)$ $(n \geqq 1)$ given by (2.9) and (2.10).

We require the following result due to Silverman [9] in order to prove our next theorem.

LEMMA 1. Let the function $f(z)$ be defined by

$$(2.15)\qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

that is, by (1.1) with $a_1 = 1$. Then $f(z)$ is analytic and univalent in the unit disk $\mathcal{U}$ if and only if

$$(2.16)\qquad \sum_{n=2}^{\infty} n\, a_n \leqq 1.$$

THEOREM 5. <u>The class</u> $\mathcal{T}_0\{b_n, z_0\}$ <u>is contained in the class</u> $\mathcal{T}$ <u>if and only if</u>

$$b_n \geqq \frac{n}{a_1} + z_0^{n-1} \tag{2.17}$$

<u>for</u> $0 < z_0 < 1$, <u>and for every integer</u> $n \geqq 2$.

PROOF. By virtue of Lemma 1, the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}$ if and only if

$$\sum_{n=2}^{\infty} n\left(\frac{a_n}{a_1}\right) \leqq 1. \tag{2.18}$$

Hence, if b_n satisfies the inequality (2.17), it is clear that

$$\mathcal{T}_0\{b_n, z_0\} \subset \mathcal{T}.$$

Conversely, we consider the function $f_n(z)$ defined by

$$f_n(z) = a_1 z - \frac{1}{b_n - z_0^{n-1}} z^n. \tag{2.19}$$

Since $f_n(z)$ satisfies (2.1), we have $f_n(z) \in \mathcal{T}_0\{b_n, z_0\}$.

Now assume that

$$b_n < \frac{n}{a_1} + z_0^{n-1} \tag{2.20}$$

for $0 < z_0 < 1$ and for some integer $n \geqq 2$. Then $f_n'(z) = 0$ at

$$z = \left[\frac{a_1(b_n - z_0^{n-1})}{n}\right]^{1/(n-1)} < 1. \tag{2.21}$$

This shows that $f_n(z)$ is not univalent, that is, $f_n(z) \notin \mathcal{T}$, and the proof of Theorem 5 is completed.

3. THE CLASS $\mathcal{T}_1\{b_n, z_0\}$

The proofs of the following results for $\mathcal{T}_1\{b_n, z_0\}$ run parallel to those of the corresponding results for $\mathcal{T}_0\{b_n, z_0\}$.

THEOREM 6. If the function $f(z)$ defined by (1.1) is in the class $\mathcal{T}_1\{b_n, z_0\}$ for $0 < z_0 < 1$, then

(3.1) $$\sum_{n=2}^{\infty} \left(b_n - nz_0^{n-1}\right) a_n \leqq 1.$$

THEOREM 7. Let the function $f(z)$ be defined by (1.1). If there exists a sequence $\{b_n\}$ of positive real numbers which satisfy (3.1) for $0 < z_0 < 1$, then

$$f(z) \in \mathcal{T}_1\{b_n, z_0\}.$$

THEOREM 8. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.5) be in the same class $\mathcal{T}_1\{b_n, z_0\}$. Then the function $F(z)$ defined by (2.6) is also in the class $\mathcal{T}_1\{b_n, z_0\}$.

THEOREM 9. Let

(3.2) $$f_1(z) = z$$

and

(3.3) $$f_n(z) = \frac{b_n z - z^n}{b_n - n z_0^{n-1}} \qquad (n \geqq 2).$$

Then $f(z)$ is in the class $\mathcal{T}_1\{b_n, z_0\}$ if and only if it can be expressed in the form (2.11).

COROLLARY 2. The extreme points of $\mathcal{T}_1\{b_n, z_0\}$ are $f_n(z)$ $(n \geqq 1)$ given by (3.2) and (3.3).

THEOREM 10. The class $\mathcal{T}_1\{b_n, z_0\}$ is contained in the class $\mathcal{T}$ if and only if

(3.4) $$b_n \geqq \frac{n}{a_1} + n\, z_0^{n-1}$$

for $0 < z_0 < 1$ and for every integer $n \geqq 2$.

4. CHARACTERIZATIONS OF THE CONVOLUTION PRODUCT

Let $f * g(z)$ denote the convolution product of two functions $f(z)$ and $g(z)$; that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by (2.5), then

(4.1) $$f * g(z) = a_1 c_1 z - \sum_{n=2}^{\infty} a_n c_n z^n.$$

In this section we prove the following theorems characterizing the convolution product $f * g(z)$ defined by (4.1).

THEOREM 11. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.5) be in the same class $\mathcal{T}_0\{b_n, z_0\}$. Then $f * g(z)$ is

<u>in the class</u> $\mathcal{T}_0\{\alpha_n, z_0\}$ <u>for</u>

$$(4.2) \qquad \alpha_n \leqq \left(b_n - z_0^{n-1}\right)^2 + z_0^{n-1}.$$

PROOF. Assuming that $f * g(z) \in \mathcal{T}_0\{\alpha_n, z_0\}$, we find from Theorem 1 and the Cauchy-Schwarz inequality that

$$(4.3) \qquad \sum_{n=2}^{\infty} \left(b_n - z_0^{n-1}\right)\sqrt{a_n c_n} \leqq 1.$$

Thus it suffices to show that

$$(4.4) \qquad \sum_{n=2}^{\infty} \left(\alpha_n - z_0^{n-1}\right) a_n c_n \leqq \sum_{n=2}^{\infty} \left(b_n - z_0^{n-1}\right)\sqrt{a_n c_n},$$

that is, that

$$(4.5) \qquad \left(\alpha_n - z_0^{n-1}\right)\sqrt{a_n c_n} \leqq b_n - z_0^{n-1}$$

for all integers $n \geqq 2$. Clearly, the condition (4.5) holds true in view of (4.3), since (4.2) implies

$$(4.6) \qquad \alpha_n - z_0^{n-1} \leqq \left(b_n - z_0^{n-1}\right)^2.$$

This evidently completes the proof of Theorem 11.

Similar is the proof of the following characterization theorem corresponding to the class $\mathcal{T}_1\{b_n, z_0\}$.

THEOREM 12. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (2.5) be in the same class $\mathcal{T}_1\{b_n, z_0\}$. Then $f * g(z)$ is in the class $\mathcal{T}_1\{\beta_n, z_0\}$ for

$$\beta_n \leqq \left(b_n - n\, z_0^{n-1}\right)^2 + n\, z_0^{n-1}. \tag{4.7}$$

5. CONVEXITY AND STARLIKENESS

A function $f(z)$ is said to be starlike of order α $(0 \leqq \alpha < 1)$ in the unit disk $\mathcal{U}$ if $f(z)$ is analytic in the unit disk $\mathcal{U}$ and satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{5.1}$$

for $0 \leqq \alpha < 1$. If $\alpha = 0$, then $f(z)$ is said to be starlike in $\mathcal{U}$. Moreover, a function $f(z)$ is said to be convex of order α $(0 \leqq \alpha < 1)$ in the unit disk $\mathcal{U}$ if $f(z)$ is analytic in the unit disk $\mathcal{U}$ and satisfies

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathcal{U}) \tag{5.2}$$

for $0 \leqq \alpha < 1$. If $\alpha = 0$, then $f(z)$ is said to be convex in $\mathcal{U}$.

Theorems 13 and 14 below determine the radii of convexity of the classes $\mathcal{T}_0\{b_n, z_0\}$ and $\mathcal{T}_1\{b_n, z_0\}$, respectively.

THEOREM 13. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{T}_0\{b_n, z_0\}$. Then $f(z)$ is convex in the disk $|z| < r_0$, where

(5.3) $$r_0 = \inf_{n\geqq 2}\left(\frac{b_n}{n^2}\right)^{1/(n-1)} .$$

The result is sharp for functions of the form

(5.4) $$f_n(z) = \frac{b_n z - z^n}{b_n - z_0^{n-1}} \qquad (n \geqq 2).$$

PROOF. It is sufficient to show that

(5.5) $$\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1$$

for $|z| \leqq r_0$.

Clearly

(5.6) $$\left|\frac{zf''(z)}{f'(z)}\right| \leqq \frac{\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}} \leqq 1,$$

if

(5.7) $$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leqq 1 + \sum_{n=2}^{\infty} a_n z_0^{n-1} - \sum_{n=2}^{\infty} n\, a_n|z|^{n-1},$$

or

(5.8) $$\sum_{n=2}^{\infty}\left(n^2|z|^{n-1} - z_0^{n-1}\right)a_n \leqq 1.$$

We note that

(5.9) $$a_n \leqq \frac{1}{b_n - z_0^{n-1}} \qquad (n \geqq 2)$$

by virtue of Theorem 1. Therefore, (5.8) holds true if

(5.10) $$n^2|z|^{n-1} \leqq b_n \qquad (n \geqq 2).$$

Hence, if

$$|z| \leqq \left(\frac{b_n}{n^2}\right)^{1/(n-1)},$$

(5.5) is satisfied, and the theorem follows.

In a similar way, we have

THEOREM 14. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{T}_1\{b_n, z_0\}$. *Then* $f(z)$ *is convex in the disk* $|z| < r_0$, *where* r_0 *is given by* (5.3). *The result is sharp for functions of the form*

(5.11) $$f_n(z) = \frac{b_n z - z^n}{b_n - nz_0^{n-1}} \qquad (n \geqq 2).$$

Finally, we determine the order of starlikeness of functions belonging to the classes $\mathcal{T}_0\{b_n, z_0\}$ and $\mathcal{T}_1\{b_n, z_0\}$. We first state

THEOREM 15. *Let*

(5.12) $$b_n \geqq \frac{n - \alpha}{1 - \alpha}$$

for every integer $n \geqq 2$ *and* $0 \leqq \alpha < 1$, *and suppose that the function* $f(z)$

defined by (1.1) *is in the class* $\mathcal{T}_0\{b_n, z_0\}$. *Then* $f(z)$ *is starlike of order* α, *with equality in* (5.12) *for the function*

$$f(z) = a_1 z - \frac{1-\alpha}{1+(1-\alpha)(1-z_0)} z^2. \tag{5.13}$$

PROOF. We need only prove that the values of $zf'(z)/f(z)$ lie in a disk with its center at 1 and radius $1-\alpha$, that is, that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1 - \alpha \qquad (z \in \mathcal{U}). \tag{5.14}$$

In fact, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{\sum_{n=2}^{\infty} (n-1) a_n |z|^{n-1}}{a_1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} \leqq \frac{\sum_{n=2}^{\infty} (n-1) a_n}{a_1 - \sum_{n=2}^{\infty} a_n} < 1 - \alpha, \tag{5.15}$$

if and only if

$$\sum_{n=2}^{\infty} \left(\frac{n-\alpha}{1-\alpha} - z_0^{n-1}\right) a_n \leqq 1, \tag{5.16}$$

and the theorem follows in view of (5.12).

Precisely this very technique would apply to prove

THEOREM 16. *Let* b_n *satisfy* (5.12) *for every integer* $n \geqq 2$ *and* $0 \leqq \alpha < 1$, *and suppose that the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{T}_1\{b_n, z_0\}$. *Then* $f(z)$ *is starlike of order* α, *with equality in* (5.12) *for the function*

$$f(z) = a_1 z - \frac{1 - \alpha}{1 + (1-\alpha)(1-2z_0)} z^2. \tag{5.17}$$

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