ERROR ESTIMATES FOR GALERKIN APPROXIMATIONS TO THE PERIODIC SCHRÖDINGER-POISSON SYSTEM

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by

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Abstract

We establish convergence rates for a Galerkin approximation to the periodic Schrödinger-Poisson problem in the unit cube $[0,1]^3$. The error estimates transform into corresponding L^∞ -error estimates for the Wigner distribution function.

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1. INTRODUCTION

We are concerned with the three-dimensional periodic Schrödinger-Poisson problem in the unit cube $Q = [0,1]^3$; this problem has been studied in Ref. 1. The equations are

$$i\,\psi_{m,t} = -\frac{1}{2}\Delta\,\psi_m + V(\Psi)\,\psi_m \tag{1.1a}$$

$$\Delta V = q(x) - n \tag{1.1b}$$

$$n = \Sigma\,\lambda_m\,|\psi_m|^2 \tag{1.1c}$$

For simplicity, we have set $\hbar = 1$ and the particle mass $M = 1$. We also assume initial charge neutrality $\int_Q n(x)\,dx = \int_Q q(x)\,dx$. In the sequel, we confine our discussion to the case $q \equiv 1$, but our method generalizes to any sufficiently smooth, one-periodic q representing the density of positive background charge. We also assume, without restricting the generality, that $\lambda_m > 0$ for all m, and that $\Sigma\,\lambda_m = 1$.

The Schrödinger-Poisson system (SP), and its relation to the Wigner-Poisson system (WP), are discussed in detail in Refs. 2-6. The periodic Vlasov-Poisson is treated in Refs. 7 and 8. In Ref. 9, the quantum periodic case is studied.

(1.1a - 1.1c) are complemented by the initial conditions

$$\psi_m(x,0) = \varphi_m(x) \tag{1.1d}$$

and periodic boundary conditions in Q. The φ_m's form an orthonormal system in $L^2(Q)$.

In Ref. [1], it was shown by a suitable Galerkin approximation that Eqns. (1.1a-d) possess a unique, global-in-time, strong solution in the space

$$C([0,T]; X^2).$$

Here, we define

$$X^k = \{\Psi = (\psi_m);\;\; \psi_m \in H^k_{\text{loc}}(\mathbb{R}^3), \quad \forall x \in Q$$

$$\forall m \in \mathbb{N} \quad \forall \ell \in \mathbb{Z}^3 \;\; \psi_m(x+\ell) = \psi_m(x)\}$$

with the norm

$$\|\Psi\|_{X^k}^2 = \sum_{\substack{m\in\mathbb{N}\\|\alpha|\le k}} \lambda_m \, \|D^\alpha \psi_m\|_{L^2(Q)}^2 \, .$$

We assume that $\Phi = (\varphi_m) \in X^2$, a necessary assumption for the existence proof of Ref. 1. Instead of X^0 we will just write X.

The objective of this paper is to obtain error estimates for Galerkin approximations of solutions of system (1.1a-d) obtained in Ref. 1. The Galerkin sequence $(\Psi^{(N)})_{N\in\mathbb{N}}$ is there defined by

$$\psi_m^{(N)}(x,t) = \sum_{|k|\le N} d_{m,k}^{(N)}(t)\, h_k(x), \tag{1.2}$$

where $h_k(x) = e^{2\pi i\, k\cdot x}$, $k \in \mathbb{Z}^3$, and

$$\left(\left[i\,\partial_t\, \psi_m^{(N)}(x,t) + \frac{1}{2}\,\Delta\psi_m^{(N)} - V^{(N)}(\Psi^{(N)})\,\psi_m^{(N)} \right], h_k \right) = 0 \tag{1.3}$$

for $|k| \le N$. Here,

$$V^{(N)}\left(\Psi^{(N)}\right) = \frac{1}{4\pi^2} \sum_{\substack{|k|\le 2N\\ k\ne 0}} \frac{n_k^{(N)}}{k^2}\, h_k \, , \tag{1.4}$$

where we define:

$$\begin{aligned} n^{(N)}(x,t) &= \sum_m \lambda_m \, |\psi_m^{(N)}(x,t)|^2 \\ &= \sum_{|k|\le 2N} n_k^{(N)}\, h_k(x). \end{aligned} \tag{1.5}$$

A short calculation shows

$$n_k^{(N)} = \sum_m \lambda_m \sum_{\substack{|\ell|\le N\\ |\ell-k|\le N}} d_{m,\ell}^{(N)}\, \bar d_{m,\ell-k}^{(N)} \, , \tag{1.6}$$

$$(|k| \le 2N, \quad k\ne 0, \text{ and } n_0^{(N)} = 1).$$

$V^{(N)}$ is the explicit solution of

$$\Delta V^{(N)} = 1 - n^{(N)} \tag{1.7}$$

with periodic boundary conditions.

From Eqn. (1.3) and the subsequent formulas one sees that the coefficients $d_{m,k}^{(N)}$ in (1.2) must satisfy the first order differential system

$$\dot{d}_{m,\ell}^{(N)}(t) = -\,2\pi^2\, i\ell^2\, d_{m,\ell}^{(N)}(t) - \frac{i}{4\pi^2} \sum_{\substack{0<|k|\leq 2N \\ |\ell-k|\leq N}} \frac{n_k^{(N)}}{k^2}\, d_{m,\ell-k}^{(N)}(t) \tag{1.8}$$

($|\ell| \leq N$) subject to the initial conditions

$$d_{m,\ell}^{(N)}(0) = (\varphi_m,\, h_\ell). \tag{1.9}$$

It is proved in Ref. 1 that for each $N \in \mathbb{N}$ the system (1.8-9) has a global unique classical solution such that $\|\Psi^{(N)}(\cdot,t)\|_{X^0} = \|\phi\|_{X^0} = 1$ (see Ref. 1, Lemma 2.1), and that the conservation law

$$\frac{d}{dt}\int_Q \left\{|\nabla\Psi^{(N)}(t)|^2 + |\nabla V^{(N)}(t)|^2\right\} dx = 0$$

(where $|\nabla\Psi^{(N)}(t)|^2 \underset{\text{def.}}{=} \sum_{m\in\mathbb{N}} \lambda_m\, |\nabla\psi_m^{(N)}|^2$) holds.

2. SOME AUXILIARY RESULTS

We begin by proving a generalization of a Lemma in Ref. 10 which is crucial for our error estimates. Recall that we are in three dimensions.

LEMMA 2.1. *Let $v \in H_p^k(Q)$, $k \in \mathbb{N}$, $k \geq 1$. Then, if*

$$v_n = \sum_{|j|\leq n} (v,\, h_j)\, h_j\,, \quad h_j = e^{2\pi i j\cdot x}, \\ \|v - v_n\|_{L^2(Q)}^2 \leq 3(2\pi)^{-2k}\left(\frac{3}{n}\right)^{2k} \|v\|_{H^k(Q)}^2. \tag{2.1}$$

Proof. Assume $v \in C_p^\infty(Q)$ (p for 1-periodic). By Pythagoras' Theorem,

$$\|v - v_n\|_{L^2(Q)}^2 = \sum_{|j|>n} |(v,\, h_j\,)|^2. \tag{2.2}$$

If $|j_1| + |j_2| + |j_3| > n$, there must be one index larger than $\frac{n}{3}$, so the sum on the right of (2.2) is bounded by

$$\sum_{j_2, j_3} \sum_{|j_1| > \frac{n}{3}} |(v, h_j)|^2 \tag{2.3}$$

plus two other sums of the same kind (with $|j_2| > \frac{n}{3}$, $|j_3| > \frac{n}{3}$ respectively). Writing

$$e^{-2\pi i j \cdot x} = \left(\partial_{x_1}^k \, e^{-2\pi i j \cdot x}\right) \cdot \frac{1}{(-2\pi i j_1)^k},$$

integrating by parts $k-1$ times, using the periodicity assumption and the continuity, the expression (2.3) can be estimated by

$$\sum_{j_2, j_3} \sum_{|j_1| > \frac{n}{3}} \left| \int_Q \partial_{x_1}^k \, v(x) \, e^{-2\pi i j \cdot x} \, dx \right|^2 \frac{1}{|2\pi \, j_1|^{2k}}$$
$$\leq \frac{1}{|2\pi \, \frac{n}{3}|^{2k}} \sum_{|j_1| > \frac{n}{3}} \sum_{j_2, j_3} \left| \left(\partial_{x_1}^k \, v, \, h_j\right) \right|^2.$$

The sum, being a sum of squares of Fourier coefficients, is bounded by $\|v\|^2_{H^k(Q)}$. The result follows for $v \in C_p^\infty(Q)$. It extends to $v \in H_p^k(Q)$ by a density argument. ∎

COROLLARY 2.2. *Let $S = [0, T]$ and $w \in L^2(S, H_p^k(Q))$, such that for all $t \in S$ $D^\alpha w(\cdot, t)$, $|\alpha| \leq k-1$, is periodic, and let $w_n(\cdot, t) = \sum_{|k| \leq n} \left(w(\cdot, t), \, h_k\right) h_k$. Then for $C = 3(2\pi)^{-2k} \, 3^{2k}$,*

$$\|w - w_n\|^2_{L^2\left(S, L^2(Q)\right)} \leq \frac{C}{n^{2k}} \, \|w\|^2_{L^2(S, H^k)} \,. \tag{2.4}$$

LEMMA 2.3. *Let $W^{(N)}(\cdot, t) = \Psi(\cdot, t) - \Psi^{(N)}(\cdot, t)$ and let*

$$\hat{\Psi}_N = \left(\sum_{|j| \leq N} \left(\psi_m(\cdot, t), \, h_j\right) h_j \right)_{m \in \mathbb{N}}$$

be the projection of the true solution onto the span of $\{h_j \,, |j| \leq N\}$. Then there are constants C and C_T (independent of N, but C_T depending on T) such that

$$\begin{aligned} \frac{d}{dt} \left\| W^{(N)}(\cdot, t) \right\|^2_X \leq C \left\| \Psi(\cdot, t) - \hat{\Psi}_N(\cdot, t) \right\|_X \\ + C_T \left\| W^{(N)}(\cdot, t) \right\|^2_X. \end{aligned} \tag{2.5}$$

Proof. Let $D_m^{(N)} = i\,\psi_{m,t}^{(N)} + \frac{1}{2}\,\Delta\psi_m^{(N)} - V^{(N)}\big(\Psi^{(N)}\big)\,\psi_m^{(N)}$ (*cf.* 1.3). Then, as $\hat{\psi}_{N,m} - \psi_m^{(N)} \in \text{span}\{h_j\,;\ |j| \le N\}$, by the definition of the Galerkin approximation (1.3) we can write

$$\begin{aligned}\big(D_m^{(N)},\, \psi_m - \psi_m^{(N)}\big) &= \big(D_m^{(N)},\, \psi_m - \hat{\psi}_{N,m} + \hat{\psi}_{N,m} - \psi_m^{(N)}\big) \\ &= \big(D_m^{(N)},\, \psi_m - \hat{\psi}_{N,m}\big)\,.\end{aligned} \tag{2.6}$$

Let $w_m^{(N)}$ be the m-th component of $W_m^{(N)}$, *i.e.* $w_m^{(N)} = \psi_m^{(N)} - \psi_m$. Note that $\frac{\partial}{\partial t}\,\psi_m^{(N)} \in \text{span}\{h_k;\ |k| \le N\}$, because the h_k's are independent of t, and that $\Delta\,\psi_m^{(N)} \in \text{span}\{h_k;\ |k| \le N\}$, because the h_k's are eigenfunctions of Δ. Also, by construction

$$\psi_m - \hat{\psi}_{N,m} \in \Big(\text{span}\{h_j\,;\ |j| \le N\}\Big)^{\perp},$$

and therefore, from (2.6)

$$\Big(D_m^{(N)},\, w_m^{(N)}\Big) = \Big(-V^{(N)}\big(\Psi^{(N)}\big)\,\psi_m^{(N)},\ \psi_m - \hat{\psi}_{N,m}\Big). \tag{2.7}$$

Using the fact that $\Psi(x,t)$ is the exact solution of (1.a-d), we can rewrite (2.7) as

$$\begin{aligned}\Big(i\,\psi_{m,t}^{(N)} + \frac{1}{2}\,\Delta\psi_m^{(N)} - V^{(N)}\big(\Psi^{(N)}\big)\,\psi_m^{(N)} \\ - i\,\psi_{m,t} - \frac{1}{2}\,\Delta\psi_m + V(\Psi)\psi_m, w_m^{(N)}\Big) \\ = -\Big(V^{(N)}\big(\Psi^{(N)}\big)\,\psi_m^{(N)},\ \psi_m - \hat{\psi}_{N,m}\Big),\end{aligned}$$

i.e.,

$$\begin{aligned}\Big(-i\,w_{m,t}^{(N)} - \frac{1}{2}\,\Delta w_m^{(N)},\, w_m^{(N)}\Big) &= \Big(V^{(N)}\big(\Psi^{(N)}\big)\psi_m^{(N)} - V(\Psi)\psi_m,\, w_m^{(N)}\Big) \\ &\quad - \Big(V^{(N)}\big(\psi^{(N)}\big)\psi_m^{(N)},\, \psi_m - \hat{\psi}_{N,m}\Big)\end{aligned} \tag{2.8}$$

Now

$$\begin{aligned}\frac{d}{dt}\,\big\|w_m^{(N)}\big\|_{L^2}^2 &= 2\Re\int \overline{w_m^{(N)}}\; w_{m,t}^{(N)}\, dx \\ &= 2\Im\int \overline{w_m^{(N)}}\; i\,w_{m,t}^{(N)}\, dx.\end{aligned}$$

As $\Im \int \Delta\, w_m^{(N)} \cdot w_m^{(N)}\, dx = 0$, it follows from the previous identity (2.8) that

$$\begin{aligned}\frac{d}{dt}\,\Big\|w_m^{(N)}\Big\|_{L^2}^2 &= 2\,\Im\int \Big(V(\Psi)\psi_m - V^{(N)}\big(\Psi^{(N)}\big)\psi_m^{(N)}\Big)\,\overline{w_m^{(N)}}\, dx \\ &\quad + 2\,\Im\int V^{(N)}\big(\Psi^{(N)}\big)\psi_m^{(N)}\,\overline{\big(\psi_m - \hat{\psi}_{N,m}\big)}\, dx,\end{aligned}$$

i.e.

$$\frac{d}{dt}\left\|w_m^{(N)}\right\|_{L^2}^2 = 2\Im\int\left(V(\Psi) - V^{(N)}(\Psi^{(N)})\right)\psi_m\,\overline{w_m^{(N)}}\,dx$$

$$+ 2\Im\int V^{(N)}(\Psi^{(N)})w_m^{(N)}\cdot\overline{w_m^{(N)}}\,dx \tag{2.9}$$

$$+ 2\Im\int V^{(N)}(\Psi^{(N)})\psi_m^{(N)}\,\overline{(\psi_m - \hat{\psi}_{N,m})}\,dx.$$

The second term on the right vanishes as $V^{(N)}(\Psi^{(N)})$ is real (see Eqn. 1.7).

We remark that $\|V^{(N)}(\Psi^{(N)})\|_{L^\infty} \leq C$, where C depends only on $\|\Phi\|_X$ (see Ref. 1, Lemma 2.3). The third term in (2.9) can therefore be estimated by

$$C\|\psi_m^{(N)}\|_{L^2}\,\|\psi_m - \hat{\psi}_{N,m}\|_{L^2},$$

(C is a generic constant) and as

$$\|\psi_m^{(N)}(\cdot,t)\|_{L^2} = \|\psi_m^{(N)}(\cdot,0)\|_{L^2}$$

$$= \left(\sum_{|j|\leq N}\left|(\varphi_m, h_j)\right|^2\right)^{\frac{1}{2}} \leq \|\varphi_m\|_{L^2} = 1,$$

we get a bound

$$C\|\psi_m - \hat{\psi}_{N,m}\|_{L^2}$$

for this term.

To estimate the first term on the right of (2.9), note that by (1.4)

$$\|V(\Psi) - V^{(N)}(\Psi^{(N)})\|_{L^\infty} \leq C\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}}\frac{\left|n_k - n_k^{(N)}\right|}{k^2},$$

where n_k is defined by

$$n(x,t) = \sum_{k\in\mathbb{Z}^3} n_k(t)\,h_k.$$

Thus

$$\begin{aligned}
&\left\|V(\Psi) - V^{(N)}(\Psi^{(N)})\right\|_{L^\infty} \\
&\le C\left(\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{1}{(k^2)^2}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}^3} \left|n_k - n_k^{(N)}\right|^2\right)^{\frac{1}{2}} \\
&\le C\left(\sum_{k\in\mathbb{Z}^3} \left|n_k - n_k^{(N)}\right|^2\right)^{\frac{1}{2}} \\
&= C\left\|n - n^{(N)}\right\|_{L^2} \\
&= C\sqrt{\int_Q \left|\Sigma\,\lambda_r\left(|\psi_r|^2 - |\psi_r^{(N)}|^2\right)\right|^2 dx} \\
&= C\sqrt{\int_Q \left|\Sigma\,\lambda_r\left(\left(\psi_r - \psi_r^{(N)}\right)\overline{\psi_r} + \left(\overline{\psi_r} - \overline{\psi_r^{(N)}}\right)\psi_r^{(N)}\right)\right|^2 dx} \\
&\le C\left(\int_Q (\Sigma\,\lambda_r|\psi_r|^2)(\Sigma\,\lambda_r|\psi_r - \psi_r^{(N)}|^2)\,dx\right)^{\frac{1}{2}} \\
&\quad + C\left(\int_Q (\Sigma\,\lambda_r|\psi_r^{(N)}|^2)(\Sigma\,\lambda_r|\psi_r - \psi_r^{(N)}|^2)\,dx\right)^{\frac{1}{2}} \\
&\le C_T\left\|\Psi - \Psi^{(N)}\right\|_X,
\end{aligned}$$

as

$$n = \Sigma\,\lambda_r|\psi_r|^2 \qquad \text{and} \qquad n^{(N)} = \Sigma\,\lambda_r\left|\psi_r^{(N)}\right|^2$$

are both bounded in the L^∞-norm on $[0,T]$. See Ref. 1, Lemma 3.2.

Summarizing, we have proven that

$$\begin{aligned}
&\Sigma\,\lambda_m\left|\int\left(V(\Psi) - V^{(N)}(\Psi^{(N)})\right)\psi_m\,\overline{w_m^{(N)}}\,dx\right| \\
&\quad\le C_T\left\|\Psi - \Psi^{(N)}\right\|_X \sum_m \int\left|\lambda_m\,\psi_m\,\overline{w_m^{(N)}}\right| dx \\
&\quad\le C_T\left\|\Psi - \Psi^{(N)}\right\|_X \left\|W^{(N)}\right\|_X = C_T\left\|W^{(N)}\right\|_X^2,
\end{aligned}$$

and as

$$\Sigma\,\lambda_m\left\|\psi_m - \hat\psi_{N,m}\right\|_{L^2} \le C\left\|\Psi - \hat\Psi_N\right\|_X,$$

it follows that

$$\frac{d}{dt}\left\|W^{(N)}(t,\cdot)\right\|_X^2 = \frac{d}{dt}\sum_m \lambda_m \left\|w_m^{(N)}(t,\cdot)\right\|_{L^2}^2$$

$$\leq C\left\|\Psi - \hat{\Psi}_N\right\|_X + C_T\left\|W^{(N)}(t,\cdot)\right\|_X^2 .$$

This completes the proof of Lemma 2.3.

3. THE MAIN THEOREM

We are now ready for the main result of this paper.

THEOREM 3.1. *If $\Phi \in X^2$, then the Galerkin sequence $\Psi^{(N)}$ associated with Φ as initial value (see (1.9)) satisfies, on $[0,T]$, the error estimates*

$$\left\|\Psi - \Psi^{(N)}\right\|_{L^\infty([0,T];X)} \leq \frac{C_T}{N},$$

where C_T depends on T and on $\|\Phi\|_{X^1}$.

Proof. We denote generic constants by C and generic constants depending on T by C_T. Note that $\Phi^{(N)} = \hat{\Phi}_N$, *i.e.* the Fourier series for Φ gives the initial data for the Galerkin approximation. From Lemma 2.3,

$$\left\|W^{(N)}(\cdot,t)\right\|_X^2 \leq \left\|W^{(N)}(\cdot,0)\right\|_X^2$$

$$+ C\int_0^t \left\|\Psi(\cdot,\tau) - \Psi(\cdot,\tau)\right\|_X d\tau$$

$$+ C_T\int_0^t \left\|W^{(N)}(\cdot,\tau)\right\|_X^2 d\tau .$$

By Lemma 2.1,

$$\left\|W^{(N)}(\cdot,0)\right\|_X^2 \leq \frac{C}{N^4}\|\Phi\|_{X^2}^2$$

and

$$C\int_0^t \left\|\Psi(\cdot,t) - \hat{\Psi}_N(\cdot,t)\right\|_X dt$$

$$\leq \frac{C}{N^2}\int_0^t \|\Psi(\cdot,t)\|_{X^2}\, dt$$

$$\leq \frac{C_T}{N^2}$$

(because $\|\Psi(\cdot,t)\|_{X^2}$ is uniformly bounded on $[0,T)$, see Ref. 1.) So

$$\left\|W^{(N)}(\cdot,t)\right\|_X^2 \le \frac{C}{N^4}\,\|\phi\|_{X^2}^2 + \frac{C_T}{N^{\frac{3}{2}}}$$

$$+\, C_T \int_0^t \left\|W^{(N)}(\cdot,\tau)\right\|_X^2 d\tau\,,$$

and by Gronwall's Lemma

$$\left\|W^{(N)}(\cdot,t)\right\|_X^2 \le \frac{C_T}{N^2}\left(1+\|\phi\|_{X^1}^2\right) e^{C_T t}.$$

This completes the proof.

Now we look at the periodic Wigner-Poisson problem (see Ref. 1) which is the system of equations

$$\partial_t \rho_{w,n} + v_n\,\nabla_x \rho_{w,n} - i\,\Theta(V)\rho_{w,n} = 0, \tag{3.1}$$

$$\Delta V = 1 - n(x,t), \tag{3.2}$$

$$\rho_{w,n}(x,0) = \rho_{w,nI}(x) \tag{3.3}$$

where $(\rho_{w,n})$ is the sequence of Wigner functions given by $\rho_w(x,v_n,t)$ ($v_n = 2\pi n$, $n \in \mathbb{Z}^3$), $\Theta(V)$ is the pseudo-differential operator

$$\Theta(V)\rho_{w,k} = \sum_{k'} \int_{Q'} \left[V\left(x+\frac{\eta}{2},t\right) - V\left(x-\frac{\eta}{2},t\right)\right]$$

$$\times\, \rho_{w,k'}(x,t)\cdot e^{2\pi i(k-k')} d\eta$$

$$= \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\psi_m\left(x-\frac{z}{2},t\right)\overline{\psi_m}\left(x+\frac{z}{2},t\right)$$

$$\times \left[V\left(x+\frac{z}{2},t\right) - V\left(x-\frac{z}{2},t\right)\right] dz,$$

$Q' = [-1,1]^3$. Here we used that $\rho_{w,k}(x,t)$ is given by the Wigner transform

$$\rho_{w,k}(x,t) = \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\psi_m\left(x-\frac{z}{2},t\right)\overline{\psi_m}\left(x+\frac{z}{2},t\right) dz. \tag{3.4}$$

We consider the Wigner transform of the Galerkin sequence

$$\rho_{w,k}^{(N)}(x,t) = \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\,\psi_m^{(N)}\left(x-\frac{z}{2},t\right)\overline{\psi_m^{(N)}}\left(x+\frac{z}{2},t\right) dz. \tag{3.5}$$

This can be computed explicitly to give

$$\rho_{w,k}^{(N)}(x,t) = \Sigma\,\lambda_m \sum_{|\ell|\le N, |2k-\ell|\le N} d_{\ell,m}^{(N)}\, \bar{d}_{2k-\ell,m}^{(N)}(t)\, e^{4\pi i x(\ell-k)} \tag{3.6}$$

where $d_{\ell,m}^{(N)}(t)$ are the coefficients of the Galerkin approximation $\psi_m^{(N)}$. Finally, we write

$$\begin{aligned}
&\rho_{w,k}(x,t) - \rho_{w,k}^{(N)}(x,t) \\
&\quad = \frac{1}{8}\sum \lambda_m \int_{Q'} \left(\psi_m\left(x-\frac{z}{2},t\right)\bar{\psi}_m\left(x+\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\psi_m^{(N)}\left(x+\frac{z}{2},t\right)\right) \\
&\qquad\quad \cdot e^{2\pi i k\cdot z}\,dz \\
&\quad = \frac{1}{8}\sum_m \lambda_m \int_{Q'} \left\{\left(\psi_m\left(x-\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right)\bar{\psi}_m\left(x+\frac{z}{2},t\right)\right) \\
&\qquad\quad + \left(\bar{\psi}_m\left(x+\frac{z}{2},t\right) - \bar{\psi}_m^{(N)}\left(x+\frac{z}{2},t\right)\right)\psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right\} e^{2\pi i k\cdot z}\,dz.
\end{aligned} \tag{3.7}$$

By the Cauchy-Schwarz inequality and periodicity we have

$$\begin{aligned}
&\int_{Q'} \left|\left(\psi_m\left(x-\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right)\cdot\bar{\psi}_m\left(x+\frac{z}{2},t\right)\right| dz \\
&\qquad \le \sqrt{8}\,\left\|\psi_m(\cdot,t) - \psi_m^{(N)}(\cdot,t)\right\|_{L^2(Q)}\sqrt{8}\,\left\|\psi_m\right\|_{L^2(Q)}
\end{aligned} \tag{3.8}$$

and a similar estimate for the other term in the right hand side of (3.7). By using Theorem 3.1 and (3.8) we arrive at

$$\left\|\rho_{w,k} - \rho_{w,k}^{(N)}\right\|_{L^\infty(Q\times S)} \le \frac{C_T}{N}$$

with the same constant C_T as in Theorem 3.1 and $S = [0,T]$.

Let us assume now that the initial function of the Wigner-Poisson system (3.1)-(3.3) is the Wigner transform of a suitable initial function $\Phi \in X^2$ of the SP system (1.1a)-(1.1d). Let us call this assumption condition (C).

It is well known (see Ref. 1) that the solution of the Wigner-Poisson system (3.1)-(3.3) is given by (3.4) where $\Psi = (\psi_m)$ is the unique strong 1-periodic solution of (1.1a)-(1.1d). Thus we have proved

THEOREM 3.2. *Let condition* (C) *be satisfied for the initial data* $(\rho_{w,k,I})$. *Then for the Wigner transform* $\rho_{w,k}^{(N)}$ *of the Galerkin sequence* $(\Psi^{(N)})$ *we have*

$$\|\rho_{w,k} - \rho_{w,k}^{(N)}\|_{L^\infty(Q\times S)} \leq \frac{C_T}{N}$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N}$ *where* $(\rho_{w,k})$ *is the unique strong 1-periodic solution sequence of the WP system, and* C_T *is the constant from Theorem* 3.1.

Remark. If the solution Ψ of the (SP) system (1.1a)-(1.1d) is actually a strong X^k-solution, then one can prove similarly the error estimate

$$\|\Psi - \Psi^{(N)}\|_{L^\infty([0,T],X^k)} \leq \frac{C_T}{N^{\frac{k}{2}}}.$$

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