

Criticality Concepts for Paired Domination in Graphs

by

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is a *paired dominating set* if every vertex in $V - S$ is adjacent to some vertex in S and $\langle S \rangle$ yields a perfect matching. The smallest cardinality of a paired dominating set is the paired domination number, denoted $\gamma_{pr}(G)$. A graph is said to be *paired domination edge-critical*, or simply γ_{pr} -edge-critical, if $\gamma_{pr}(G + e) < \gamma_{pr}(G)$ for any $e \in E(\overline{G})$. Similarly, a graph is said to be *paired domination vertex-critical*, or γ_{pr} -vertex-critical, if $\gamma_{pr}(G - v) < \gamma_{pr}(G)$ for any $v \in V(G)$. More generally, for a graph theoretic parameter π , a graph G is called π -critical if, under some operation, the value $\pi(G)$ changes for any action under the operation. In addition, the graph is called π -stable if, under some operation, the value $\pi(G)$ remains the same for any action under the operation.

We open by establishing the existence or nonexistence of various types of critical and stable graphs. We then concentrate on γ_{pr} -edge-critical graphs, and in particular, those with $\gamma_{pr} = 4$. We give a characterisation of γ_{pr} -edge-critical graphs with $\gamma_{pr} = 4$ that contain a cut-vertex. We next shift attention to γ_{pr} -vertex-critical graphs. In particular, we show that G is a γ_{pr} -vertex-critical graph if every block of G is a γ_{pr} -vertex-critical graph. For a fixed value of γ_{pr} , we determine bounds on the diameter for both γ_{pr} -edge-critical graphs and γ_{pr} -vertex-critical graphs. Finally, we close with a presentation of open problems in the area.

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Chapter 1

Introduction, Definitions and Summary of the Problem

1.1 Terminology and Definitions

Let $G = (V(G), E(G)) = (V, E)$ be a graph with order $|V| = n$. The *open neighbourhood* of a vertex v , denoted by $N(v)$, is the set of vertices adjacent to v . That is, $N(v) = \{w : vw \in E\}$. The *closed neighbourhood* of a vertex v , denoted by $N[v]$, is the open neighbourhood of v and v itself. That is, $N[v] = N(v) \cup \{v\}$. For $S \subseteq V$, similar definitions are given for the *open neighbourhood* and *closed neighbourhood* of S , denoted $N(S)$ and $N[S]$, respectively. That is, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v] = N(S) \cup S$. For $x \in S$, the *private neighbourhood of x relative to S* is the set of all vertices in the closed neighbourhood of x that are not in the closed neighbourhood of

$S - \{x\}$, and is denoted by $\text{pn}(x, S)$. That is, $\text{pn}(x, S) = N[x] - N[S - \{x\}]$. If $y \in \text{pn}(x, S)$ then y is called a *private neighbour of x relative to S* , or simply a *private neighbour of x* if confusion is unlikely. For a set $S' \subseteq S$, a similar definition is given for the *private neighbourhood of S' relative to S* , denoted by $\text{pn}(S', S)$. That is, $\text{pn}(S', S) = N[S'] - N[S - S']$.

A set S is a *dominating set*, respectively *total dominating set*, if $N[S] = V(G)$, respectively $N(S) = V(G)$. The cardinality of a minimum dominating set, respectively minimum total dominating set, is called the *domination number of G* , respectively *total domination number of G* , and is denoted by $\gamma(G)$, respectively $\gamma_t(G)$. Notice that the domination number exists for any graph G as $V(G)$ is a dominating set; similarly, the total domination number exists for any graph without isolated vertices. If S is a dominating set (total dominating set) with cardinality $\gamma(G)$ ($\gamma_t(G)$), then S is called a γ -set (γ_t -set). A set S is a *paired dominating set* if S is a dominating set such that $\langle S \rangle$ has a perfect matching. Furthermore, we say that S *pairwise dominates G* if S is a paired dominating set of G . The minimum cardinality of a paired dominating set is called the *paired domination number of G* and is denoted by $\gamma_{pr}(G)$. Notice that the paired domination number exists for any graph G without isolated vertices as the end-vertices of any maximum matching of G is a paired dominating set. Also notice a parity restriction on the cardinality of any paired dominating set, that is, any paired dominating set has even cardinality. If S is a paired dominating set with cardinality $\gamma_{pr}(G)$, then S is called a γ_{pr} -set. Since any paired dominating set is a total dominating set,

which in turn is a dominating set, it is obvious that $\gamma(G) \leq \gamma_t(G) \leq \gamma_{pr}(G)$ for any graph G .

A set S is a *minimal paired dominating set* if S is a paired dominating set and no proper subset of S is a paired dominating set. For a paired dominating set S with vertices $u, v \in S$, we say that uv is a *pair* in S if for some perfect matching M of $\langle S \rangle$, uv is an edge of M . Furthermore, we say that u is the *partner* of v in S if uv is a pair in S . A paired dominating set S *precisely dominates* $G - X$ if S is a paired dominating set of $G - X$, but S does not dominate any vertex in X .

If G is a graph with $\text{diam}(G) = k$ and $d(u, v) = k$, then we say that u and v are *peripheral vertices* and a shortest $u - v$ path is called a *diametrical path*. A vertex $v \in V(G)$ with $\text{deg}(v) = 1$ is called an *end-vertex*, and a vertex adjacent to an end-vertex is called a *support vertex*. If some vertex $v \in V(G)$ has degree zero, then we say that v is an *isolate*. A graph that contains no isolates is called *isolate-free*.

For a pair of vertices $u, v \in V(G)$ we define $G.uv$ as the graph obtained by identifying vertex u with vertex v . In $G.uv$ the vertex (uv) is the identified vertex. Thus $G.uv$ may be viewed as the graph obtained from G by deleting the vertices v and u and appending a new vertex (uv) that is adjacent to all the vertices in $(N(u) \cup N(v)) - \{u, v\}$. In the case that $uv \in E(G)$, $G.uv$ is the graph obtained by contracting the edge uv .

Domination-related concepts not defined here can be found in [4] and [5], whereas we follow [2] for general graph-theoretic notation and terminology.

We now define several criticality concepts and their “opposites”, called stability concepts, with respect to some graph parameter π . For our purposes we define $\pi(G) = \infty$ if no finite value of $\pi(G)$ exists.

For the parameter π , a graph G is

π^- - critical	if $\pi(G - v) < \pi(G)$ for all $v \in V_G$
π^+ - critical	if $\pi(G - v) > \pi(G)$ for all $v \in V_G$
π - stable	if $\pi(G - v) = \pi(G)$ for all $v \in V_G$
π - edge - critical	if $\pi(G + e) < \pi(G)$ for all $e \in E_{\overline{G}} \neq \phi$
π^+ - edge - critical	if $\pi(G + e) > \pi(G)$ for all $e \in E_{\overline{G}} \neq \phi$
π - edge - stable	if $\pi(G + e) = \pi(G)$ for all $e \in E_{\overline{G}} \neq \phi$
π - ER - critical	if $\pi(G - e) > \pi(G)$ for all $e \in E_G \neq \phi$
π^- - ER - critical	if $\pi(G - e) < \pi(G)$ for all $e \in E_G \neq \phi$
π - ER - stable	if $\pi(G - e) = \pi(G)$ for all $e \in E_G \neq \phi$
π - dot - critical	if $\pi(G.uv) < \pi(G)$ for all $uv \in E_G \neq \phi$
π^+ - dot - critical	if $\pi(G.uv) > \pi(G)$ for all $uv \in E_G \neq \phi$
π - dot - stable	if $\pi(G.uv) = \pi(G)$ for all $uv \in E_G \neq \phi$
π - totally - dot - critical	if $\pi(G.uv) < \pi(G)$ for all $u, v \in V_G$
π^+ - totally - dot - critical	if $\pi(G.uv) > \pi(G)$ for all $u, v \in V_G$
π - totally - dot - stable	if $\pi(G.uv) = \pi(G)$ for all $u, v \in V_G$

For the remainder of this work, the parameter we focus on is the paired domination number, γ_{pr} . In particular, we concentrate on two types of critical graphs: γ_{pr} -edge-critical graphs and γ_{pr} -vertex-critical graphs. A γ_{pr} -edge-critical graph (γ_{pr} -vertex-critical graph) G with $\gamma_{pr}(G) = k$ is also called a k_{pr} -edge-critical graph (k_{pr} -vertex-critical graph).

1.2 Minimal Paired Dominating Sets

Let P be a graph theoretic property. We say that P is *hereditary* if for any graph G (or any $S \subseteq V(G)$, or any $F \subseteq E(G)$), whenever G (or S or F) has

Property P , then every subgraph of G (or subset of S or F) has Property P . We furthermore say that P is *superhereditary* if, for any graph G (or any $S \subseteq V(G)$, or any $F \subseteq E(G)$), whenever G (or S or F) has Property P , then every supergraph of G (or superset of S or F) has Property P .

In particular, the property that $D \subseteq V(G)$ is a dominating set of G is superhereditary. So, to check whether D is a minimal dominating set of G , we only have to check that $D - \{d\}$ is not a dominating set for each $d \in D$. If this is true, it also follows that no smaller set is dominating, because if (say) $D - \{d, d'\}$ is a dominating set, then by the superhereditary property of dominating sets, $D - \{d\}$ is a dominating set, a contradiction.

However, the property that $D \subseteq V(G)$ is a paired dominating set is not superhereditary because if D is a paired dominating set, then $D \cup \{d\}$ is not a paired dominating set for all $d \in V(G) - D$; and $D \cup \{d, d'\}$ is only a paired dominating set if $\langle D \cup \{d, d'\} \rangle$ has a perfect matching. In what follows we attempt to obtain some form of a superhereditary property for paired dominating sets.

Proposition 1.1 *Let G be a connected graph with a perfect matching. If there exists a set $S \subseteq V(G)$ with even cardinality such that $G - S$ has a perfect matching, then there exist $u, v \in S$ such that $G - (S - \{u, v\})$ has a perfect matching.*

Proof. Let M' be a perfect matching of $G' = G - S$. If $\langle S \rangle$ contains an edge $e = uv$, then $M' \cup \{e\}$ is a perfect matching of $G - (S - \{u, v\})$. Assume

therefore that S is independent and say $S = \{s_1, \dots, s_t\}$. Since G has a perfect matching M , there exists a vertex $u_1 \in V(G')$ such that $s_1u_1 \in M$. Since M' is a perfect matching of G' , there exists a vertex $u_2 \in V(G')$ such that $u_1u_2 \in M'$; necessarily $u_1u_2 \in M' - M$. Since G has the perfect matching M , there exists a vertex $u_3 \in V(G) - \{s_1\}$ such that $u_2u_3 \in M$. If $u_3 \in S$, say $u_3 = s_3$, then $(M' - \{u_1u_2\}) \cup \{s_1u_1, u_2s_2\}$ is a perfect matching of $G - (S - \{s_1, s_2\})$. If $u_3 \notin S$, then $u_3 \in V(G') - \{u_1, u_2\}$ and since G' has the perfect matching M' , there exists a vertex $u_4 \in V(G) - S$ such that $u_3u_4 \in M'$.

Now since G has the perfect matching M , there exists a vertex u_5 such that $u_4u_5 \in M$. We can repeat the same argument as above as either $u_5 \in S$ or $u_5 \in V(G') - \{u_1, u_2, u_3, u_4\}$.

Because of the finiteness of G and because of the parity restriction on $|V(G')|$ we cannot continue indefinitely with the second case of the above argument. Hence eventually we encounter an edge $u_{2i}u_{2i+1} \in M$ with $u_{2i+1} \in S$, say $u_{2i+1} = s_k$. Then

$$(M - \{u_1u_2, \dots, u_{2i-1}u_{2i}\}) \cup \{s_1u_1, u_2u_3, \dots, u_{2i}s_k\}$$

is a perfect matching of $G - (S - \{s_1, s_t\})$. ■

Proposition 1.2 *Let G be a connected graph with a perfect matching. If there exists a set $S \subseteq V(G)$ with even cardinality such that $G - S$ has a*

perfect matching, then there exist $u, v \in S$ such that $G - \{u, v\}$ has a perfect matching.

Proof. Let G be a connected graph with a perfect matching M and suppose that there exists a set $S \subseteq V(G)$ such that $|S| = 2k$ and $G - S$ has a perfect matching M' . Then by Proposition 1.1 there exist vertices $u_1, u_2 \in S$ such that $G - (S - \{u_1, u_2\})$ has a perfect matching. Let $S^{(1)} = S - \{u_1, u_2\}$. Then $|S^{(1)}| = 2k - 2$ and $G - S^{(1)}$ has a perfect matching. Then again by Proposition 1.1 there exist vertices $u_3, u_4 \in S^{(1)}$ such that $G - (S^{(1)} - \{u_3, u_4\})$ has a perfect matching. Let $S^{(2)} = S^{(1)} - \{u_3, u_4\}$. Then $|S^{(2)}| = 2k - 4$ and $G - S^{(2)}$ has a perfect matching.

By the finiteness of G , under repeated applications of Proposition 1.1 we eventually arrive at a set $S^{(k-1)} \subseteq S$ with $|S^{(k-1)}| = 2$ such that $G - S^{(k-1)}$ has a perfect matching. Thus there exist vertices $u, v \in S$ (namely $\{u, v\} = S^{(k-1)}$) such that $G - \{u, v\}$ has a perfect matching. ■

It follows from this proposition and the fact that dominating sets are superhereditary, that to check whether D is a minimal paired dominating set of G , we only need to check that $D - \{d, d'\}$ is not a paired dominating set for any two vertices $d, d' \in D$.

The following proposition comes from [7], but the proof given there is incomplete. We give a complete proof here.

Proposition 1.3 *Let G be a graph with no isolated vertices and S be a paired dominating set of G . Then S is a minimal paired dominating set if and only*

if any two vertices x, y in S satisfy one of the following conditions:

1. $\langle S - \{x, y\} \rangle$ does not contain a perfect matching,
2. without loss of generality, x is an end-vertex in $\langle S \rangle$ adjacent to y ,
3. there exists a vertex $u \in V - S$ such that $N(u) \cap S \subseteq \{x, y\}$.

Proof. Let G be a graph with no isolated vertices and S be a paired dominating set of G . Then $|S|$ is even. Suppose that S is a minimal paired dominating set. Then for any vertices x and y of S , $S - \{x, y\}$ is not a paired dominating set. Thus either $S - \{x, y\}$ does not contain a perfect matching and hence we have Condition 1, or there exists $u \in V$ not pairwise dominated by $S - \{x, y\}$. Now, if $u \in \{x, y\}$, without loss of generality say $u = x$, then x is an end-vertex in $\langle S \rangle$ adjacent to y . If $u \in S - \{x, y\}$, we have Condition 1 since u is an isolated vertex in $\langle S - \{x, y\} \rangle$. If $u \in V - S$, then $N(u) \cap S \subseteq \{x, y\}$.

Conversely, suppose that any two vertices in S satisfies one of the three conditions of the proposition and assume that S is not a minimal pair dominating set of G . Then there exists a proper subset S' of S such that S' is a paired dominating set of G . By Proposition 1.2, we may assume that $|S'| = |S| - 2$. Thus there exist two vertices $x, y \in S$ such that $S - \{x, y\}$ is a paired dominating set of G . Clearly, Conditions 1 and 2 do not hold for the vertices x, y . Furthermore, since $S - \{x, y\}$ dominates G , no vertex y in $V - S$ has $N(y) \cap S = \{x, y\}$. Hence Condition 3 does not hold for x and y , a contradiction. This completes the proof. ■

Corollary 1.4 *If S is a minimal paired dominating set of G , then for each pair of vertices $u, v \in S$ such that $\langle S - \{u, v\} \rangle$ has a perfect matching, $\text{pn}(\{u, v\}, S) \neq \phi$.*

Proof. Suppose that S is a minimal paired dominating set of G . Consider vertices $u, v \in S$ such that $\langle S - \{u, v\} \rangle$ has a perfect matching. If $\text{pn}(\{u, v\}, S) = \phi$ then every vertex in $N[\{u, v\}]$ is in $N[S - \{u, v\}]$ and thus $S - \{u, v\}$ is also a paired dominating set of G , contradicting that S is minimal. Thus $\text{pn}(\{u, v\}, S) \neq \phi$. ■

Corollary 1.5 *If S is a minimal paired dominating set of G , then for each matching M of $\langle S \rangle$ and each edge $uv \in M$, $\text{pn}(\{u, v\}, S) \neq \phi$.*

The converse of Corollary 1.5 is not true. Take P_4 with vertex sequence a, b, c, d . Join a vertex u to a and b and vertex v to c and d . Then $S = \{a, b, c, d\}$ is a paired dominating set and $\langle S \rangle$ has a unique perfect matching $M = \{ab, cd\}$. Now $\text{pn}(\{a, b\}, S) = \{u, a\}$ and $\text{pn}(\{c, d\}, S) = \{v, d\}$, but S is not a minimal paired dominating set because $\{b, c\}$ is also a paired dominating set.

Proposition 1.6 *Let $v \in V(G)$ with $\deg(v) \geq 1$. If there exists a set S with perfect matching M such that S precisely dominates $G - v$, then there exists a paired dominating set S' with perfect matching $M' = M \cup \{vv'\}$ such that $\text{pn}(v', S') = \phi$.*

Proof. Since S precisely dominates $G - v$, $x \notin S$ for all $x \in N(v)$. Let $v' \in N(v) \neq \phi$. Then $S' = S \cup \{v, v'\}$ with perfect matching $M' = M \cup \{vv'\}$ is a paired dominating set of G and $\text{pn}(v', S') = \phi$. ■

1.3 Motivation and Summary of the Problem

The concept of paired domination was introduced by Haynes and Slater in [6]. As suggested, one application could be to model the situation where a paired dominating set is a set of guards in a network where each guard has a designated (adjacent) backup. The problem of finding a paired dominating set can also be viewed as the problem of finding a set of independent edges such that every vertex in the graph is either incident to one of these edges or adjacent to at least one of the incident vertices.

Criticality has been studied both with respect to the domination number and the total domination number. Edge-criticality was first introduced by Sumner and Blich in [8] and vertex-criticality was first introduced by Brigham, Chinn, and Dutton in [1]. In both cases, criticality is studied with respect to the domination number. In his Ph.D. dissertation [9], Van der Merwe gives an overview of total domination edge-critical graphs.

Thus far, no work has been published concerning criticality with respect to the paired domination number. The main goal of this work is to present results on paired domination critical graphs that are analogous to those known for domination and total domination critical graphs. In Chapter 2 we estab-

lish the existence or nonexistence of the critical and stable graphs defined on page 4 with respect to the paired domination number. Chapter 3 presents results on γ_{pr} -edge-critical graphs, and in Chapter 4 we narrow the focus to 4_{pr} -edge-critical graphs. Chapter 4 concludes with a characterisation of 4_{pr} -edge-critical graphs that contain a cut-vertex. The contents of Chapter 5 concern properties of γ_{pr} -vertex-critical graphs. Chapter 6 investigates the maximum value for the diameter of both γ_{pr} -edge-critical and γ_{pr} -vertex-critical graphs where γ_{pr} is fixed. We finish in Chapter 7 by presenting open problems in the research area.

Chapter 2

The Existence of Paired Domination Critical and Stable Graphs

In this chapter we establish the existence or nonexistence of the critical and stable graphs defined on page 4 with respect to the paired domination number.

2.1 Critical Graphs

We discuss the existence or nonexistence of the various types of paired domination critical graphs. Before we begin, however, we state the following result which will be used frequently.

Proposition 2.1 [6] $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2 \lceil \frac{n}{4} \rceil$.

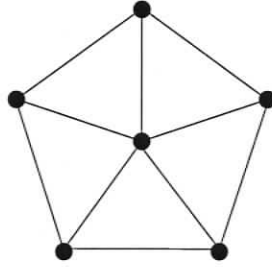
Proposition 2.2 *If $\gamma_{pr}(G - v) > \gamma_{pr}(G)$ for some vertex $v \in V(G)$, then v is in every minimum paired dominating set of G .*

Proof. Suppose v is not in every minimum paired dominating set of G . Consider a minimum paired dominating set S such that $v \notin S$. Then S dominates every vertex in $G - v$, and furthermore the pairing of vertices in S remains unchanged. Therefore S is a paired dominating set of $G - v$ and $\gamma_{pr}(G - v) \leq \gamma_{pr}(G)$, a contradiction. ■

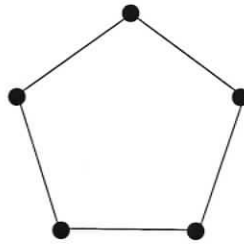
It follows from this result that if G is a γ_{pr}^+ -vertex-critical graph, then every vertex $v \in V(G)$ is in every minimum paired dominating set and $\gamma_{pr}(G) = 2m = |V(G)|$. However, it is shown in [6] that the only such graph is $G = mK_2$, hence the graphs mK_2 , $m \geq 1$, are the only γ_{pr}^+ -vertex-critical graphs. We shall call these graphs *infinitely γ_{pr}^+ -vertex-critical* graphs, as $\gamma_{pr}(G - v) = \infty$ for each $v \in V(G)$.

Even though finitely γ_{pr}^+ -vertex-critical graphs do not exist, there are graphs for which the removal of some vertex can increase the paired domination number by an arbitrary finite amount. For example, let W_n be the graph $C_n + v$. (The graph W_5 is shown in Figure 2.1.) Then $\gamma_{pr}(W_n) = 2$ and

$$\gamma_{pr}(W_n - v) = \begin{cases} \gamma_{pr}(C_n) = 2 \lceil \frac{n}{4} \rceil & \text{if } \deg v = n \\ 2 & \text{otherwise.} \end{cases}$$

Figure 2.1: The graph W_5 .

It is easy to verify that the graph C_5 is γ_{pr} -vertex-critical, and thus the existence of γ_{pr} -vertex-critical graphs is established.

Figure 2.2: A γ_{pr} -vertex-critical graph.

The following proposition gives an infinite family of γ_{pr} -vertex-critical graphs.

Proposition 2.3 *The cycle C_n is γ_{pr} -vertex-critical if and only if $n \equiv 1 \pmod{4}$. Otherwise C_n is γ_{pr} -vertex-stable.*

Proof. For any $n \geq 3$ and any $v \in V(C_n)$, the graph $C_n - v = P_{n-1}$. Recall from Proposition 2.1 that $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2 \lceil \frac{n}{4} \rceil$. Thus the only instance when $\gamma_{pr}(P_{n-1}) < \gamma_{pr}(C_n)$ is if $n \equiv 1 \pmod{4}$. Otherwise $\gamma_{pr}(P_{n-1}) = \gamma_{pr}(C_n)$. ■

Other examples of γ_{pr} -vertex-critical graphs are given in Figure 2.3. It is routine to verify that these graphs are γ_{pr} -vertex-critical and the proofs are omitted.



Figure 2.3: More γ_{pr} -vertex-critical graphs.

We now investigate the existence of graphs that are critical under the addition or deletion of an edge. If G has an isolated vertex, then $\gamma_{pr}(G) = \infty$ and so $\gamma_{pr}(G + e) \leq \gamma_{pr}(G)$ for each $e \in E(\overline{G})$. Moreover, $\gamma_{pr}(G + e)$ is finite for each $e \in E(\overline{G})$ if and only if the addition of each edge eliminates all isolated vertices, and this happens if and only if $G = K_n \cup K_1$ for some $n \geq 1$. We shall call these graphs *infinitely γ_{pr} -edge-critical*, and reserve the term “ γ_{pr} -edge-critical” for graphs G where $\gamma_{pr}(G)$ is finite. Similarly, if $\gamma_{pr}(G - e) > \gamma_{pr}(G)$ for all $e \in E(G)$, but $\gamma_{pr}(G - e_1) = \infty$ for some $e_1 \in E(G)$, we say that G is *infinitely γ_{pr} -ER-critical*, and reserve the term “ γ_{pr} -ER-critical” for graphs G such that $\gamma_{pr}(G - e)$ is finite for all $e \in E(G)$.

Proposition 2.4 *For every graph G without isolated vertices,*

1. $\gamma_{pr}(G + e) \leq \gamma_{pr}(G)$ for each $e \in E(\overline{G})$ and
2. $\gamma_{pr}(G - e) \geq \gamma_{pr}(G)$ for each $e \in E(G)$.

Proof. Consider a γ_{pr} -set S of G . Since S pairwise dominates all the vertices in G , S pairwise dominates all the vertices in $G + e$. Thus S is a paired dominating set of $G + e$ and $\gamma_{pr}(G + e) \leq |S|$.

If $G - e$ has an isolated vertex, then the result holds, so assume this is not the case. Consider a γ_{pr} -set S' of $G - e$. Since S' pairwise dominates all the vertices in $G - e$, S' pairwise dominates all the vertices in G . Thus S' is a paired dominating set of G and $\gamma_{pr}(G) \leq |S'|$. ■

From this result it follows that there does not exist a γ_{pr}^+ -edge-critical graph and that there does not exist a γ_{pr}^- -ER-critical graph.

The graph C_5 also gives an example of a γ_{pr} -edge-critical graph. Other examples are given in Figure 2.4; again the proofs are omitted.

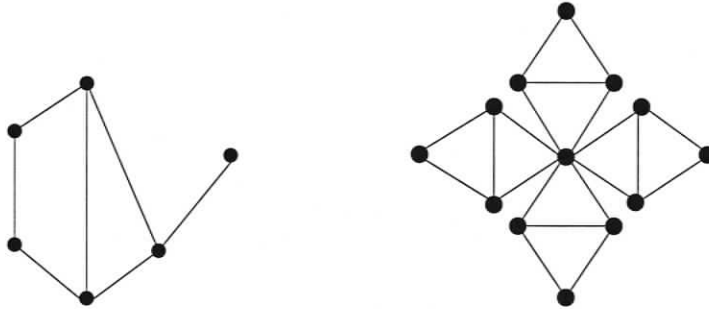


Figure 2.4: More γ_{pr} -edge-critical graphs.

Consider the graph P_4 . By Proposition 2.1, $\gamma_{pr}(P_4) = 2$. Notice that the removal of any edge $e \in E(P_4)$ results in $\gamma_{pr}(P_4 - e) = 4$ or $\gamma_{pr}(P_4 - e) = \infty$.

Thus infinitely γ_{pr} -ER-critical graphs exist.

Proposition 2.5 *If $\infty > \gamma_{pr}(G - uv) > \gamma_{pr}(G)$ for some edge $uv \in E(G)$, then $\{u, v\} \cap S \neq \emptyset$ for every minimum paired dominating set S of G .*

Proof. Suppose S is a γ_{pr} -set of G such that $\{u, v\} \cap S = \emptyset$. Then S dominates $G - uv$ and S is paired in $G - uv$. Therefore $\gamma_{pr}(G - uv) \leq \gamma_{pr}(G)$, a contradiction. ■

Proposition 2.6 *There does not exist a γ_{pr} -ER-critical graph.*

Proof. Suppose to the contrary that G is a γ_{pr} -ER-critical graph. Then, by Proposition 2.5, $\{u, v\} \cap S \neq \emptyset$ for every γ_{pr} -set S of G and every edge $uv \in E(G)$. Consider a γ_{pr} -set S' of G and an edge $xy \in E(G)$. First, notice that G does not contain any vertices of degree one. Otherwise, the removal of the incident edge would result in an isolated vertex and the paired domination number of this new graph would be infinite. Now, by Proposition 1.5, since S' is a minimal paired dominating set of G , $\text{pn}(\{x, y\}, S') \neq \emptyset$. Let $z \in \text{pn}(\{x, y\}, S')$. Notice that $z \notin S'$. If $N(z) \subseteq \{x, y\}$, then S' dominates $G - xz$ and S' is paired in $G - xz$. Thus $\gamma_{pr}(G - xz) \leq \gamma_{pr}(G)$, a contradiction. Thus $N(z) \not\subseteq \{x, y\}$. Let $z' \in N(z) - \{x, y\}$. Since $z \in \text{pn}(\{x, y\}, S')$, $z' \notin S'$. Then S' dominates $G - zz'$ and S' is paired in $G - zz'$. Thus $\gamma_{pr}(G - zz') \leq \gamma_{pr}(G)$, a contradiction. ■

Recall from Chapter 1 that the graph $G.uv$ is obtained by identifying vertex u with vertex v . Note that if K_2 is a component of G and u and v are

its vertices, then $G.uv$ has an isolated vertex, hence $\gamma_{pr}(G.uv) = \infty$. It is easy to see that $\gamma_{pr}(G)$ is finite while $\gamma_{pr}(G.uv)$ is infinite for all $uv \in E(G)$ if and only if $G = mK_2$ for some $m \geq 1$. Therefore we shall call these graphs *infinitely γ_{pr}^+ -dot-critical*, reserving the term “ γ_{pr} -dot-critical” for graphs such that $\gamma_{pr}(G.uv)$ is finite for all $uv \in E(G)$.

We present the following result which establishes the nonexistence of γ_{pr}^+ -dot-critical graphs.

Proposition 2.7 *For any graph G without isolated vertices and all $uv \in E(G)$ such that $\langle\{u, v\}\rangle$ is not a component of G , $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.*

Proof. Let S be a γ_{pr} -set of G and suppose we contract the edge $uv \in E(G)$.

If $\{u, v\} \cap S = \emptyset$, then the vertex u is dominated by a vertex in $S \cap N(u)$ and v is dominated by a vertex in $S \cap N(v)$. Therefore if we use the set S in $G.uv$, then (uv) is dominated by a vertex in $N((uv))$. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.

If $u \in S, v \notin S$, then the vertex v is dominated by u and u is dominated by the partner $u' \in S \cap N(u)$ of u . If we use the set $(S - \{u\}) \cup \{(uv)\}$ in $G.uv$, then (uv) is paired with $u' \in N((uv))$. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.

Suppose $u, v \in S$ and uv is a pair in S . Since $\langle\{u, v\}\rangle$ is not a component of G , at least one of u and v has another neighbour in G . If all the neighbours of u and v are also in S , then $S - \{u, v\}$ is a paired dominating set of $G.uv$. If some neighbour x of (without loss of generality) u is not in S , then $(S - \{u, v\}) \cup \{(uv), x\}$ is a paired dominating set of $G.uv$. Hence

$$\gamma_{pr}(G.uv) \leq \gamma_{pr}(G).$$

Suppose $u, v \in S$ and uv is not a pair in S . Say that uu' and vv' are pairs in S where all four vertices are distinct. If all the neighbours of (say) v' are in S , then $(S - \{v', u, v\}) \cup \{(uv)\}$ is a paired dominating set of $G.uv$ (where (uv) is paired with u'). If some neighbour x of v' is not in S , then $(S - \{u, v\}) \cup \{(uv), x\}$ is a paired dominating set of $G.uv$ where (uv) and u' , and v' and x , are partners. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$. ■

Using the same idea as in the proof of Proposition 2.7 we also show the nonexistence of γ_{pr}^+ -totally-dot-critical graphs, K_2 obviously being the only infinitely γ_{pr}^+ -totally-dot-critical graph (defined similarly to the above).

Proposition 2.8 *For any graph G without isolated vertices and all $\{u, v\} \subseteq V(G)$ such that $\langle \{u, v\} \rangle$ is not a component of G , $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.*

Proof. By Proposition 2.7, we only need to consider the case where u and v are not adjacent in G . Let S be a γ_{pr} -set of G and suppose we identify the vertices u and v .

If $\{u, v\} \cap S = \emptyset$, then the vertex u is dominated by a vertex in $N(u) \cap S$ and v is dominated by a vertex in $N(v) \cap S$. Therefore if we use the set S in $G.uv$ then (uv) is dominated by a vertex in $N((uv))$. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.

If $u \in S, v \notin S$, then the vertex v is dominated by v' in $N(v) \cap S$ and u is dominated by its partner in $N(u) \cap S$. If we use the dominating set

$(S - \{u\}) \cup \{(uv)\}$ in $G.uv$, then (uv) is paired with the vertex $u' \in N((uv))$. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$.

If $u \in S, v \in S$, then uu' and vv' are pairs in S where all four vertices are distinct. If all the neighbours of (say) v' are in S , then $(S - \{v', u, v\}) \cup \{(uv)\}$ is a paired dominating set of $G.uv$. If some neighbour x of v' is not in S , then $(S - \{u, v\}) \cup \{(uv), x\}$ is a paired dominating set of $G.uv$. Hence $\gamma_{pr}(G.uv) \leq \gamma_{pr}(G)$. ■

Notice that by definition any graph that is γ_{pr} -totally-dot-critical is also γ_{pr} -dot-critical. If $G \neq K_1$ and $\gamma_{pr}(G)$ is infinite, then $\gamma_{pr}(G.uv)$ is infinite for all $uv \in E(G)$; moreover, if $E(G) = \phi$ (and $G \neq K_1$) then $\gamma_{pr}(G.uv)$ is also infinite for all $u, v \in V(G)$. Hence for γ_{pr} -(totally)-dot-critical graphs we shall assume that G has no isolated vertices.

Again the graph C_5 is used as an example to show the existence of γ_{pr} -totally-dot-critical graphs (and thus γ_{pr} -dot-critical graphs). The following proposition gives an infinite family of γ_{pr} -dot-critical graphs.

Proposition 2.9 *The cycle C_n is γ_{pr} -dot-critical if and only if $n \equiv 1 \pmod{4}$.*

Proof. For any $n \geq 3$ and any $uv \in E(C_n)$, the graph $C_n.uv = C_{n-1}$. Recall from Proposition 2.1 that $\gamma_{pr}(C_n) = 2 \lceil \frac{n}{4} \rceil$. Thus the only instance when $\gamma_{pr}(C_{n-1}) < \gamma_{pr}(C_n)$ is if $n \equiv 1 \pmod{4}$. Otherwise $\gamma_{pr}(C_{n-1}) = \gamma_{pr}(C_n)$. ■

The results of this section are summarized in Table 2.1, where we disregard all cases where γ_{pr} is infinite.

Type	Existence?
γ_{pr} -vertex-critical	yes
γ_{pr}^+ -vertex-critical	no
γ_{pr} -edge-critical	yes
γ_{pr}^+ -edge-critical	no
γ_{pr} -ER-critical	no
γ_{pr}^- -ER-critical	no
γ_{pr} -dot-critical	yes
γ_{pr}^+ -dot-critical	no
γ_{pr} -totally-dot-critical	yes
γ_{pr}^+ -totally-dot-critical	no

Table 2.1: Summary of Criticality Results

2.2 Stable Graphs

For all types of stable graphs defined on page 4, the question of existence has a positive answer. In this section we give examples of such stable graphs. The case where $\gamma_{pr}(G)$ is infinite is simple - consider graphs with as many isolated vertices as necessary - hence we only consider graphs without isolated vertices.

Recall that for $n \geq 3$, $\gamma_{pr}(C_n) = \gamma_{pr}(P_n) = 2 \lceil \frac{n}{4} \rceil$. Thus since $C_n - e = P_n$ for any $e \in E(C_n)$, C_n is γ_{pr} -ER-stable.

Notice that for $n \geq 2$, $\gamma_{pr}(K_n) = 2$ and so for $n \geq 3$, $\gamma_{pr}(K_n) = \gamma_{pr}(K_n - v) = \gamma_{pr}(K_n - e) = 2$ for any $v \in V(K_n)$ and any $e \in E(K_n)$. Therefore $\{G : G = K_n, n \geq 3\}$ is an infinite family of γ_{pr} -vertex-stable and γ_{pr} -ER-stable graphs. Another infinite family of γ_{pr} -ER-stable graphs is given by $\{G : G = W_n, n \geq 3\}$ since $\gamma_{pr}(W_n) = \gamma_{pr}(W_n - e) = 2$ for any $e \in E(W_n)$, $n \geq 3$.

The following proposition gives an infinite family of γ_{pr} -edge-stable graphs.

Proposition 2.10 *The graph C_n , $n \geq 4$, is γ_{pr} -edge-stable if and only if $n \equiv 0, 3 \pmod{4}$.*

Proof. Let the vertices of C_n be labelled cyclically as v_1, v_2, \dots, v_n . Suppose $n \equiv 1 \pmod{4}$. Then $C_n + v_1v_3$ can be pairwise dominated with $2 \lceil \frac{n-1}{4} \rceil$ vertices and in this case $2 \lceil \frac{n-1}{4} \rceil < 2 \lceil \frac{n}{4} \rceil = \gamma_{pr}(C_n)$. Suppose $n \equiv 2 \pmod{4}$. Then $C_n + v_1v_4$ can be pairwise dominated with $2 \lceil \frac{n-2}{4} \rceil$ vertices and in this case $2 \lceil \frac{n-2}{4} \rceil < 2 \lceil \frac{n}{4} \rceil = \gamma_{pr}(C_n)$.

Suppose $n \equiv 0 \pmod{4}$. The result is trivially true if $n = 4$, so assume that $n \geq 8$. Then by Proposition 2.1, $\gamma_{pr}(C_n) = 2 \lceil \frac{4k}{4} \rceil = 2k$ for some $k \in \mathbb{Z}^+$ and any γ_{pr} -set consists of k pairs. Suppose $\gamma_{pr}(C_n + e) = 2(k-1)$ for some $e \in E(\overline{C_n})$. Let S be a γ_{pr} -set of $C_n + e$. Then we have that:

(i) either $k-2$ pairs in S dominate at most four vertices each, and 1 pair contains both vertices of degree 3 and hence dominates at most six vertices, or

(ii) $k-3$ pairs in S dominate at most four vertices each, and 2 pairs dominate at most five vertices each.

But in either case, at most $4k-2$ vertices are dominated, a contradiction.

A similar argument follows for the case when $n \equiv 3 \pmod{4}$. ■

Since $C_n \cdot uv = C_{n-1}$ for any $uv \in E(C_n)$, a similar proof as in Proposition 2.3 gives an infinite family of γ_{pr} -dot-stable graphs. That is, $\{G : G = C_n, n \not\equiv 1 \pmod{4}\}$ is γ_{pr} -dot-stable. Also, notice that $\gamma_{pr}(C_4 \cdot uv) =$

$\gamma_{pr}(C_4) = 2$ for any two vertices $u, v \in V(C_4)$ and hence the existence of γ_{pr} -totally-dot-stable graphs is established.

The results of this section are summarized in Table 2.2.

Type	Existence?
γ_{pr} -vertex-stable	yes
γ_{pr} -edge-stable	yes
γ_{pr} -ER-stable	yes
γ_{pr} -dot-stable	yes
γ_{pr} -totally-dot-stable	yes

Table 2.2: Summary of Stability Results

Now that we have established the existence or nonexistence of the critical and stable graphs mentioned, we focus solely on two types of criticality: γ_{pr} -edge-criticality and γ_{pr} -vertex-criticality. The remaining chapters are devoted to results concerning such graphs.

Chapter 3

Properties of γ_{pr} -Edge-Critical Graphs

We begin with a result that is used extensively throughout the next two chapters.

Proposition 3.1 *For any graph G , any edge $uv \in E(\overline{G})$ such that $\gamma_{pr}(G + uv) < \gamma_{pr}(G)$ and any γ_{pr} -set S of $G + uv$, $S \cap \{u, v\} \neq \emptyset$.*

Proof. Suppose $u, v \notin S$. This implies that S is paired in G . But S dominates G and so there exists a paired dominating set of G with cardinality less than $\gamma_{pr}(G)$, a contradiction. ■

If $\gamma_{pr}(G + uv) < \gamma_{pr}(G)$, we can say something about the structure of a perfect matching in a γ_{pr} -set of $G + uv$ depending on which of u and v is in the γ_{pr} -set.

Proposition 3.2 *Let G be a graph such that $\gamma_{pr}(G + uv) < \gamma_{pr}(G)$ for $uv \in E(\overline{G})$. If S is a γ_{pr} -set of $G + uv$, then*

1. $\{u, v\} \subseteq S$ implies that u and v are paired in S , S dominates G and $S - \{u, v\}$ is paired in G .
2. (without loss of generality) $u \in S, v \notin S$ implies that S is paired in G and S precisely dominates $G - v$.

Proof. 1) Let $\{u, v\} \subseteq S$ and suppose that u and v are not paired in S . Then u is paired with $u' \in N_G(u)$ and v is paired with $v' \in N_G(v)$ where $u' \neq v'$. Thus S is paired in G . But S dominates G and so S is a paired dominating set of G with smaller cardinality than $\gamma_{pr}(G)$, a contradiction. Therefore u and v are paired in S , $S - \{u, v\}$ is paired in G and S dominates G .

2) Say (without loss of generality) that $u \in S$ and $v \notin S$. Then S is paired in G and S does not dominate G , for otherwise we would have a paired dominating set of cardinality less than $\gamma_{pr}(G)$. Since S pairwise dominates $G + uv$, the only undominated vertex in G is v and S precisely dominates $G - v$. ■

In all previous examples, if the addition of an edge decreases the paired domination number, then it decreases by two. This is in fact the most that the paired domination number will decrease upon the addition of an edge.

Proposition 3.3 For any edge $uv \in E(\overline{G})$,

$$\gamma_{pr}(G) - 2 \leq \gamma_{pr}(G + uv) \leq \gamma_{pr}(G)$$

Proof. As shown in Proposition 2.4, adding an edge does not increase the paired domination number, so the upper bound holds.

Let uv be an edge of \overline{G} such that $\gamma_{pr}(G + uv) < \gamma_{pr}(G)$. From Proposition 3.1 we know that at least one of u and v is in any γ_{pr} -set of $G + uv$. Suppose there is a γ_{pr} -set S of $G + uv$ such that $u, v \in S$. Then S is a dominating set of G . Furthermore, the only possible unpaired vertices in the induced subgraph $\langle S \rangle$ of G are u and v . Since G has no isolated vertices, there exist $x \in N_G(u)$ and $y \in N_G(v)$.

If $x \in N(u) - S$ and $y \in N(v) - (S - \{x\})$, then $S \cup \{x, y\}$ is a paired dominating set of G and $\gamma_{pr}(G) \leq \gamma_{pr}(G + uv) + 2$.

If $N_G(u) \subseteq S$ and $N_G(v) \subseteq S$, then by Proposition 3.2 u and v are paired in S and $S - \{u, v\}$ is a paired dominating set in G . But $|S - \{u, v\}| < |S|$, a contradiction since adding an edge does not increase the paired domination number.

If (without loss of generality) $y \in N_G(v) - S$ and $N_G(u) - \{y\} \subseteq S$, then $(S - \{u\}) \cup \{y\}$ is a paired dominating set of G and $\gamma_{pr}(G) \leq \gamma_{pr}(G + uv)$.

Thus assume (without loss of generality) that $u \in S$ and $v \notin S$ for a given γ_{pr} -set S of $G + uv$. Now by Proposition 3.2, S is a paired dominating set of

$G - v$. But G has no isolated vertices, therefore v has at least one neighbour in G . If $N(v) \cap S \neq \emptyset$ then S also dominates v and $\gamma_{pr}(G + uv) \leq \gamma_{pr}(G)$. If v has no neighbours in G that are in S then for some $w \in N(v)$, $S \cup \{v, w\}$ is a paired dominating set of G . Therefore $\gamma_{pr}(G + uv) \geq \gamma_{pr}(G) - 2$.

Hence in all cases $\gamma_{pr}(G) - 2 \leq \gamma_{pr}(G + uv) \leq \gamma_{pr}(G)$. ■

Upon the inspection of previous examples of γ_{pr} -edge-critical graphs, we see that every vertex in the graph is in some γ_{pr} -set. This result is true for any γ_{pr} -edge-critical graph.

Theorem 3.4 *In a γ_{pr} -edge-critical graph G , every vertex is in some γ_{pr} -set of G .*

Proof. Suppose G is a γ_{pr} -edge-critical graph. Then no vertex of G is adjacent to all other vertices of G (for otherwise $\gamma_{pr}(G) = 2$ and G is not critical). Therefore for every $v \in V(G)$, there exists a vertex $u \in V(G)$ such that $uv \in E(\overline{G})$.

Consider the graph $G + uv$. Since G is critical, $\gamma_{pr}(G + uv) = \gamma_{pr}(G) - 2$. We want to show that there is a γ_{pr} -set S' of G such that $v \in S'$. Say that S is a γ_{pr} -set of $G + uv$. By Proposition 3.1, at least one of u and v is in S .

If $u \in S$ and $v \notin S$, then u is paired in S with the vertex u' where $uu' \in E(G)$. In G , the set S precisely dominates $G - v$. Thus no $v' \in N_G(v) \neq \emptyset$ is in S , otherwise S is a paired dominating set of G (a contradiction to the criticality of G). Then for any $v' \in N_G(v)$, $S' = S \cup \{v, v'\}$ is a paired dominating set of G and therefore a γ_{pr} -set of G with $v \in S'$.

Notice that S' contains both u and v and so the desired result also holds if $u \notin S, v \in S$.

If $u \in S$ and $v \in S$, then by Proposition 3.2, this means that $uv \in M$ for any perfect matching M of $\langle S \rangle$.

Case i: $N_G(u) \cup N_G(v) \subseteq S$.

Then $S' = S - \{u, v\}$ is a paired dominating set of $G + uv$, a contradiction.

Case ii: Without loss of generality, there exists $u' \in N_G(u) - S$, while $N_G(v) - \{u'\} \subseteq S$.

Then $S' = (S - \{v\}) \cup \{u'\}$ is a paired dominating set, hence a γ_{pr} -set, of G . But this means that $|S'| = |S|$ and thus $\gamma_{pr}(G) \leq \gamma_{pr}(G + uv)$ and from the bounds given in a Proposition 3.3 this means $\gamma_{pr}(G) = \gamma_{pr}(G + uv)$, a contradiction.

Case iii: there exists $u' \in N_G(u) - S$ and $v' \in N_G(v) - (S \cup \{u'\})$.

Let M be a perfect matching of $\langle S \rangle$. Then $S' = S \cup \{u', v'\}$, with matching $M' = (M - \{uv\}) \cup \{uu', vv'\}$, is a γ_{pr} -set of G with $v \in S'$. ■

We now show that any graph can be extended to produce a γ_{pr} -edge-critical graph.

Proposition 3.5 *If $\gamma_{pr}(G) = k \geq 4$, then G is a spanning subgraph of some k_{pr} -edge-critical graph.*

Proof. If G is k_{pr} -edge-critical then we are done. Thus suppose that G is not k_{pr} -edge-critical. Then by definition and by Proposition 2.4, there exists an

edge $e_1 \in E(\overline{G})$ such that $\gamma_{pr}(G+e_1) = \gamma_{pr}(G)$. Let $G_1 = G+e_1$. If G_1 is k_{pr} -edge-critical then we are done, otherwise there exists an edge $e_2 \in E(\overline{G_1})$ such that $\gamma_{pr}(G_1+e_2) = \gamma_{pr}(G_1)$. Let $G_2 = G_1+e_2$. Continuing in this process we will eventually obtain a graph G_i such that $\gamma_{pr}(G) = \gamma_{pr}(G_1) = \dots = \gamma_{pr}(G_i)$ and for any $e' \in E(\overline{G_i})$, $\gamma_{pr}(G_i+e') < \gamma_{pr}(G_i)$. Thus G_i is a γ_{pr} -edge-critical graph such that G is a spanning subgraph of G_i and $\gamma_{pr}(G_i) = \gamma_{pr}(G)$. ■

The graph C_5 is used throughout Chapter 2 as an example for various types of criticality. In particular, C_5 is γ_{pr} -edge-critical. We show that this is the only γ_{pr} -edge-critical cycle.

Proposition 3.6 *The only γ_{pr} -edge-critical cycle is C_5 .*

Proof. By inspection C_5 is γ_{pr} -edge-critical while C_4, C_6, C_7 and C_8 are not. Let C_n be a cycle of order $n \geq 9$ with vertex sequence v_1, v_2, \dots, v_n , and suppose C_n is γ_{pr} -edge-critical.

Consider $G = C_n + v_1v_6$ and let S be an arbitrary γ_{pr} -set of G . By Proposition 3.1, $S \cap \{v_1, v_6\} \neq \emptyset$. Let $T = S \cap \{v_2, v_3, v_4, v_5\}$. If $\{v_1, v_6\} \subseteq S$, then v_1 and v_6 are paired in any matching of $\langle S \rangle$. To pairwise dominate v_3 and v_4 , $|T| \geq 2$. But then $S' = (S - T) \cup \{v_2, v_5\}$ pairwise dominates C_n and $|S'| \leq |S| < \gamma_{pr}(C_n)$, a contradiction. Hence we may assume, without loss of generality, that $S \cap \{v_1, v_6\} = \{v_1\}$. If $v_5 \in S$, then $v_4 \in S$ because $v_6 \notin S$, and so S pairwise dominates C_n , a contradiction. Hence $v_5 \notin S$. Therefore to pairwise dominate v_5 , $\{v_4, v_3\} \subseteq T$. But then $S'' = (S - \{v_3\}) \cup \{v_5\}$ pairwise dominates C_n and $|S''| = |S| < \gamma_{pr}(C_n)$, a contradiction. ■

If the paired domination number and total domination number of a γ_{pr} -edge-critical graph are “close enough” then we can draw conclusions about the edge-criticality of the graph with respect to the total domination number.

We say that a graph G is γ_t -edge-critical if $\gamma_t(G + e) = \gamma_t(G) - 1$ for any $e \in E(\overline{G})$ and that G is γ_t -super-edge-critical if $\gamma_t(G + e) = \gamma_t(G) - 2$ for any $e \in E(\overline{G})$.

Theorem 3.7 *If $2k < \gamma_t(G) \leq 2(k + 1) = \gamma_{pr}(G)$ and G is γ_{pr} -edge-critical, then G is γ_t -edge-critical or γ_t -super-edge-critical.*

Proof. Suppose $2k < \gamma_t(G) \leq 2(k + 1) = \gamma_{pr}(G)$ and G is γ_{pr} -edge-critical. For an edge $e \in E(\overline{G})$ consider the graph $G + e$. Since G is $2(k + 1)_{pr}$ -edge-critical, $\gamma_{pr}(G + e) = 2k$. But by Proposition 2.4, $\gamma_t(G + e) \leq \gamma_{pr}(G + e) = 2k$. Thus $\gamma_t(G + e) < \gamma_t(G)$ and G is either γ_t -edge-critical or γ_t -super-edge-critical. ■

This result allows us to relate 4_{pr} -edge-critical graphs to 3_t -edge-critical graphs and 4_t -super-edge-critical graphs. First, though, a quick result.

Proposition 3.8 *If $\gamma_t(G) = 3$, then $\gamma_{pr}(G) = 4$.*

Proof. Suppose $\gamma_t(G) = 3$ and consider a γ_t -set S . The only possibilities for $\langle S \rangle$ are $\langle S \rangle = K_3$ or $\langle S \rangle = P_3$. Notice also that all vertices in $V(G) - S \neq \phi$ are adjacent to some vertex in S . Suppose that $\langle S \rangle = K_3$. For any $x \in V(G) - S$, $S \cup \{x\}$ is a paired dominating set of G and thus $\gamma_{pr}(G) = 4$ since

$\gamma_{pr}(G) \geq \gamma_t(G)$. Suppose $\langle S \rangle = P_3$ and let the vertices be labelled v_1, v_2, v_3 where v_2 is adjacent to v_1 and v_3 in $\langle S \rangle$. If there exists a vertex $x \in V(G) - S$ such that x is adjacent to v_1 or v_3 , then $S \cup \{x\}$ is a paired dominating set of G and thus $\gamma_{pr}(G) = 4$. Suppose that $N(v_1) \cup N(v_3) = \{v_2\}$. Then all vertices in $V(G) - S$ are adjacent to v_2 and in fact $\gamma_t(G) = 2$, a contradiction. ■

Corollary 3.9 *The class of 4_{pr} -edge-critical graphs is exactly the union of the class of all 3_t -edge-critical graphs and the class of all 4_t -super-edge-critical graphs.*

Proof. Suppose G is 4_{pr} -edge-critical and $\gamma_t(G) = 2$. The only possibility for a total dominating set of cardinality 2 is that the set consists of two adjacent vertices u and v . But then $\{u, v\}$ is also a paired dominating set, implying that $\gamma_{pr}(G) = 2$, a contradiction. Thus $4 = \gamma_{pr}(G) \geq \gamma_t(G) \geq 3$, and by Proposition 3.7, G is either 3_t -edge-critical or 4_t -super-edge-critical.

Conversely, if G is 3_t -edge-critical, then by Proposition 3.8, $\gamma_{pr}(G) = 4$ and G is 4_{pr} -edge-critical, because $\gamma_t(G) = 2$ implies $\gamma_{pr}(G) = 2$. In [9] it was shown that G is 4_t -super-edge-critical if and only if G is the disjoint union of two nontrivial complete graphs and so G is also 4_{pr} -edge-critical. ■

We further note in passing that if $\gamma_t(G) = 2k + 1$, $k \geq 2$, then it does not follow that $\gamma_{pr}(G) = 2(k + 1)$; in fact,

$$2 \left\lceil \frac{\gamma_t(G)}{2} \right\rceil \leq \gamma_{pr}(G) \leq 2\gamma_t(G) - 2$$

for all graphs G without isolated vertices, and the bounds are best possible. The proof is not difficult, but not relevant here and hence omitted.

We now focus on structural properties of γ_{pr} -edge-critical graphs.

Proposition 3.10 *Every vertex in a γ_{pr} -edge-critical graph is adjacent to at most one end-vertex.*

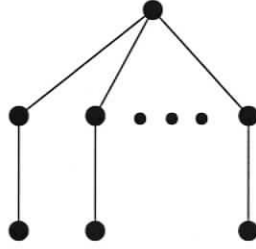
Proof. Let G be a γ_{pr} -edge-critical graph and suppose some vertex v is adjacent to end-vertices x and y . Let S be a γ_{pr} -set of $G + xy$. By Proposition 3.1, at least one of x and y is in S .

If $\{x, y\} \subseteq S$ then x and y are paired in S and $v \notin S$, for otherwise $S - \{x, y\}$ is a paired dominating set in G and in $G + xy$. However, then $S' = (S - \{y\}) \cup \{v\}$ is a paired dominating set of G . Therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G + xy)$.

If $\{x, y\} \not\subseteq S$ then we can assume that $\{v, x\} \subseteq S$ and thus S is a paired dominating set of G . Therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G + xy)$.

In either case $\gamma_{pr}(G) \leq \gamma_{pr}(G + xy)$ and by Proposition 2.4 this implies that $\gamma_{pr}(G) = \gamma_{pr}(G + xy)$, a contradiction. ■

In particular, this result implies that in a γ_{pr} -edge-critical tree every support vertex is adjacent to exactly one leaf. A family of γ_{pr} -edge-critical trees is given in Figure 3.1, where the three smaller dots represent the addition of any number of branches of length two to the vertex of highest degree. It is easy to verify that these graphs are γ_{pr} -edge-critical and the proof is omitted.

Figure 3.1: A family of γ_{pr} -edge-critical trees.

As used in [2], $k(G)$ denotes the number of components of G . For any graph, the number of components is bounded above by the paired domination number.

Proposition 3.11 *If $\gamma_{pr}(G) = 2l$, then $k(G) \leq l$.*

Proof. Since $\gamma_{pr}(G) = 2l$, G is dominated by l pairs of vertices. Since any component of G is dominated by at least one pair of vertices, this implies that $k(G) \leq l$. ■

This bound is sharp, as can be seen by taking the disjoint union of l nontrivial complete graphs.

Corollary 3.12 *If G is $2l_{pr}$ -edge-critical, then $k(G + uv) \leq l - 1$ for any $u, v \in V(G)$.*

Proof. Since G is γ_{pr} -edge-critical, $\gamma_{pr}(G + uv) = 2l - 2 = 2(l - 1)$ for any $u, v \in V(G)$. By Proposition 3.11 this implies that $k(G + uv) \leq l - 1$. ■

Again, the graphs that consist of l disjoint nontrivial complete graphs show that this bound is the best possible. It is also obvious that these are the only graphs for which equality holds.

Given a vertex $x \in V(G)$, define $Q_x = \{u \in N[x] : N[u] \subsetneq N[x]\}$. We now generalise Proposition 3.10. (Note that if u is a leaf adjacent to x , then $u \in Q_x$.)

Proposition 3.13 *If G is a γ_{pr} -edge-critical graph and x is any vertex of G , then $\langle Q_x \rangle$ is complete.*

Proof. If $Q_x = \{y\}$ for some $y \in N(x)$, the result is trivial, so suppose $u, v \in Q_x$, $u \neq v$, and $uv \in E(\overline{G})$. By Proposition 3.10, not both u and v are end-vertices of G , hence without loss of generality there exists $w \in N(u) - \{x\}$. Since $u \in Q_x$, $w \in N(x)$. Let S be any γ_{pr} -set of $G + uv$. By Proposition 3.1, $S \cap \{u, v\} \neq \emptyset$. If $u, v \in S$, then by Proposition 3.2, u and v are paired in S , and $x \notin S$, otherwise uv does not satisfy Corollary 1.5. But then $(S - \{u, v\}) \cup \{w, x\}$ is a paired dominating set of G , because x dominates u and v and all their neighbours, contradicting the criticality of G .

Hence $|S \cap \{u, v\}| = 1$; say $y \in S$ for $y \in \{u, v\}$. Then S precisely dominates $G - (\{u, v\} - \{y\})$ and y is paired with $y' \in N(y) - \{x\}$, and $y' \in N(x)$ because $y \in Q_x$. However, now $(S - \{y\}) \cup \{x\}$ is a paired dominating set of G , again a contradiction. ■

Proposition 3.14 *If G is γ_{pr} -edge-critical and x is a cut-vertex of G , then $G - x$ has at most one component G_1 such that each vertex of G_1 is adjacent to x in G , and in this case, $G_1 = K_1$.*

Proof. Let G_1, \dots, G_t be the components of $G - x$ such that every vertex of G_i , $i = 1, \dots, t$, is adjacent to x in G . For any $u \in V(G_i)$, u is adjacent, in G , only to x and vertices of G_i , hence $u \in Q_x$. Thus $V(G_i) \subseteq Q_x$ for each i . By Proposition 3.13, $\langle Q_x \rangle$ is complete and so $t = 1$.

Suppose $u, w \in V(G_1)$, $u \neq w$. Since x is a cut-vertex, there exists a vertex $v \in N(x) - G_1$. Let S be a γ_{pr} -set of $G + uv$. Then $S \cap \{u, v\} \neq \emptyset$.

If $\{u, v\} \subseteq S$, then u and v are paired in S . Hence if $x \notin S$, then $(S - \{u\}) \cup \{x\}$ is a paired dominating set of G , a contradiction to G being γ_{pr} -edge-critical. Suppose $x \in S$. If $N(v) \subseteq S$, then $\text{pn}(\{u, v\}, S) = \emptyset$, contradicting Proposition 1.5. Let $v' \in N(v) - S$. Then $(S - \{u\}) \cup \{v'\}$ is a paired dominating set of G , again a contradiction.

If $\{u, v\} \cap S = \{u\}$, then u is paired in S with $u' \in N(u) \subseteq N[x]$ and S precisely dominates $G - v$, which implies that $x \notin S$. But then $(S - \{u'\}) \cup \{x\}$ pairwise dominates G , a contradiction as before.

Finally, if $\{u, v\} \cap S = \{v\}$, then v is paired in S with $v' \in N(v)$, and S precisely dominates $G - u$, which implies that $x \notin S$. Thus $v' \in V(G) - G_1$. But S pairwise dominates w , so $\{y, y'\} \subseteq S$, where y and y' are paired in S and $\{y, y'\} \subseteq V(G_1) \subseteq N(x)$. However, $(S - \{y\}) \cup \{x\}$ is a paired dominating set of G , contradicting G being γ_{pr} -edge-critical. ■

Proposition 3.15 *If G is $2l_{pr}$ -edge-critical, then $k(G - x) \leq l$ for any cut-vertex $x \in V(G)$.*

Proof. Suppose G is $2l_{pr}$ -edge-critical and let x be a cut-vertex of G . Since $\gamma_{pr}(G) = 2l$, G is dominated by l pairs of vertices. Consider S , a γ_{pr} -set of G with matching M .

If $x \notin S$ then no component of $G - x$ is an isolate, and for any edge in M , both incident vertices lie within the same component of $G - x$. Also, since S dominates G , S dominates $G - x$. But each component of $G - x$ is dominated by at least one edge in M , and thus $k(G - x) \leq l$.

Thus we may assume that $x \in S$. By Proposition 3.14, if $G - x$ has a component G_1 such that every vertex in G_1 is adjacent to x , then $|V(G_1)| = 1$. Suppose x is adjacent to an end-vertex v and suppose $xx' \in M$ where $x' \neq v$, $x' \in V(G_2)$ and G_2 is a component of $G - \{x, v\}$. Suppose further that $G_2 - (N[x] \cup N[x']) = \phi$, i.e. every vertex in G_2 is adjacent to either x or x' . Consider the graph $G + vx'$ with γ_{pr} -set S' . Suppose $\{v, x'\} \subseteq S'$. Then v and x' are paired in S' . If $x \notin S'$ then $(S' - \{v\}) \cup \{x\}$ is a γ_{pr} -set of G , a contradiction to G being critical. Thus $x \in S'$. By Corollary 1.5, $\text{pn}(\{x', v\}, S') \neq \phi$. Thus there exists a vertex $w \in \text{pn}(\{x', v\}, S')$. But then $(S' - \{v\}) \cup \{w\}$ is a γ_{pr} -set of G , a contradiction to G being critical. Suppose $\{v, x\} \subseteq S'$. Then by Proposition 3.2, S' precisely dominates $G - x'$. However, x' is dominated by x , a contradiction. We may therefore assume that $\{x', y\} \subseteq S'$, where $y \in N_G(x')$. Then by Proposition 3.2, S' precisely dominates $G - v$, which implies that $x \notin S'$. But then $(S' - \{y\}) \cup \{x\}$ is a γ_{pr} -set of G since all the vertices in G_2 are adjacent to either x' or x , a contradiction.

Thus either $G_2 - (N[x] \cup N[x']) \neq \phi$, or $xx' \notin M$ for any $x' \in G_2$ (or both). In the former case there exists $w \in V(G_2) - N[\{x, x'\}]$ and at least two edges of M are required to pairwise dominate $\{v, x, x', w\}$. Moreover, each component G_i , $i > 2$, of $G - x$ contains a vertex nonadjacent to x (by Proposition 3.14) and requires at least one new pair of vertices in S to dominate it. Hence $k(G) \leq l$. In the latter case, $vx \in M$ to dominate v , and again by Proposition 3.14, each component G_i , $i \geq 2$, of $G - x$ requires a new pair of vertices in S to dominate it; again $k(G) \leq l$.

Finally, suppose no component of $G - x$ has all its vertices adjacent to x . Then x is paired in S by $x' \in V(G_i)$ for some i , say $i = 1$, and for each $i \geq 2$, at least one new pair of vertices in S is required to dominate G_i , and the result follows. ■

This bound can be achieved, as seen by the graphs given in Figure 3.1, and thus is the best possible.

We next describe a construction for a disconnected γ_{pr} -edge-critical graph. First, though, we give a result which limits the possibilities for the components of such a graph.

Proposition 3.16 *If G is a disconnected γ_{pr} -edge-critical graph, then every component of G is either complete or is a γ_{pr} -edge-critical graph.*

Proof. Suppose G is a disconnected γ_{pr} -edge-critical graph with components G_1, G_2, \dots, G_k and γ_{pr} -set S . Let G_i , $i \in \{1, 2, \dots, k\}$, be a component of G that is not complete. Thus there exists an edge $uv \in E(\overline{G_i})$. Consider the

graph $G + uv$ with γ_{pr} -set S' ($|S'| = |S| - 2$). If $|S' \cap V(G_i)| = |S \cap V(G_i)|$, then $|S' \cap V(G_j)| = |S \cap V(G_j)| - 2$ for some component G_j , $j \neq i$. But then $S'' = (S - (S \cap V(G_j))) \cup (S' \cap V(G_j))$ is a γ_{pr} -set of G with $|S''| = |S| - 2$, a contradiction. Thus $|S' \cap V(G_i)| = |S \cap V(G_i)| - 2$ and G_i is γ_{pr} -edge-critical. Therefore every component of G is either complete or is a γ_{pr} -edge-critical graph. ■

We now give sufficient conditions for a disconnected γ_{pr} -edge-critical graph.

Theorem 3.17 *If one of the following conditions holds, then G is a disconnected γ_{pr} -edge-critical graph:*

1. *Every component G_i of G is a nontrivial complete graph, or a γ_{pr} -edge-critical graph such that for every vertex $v \in V(G_i)$ there exists a set S , $|S| = \gamma_{pr}(G_i) - 2$, such that S precisely dominates $G_i - v$.*
2. *One component G_1 of G is a γ_{pr} -edge-critical graph with a vertex $u \in V(G_1)$ such that there does not exist a set S , $|S| = \gamma_{pr}(G_1) - 2$, that precisely dominates $G_1 - u$, and every other component G_i , $i \neq 1$, of G is a γ_{pr} -edge-critical graph such that for every vertex $v \in V(G_i)$ there exists a set S , $|S| = \gamma_{pr}(G_i) - 2$, such that S precisely dominates $G_i - v$.*

Proof. Notice that the paired domination number of a disconnected graph is the sum of the paired domination numbers of its components. Let S be a γ_{pr} -set of G .

Suppose firstly that G is a graph that satisfies Condition 1. Let G_1, \dots, G_t be the components that are nontrivial complete graphs and G_{t+1}, \dots, G_k be the other components of G .

Consider the graph $G + uv$ where u and v lie in the same component G_i , $t+1 \leq i \leq k$. Then $\gamma_{pr}(G_i + uv) = \gamma_{pr}(G_i) - 2$. Let S_i be a γ_{pr} -set of $G_i + uv$. Then $(S - V(G_i)) \cup S_i$ is a paired dominating set of $G + uv$ of cardinality $\gamma_{pr}(G) - 2$.

Next consider the case where $u \in V(G_i)$, $1 \leq i \leq t$, and $v \in V(G_j)$, $t+1 \leq j \leq k$. By the hypothesis, there exists a set S_j of cardinality $\gamma_{pr}(G_j) - 2$ that precisely dominates $G_j - v$. Then $(S - (V(G_i) \cup V(G_j))) \cup S_j \cup \{u, v\}$ is a paired dominating set of $G + uv$ of cardinality $\gamma_{pr}(G) - 2$.

Finally, consider the case where $u \in V(G_i)$ and $v \in V(G_j)$, $t+1 \leq i < j \leq k$. By the hypothesis, there exist a set S_i of cardinality $\gamma_{pr}(G_i) - 2$ and a set S_j of cardinality $\gamma_{pr}(G_j) - 2$ such that S_i precisely dominates $G_i - u$ and S_j precisely dominates $G_j - v$. Then $(S - (V(G_i) \cup V(G_j))) \cup S_i \cup S_j \cup \{u, v\}$ is a paired dominating set of $G + uv$ of cardinality $\gamma_{pr}(G) - 2$. Therefore G is γ_{pr} -edge-critical.

Now suppose that G is a graph that satisfies Condition 2. Let G_1 be the component such that for some vertex $x \in V(G_1)$, there does not exist a set S' , $|S'| = \gamma_{pr}(G_1) - 2$, that precisely dominates $G_1 - x$ and let G_2, \dots, G_k be the other components of G .

Consider the graph $G + uv$ where u and v lie in the same component G_i , $1 \leq i \leq k$. Then $\gamma_{pr}(G_i + uv) = \gamma_{pr}(G_i) - 2$. Let S_i be a γ_{pr} -set of $G_i + uv$.

Then $(S - V(G_i)) \cup S_i$ is a paired dominating set of $G + uv$ of cardinality $\gamma_{pr}(G) - 2$.

Finally, consider the case where $u \in V(G_i)$ and $v \in V(G_j)$, $1 \leq i < j \leq k$. Then by the hypothesis, there exists a set S_j of cardinality $\gamma_{pr}(G_j) - 2$ that precisely dominates $G_j - v$. Let S_i be a γ_{pr} -set of G_i such that $v \in S_i$ (such a set exists by Theorem 3.4). Then $(S - (V(G_i) \cup V(G_j))) \cup S_i \cup S_j$ is a paired dominating set of $G + uv$ of cardinality $\gamma_{pr}(G) - 2$. Therefore G is γ_{pr} -edge-critical. ■

We will see in Chapter 5 that Theorem 3.17 is equivalent to the following theorem.

Theorem 3.18 *If one of the following conditions holds, then G is a disconnected γ_{pr} -edge-critical graph:*

1. *Every component G_i of G is a nontrivial complete graph or a γ_{pr} -edge-critical graph that is also γ_{pr} -vertex-critical.*
2. *One component G_1 of G is a γ_{pr} -edge-critical graph that is not γ_{pr} -vertex-critical, and every other component G_i , $i \neq 1$, of G is a γ_{pr} -edge-critical graph that is also γ_{pr} -vertex-critical.*

As mentioned in Chapter 2, the graph C_5 is an example of a graph that is both γ_{pr} -edge-critical and γ_{pr} -vertex-critical. An example of a graph that is γ_{pr} -edge-critical but not γ_{pr} -vertex-critical is given in Figure 3.2, for if we remove the vertex v , where $\deg(v) = 6$, then $\gamma_{pr}(G - v) = \gamma_{pr}(G) = 6$.

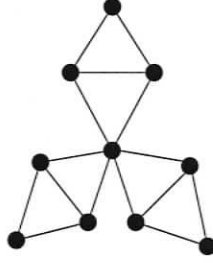


Figure 3.2: A graph that is γ_{pr} -edge-critical but not γ_{pr} -vertex-critical.

Theorem 3.17 gives a method of constructing a γ_{pr} -edge-critical graph out of edge-critical graphs with smaller paired domination numbers, however the result is a disconnected graph. We now give a method of constructing a connected γ_{pr} -edge-critical graph from another edge-critical graph.

For a graph G and vertex $v \in V(G)$ define the *expansion of G via v* , denoted $G_{[v]}$, as the graph with vertex set $V(G) \cup \{v'\}$ and edge set $E(G) \cup \{xv' : x \in N_G[v]\}$. Thus, $G_{[v]}$ is the graph obtained by adding a new vertex v' that has the same closed neighbourhood as v .

Proposition 3.19 *Let G be γ_{pr} -edge-critical and v be a vertex of G such that for each $u \in V(G) - N[v]$, there exists a γ_{pr} -set S of $G + uv$ with $v \in S$. Then $G_{[v]}$ is γ_{pr} -edge-critical.*

Proof. Note firstly that $\gamma_{pr}(G) = \gamma_{pr}(G_{[v]})$.

Let $x, y \in V(G_{[v]})$ such that $xy \in E(\overline{G})$. Consider the graph $G_{[v]} + xy$. (Note that $\{x, y\} \neq \{v', z\}$ where $z \in N_G[v]$.) Say $v' \notin \{x, y\}$, and suppose S is a γ_{pr} -set of $G + xy$. Since G is γ_{pr} -edge-critical, $|S| = \gamma_{pr}(G) - 2 = \gamma_{pr}(G_{[v]}) - 2$. In particular, S dominates v in $G + xy$. So there exists some

vertex $z \in N_G[v] \cap S$, and by the construction of $G_{[v]}$, z also dominates v' . Hence $\gamma_{pr}(G_{[v]} + xy) \leq |S| < \gamma_{pr}(G_{[v]})$.

Say $v' \in \{x, y\}$ and without loss of generality $v' = x$. Then $v \neq y$ and $v \notin \{x, y\}$. By construction of $G_{[v]}$, $vy \notin E(G)$. Hence by the criticality of G , $\gamma_{pr}(G + vy) = \gamma_{pr}(G) - 2 = \gamma_{pr}(G_{[v]}) - 2$. Let S be any γ_{pr} -set of $G + vy$ such that $v \in S$. Then S pairwise dominates $G_{[v]} + vy \cong G_{[v]} + v'y$ and so $\gamma_{pr}(G_{[v]} + v'y) < \gamma_{pr}(G_{[v]})$. ■

Any γ_{pr} -edge-critical graph with a support vertex v is an example of a graph G and vertex $v \in V(G)$ that satisfies the hypothesis of Proposition 3.19. Such graphs with $\gamma_{pr} = 4$ are characterised in Theorem 4.9.

Chapter 4

Properties of 4_{pr} -Edge-Critical Graphs

As shown in Chapter 3, Corollary 3.9, the class of all 4_{pr} -edge-critical graphs is exactly the class of all 3_t -edge-critical graphs combined with the class of all 4_t -super-edge-critical graphs. In [9] it is shown that a graph G is γ_t -super-edge-critical if and only if G is the union of two or more nontrivial complete graphs. Classes of 3_t -edge-critical graphs are also characterised in [9], however there remain some classes of 3_t -edge-critical graphs for which a characterisation is unknown. Although the analogous results of this chapter have been presented in [9], the proofs in the paired domination case are different from those for total domination.

We first give a characterisation of disconnected 4_{pr} -edge-critical graphs.

Proposition 4.1 *The graph G is a disconnected 4_{pr} -edge-critical graph if*

and only if G is the union of two nontrivial complete graphs.

Proof. It is clear that the union of two nontrivial complete graphs is 4_{pr} -edge-critical.

Conversely, suppose that vertices u and v lie in the same component of G and $d(u, v) \geq 2$. The graph $G + uv$ is a disconnected graph and thus has no paired dominating of cardinality two. Therefore $\gamma_{pr}(G + uv) \geq 4$, a contradiction to G being critical. Thus every component of G is complete. Moreover, if G is the union of n nontrivial complete graphs, then $\gamma_{pr}(G) = 2n$. Hence $n = 2$ and G has only two components. ■

4.1 4_{pr} -Edge-Critical Graphs with End-Vertices

By Proposition 3.10, any vertex of a γ_{pr} -edge-critical graph is adjacent to at most one end-vertex. We next show that a stronger result holds for 4_{pr} -edge-critical graphs.

Proposition 4.2 *Any 4_{pr} -edge-critical graph G contains at most one end-vertex.*

Proof. Suppose G contains at least two end-vertices x and y . By Proposition 3.10, x and y are adjacent to x' and y' respectively where $x' \neq y'$. Consider the graph $G + xy$. Since G is critical, $\gamma_{pr}(G + xy) = 2$ and at least one of x and y is in any γ_{pr} -set of $G + xy$. Consider any γ_{pr} -set S of $G + xy$.

Suppose (without loss of generality) $x \in S$, $y \notin S$. Thus $\{x, x'\} = S$ and every vertex in $G - y$ is adjacent to x' . But y' is in $G - y$ and so y' is adjacent to x' . Then $\{x', y'\}$ is a paired dominating set of G , a contradiction to $\gamma_{pr}(G) = 4$.

Suppose that $\{x, y\} = S$. This implies that every vertex in G is adjacent to x or y . Since we are only considering connected graphs, this implies that G is the path x, x', y', y , and $\{x', y'\}$ is a paired dominating set of G , a contradiction to $\gamma_{pr}(G) = 4$. ■

Corollary 4.3 *There are no 4_{pr} -edge-critical trees.*

Thus any γ_{pr} -edge-critical tree T has $\gamma_{pr}(T) \geq 6$. In fact, in Chapter 3, Figure 3.1 shows a family of γ_{pr} -edge-critical trees and it follows from this figure that for every $k \geq 3$ there exists a $2k_{pr}$ -edge-critical tree.

Figure 4.1 illustrates a 4_{pr} -edge-critical graph with an end-vertex. In general, we can construct a 4_{pr} -edge-critical graph with an end-vertex as follows:

- (1) Begin with a K_t , $t \geq 3$, a K_s , $s \geq 2$, and a vertex u .
- (2) Add the edge uv for a vertex $v \in V(K_t)$. Then add edges between K_t and K_s such that every vertex in $K_t - \{v\}$ is adjacent to $|s| - 1$ vertices in K_s and every vertex in K_s is adjacent to at least one vertex in $K_t - \{v\}$.

We can see by this construction that there exist 4_{pr} -edge-critical graphs of order n for every $n \geq 6$. Furthermore, this construction shows that there

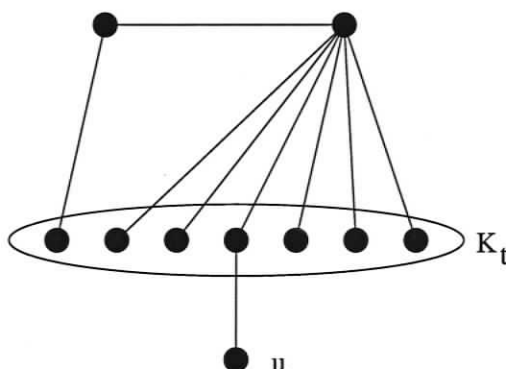


Figure 4.1: A 4_{pr} -edge-critical graph with an end-vertex.

exists a 4_{pr} -edge-critical graph of order n with exactly one end-vertex for every $n \geq 6$. We show in Theorem 4.9 that the graphs produced from this construction are precisely the 4_{pr} -edge-critical graphs with a cut-vertex.

4.2 Characterisation of 4_{pr} -Edge-Critical Graphs with a Cut-Vertex

Proposition 3.15 states that if G is a 4_{pr} -edge-critical graph with a cut-vertex x , then $G - x$ has exactly two components. We now show that any 4_{pr} -edge-critical graph has at most one cut-vertex, and any cut-vertex is a support vertex. We do this in several stages.

Proposition 4.4 *Any 4_{pr} -edge-critical graph G has at most two cut-vertices.*

Proof. Suppose G is a 4_{pr} -edge-critical graph. Let

$$X = \{x \in V(G) : x \text{ is a cut-vertex}\}$$

and suppose that $|X| \geq 3$.

Suppose G has at least three end-blocks. Let G_1, G_2 and G_3 be three of these end-blocks and let $G_i \cap X = \{x_i\}$, $i = 1, 2, 3$. By Proposition 3.15, for any $x \in X$, $k(G - x) = 2$. Thus x_1, x_2 , and x_3 are all distinct vertices. Since G_1, G_2 , and G_3 are end-blocks there exist vertices $u_i \in V(G_i - X)$, $i = 1, 2, 3$. Notice that $d(u_i, u_j) \geq 3$, for $i \neq j$. Also, since u_1 and u_2 are in different end-blocks, $u_1 u_2 \in E(\overline{G})$. Consider the graph $G + u_1 u_2$. By Proposition 3.1 at least one of u_1 and u_2 is in any γ_{pr} -set of $G + u_1 u_2$. Consider any γ_{pr} -set S of $G + u_1 u_2$.

If $\{u_1, u_2\} = S$, or (without loss of generality) $\{u_1, u'_1\} = S$ where $u'_1 \in N_G(u_1)$, then S does not dominate u_3 since $d(u_i, u_3) \geq 3$, $i = 1, 2$, a contradiction.

Therefore G has two end-blocks. Since $|X| \geq 3$, G is a chain of at least four blocks with end-blocks G_1 and G_2 . Since G_1 and G_2 are end-blocks, there exists a vertex $u_i \in V(G_i - X)$ for $i = 1, 2$. Let $V(G_1) \cap X = \{x_1\}$ and $V(G_2) \cap X = \{x_2\}$ and let x_3 be any other cut-vertex. Notice that $d(u_1, x_3) \geq 2$, $d(u_2, x_3) \geq 2$, and $d(x_1, x_2) \geq 2$. Consider the graph $G + u_1 u_2$ with γ_{pr} -set S . Then at least one of u_1 and u_2 is in S .

If $\{u_1, u_2\} = S$, then x_3 is undominated since $d(u_1, x_3) \geq 2$ and $d(u_2, x_3) \geq 2$. Thus $\{u_1, u_2\}$ is not a paired dominating set of $G + u_1 u_2$.

If (without loss of generality), $\{u_1, u'_1\} = S$ where $u'_1 \in N_G(u_1)$, then x_2 is left undominated since $d(x_1, x_2) \geq 2$ and therefore $d(u'_1, x_2) \geq 2$. Thus $\{u_1, u'_1\}$ is not a paired dominating set of $G + u_1 u_2$.

In any of the cases above there is no γ_{pr} -set S of $G + u_1u_2$ with $|S| = 2$ such that at least one of u_1 and u_2 is in S , a contradiction. Thus $|X| \leq 2$. ■

Proposition 4.5 *If G is a 4_{pr} -edge-critical graph with two cut-vertices, then any end-block of G is complete.*

Proof. Let $X = \{x_1, x_2\}$ be the set of cut-vertices of G and let G_1 and G_2 be end-blocks of G such that $V(G_i) \cap X = \{x_i\}$, $i = 1, 2$. Since G_1 and G_2 are end-blocks, there exist $u \in V(G_1 - X)$ and $v \in V(G_2 - X)$. Suppose that G_1 is not complete. Then we may assume without loss of generality that u has been chosen such that $uw \in E(\overline{G})$ for some $w \in V(G_1)$. By Proposition 3.15, for any $x \in X$, $k(G - x) = 2$. Thus any cut-vertex is in at most one end-block. Therefore no vertex in G_2 is adjacent to any vertex in G_1 , with the possible exception of x_2 being adjacent to x_1 . Consider the graph $G + uw$ and any γ_{pr} -set S . By Proposition 3.1, at least one of u and w is in S . Hence $S = \{u, w\}$ or (without loss of generality) $S = \{u, u'\}$, where $u' \in N_G(u)$. But in all cases, v is undominated and so S is not a paired dominating set of $G + uw$, a contradiction. Therefore any end-block of G is complete. ■

Proposition 4.6 *Any 4_{pr} -edge-critical graph G has at most one cut-vertex.*

Proof. By Proposition 4.4, G has at most two cut-vertices. Suppose $X = \{x_1, x_2\}$ is the set of cut-vertices of G and let G_1 and G_2 be end-blocks of

G such that $V(G_i) \cap X = \{x_i\}$, $i = 1, 2$. Thus there exist vertices $u_i \in V(G_i - X)$, $i = 1, 2$. By Proposition 4.5, G_1 and G_2 are complete. Consider $G + u_1u_2$. Since G is critical, $\gamma_{pr}(G + u_1u_2) = 2$ and at least one of u_1 and u_2 is in any γ_{pr} -set of $G + u_1u_2$. Consider any γ_{pr} -set S of $G + u_1u_2$.

If $\{u_1, u_2\} \subseteq S$, then every vertex in G is adjacent to u_1 or u_2 . Since we are only considering connected graphs, G is exactly the graph $(K_n \cup K_m) + x_1x_2$ where $x_1 \in V(K_n)$, $x_2 \in V(K_m)$, and m and n are any positive integers. But then $\{x_1, x_2\}$ is a paired dominating set of G , a contradiction to $\gamma_{pr}(G) = 4$.

Suppose without loss of generality that $S = \{u_1, w\}$ with $w \in N(u_1)$. To dominate $x_2 \neq u_2$, $w = x_1$ and $x_1x_2 \in E(G)$. Hence every vertex in $G - u_2$ is adjacent to u_1 or x_1 . But the only vertices in $G - u_2$ that are adjacent to u_1 are all those in G_1 . Thus every vertex in $G - G_1 - u_2$ is adjacent to x_1 . But then $\{x_1, x_2\}$ is a paired dominating set of G , a contradiction to $\gamma_{pr}(G) = 4$.

Therefore $|X| \leq 1$, and any 4_{pr} -edge-critical graph G has a most one cut-vertex. ■

From this we conclude that any 4_{pr} -edge-critical graph has at most two blocks. Suppose G is a 4_{pr} -edge-critical graph with a cut-vertex x . Let G_1 and G_2 be the blocks of G and let $A = N(x) \cap G_1$ and $B = N(x) \cap G_2$.

Proposition 4.7 *If G is a 4_{pr} -edge-critical graph with a cut-vertex x and blocks G_1 and G_2 , then $\langle N(x) \cap V(G_1) \rangle$ and $\langle N(x) \cap V(G_2) \rangle$ are complete.*

Proof. Let $A = N(x) \cap V(G_1)$ and suppose that $\langle A \rangle$ is not complete. Thus there exist vertices $y, z \in A$ such that $yz \in E(\overline{G})$. Consider the graph $G + yz$

and any γ_{pr} -set S of $G + yz$. By Proposition 3.1, at least one of y and z is in S . However $x \in S$ as x is the only vertex in $V(G_1)$ that dominates vertices in G_2 . Without loss of generality, say that $S = \{y, x\}$. Then, by Proposition 3.2, S precisely dominates $G - z$. However, $xz \in E(G)$ and so S also dominates z , a contradiction. Therefore, $\langle A \rangle$ is complete and likewise $\langle N(x) \cap V(G_2) \rangle$ is complete. ■

Proposition 4.8 *Let G be a 4_{pr} -edge-critical graph with a cut-vertex x . Then x is adjacent to an end-vertex.*

Proof. Let G_1 and G_2 be the blocks of G , $A = N(x) \cap V(G_1)$ and $B = N(x) \cap V(G_2)$ and suppose that $|A| \geq 2$ and $|B| \geq 2$. Let $\{y_1, y_2\} \subseteq A$ and $\{z_1, z_2\} \subseteq B$. By the Proposition 4.7, $\langle A \rangle$ and $\langle B \rangle$ are complete. Consider the graph $G + y_1z_1$ and consider any γ_{pr} -set S of $G + y_1z_1$.

Suppose firstly that $\{y_1, z_1\} = S$. Then every vertex of G is adjacent to at least one of y_1 or z_1 . Thus $S' = \{y_2, y_1, x, z_1\}$ is a paired dominating set of G with the unique perfect matching $M = \{y_2y_1, xz_1\}$. By Corollary 1.5, $\text{pn}(\{y_1, y_2\}, S') \neq \emptyset$. Let $u_1 \in \text{pn}(\{y_1, y_2\}, S')$. Thus $u_1x \in E(\overline{G})$. However, $S'' = \{y_1, x, z_1, z_2\}$ is also a γ_{pr} -set of G with the unique perfect matching $M' = \{y_1x, z_1z_2\}$. Also, $\text{pn}(\{z_1, z_2\}, S'') \neq \emptyset$. Thus there exists a vertex $v_1 \in \text{pn}(\{z_1, z_2\}, S'')$, which implies that $v_1x \in E(\overline{G})$. Notice that $u_1v_1 \in E(\overline{G})$ since $u_1 \in V(G_1) - \{x\}$ and $v_1 \in V(G_2) - \{x\}$. Consider the graph $G + u_1v_1$ and any γ_{pr} -set S''' of $G + u_1v_1$.

Since x is undominated by $\{u_1, v_1\}$, $S''' \neq \{u_1, v_1\}$. Hence by symmetry we may assume without loss of generality that $S''' = \{u_1, w\}$ where $w \in N_G(u_1)$.

Then z_1 is undominated as $d(u_1, z_1) \geq 3$. Thus $\{u_1, w\}$ is not a γ_{pr} -set of $G + u_1v_1$, a contradiction.

Thus we may assume without loss of generality that $\{y_1, w\} = S$, where $w \in N_G(y_1)$. To dominate z_2 , $w = x$. Then S precisely dominates $G - z_1$. However, $z_1x \in E(G)$ and so S also dominates z_1 , a contradiction.

In all the above cases we reach a contradiction and so not both $|A| \geq 2$ and $|B| \geq 2$. Say $|A| = 1$ and let $y \in A$. If y is adjacent to any vertex in $G_1 - x$, then y is a cut-vertex of G , contradicting Proposition 4.6. It follows that y is an end-vertex. Thus if G has a cut-vertex x then x is adjacent to an end-vertex. ■

We now give a characterisation of 4_{pr} -edge-critical graphs with a cut-vertex.

Theorem 4.9 *Let G be a graph with a pendant edge xv (with cut-vertex x and end-vertex v) and let $A = N(x) - \{v\}$ and $B = V(G) - N[x]$. Then G is 4_{pr} -edge-critical if and only if*

1. $\langle A \rangle$ is complete and $|A| \geq 2$,
2. $\langle B \rangle$ is complete and $|B| \geq 2$, and

3. every vertex in A is adjacent to $|B| - 1$ vertices in B and every vertex in B is adjacent to at least one vertex in A .

Proof. Let G be a 4_{pr} -edge-critical graph with pendant edge xv and let A and B be as defined above. Since $G \neq K_2$, $A \neq \phi$.

1) By Proposition 4.7, A is complete. Suppose $|A| = 1$. Then the vertex in A is a cut-vertex, a contradiction to Proposition 4.6 as any 4_{pr} -edge-critical graph has at most one cut-vertex. Thus $|A| \geq 2$.

2) Let $y \in A$ be a vertex such that there exists a vertex $u \in N(y) \cap B$. Such a vertex exists, otherwise $B = \phi$ and $\{x, v\}$ is a paired dominating set of G , a contradiction. If $B \subseteq N(y)$, then $\{y, x\}$ is a paired dominating set of G , which is not the case. Thus there exists a vertex $z \in B$ such that $z \notin N(y)$ and thus $|B| \geq 2$. Suppose B is not complete. Then there exist vertices $w_1, w_2 \in B$ such that $w_1w_2 \in E(\overline{G})$. Consider the graph $G' = G + w_1w_2$ and consider any γ_{pr} -set S of $G + w_1w_2$.

Without loss of generality $S = \{w_1, w\}$ where $w \in N_{G'}(w_1)$. But then v is undominated as $d(w_1, v) \geq 3$ and $d(w, v) \geq 2$ for all $w \in N_{G'}(w_1)$. Therefore B is complete.

3) For any $a \in A$, let $b \in B$ such that $ab \in E(\overline{G})$. Such a vertex exists for otherwise $\gamma_{pr}(G) = 2$. For any γ_{pr} -set S of $G + ab$, $x \in S$ to dominate v . Hence $S = \{a, x\}$ by Proposition 3.1. Then $\{a, x\}$ precisely dominates $G - b$, implying that every vertex in $B - \{b\}$ is adjacent to a . Thus for every $a \in A$ there is exactly one vertex $b \in B$ not adjacent to a . Suppose there is a vertex $b \in B$ such that $ab \in E(\overline{G})$ for all $a \in A$. Consider the graph

$G + bv$, and consider any γ_{pr} -set S of $G + ab$. Then $S \neq \{b, v\}$, otherwise a is undominated as $d(b, a) \geq 2$ and $d(v, a) \geq 2$. If $S = \{b, b'\}$ where $b' \in N_G(b)$, then x is undominated as $d(b, x) \geq 2$ and $d(b', x) \geq 2$, and if $S = \{v, x\}$, then there exists $c \in B - \{b\}$ and c is undominated as $d(c, v) \geq 2$ and $d(c, x) \geq 2$.

Therefore every $b \in B$ is adjacent to at least one $a \in A$.

Conversely it is easy to check that any graph satisfying the three given conditions is 4_{pr} -edge-critical. ■

Corollary 4.10 *If G is a 4_{pr} -edge-critical graph with a cut-vertex, then $\text{diam}(G) = 3$.*

Corollary 4.11 *If G is a 4_{pr} -edge-critical graph with a cut-vertex x , then x is in any γ_{pr} -set of G .*

By Proposition 3.19 and Corollary 4.11, if G is 4_{pr} -edge-critical and x is a cut-vertex of G , then $G_{[x]}$ is 4_{pr} -edge-critical. Moreover, $\text{diam}(G_{[x]}) = 3$, and $G_{[x]}$ is 2-connected.

Chapter 5

Properties of

γ_{pr} -Vertex-Critical Graphs

In this chapter we shift our attention to γ_{pr} -vertex-critical graphs and give properties of such graphs. We begin with an alternative to the definition of γ_{pr} -vertex-criticality.

Proposition 5.1 *The graph G is γ_{pr} -vertex-critical if and only if for every $v \in V(G)$ there exists a set S , $|S| \leq \gamma_{pr}(G) - 2$, such that S precisely dominates $G - v$.*

Proof. Suppose G is γ_{pr} -critical. Then by definition, $\gamma_{pr}(G - v) \leq \gamma_{pr}(G) - 2$ for every $v \in V(G)$. Thus for every $v \in V(G)$ there exists a set S , $|S| \leq \gamma_{pr}(G) - 2$, such that S precisely dominates $G - v$.

Conversely, suppose that for every $v \in V(G)$ there exists a set S , $|S| \leq \gamma_{pr}(G) - 2$, such that S precisely dominates $G - v$. Then $\gamma_{pr}(G - v) \leq |S| <$

$\gamma_{pr}(G)$ for every $v \in V(G)$ and so G is γ_{pr} -vertex-critical. ■

As we saw in Chapter 3, Proposition 3.3, the addition of an edge decreases the paired domination number by at most two. We give an analogous result for vertex deletion.

Proposition 5.2 *If G is γ_{pr} -vertex-critical, then $\gamma_{pr}(G - v) = \gamma_{pr}(G) - 2$ for any $v \in V(G)$.*

Proof. Let G be a γ_{pr} -vertex-critical graph. Consider the graph $G - v$, $v \in V(G)$. By Proposition 5.1, there exists a set S with $|S| \leq \gamma_{pr}(G) - 2$ such that S precisely dominates $G - v$. This implies that $x \notin S$ for every $x \in N(v)$. Let $v' \in N(v)$. Then $S' = S \cup \{v, v'\}$ is a paired dominating set of G . Therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G - v) + 2$. ■

As well, we also have a vertex deletion analogue to Proposition 3.4.

Proposition 5.3 *If G is γ_{pr} -vertex-critical, then for every $v \in V(G)$ there exists a γ_{pr} -set S such that $v \in S$.*

Proof. By Propositions 5.1 and 5.2, for every vertex $v \in V(G)$ there exists a set $S' \subseteq V(G)$ with $|S'| = \gamma_{pr}(G) - 2$ such that S' precisely dominates $G - v$. Thus no neighbour of v is in S' . Therefore, for any $x \in N_G(v)$, the set $S = S' \cup \{x, v\}$ is a paired dominating set of G of cardinality $\gamma_{pr}(G)$ that contains v . ■

Proposition 5.4 *If G has two vertices u and v such that $N[u] \subseteq N[v]$, then G is not γ_{pr} -vertex-critical.*

Proof. If v is a support vertex, then $\gamma_{pr}(G - v) = \infty$ because $G - v$ has an isolated vertex. Thus G is not γ_{pr} -vertex-critical. If v is not a support vertex, then $\gamma_{pr}(G - v)$ is finite. Any γ_{pr} -set of $G - v$ contains a vertex in $N[u]$ and hence is a paired dominating set of G . In this case, too, G is not γ_{pr} -vertex-critical. ■

Corollary 5.5 *If G has a (nonisolated) vertex v such that $\langle N(v) \rangle$ is complete, then G is not γ_{pr} -vertex-critical.*

Proof. For any $u \in N[v]$, $N[v] \subseteq N[u]$ and the result follows from Proposition 5.4. ■

Since each chordal graph has a vertex v such that $N(v)$ is complete (see for example Exercise 8.13 in [2]), we have the following corollary.

Corollary 5.6 *No chordal graph is γ_{pr} -vertex-critical.*

The 4_{pr} -edge-critical graphs with cut-vertices are characterised in Theorem 4.9. In contrast, there are no 4_{pr} -vertex-critical graphs with cut-vertices.

Proposition 5.7 *If G is 4_{pr} -vertex-critical, then G is 2-connected.*

Proof. Suppose G is 4_{pr} -vertex-critical with a cut-vertex x . Then, since G is vertex-critical, $\gamma_{pr}(G - x) = 2$. But $k(G - x) = 2$, a contradiction. ■

We have also seen that the class of all 4_{pr} -edge-critical graphs is exactly the union of the class of all 3_t -edge-critical graphs and 4_t -super-edge-critical graphs. For γ_{pr} -vertex-critical graphs we obtain a similar result.

We say that a graph G is γ_t -vertex-critical if $\gamma_t(G - v) = \gamma_t(G) - 1$ for any $v \in V(G)$ and that G is γ_t -super-vertex-critical if $\gamma_t(G - v) = \gamma_t(G) - 2$ for any $v \in V(G)$.

Proposition 5.8 *If G is 4_{pr} -vertex-critical, then G is 3_t -vertex-critical or 4_t -super-vertex-critical.*

Proof. Recall that $\gamma_t(G) \leq \gamma_{pr}(G)$. If $\gamma_{pr}(G) = 4$, then $\gamma_t(G) = 3$ or $\gamma_t(G) = 4$ since if $\gamma_t(G) = 2$ then $\gamma_{pr}(G) = 2$. Since G is γ_{pr} -vertex-critical, $\gamma_{pr}(G - v) = 2$, and so $\gamma_t(G - v) = 2$ since $2 \leq \gamma_t(G - v) \leq \gamma_{pr}(G - v) = 2$. ■

In fact, there do not exist graphs that are γ_t -super-vertex-critical, as shown in the following proposition.

Proposition 5.9 *For any graph G without isolated vertices and any vertex v of G such that v is not a support vertex, $\gamma_t(G) - 1 \leq \gamma_t(G - v)$.*

Proof. Consider the graph $G - v$ with γ_t -set S (where S exists since v is not a support vertex). Since G is isolate-free, $N(v) \neq \emptyset$. If there exists a vertex

$x \in N(v) \cap S$, then $\gamma_t(G) \leq \gamma_t(G-v)$. Suppose that $N(v) \cap S = \emptyset$ and consider a vertex $y \in N(v)$. Since S is a γ_t -set of $G-v$ there exists a $z \in N(y) \cap S$. Thus $S \cup \{y\}$ is a total dominating set of G and $\gamma_t(G) \leq \gamma_t(G-v) + 1$. ■

Additionally, it follows from this result and from Proposition 5.8 that if G is 4_{pr} -vertex-critical, then G is 3_t -vertex-critical. The converse is also true and follows from Propositions 3.8, 5.8 and 5.9. Therefore the set of all 4_{pr} -vertex-critical graphs is exactly the set of all 3_t -vertex-critical graphs.

We now give the vertex deletion analogue of Theorem 3.7.

Theorem 5.10 *If $\gamma_t(G) = \gamma_{pr}(G) - 1$ and G is γ_{pr} -vertex-critical, then G is γ_t -vertex-critical. If $\gamma_t(G) = \gamma_{pr}(G)$ and G is γ_{pr} -vertex-critical, then G is not γ_t -vertex-critical.*

Proof. Suppose that $\gamma_t(G) = \gamma_{pr}(G) - 1$ and G is γ_{pr} -vertex-critical. Consider the graph $G-v$, $v \in V(G)$. Since G is γ_{pr} -vertex-critical, $\gamma_{pr}(G-v) = \gamma_{pr}(G) - 2$. But $\gamma_t(G-v) \leq \gamma_{pr}(G-v) = \gamma_t(G) - 1$, and by Proposition 5.9, $\gamma_t(G) - 1 = \gamma_t(G-v)$. Thus G is γ_t -vertex-critical.

Now suppose that $\gamma_t(G) = \gamma_{pr}(G)$ and G is γ_{pr} -vertex-critical. Then $\gamma_{pr}(G-v) = \gamma_{pr}(G) - 2 = \gamma_t(G) - 2$, and $\gamma_t(G-v) \leq \gamma_{pr}(G-v) = \gamma_t(G) - 2$, a contradiction to Proposition 5.9. Thus G is not γ_t -vertex-critical. ■

As in [1], we define a *coalescence* of disjoint graphs G and H as the graph $G.H$ obtained by identifying one vertex of H and one vertex of G . The identified vertex thus becomes a cut-vertex of $G.H$.

Proposition 5.11 *For any graphs G and H ,*

$$\gamma_{pr}(G) + \gamma_{pr}(H) - 2 \leq \gamma_{pr}(G.H) \leq \gamma_{pr}(G) + \gamma_{pr}(H).$$

Proof. Let v be the vertex that is common to G and H and let S be a γ_{pr} -set of $G.H$. If $v \notin S$, then $S \cap V(G)$ pairwise dominates G and $S \cap V(H)$ pairwise dominates H and so in this case $\gamma_{pr}(G) + \gamma_{pr}(H) \leq \gamma_{pr}(G.H)$.

Say $v \in S$ and let v' be the partner of v in S . Without loss of generality say that $v' \in V(H)$. If $N_G(v) \subseteq S$, then $S \cap (V(G) - \{v\})$ pairwise dominates G and $S \cap V(H)$ pairwise dominates H and so in this case $\gamma_{pr}(G) + \gamma_{pr}(H) \leq \gamma_{pr}(G.H)$. If $N_G(v) \not\subseteq S$, then there exists a vertex $u \in N_G(v)$ such that $u \notin S$. Then $S \cap V(H)$ pairwise dominates H and $(S \cap V(G)) \cup \{u\}$ pairwise dominates G and in this case $\gamma_{pr}(G) + \gamma_{pr}(H) \leq \gamma_{pr}(G.H) + 2$.

For the upper bound, let S_1 be a γ_{pr} -set of G and S_2 be a γ_{pr} -set of H . If $v \notin S_1 \cap S_2$, then $S_1 \cup S_2$ is a paired dominating set of $G.H$ and thus $\gamma_{pr}(G.H) \leq \gamma_{pr}(G) + \gamma_{pr}(H)$. Say $v \in S_1 \cap S_2$ and let u_1 be the partner of v in S_1 and u_2 be the partner of v in S_2 . If $N_G(u_1) \subseteq S_1$, then $(S_1 - \{u_1\}) \cup S_2$ is a paired dominating set of $G.H$ and so $\gamma_{pr}(G.H) \leq \gamma_{pr}(G) + \gamma_{pr}(H) - 1$. A similar result follows if $N_H(u_2) \subseteq S_2$. Thus assume that both $N_G(u_1) \not\subseteq S_1$ and $N_H(u_2) \not\subseteq S_2$. Let $y \in N_G(u_1) - \{v\}$ such that $y \notin S_1$. Then $(S_1 - \{v\}) \cup \{y\} \cup S_2$ is a pairwise dominating set of $G.H$ and so $\gamma_{pr}(G.H) \leq \gamma_{pr}(G) + \gamma_{pr}(H)$. ■

Proposition 5.12 *If G and H are γ_{pr} -vertex-critical graphs, then*

$$\gamma_{pr}(G.H) = \gamma_{pr}(G) + \gamma_{pr}(H) - 2.$$

Proof. Let v be the vertex common to both G and H in $G.H$. Since G is γ_{pr} -vertex-critical there exists a γ_{pr} -set S_1 of cardinality $\gamma_{pr}(G) - 2$ that precisely dominates $G - v$. Since H is critical there exists a γ_{pr} -set S_2 of cardinality $\gamma_{pr}(H) - 2$ that precisely dominates $H - v$. Thus $N_G(v) \cap S_1 = \phi$. Let $v' \in N_G(v)$. Then $S_1 \cup S_2 \cup \{v, v'\}$ is a paired dominating set of $G.H$. Therefore $\gamma_{pr}(G.H) \leq \gamma_{pr}(G) - 2 + \gamma_{pr}(H) - 2 + 2 = \gamma_{pr}(G) + \gamma_{pr}(H) - 2$ and by Proposition 5.11, $\gamma_{pr}(G) + \gamma_{pr}(H) - 2 \leq \gamma_{pr}(G.H)$. The result follows. ■

Proposition 5.13 *If both G and H are γ_{pr} -vertex-critical graphs, then $G.H$ is γ_{pr} -vertex-critical.*

Proof. Let v be the vertex common to both G and H in $G.H$. Say that both G and H are γ_{pr} -vertex-critical. Thus by Proposition 5.12, $\gamma_{pr}(G.H) = \gamma_{pr}(G) + \gamma_{pr}(H) - 2$. For an arbitrary $x \in V(G) \cup V(H)$, consider the graph $G.H - x$. If $x = v$, then $G.H - v$ is the disjoint union of $G - v$ and $H - v$ and since both G and H are γ_{pr} -vertex-critical, $\gamma_{pr}(G.H - v) = \gamma_{pr}(G - v) + \gamma_{pr}(H - v) = \gamma_{pr}(G) - 2 + \gamma_{pr}(H) - 2 = \gamma_{pr}(G.H) - 2$.

If $x \neq v$, then, without loss of generality, say $x \in V(G)$. Note that $G.H - x = (G - x).H$. Since G is γ_{pr} -vertex-critical, there exists a set S_1 of cardinality $\gamma_{pr}(G) - 2$ that pairwise dominates $G - x$ and since H is γ_{pr} -

vertex-critical, there exists a set S_2 of cardinality $\gamma_{pr}(H) - 2$ that precisely dominates $H - v$. Thus $S_1 \cup S_2$ is a paired dominating set of $G.H - x$ of cardinality $\gamma_{pr}(G) + \gamma_{pr}(H) - 4 = \gamma_{pr}(G.H) - 2$. Thus $G.H$ is γ_{pr} -vertex-critical. ■

Proposition 5.14 *If every block of G is a γ_{pr} -vertex-critical graph, then G is a γ_{pr} -vertex-critical graph.*

Proof. We use induction on the number of blocks n . The result is trivially true when $n = 1$. Let G have blocks $G_1, G_2, \dots, G_n, G_{n+1}$, indexed so that G_{n+1} contains only one cut-vertex of G . Then $G = H.G_{n+1}$, where H is the subgraph composed of the blocks G_1, G_2, \dots, G_n . The result now follows by induction and Proposition 5.13. ■

Hence we have a method of constructing a connected γ_{pr} -vertex-critical graph from smaller γ_{pr} -vertex-critical graphs.

Chapter 6

The Diameter of Critical Graphs

It is intuitively clear that graphs with fixed paired domination number cannot have arbitrary diameter. This idea also suggests that k_{pr} -edge-critical and k_{pr} -vertex-critical graphs have smaller diameter than the maximum diameter realized by graphs with $\gamma_{pr} = k$. In this chapter we show that these notions are indeed correct.

We first bound the diameter of graphs with $\gamma_{pr} = k$. We begin by defining some notation and proving a lemma.

For any graph G with $\text{diam}(G) = m$, let P be a diametrical path of G with end-vertex x . Define the *levels* V_0, V_1, \dots, V_m of G (with respect to x) by $V_i = \{v \in V(G) : d(x, v) = i\}$. (Notice that $V_0 = \{x\}$ and $V_m \neq \phi$.)

Lemma 6.1 *If S is a paired dominating set of a graph G and a and b are*

paired in S , then $\{a, b\}$ dominates at most four levels of G .

Proof. Suppose there exist vertices $a, b \in S$ such that a and b are paired in S and that these two vertices dominate at least five levels of G , call them $V_j, V_{j+1}, \dots, V_{j+l}$ ($l \geq 4$). Thus every vertex in $V_j, V_{j+1}, \dots, V_{j+l}$ is adjacent to at least one of a or b . Take $u \in V_j$ and $v \in V_{j+l}$. This implies that $d(u, v) \geq l \geq 4$. But u is adjacent to at least one of a and b and v is also adjacent to at least one of a and b . Thus $d(u, v) \leq 3$, a contradiction. ■

Theorem 6.2 For any connected graph G , $\text{diam}(G) \leq 2\gamma_{pr}(G) - 1$.

Proof. Suppose $\gamma_{pr}(G) = 2k$ and let S be a γ_{pr} -set of G . Let $m = \text{diam}(G)$ and let V_0, V_1, \dots, V_m be the levels of G with respect to a peripheral vertex x .

By Lemma 6.1, every pair of vertices $a, b \in S$ dominates at most four levels of G . Since there are k such pairs in S , this implies that at most $4k$ levels of G are dominated. Hence if $\text{diam}(G) \geq 4k$, at least one level of G is undominated. Therefore $\text{diam}(G) \leq 4k - 1$. ■

6.1 The Diameter of γ_{pr} -Edge-Critical Graphs

In general, one can do better than the bound in Theorem 6.2 if the graph G is γ_{pr} -edge-critical. As we have seen following the result of Theorem 4.9, if G is a 4_{pr} -edge-critical graph with a cut-vertex, then $\text{diam}(G) = 3$. This in fact is the maximum diameter of a 4_{pr} -edge-critical graph.

Proposition 6.3 *If G is a connected 4_{pr} -edge-critical graph, then*

$$\text{diam}(G) \leq 3.$$

Proof. Let G be a 4_{pr} -edge critical graph of order n . Suppose $\text{diam}(G) \geq 4$ and consider vertices $u, v \in V(G)$ such that $d(u, v) = 4$. Then there exists a shortest $u - v$ path u, x_1, x_2, x_3, v in G . Consider the graph $G + uv$. Then $\gamma_{pr}(G + uv) = 2$ and for any γ_{pr} -set S of $G + uv$ at least one of u and v is in S . If $S = \{u, v\}$, then x_2 is undominated, a contradiction. If, without loss of generality, $S = \{u, x\}$, where $x \neq v$, then x_3 is undominated, a contradiction. Therefore $\text{diam}(G) \leq 3$. ■

This idea can be generalised for $\gamma_{pr} \geq 4$.

Proposition 6.4 *If G is a connected k_{pr} -edge-critical graph, then $\text{diam}(G) \leq 2k - 5$.*

Proof. Since G is k_{pr} -edge-critical, $\gamma_{pr}(G + e) = k - 2$ for any $e \in E(\overline{G})$ and so by Proposition 6.2,

$$\text{diam}(G + e) \leq 2k - 5. \quad (6.1)$$

Suppose firstly that $\text{diam}(G) \geq 2k - 3$ and consider the nonempty levels V_0, \dots, V_{2k-3} with respect to a peripheral vertex v . Let $v' \in V_2$. Then $vv' \in E(\overline{G})$ and $\gamma_{pr}(G + vv') = k - 2$. However, for any $u \in V_{2k-3}$, $d_{G+vv'}(v, u) \geq 2k - 4$, contradicting (6.1). Therefore $\text{diam}(G) \leq 2k - 4$.

Suppose that $\text{diam}(G) = 2k - 4$, let $k = 2l$ and consider the nonempty levels V_0, \dots, V_{4l-4} with respect to a peripheral vertex v . Choose $u \in V_{4l-4}$. Let $G' = G + uv$. Then $\gamma_{pr}(G') = 2(l - 1)$ and any γ_{pr} -set D of G' contains at least one of u and v , where u and v are paired in D if D contains both.

Say that both u and v are in D . Then this pair dominates at most the vertices in V_{4l-4}, V_{4l-5}, V_1 and $V_0 = \{v\}$. So there are $2l - 4$ vertices, or $l - 2$ pairs, left in D to dominate $4l - 3 - 4 = 4l - 7$ levels. However, by Lemma 6.1, each pair dominates at most four levels, and so the $l - 2$ pairs dominate at most $4(l - 2) = 4l - 8 < 4l - 7$ levels. Therefore at least one level is left undominated.

Now suppose that $u \in D$ and $v \notin D$.

Say that in D , u is paired with $u' \in V_{4l-4} \cup V_{4l-5}$. Then this pair dominates at most the levels $V_{4l-4}, V_{4l-5}, V_{4l-6}$ and $V_0 = \{v\}$. So there are $2l - 4$ vertices, or $l - 2$ pairs, left in D to dominate $4l - 3 - 4 = 4l - 7$ levels. This is impossible as shown above. Therefore at least one level is left undominated.

Finally, suppose that $u \notin D$ and $v \in D$. Then since $V_0 = \{v\}$, v is partnered with a vertex $v' \in V_1$. Then this pair dominates at most the levels V_{4l-4}, V_2, V_1 and $V_0 = \{v\}$. So there are $2l - 4$ vertices, or $l - 2$ pairs, left in D to dominate $4l - 3 - 4 = 4l - 7$ levels. As before, this is impossible. Therefore at least one level is left undominated.

Hence $\text{diam}(G) \leq 2k - 5$. ■

We already have shown that this bound can be achieved when $k = 4$, that is, there exists a 4_{pr} -edge-critical graph of diameter 3. Such a graph can be

seen in Figure 2.4. For $k > 4$, the bound in Proposition 6.4 can be improved.

Theorem 6.5 *If G is a connected k_{pr} -edge-critical graph with $k \geq 6$, then $\text{diam}(G) \leq 2k - 6$.*

Proof. Let G be a k_{pr} -edge critical graph where $k > 4$. We know from Proposition 6.4 that $\text{diam}(G) \leq 2k - 5$ and so suppose that $\text{diam}(G) = 2k - 5$. Since G is k_{pr} -edge-critical, $\gamma_{pr}(G + e) = k - 2$ for any $e \in E(\overline{G})$. Let $k = 2l$ and consider the nonempty levels $V_0, V_1, \dots, V_{2k-5}$ with respect to a peripheral vertex v . Let $u \in V_4$. Then $vu \in E(\overline{G})$, $\gamma_{pr}(G + uv) = 2l - 2$ and any γ_{pr} -set D of $G + uv$ contains at least one of u and v , and u and v are paired in D if D contains both.

Suppose first that both u and v are in D . Then the pair u, v dominates at most V_0, V_1, V_3, V_4 and V_5 . Let u', v' be paired in D such that $\{u', v'\}$ dominates vertices in V_2 . Then by Lemma 6.1, $\{u', v'\}$ does not dominate any vertices in $V_t, t \geq 6$. Hence the remaining $2(l - 3)$ vertices in D dominate all of $\bigcup_{t=6}^{2k-5} V_t$, a total of $4l - 10$ levels of G . (Note that $4l - 10 > 0$ because $l \geq 3$.) Again, using Lemma 6.1 and the pigeonhole principle, we see that this is impossible.

Suppose that $v \notin D, u \in D$ and u is paired with the vertex $u' \in V_3$. Then the pair $\{u, u'\}$ dominates at most the levels V_0, V_2, V_3, V_4 and V_5 . At least one more pair of vertices in D is needed to dominate V_1 , and by Lemma 6.1, no such pair dominates any vertex in $V_t, t \geq 6$. This leads to a contradiction as above.

Suppose next that $v \notin D$, $u \in D$ and u is paired with the vertex $u' \in V_5$. Then the pair u, u' dominates at most the levels V_0, V_3, V_4, V_5 and V_6 . Again at least one more pair is needed to dominate V_1 and V_2 , and by Lemma 6.1, no such pair dominates any vertex in V_t , $t \geq 7$. Hence the remaining $2(l-3)$ vertices in D dominate at least $4l-11$ levels. By the pigeonhole principle, at least one pair dominates at least five levels, contradicting Lemma 6.1.

If $v \notin D$, $u \in D$ and u is paired with the vertex $u' \in V_4$, we also obtain a contradiction by using the same arguments as above. Hence we conclude that $u \notin D$ and so $v \in D$.

Necessarily, v is paired with the vertex $v' \in V_1$. Then the pair $\{v, v'\}$ dominate at most the levels V_0, V_1, V_2 and V_4 . At least one more pair is needed to dominate V_3 , and by Lemma 6.1, no such pair dominates any vertices in V_t , $t \geq 7$. This leads, once again, to a contradiction as before.

Therefore $\text{diam}(G) \leq 2k - 6$. ■

It is not known whether the bound in Theorem 6.5 is the best possible if G is 6_{pr} -edge-critical. However, it can be shown that if G is k_{pr} -edge-critical, $k \geq 8$, then $\text{diam}(G) \leq 2k - 7$. However, even this bound may not be the best possible, and we leave it as an open problem to determine a sharp upper bound for the diameter of γ_{pr} -edge-critical graphs.

Thus far we have only presented results concerning the maximum diameter of γ_{pr} -edge-critical graphs. However one could request a lower bound on the diameter of γ_{pr} -edge-critical graphs. Clearly, if G is a γ_{pr} -edge-critical graph, then $\text{diam}(G) > 1$ since the only graphs with diameter 1 are the com-

plete graphs, and these are not γ_{pr} -edge-critical. Thus the first possibility for the diameter of a γ_{pr} -edge-critical graph is 2. This is in fact the best possible lower bound, as we show next.

Theorem 6.6 *For any $k \geq 4$, there exists a k_{pr} -edge-critical graph of diameter 2.*

Proof. For $t = 2l$, $l \geq 2$, consider the Cartesian product of the graph K_t with itself, i.e. the graph $G_t = K_t \times K_t$. We can think of G_t as having t disjoint copies of K_t in “rows” and t disjoint copies of K_t in “columns”. In other words, we consider the vertices of G_t as a matrix, where vertex v_{ij} is in the i^{th} row (copy of K_t) and the j^{th} column (copy of K_t). For ease of discussion, we will use the words row and column to mean a “copy of K_t ”.

We show first that $\gamma_{pr}(G_t) = t$. Since $\{v_{11}, v_{21}, \dots, v_{t1}\}$ is a dominating set with a perfect matching, we have $\gamma_{pr}(G_t) \leq t$. Suppose $\gamma_{pr}(G_t) \leq t - 2$. Then for any γ_{pr} -set S there exists an i such that S does not have a vertex in row i . Now any vertex in S dominates only one vertex of row i , implying that at most $t - 2$ of the t vertices of row i are dominated. Thus $\gamma_{pr}(G_t) = t$.

We now show that $\text{diam}(G) = 2$. Clearly, for $t \geq 2$, G_t is not complete and so $\text{diam}(G) \geq 2$. Consider vertices $x, y \in V(G_t)$. If x and y are in the same row copy of K_t , i.e. $x = v_{ij}$ and $y = v_{ik}$, then $d(x, y) = 1$. If x and y are in the same column copy of K_t , i.e. $x = v_{ij}$ and $y = v_{kj}$, then $d(x, y) = 1$. If x and y are not in the same row or column copy of K_t , i.e. $x = v_{hi}$ and $y = v_{jk}$ where $h \neq j$ and $i \neq k$, then let $z = v_{hk}$. Thus $d(x, z) = 1$ and

$d(z, y) = 1$ and so $d(x, y) \leq 2$. It follows that $\text{diam}(G_t) = 2$.

Now if G_t is γ_{pr} -edge-critical, then we are finished. Otherwise, by Proposition 3.5, G_t is a spanning subgraph of some γ_{pr} -edge-critical graph G' . It follows that $\gamma_{pr}(G') = \gamma_{pr}(G_t)$ and, since G' is not complete, the diameter of G' is 2. ■

6.2 The Diameter of γ_{pr} -Vertex-Critical Graphs

As with γ_{pr} -edge-critical graphs, no γ_{pr} -vertex-critical graph realizes the upper bound on diameter as given in Proposition 6.2. This simple result is given in the following proposition.

Proposition 6.7 *If G is a connected γ_{pr} -vertex-critical graph, then*

$$\text{diam}(G) \leq 2\gamma_{pr}(G) - 4.$$

Proof. Consider the graph $G - v$ for some $v \in V(G)$. Since G is γ_{pr} -vertex-critical, $\gamma_{pr}(G - v) = \gamma_{pr}(G) - 2$, as shown in Proposition 5.2.

Firstly, suppose that $G - v$ is connected. Then by Theorem 6.2, $\text{diam}(G - v) \leq 2\gamma_{pr}(G) - 5$. Thus in the graph G , we increase the diameter of the graph by at most one, the possible longer distance being between v and some vertex in $G - v$, and so $\text{diam}(G) \leq 2\gamma_{pr}(G) - 4$.

Hence suppose that $G - v$ is disconnected. Clearly, $\gamma_{pr}(G - v) = \gamma_{pr}(G) - 2 = \sum_{i=1}^k \gamma_{pr}(G_i)$ where G_1, G_2, \dots, G_k are the blocks of $G - v$. Further-

more, by Theorem 6.2, $\text{diam}(G_i) \leq 2\gamma_{pr}(G_i) - 1$ and so $\sum_{i=1}^k \text{diam}(G_i) \leq \sum_{i=1}^k (2\gamma_{pr}(G_i) - 1) = 2(\gamma_{pr}(G) - 2) - k \leq 2\gamma_{pr}(G) - 6$. Now, the diameter of G is at most two more than the sum of the diameters of the components of $G - v$, the possible longer distance being between some vertex $x \in V(G_i)$ and $y \in V(G_j)$, for $j \neq i$, and so $\text{diam}(G) \leq 2\gamma_{pr}(G) - 4$. ■

No γ_{pr} -vertex-critical graphs are known to meet the bound given in Proposition 6.7, and so we leave it as an open problem to determine a sharp upper bound for the diameter of γ_{pr} -vertex-critical graphs.

Chapter 7

Concluding Remarks

We conclude by summarizing the results presented through a series of open questions.

In Chapter 2 we decided upon the existence or nonexistence of various types of critical and stable graphs with respect to the paired domination number. In particular, we showed that γ_{pr} -edge-critical graphs, γ_{pr} -vertex-critical graphs, γ_{pr} -dot-critical graphs, and γ_{pr} -totally-dot-stable graphs are exactly those for which the question of existence has a positive answer. We investigated the first two of these types in Chapters 3 and 5 respectively, and found analogous properties of such graphs to those known with respect to the domination number and the total domination number. However, in the cases of dot-criticality and totally-dot-criticality little is known, even with respect to the domination number and total domination number. Thus, for dot-criticality and total-dot-criticality, can we find properties for such

critical graphs that are analogous to the cases of edge-criticality and vertex-criticality?

We also determined in Chapter 2 that for all types of stability, the question of existence has a positive answer. Again, little is known about such graphs, even with respect to the domination number and total domination number. We remark that this could be a fruitful area of research.

At the conclusion of Chapter 3 we presented Theorem 3.17, which gave sufficient conditions for a disconnected graph to be γ_{pr} -edge-critical. We ask then, is it true that these sufficient conditions are also necessary? Through results of Chapter 5, we showed that these conditions are equivalent to using graphs that are both γ_{pr} -edge-critical and γ_{pr} -vertex-critical, and thus there is a relationship between edge-criticality and vertex-criticality. Is it possible then to find other relationships between γ_{pr} -edge-critical and γ_{pr} -vertex-critical graphs?

Chapter 4 focused on 4_{pr} -edge-critical graphs. Earlier in Chapter 3 we saw that such graphs are exactly those that are 3_t -edge-critical and those that are 4_t -super-edge-critical. In [9], Van der Merwe showed that G is a γ_{pr} -super-edge-critical graph if and only if G is the disjoint union of nontrivial complete graphs. Thus the only cases where a characterisation remains unknown are for some classes of 3_t -edge-critical graphs. Using the knowledge that 3_t -edge-critical graphs are the same as 4_{pr} -edge-critical graphs, can we find a characterisation for those unknown cases?

Unlike the edge-critical case, we know of few classes of γ_{pr} -vertex-critical

graphs. Using the results presented in Chapter 5, can we find characterisations for classes of γ_{pr} -vertex-critical graphs?

Finally, in Chapter 6 we determined bounds on the diameter of γ_{pr} -edge-critical graphs and γ_{pr} -vertex-critical graphs. In particular, we showed that if G is 4_{pr} -edge-critical, then $\text{diam}(G) \leq 3$, and if G is k_{pr} -edge-critical, $k \geq 6$, then $\text{diam}(G) \leq 2\gamma_{pr} - 6$. However, it is not known that this bound is the best possible, even in the case where $k = 6$, and we remarked that for $k \geq 8$ a better bound can be achieved. Let d_k be the maximum value of the diameter for a k_{pr} -edge-critical graph. Can we find a sharp upper bound for d_k ?

We also showed in Chapter 6 that the minimum value for the diameter of a γ_{pr} -edge-critical graph is two, and that γ_{pr} -edge-critical graphs satisfying this diameter exist for all $\gamma_{pr} \geq 4$. We ask then about the spectrum of diameters for k_{pr} -edge-critical graphs. In particular, is it true that there exists a k_{pr} -edge-critical graph of diameter l for every $2 \leq l \leq d_k$?

Lastly, in Chapter 6 we gave the open problem of determining a sharp upper bound for the diameter of γ_{pr} -vertex-critical graphs.

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