ORTHOGONALITY RELATIONS AND GENERATING FUNCTIONS FOR JACOBI POLYNOMIALS AND RELATED HYPERGEOMETRIC FUNCTIONS

MING-PO CHEN
and
H.M. SRIVASTAVA

DMS-659-IR **April 1994**

ORTHOGONALITY RELATIONS AND GENERATING FUNCTIONS FOR JACOBI POLYNOMIALS AND RELATED HYPERGEOMETRIC FUNCTIONS

MING-PO CHEN
Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529
Taiwan
E-Mail: MAAPO@CCVAX.SINICA.EDU.TW

AND

H.M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada
E-Mail: HMSRI@UVVM.UVIC.CA

Abstract

The authors begin by examining the validity of some orthogonality relations and expansion formulas (asserted recently by S.D. Bajpai [6]) involving a class of hypergeometric polynomials which are essentially certain modified Jacobi polynomials. The corrected version of each of these orthogonality relations is shown to follow readily from the familiar orthogonality property of the classical Jacobi polynomials. A brief discussion is then presented about the applicability of an orthogonality property for the first few Jacobi polynomials, but over a semi-infinite interval, which was considered by V. Romanovski [29] and (more recently) by S.D. Bajpai [5]. Several families of generating functions for Jacobi and Laguerre polynomials, and for various related hypergeometric functions in one and more variables, are also considered systematically.

1991 *Mathematics Subject Classification.* Primary 33C25, 33C20; Secondary 33C05.

1. Introduction and Definitions

In the usual notation, let

$$P_n^{(\alpha,\beta)}(x) := \sum_{k=0}^{n} \binom{\alpha+n}{n-k}\binom{\beta+n}{k}\left(\frac{x+1}{2}\right)^{n-k}\left(\frac{x-1}{2}\right)^{k}, \tag{1.1}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial of degree n in x (and with parameters or indices α and β). These polynomials are orthogonal over the interval $(-1,1)$ with respect to the weight function:

$$w(x) := (1-x)^{\alpha}\,(1+x)^{\beta}; \tag{1.2}$$

in fact, we have (*cf.*, *e.g.*, Szegö [43])

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, P_m^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(x)\,dx$$

$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{mn} \tag{1.3}$$

$$\left(\min\{\Re(\alpha),\Re(\beta)\} > -1; \quad m,n\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}; \quad \mathbb{N} := \{1,2,3,\ldots\}\right),$$

where (and in what follows) δ_{mn} denotes the Kronecker delta.

Just as the other members of the family of classical orthogonal polynomials (*e.g.*, Hermite polynomials, Laguerre polynomials, Gegenbauer (or ultraspherical) polynomials, Legendre (or spherical) polynomials, and the Chebyshev polynomials of the first and second kinds), the Jacobi polynomials can be expressed as a hypergeometric function:

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n}\,{}_2F_1\left(-n,\ \alpha+\beta+n+1;\ \alpha+1;\ \frac{1-x}{2}\right), \tag{1.4}$$

where ${}_2F_1$ is the Gaussian hypergeometric function which corresponds to the special case

$$u-1=v=1$$

of the generalized hypergeometric function ${}_uF_v$ (with u numerator and v denominator parameters) defined by

$$\begin{aligned}
&{}_uF_v(\alpha_1,\ldots,\alpha_u;\ \beta_1,\ldots,\beta_v;\ z)\\
&\equiv {}_uF_v\left[\begin{matrix}\alpha_1,\ldots,\alpha_u;\\ \beta_1,\ldots,\beta_v;\end{matrix}\ z\right]\\
&=: \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_u)_k}{(\beta_1)_k \cdots (\beta_v)_k} \frac{z^k}{k!}
\end{aligned} \tag{1.5}$$

$$\big(u,v\in\mathbb{N}_0;\quad u<v+1,\ |z|<\infty;\quad u=v+1,\ z\in\mathcal{U}:=\{z:|z|<1\};$$

$$u=v+1,\ z\in\partial\mathcal{U}:=\{z:|z|=1\},\ \Re(\omega)>0\big),$$

provided that no zeros appear in the denominator; here $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k := \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0)\\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N}),\end{cases} \tag{1.6}$$

and (for convenience)

$$\omega := \sum_{j=1}^{v} \beta_j - \sum_{j=1}^{u} \alpha_j\,. \tag{1.7}$$

It follows readily from (1.6) that

$$(-n)_k = 0 \quad \big(k=n+1,n+2,n+3,\ldots;n\in\mathbb{N}_0\big), \tag{1.8}$$

which accounts for the fact that a hypergeometric function ${}_uF_v$ would reduce to a polynomial whenever a numerator parameter is a non-positive integer. More importantly, these so-called hypergeometric polynomials satisfy the identity:

$$\begin{aligned}
{}_{u+1}F_v\left[\begin{matrix}-n,\ \alpha_1,\ldots,\alpha_u\,;\\ \beta_1,\ldots,\beta_v;\end{matrix}\ z\right] &= \frac{(\alpha_1)_n\cdots(\alpha_u)_n}{(\beta_1)_n\cdots(\beta_v)_n}(-z)^n\\
&\cdot\ {}_{v+1}F_u\left[\begin{matrix}-n,\ 1-\beta_1-n,\ldots,1-\beta_v-n;\\ 1-\alpha_1-n,\ldots,1-\alpha_u-n;\end{matrix}\ \frac{(-1)^{u+v}}{z}\right]\ (n\in\mathbb{N}_0),
\end{aligned} \tag{1.9}$$

which results when we reverse the order of terms in the finite sum for either side of (1.9).

Recently, Bajpai [6] claimed that the hypergeometric polynomials $B_n(a,b;x)$ defined by

$$B_n(a,b;x) := {}_2F_1\left[\begin{matrix}-n,\ \ 1-b-n;\\ 1-a-n;\end{matrix}\ x\right] \quad (n\in\mathbb{N}_0) \tag{1.10}$$

are orthogonal and asserted that these polynomials satisfy each of the following orthogonality relations (*cf.* [6, Section 3]):

$$\int_0^1 x^{-a-n}(1-x)^{a-b-n}\, B_m(a,b;x)\, B_n(a,b;x)\, dx$$

$$= \frac{n!\,(b)_n\,\Gamma(1-a-n)\Gamma(a-b+1)}{(a)_n\,\Gamma(2-b)}\,\delta_{mn} \tag{1.11}$$

$$(\Re(a)<1;\quad \Re(a-b)>-1;\quad b\neq 0,\pm1,\pm2,\cdots);$$

$$\int_1^\infty x^{-a-n}(x-1)^{a-b-n}\, B_m(a,b;x)\, B_n(a,b;x)\, dx$$

$$= \frac{n!\,(b)_n\,\Gamma(b-1)\Gamma(a-b+1)}{(a)_n\,\Gamma(a+n)}\,\delta_{mn} \tag{1.12}$$

$$(\Re(a-b)>-1;\quad 1+m-n<\Re(b)<1-m+n\ (m\neq n);\quad \Re(b)\neq 1\ (m=n));$$

$$\int_0^\infty x^{-a-n}(1+x)^{a-b-n}\, B_m(a,b;-x)\, B_n(a,b;-x)\, dx$$

$$= (-1)^n\,\frac{n!(b)_n\,\Gamma(b-1)\,\Gamma(1-a-n)}{(a)_n\,\Gamma(b-a)}\,\delta_{mn} \tag{1.13}$$

$$(\Re(a)<1;\quad 1+m-n<\Re(b)<1-m+n\ (m\neq n);\quad \Re(b)\neq 1\ (m=n)).$$

In Sections 2 and 3 of this paper we show that the polynomials $B_n(a,b;x)$ are not at all orthogonal, as claimed by Bajpai [6], and that the *corrected* versions of the integral relations (1.11), (1.12), and (1.13) are rather straightforward consequences of the familiar orthogonality property (1.3) for Jacobi polynomials. We also observe that all of the expansion formulas, which were derived by Bajpai [6, Sections 4 and 5] by applying the *invalid* results (1.11), (1.12), and (1.13), do not hold true as stated. Section 4 deals with an orthogonality property for the first few Jacobi polynomials, but over a semi-infinite interval, which was considered by Romanovsky [29] and (more recently) by Bajpai [5].

The remainder of this paper is concerned with several families of linear, bilinear, and mixed multilateral generating functions for the Jacobi polynomials, for the Laguerre polynomials $L_n^{(\alpha)}(x)$ defined by

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{\alpha+n}{n-k} \frac{(-x)^k}{k!} \tag{1.14}$$

or, equivalently, by

$$L_n^{(\alpha)}(x) := \binom{\alpha+n}{n} {}_1F_1(-n;\alpha+1;x), \tag{1.15}$$

and indeed also for various related hypergeometric functions in one and more variables (see also Bailey [4, p. 247]).

2. Orthogonality of Hypergeometric Polynomials

We begin by recalling the following known result (Theorem A below) which provides an interesting characterization of the classical Laguerre polynomials (*cf.* [15] and [1]):

Theorem A. *The only orthogonal polynomials of the hypergeometric form:*

$${}_{u+1}F_v \left[\begin{matrix} -n, & \alpha_1, \cdots, \alpha_u; \\ & \beta_1, \cdots, \beta_v; \end{matrix} \; x \right] \quad (n \in \mathbb{N}_0), \tag{2.1}$$

where the parameters

$$\alpha_j \quad (j = 1, \cdots, u) \qquad \text{and} \qquad \beta_j \quad (j = 1, \cdots, v)$$

are independent of x and n, are the Laguerre polynomials (in which case $u = 0$ and $v = 1$).

Since

$$B_n(a,b;x) = \frac{(b)_n}{(a)_n}(-x)^n \; {}_2F_1 \left[\begin{matrix} -n, & a; \\ & b; \end{matrix} \; \frac{1}{x} \right] \qquad (n \in \mathbb{N}_0), \tag{2.2}$$

which follows easily from (1.9) and (1.10), the polynomials $B_n(a,b;x)$ are essentially of the hypergeometric form (2.1), but with

$$u = v = 1.$$

Thus, by Theorem A, the polynomials $B_n(a,b;x)$ are *not* orthogonal. This assertion can indeed by reinforced by means of the following counter-examples for the integral relations (1.11), (1.12), and (1.13).

Example 1. Denoting, for convenience, the first member of (1.11) by $I^{(1)}_{m,n}$, we have

$$I^{(1)}_{1,0} = \int_0^1 x^{-a}\,(1-x)^{a-b}\left(1-\frac{b}{a}\,x\right)dx$$

$$= \frac{\Gamma(1-a)\,\Gamma(a-b+1)}{\Gamma(3-b)}\left(2-\frac{b}{a}\right) \tag{2.3}$$

$$\neq 0 \qquad (\Re(a)<1;\quad \Re(a-b)>-1),$$

where we have made use of the Eulerian integral:

$$\int_0^1 t^{\lambda-1}(1-t)^{\mu-1}\,dt = \frac{\Gamma(\lambda)\,\Gamma(\mu)}{\Gamma(\lambda+\mu)} \tag{2.4}$$

$$(\min\{\Re(\lambda),\ \Re(\mu)\}>0).$$

Example 2. Let $I^{(2)}_{m,n}$ denote the integral occurring on the left-hand side of (1.12). Then, setting $x=t^{-1}$ and applying the Eulerian integral (2.4), we have

$$I^{(2)}_{1,0} = \int_1^\infty x^{-a}(x-1)^{a-b}\left(1-\frac{b}{a}\,x\right)dx$$

$$= \int_0^1 t^{b-3}(1-t)^{a-b}\left(t-\frac{b}{a}\right)dt \tag{2.5}$$

$$= \frac{\Gamma(b-2)\,\Gamma(a-b+1)}{\Gamma(a)}\left(\frac{b}{a}-2\right)$$

$$\neq 0 \qquad (\Re(a-b)>-1;\quad \Re(b)>2).$$

Example 3. If we denote the first member of the integral relation (1.13) by $I^{(3)}_{m,n}$ and make use of the Eulerian integral (2.4) once again, after the substitution $x=t\ (1-t)^{-1}$, we obtain

$$I^{(3)}_{1,0} = \int_0^\infty x^{-a}\,(1+x)^{a-b}\left(1+\frac{b}{a}\,x\right)dx$$

$$= \int_0^1 t^{-a}\,(1-t)^{b-2}\left(1+\frac{b}{a}\,\frac{t}{1-t}\right)dt \tag{2.6}$$

$$= \frac{\Gamma(-a)\,\Gamma(b-2)}{\Gamma(b-a)}\,(2a-b)$$

$$\neq 0 \qquad (\Re(a)<1;\quad a\neq 0;\quad \Re(b)>2).$$

It may be of interest to examine rather carefully Bajpai's proofs (*cf.* [6, p. 99]) of his claimed assertions (1.11), (1.12), and (1.13). In fact, by essentially using term-by-term integration and the Eulerian integral (2.4), Bajpai [6, p. 99] first showed that

$$I_{m,n}^{(j)} = 0 \qquad (m < n; \quad m,n \in \mathbb{N}_0; \quad j \in \{1,2,3\}), \tag{2.7}$$

and (assuming, in each case, that $I_{m,n}^{(j)}$ is symmetrical in m and n) he then concluded further that

$$I_{m,n}^{(j)} = 0 \qquad (m > n; \quad m,n \in \mathbb{N}_0; \quad j \in \{1,2,3\}). \tag{2.8}$$

Notice, however, that the integrand in $I_{m,n}^{(j)}$ is *not* symmetrical in m and n for *each* $j \in \{1,2,3\}$. Obviously, therefore, we have

$$I_{m,n}^{(j)} \neq 0 \qquad (m > n; \quad m,n \in \mathbb{N}_0; \quad j \in \{1,2,3\}), \tag{2.9}$$

which contradicts Bajpai's claim (2.8) and is supported by numerous counter-examples including those considered already in Examples 1, 2, and 3 above.

3. Corrected Versions of the Integral Relations (1.11), (1.12), and (1.13)

Making use of the hypergeometric representation (1.4) in the familiar orthogonality property (1.3), and setting

$$x = 1 - 2t \qquad (0 \leq t \leq 1),$$

we obtain

$$\begin{aligned} \int_0^1 & t^{\alpha}(1-t)^{\beta}\ {}_2F_1(-m, \alpha+\beta+m+1; \alpha+1; t) \\ & \cdot {}_2F_1(-n, \alpha+\beta+n+1; \alpha+1; t)\, dt \\ &= \frac{n!\,\{\Gamma(\alpha+1)\}^2\,\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)\,\Gamma(\alpha+n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{mn} \\ & (\min\{\Re(\alpha),\ \Re(\beta)\} > -1; \quad m,n \in \mathbb{N}_0), \end{aligned} \tag{3.1}$$

which, for

$$\alpha = -a - n \qquad \text{and} \qquad \beta = a - b - n \quad (n \in \mathbb{N}_0), \tag{3.2}$$

immediately yields the following *corrected* version of (1.11):

$$\int_0^1 x^{-a-n}(1-x)^{a-b-n}\ {}_2F_1(-m,\,1-b+m-2n;\,1-a-n;\,x)$$
$$\cdot\ {}_2F_1(-n,\,1-b-n;\,1-a-n;\,x)\,dx$$
$$=\frac{n!\,(b)_n\,\Gamma(1-a-n)\,\Gamma(a-b+1)}{(a)_n\,\Gamma(2-b)}\,\delta_{mn} \tag{3.3}$$
$$\big(\Re(a)<1-n;\ \Re(a-b)>n-1;\quad m,n\in\mathbb{N}_0\big).$$

or, in the notation of (1.10),

$$\int_0^1 x^{-a-n}(1-x)^{a-b-n}\ B_m(a-m+n,\,b-2(m-n);\,x)\ B_n(a,b;x)\,dx$$
$$=\frac{n!\,(b)_n\,\Gamma(1-a-n)\,\Gamma(a-b+1)}{(a)_n\,\Gamma(2-b)}\,\delta_{mn} \tag{3.4}$$
$$\big(\Re(a)<1-n;\ \Re(a-b)>n-1;\quad m,n\in\mathbb{N}_0\big).$$

Next, by applying the polynomial identity (1.9) to the hypergeometric representation (1.4), we have (*cf.*, *e.g.*, Srivastava and Manocha [39, p. 91, Problem 16(v)])

$$P_n^{(\alpha,\beta)}(x)=\binom{\alpha+\beta+2n}{n}\left(\frac{x-1}{2}\right)^n\ {}_2F_1\left[\begin{matrix}-n,\,-\alpha-n; \\ -\alpha-\beta-2n;\end{matrix}\ \frac{2}{1-x}\right]. \tag{3.5}$$

In view of the hypergeometric representation (3.5), we find from the orthogonality property (1.3) with

$$x=1-\frac{2}{t}\qquad(1\leq t<\infty)$$

that

$$\int_1^\infty t^{-\alpha-\beta-m-n-2}(t-1)^{\beta}\ {}_2F_1(-m,\,-\alpha-m;\,-\alpha-\beta-2m;\,t)$$
$$\cdot\ {}_2F_1(-n,\,-\alpha-n;\,-\alpha-\beta-2n;\,t)\,dt$$
$$=\frac{n!\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)\,\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+2n+2)}\,\delta_{mn} \tag{3.6}$$
$$\big(\min\{\Re(\alpha),\,\Re(\beta)\}>-1;\quad m,n\in\mathbb{N}_0\big),$$

which, for

$$\alpha = b - m + n - 2 \qquad \text{and} \qquad \beta = a - b - n \quad (m, n \in \mathbb{N}_0), \tag{3.7}$$

yields the following *corrected* version of (1.12):

$$\begin{aligned} \int_1^\infty x^{-a-n}(x-1)^{a-b-n} \; {}_2F_1(-m,\, 2-b-n;\, 2-a-m;\, x) \\ \cdot \; {}_2F_1(-n,\, 2-b+m-2n;\, 2-a+m-2n;\, x)\, dx \\ = \frac{n!\; \Gamma(b+n-1)\; \Gamma(a-b+1)}{(a-1)_n\; \Gamma(a+n)}\; \delta_{mn} \\ \big(\Re(a-b) > n-1; \quad \Re(b) > m-n+1; \quad m, n \in \mathbb{N}_0\big) \end{aligned} \tag{3.8}$$

or, in the notation of (1.10),

$$\begin{aligned} \int_1^\infty x^{-a-n}\,(x-1)^{a-b-n}\; B_m(a-1,\, b-m+n-1;\, x) \\ \cdot\, B_n(a-m+n-1,\, b-m+n-1;\, x)\, dx \\ = \frac{n!\; \Gamma(b+n-1)\,\Gamma(a-b+1)}{(a-1)_n\; \Gamma(a+n)}\; \delta_{mn} \\ \big(\Re(a-b) > n-1; \quad \Re(b) > m-n+1; \quad m, n \in \mathbb{N}_0\big). \end{aligned} \tag{3.9}$$

Finally, by means of the Pfaff-Kummer transformation (*cf.* [45, p. 286]):

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\; {}_2F_1\left(a,\; c-b;\; c;\; \frac{z}{z-1}\right) \tag{3.10}$$

$$\big(|\arg(1-z)| \leq \pi - \epsilon \quad (0 < \epsilon < \pi)\big),$$

we can rewrite the hypergeometric representation (1.4) in its *equivalent* form (*cf.*, *e.g.*, [39, p. 91, Problem 16(ii)]):

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+n}{n} \left(\frac{x+1}{2}\right)^n \;\; {}_2F_1\left[\begin{matrix} -n,\; -\beta-n; \\ \alpha+1; \end{matrix}\; \frac{x-1}{x+1}\right]. \tag{3.11}$$

Thus, if we employ the hypergeometric representation (3.11) on the left-hand side of the orthogonality property (1.3) and set

$$x = \frac{1-t}{1+t} \qquad (0 \leq t < \infty),$$

we get

$$\begin{aligned} \int_0^\infty t^\alpha (1+t)^{-\alpha-\beta-m-n-2} \, {}_2F_1(-m, -\beta-m; \alpha+1; -t) \\ \cdot \, {}_2F_1(-n, -\beta-n; \alpha+1; -t) \, dt \\ = \frac{n! \, \{\Gamma(\alpha+1)\}^2 \, \Gamma(\beta+n+1)}{(\alpha+\beta+2n+1) \, \Gamma(\alpha+n+1) \, \Gamma(\alpha+\beta+n+1)} \, \delta_{mn} \\ \left(\min\{\Re(\alpha), \, \Re(\beta)\} > -1; \quad m, n \in \mathbb{N}_0\right), \end{aligned} \tag{3.12}$$

which, for

$$\alpha = -a-n \qquad \text{and} \qquad \beta = b-m+n-2 \quad (m, n \in \mathbb{N}_0), \tag{3.13}$$

readily yields the following *corrected* version of (1.13):

$$\begin{aligned} \int_0^\infty x^{-a-n} (1+x)^{a-b-n} \, {}_2F_1(-m, 2-b-n; 1-a-n; -x) \\ \cdot \, {}_2F_1(-n, 2-b+m-2n; 1-a-n; -x) \, dx \\ = (-1)^n \frac{n! (b-1)_n \, (b-a-1)_n \, \Gamma(b-1) \, \Gamma(1-a-n)}{(a)_n \, (b-a)_n \, \Gamma(b-a)} \, \delta_{mn} \\ \left(\Re(a) < 1-n; \quad \Re(b) > m-n+1; \quad m, n \in \mathbb{N}_0\right) \end{aligned} \tag{3.14}$$

or, in the notation of (1.10),

$$\begin{aligned} \int_0^\infty x^{-a-n} (1+x)^{a-b-n} \, B_m(a-m+n, \, b-m+n-1; -x) \\ \cdot \, B_n(a, \, b-m+n-1; -x) \, dx \\ = (-1)^n \frac{n! (b-1)_n \, (b-a-1)_n \, \Gamma(b-1) \, \Gamma(1-a-n)}{(a)_n \, (b-a)_n \, \Gamma(b-a)} \, \delta_{mn} \\ \left(\Re(a) < 1-n; \quad \Re(b) > m-n+1; \quad m, n \in \mathbb{N}_0\right). \end{aligned} \tag{3.15}$$

We remark in passing that, since the hypergeometric polynomials $B_n(a,b;x)$ are *not* orthogonal, we cannot validate the various Fourier expansions in series of these polynomials, which were obtained by Bajpai [6, Sections 4 and 5] by applying his erroneous assertions (1.11), (1.12), and (1.13) as orthogonality relations for $B_n(a,b;x)$.

4. Orthogonality of Jacobi Polynomials Over Semi-Infinite Intervals

Since [43, p. 64, Equation (4.22.1)]

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\beta)}\left(\frac{x+3}{x-1}\right) \\ &= \left(\frac{x-1}{2}\right)^n P_n^{(\beta,-\alpha-\beta-2n-1)}\left(-\frac{x+3}{x-1}\right), \end{aligned} \tag{4.1}$$

where we have also used the familiar identity:

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{4.2}$$

it is not difficult to obtain yet another equivalent form of the orthogonality property (1.3) by setting

$$x = \frac{t-3}{t+1} \qquad (1 \leq t < \infty).$$

We thus obtain the following alternative form of the familiar result (1.3):

$$\begin{aligned} \int_1^\infty (t-1)^\beta (t+1)^{-\alpha-\beta-m-n-2} P_m^{(\beta,-\alpha-\beta-2m-1)}(t)\, P_n^{(\beta,-\alpha-\beta-2n-1)}(t)\, dt \\ = \frac{2^{-\alpha-2n-1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)}\,\delta_{mn} \end{aligned} \tag{4.3}$$

$$\left(\min\{\Re(\alpha),\,\Re(\beta)\} > -1; \quad m,n \in \mathbb{N}_0\right),$$

which may be compared with the orthogonality relations (3.6), (3.8), and (3.9).

We now turn to an orthogonality property for the first few Jacobi polynomials, but over a semi-infinite interval $\xi \leq x < \infty$, which was given by Romanovsky [29, p. 1025]. We choose to recall here Romanovsky's result in the familiar notation for Jacobi polynomials:

$$\int_{\xi}^{\infty} (x-\xi)^{\alpha}\, x^{\beta}\, P_m^{(\alpha,\beta)}\left(\frac{2x-\xi}{\xi}\right)\, P_n^{(\alpha,\beta)}\left(\frac{2x-\xi}{\xi}\right)\, dx$$

$$= \frac{(-1)^{n+1}\,\xi^{\alpha+\beta+1}\,(\beta+1)_n\,\Gamma(\alpha+n+1)\Gamma(-\alpha-\beta-n)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(-\beta)}\,\delta_{mn} \tag{4.4}$$

$$\big(\Re(\alpha) > -1;\quad \Re(\alpha+\beta+m+n) < -1;\quad m,n \in \mathbb{N}_0\big).$$

The case $\xi = 1$ of Romanovsky's result (4.4) (with x replaced trivially by $x+1$) was cited subsequently by Askey [3, p. 30] in the form:

$$\int_{0}^{\infty} x^{\alpha}(1+x)^{\beta}\, R_m^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(x)\, dx$$

$$= \frac{(-1)^{n+1}\,(\beta+1)_n\,\Gamma(\alpha+n+1)\,\Gamma(-\alpha-\beta-n)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(-\beta)}\,\delta_{mn} \tag{4.5}$$

$$\big(\Re(\alpha) > -1;\quad \Re(\alpha+\beta+m+n) < -1;\quad m,n \in \mathbb{N}_0\big),$$

where, for convenience,

$$R_n^{(\alpha,\beta)}(x) := \binom{\alpha+n}{n}\,{}_2F_1(-n,\alpha+\beta+n+1;\ \alpha+1;\ -x). \tag{4.6}$$

Comparing the definition (4.6) with the hypergeometric representation in (1.4), we immediately have

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x+1) \qquad (n \in \mathbb{N}_0). \tag{4.7}$$

In view of the relationship (4.7), upon setting

$$x = \frac{t-1}{2}$$

and observing that

$$\Gamma(-\alpha-\beta-n) = \frac{(-1)^{n+1}\,\Gamma(\alpha+\beta+2)\,\Gamma(-\alpha-\beta-1)}{\Gamma(\alpha+\beta+n+1)} \qquad (n \in \mathbb{N}_0), \tag{4.8}$$

Romanovsky's result (4.5) can easily be rewritten in the form [*cf.* Equations (1.3) *and* (4.3)]:

$$\int_1^\infty (t-1)^\alpha (t+1)^\beta P_m^{(\alpha,\beta)}(t)\, P_n^{(\alpha,\beta)}(t)\, dt$$
$$= \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+n+1)\,\Gamma(\beta+n+1)}{n!\,(\alpha+\beta+2n+1)\,\Gamma(\alpha+\beta+n+1)} \cdot \frac{\Gamma(\alpha+\beta+2)\,\Gamma(-\alpha-\beta-1)}{\Gamma(\beta+1)\,\Gamma(-\beta)}\,\delta_{mn} \tag{4.9}$$
$$\big(\Re(\alpha) > -1;\quad \Re(\alpha+\beta+m+n) < -1;\quad m,n \in \mathbb{N}_0\big),$$

which obviously follows *directly* from Romanovsky's result (4.4) by setting $\xi = 1$ *and* $x = \frac{1}{2}(t+1)$, and applying the identity (4.8).

Romanovsky's result (4.4) or (4.5) in its alternative form (4.9) was rederived recently by Bajpai [5], who made use of the *Pfaff-Saalschütz theorem* (*cf.*, *e.g.*, Slater [32, p. 243, Equation (III.2)]):

$$ {}_3F_2(-n, a, b;\ c,\ a+b-c-n+1;\ 1) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n} \qquad (n \in \mathbb{N}_0) \tag{4.10}$$

and the special case $\xi = 1$ of the following *Weyl fractional integral* [14, p. 201, Entry 13.2(6)]:

$$\frac{1}{\Gamma(\mu)} \int_\xi^\infty (x-\xi)^{\mu-1}\, x^{-\lambda}\, dx = \frac{\Gamma(\lambda-\mu)}{\Gamma(\lambda)}\, \xi^{\mu-\lambda} \tag{4.11}$$
$$\big(0 < \Re(\mu) < \Re(\lambda)\big).$$

The second condition for the convergence of Romanovsky's result (4.4), (4.5) or (4.9), *viz*

$$\Re(\alpha+\beta+m+n) < -1 \qquad (m,n \in \mathbb{N}_0), \tag{4.12}$$

would obviously render the orthogonality relation (4.4), (4.5) or (4.9) useless for expansions of analytic functions in Jacobi series. Consequently, the so-called Fourier expansions involving Jacobi polynomials (which were obtained by Bajpai [5, p. 225, Equation (12); p. 226, Equation (14)] by applying (4.9) *without* satisfying the condition (4.12) above) do not hold true as claimed by him.

5. Linear Generating Functions

One of the earliest known generating functions for the Jacobi polynomials is Jacobi's generating function:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n = 2^{\alpha+\beta}\, R^{-1}\,(1-t+R)^{-\alpha}\,(1+t+R)^{-\beta} \tag{5.1}$$

$$\left(R := (1-2xt+t^2)^{\frac{1}{2}}\right),$$

for which several interesting proofs are available in the mathematical literature (see, for example, Szegö [43, Section 4.4], Rainville [28, Section 140], Carlitz [7], Askey [2], Foata and Leroux [16], Srivastava [34], and Parnes and Ekhad [27]).

The classical result (5.1), and numerous analogous generating functions in which the Jacobi polynomials' indices α and β also depend *linearly* upon the summation index n, are all contained in the following generating function obtained, over two decades ago, by Srivastava and Singhal [41] (see also Srivastava [36]):

$$\sum_{n=0}^{\infty} P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)\, t^n$$
$$= (1+\xi)^{\alpha+1}(1+\eta)^{\beta+1}\,\left\{1-\lambda\xi-\mu\eta-(1+\lambda+\mu)\xi\eta\right\}^{-1}, \tag{5.2}$$

where ξ and η are functions of x and t defined implicitly by

$$(x+1)^{-1}\,\xi = (x-1)^{-1}\,\eta = \frac{1}{2}\,t(1+\xi)^{\lambda+1}\,(1+\eta)^{\mu+1}. \tag{5.3}$$

It may be remarked in passing that the Srivastava-Singhal generating function (5.2) was applied recently by Chen and Ismail [9] in order to determine the asymptotic behaviour of the Jacobi polynomials:

$$P_n^{(\alpha+\lambda n,\beta+\mu n)}(x)$$

when $n \to \infty$, and α, β, λ, μ and x remain fixed. (See also an interesting combinatorial proof of the Srivastava-Singhal result (5.2) by Strehl [42].)

Generating functions in which the summation index appears only in the Jacobi polynomials' indices α and β happen to be the subject of study in a number of recent works demonstrating the usefulness of the familiar group-theoretic method of Louis Weisner (1899-1988), which is described and illustrated fairly adequately by Miller [24], McBride [23], and Srivastava and Manocha [39]. We choose first to recall here the set of generating functions considered (among others) by Ghosh [19] (and, more recently, by Sharma and Chongdar [30]) in the following (*slightly modified*) forms:

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\,\beta+k)}(x)\, t^k$$
$$= (1-t)^{\alpha}\; P_n^{(\alpha,\beta)}\big(t+(1-t)x\big) \quad (n \in \mathbb{N}_0;\, |t|<1); \tag{5.4}$$

$$\sum_{k=0}^{\infty} \binom{k-\beta-n-1}{k} P_n^{(\alpha+k,\,\beta-k)}(x)\, (-t)^k$$
$$= (1+t)^{\beta}\; P_n^{(\alpha,\beta)}\big(t+(1+t)x\big) \quad (n \in \mathbb{N}_0;\, |t|<1); \tag{5.5}$$

$$\sum_{k,\ell=0}^{\infty} \binom{k-\beta-\ell-n-1}{k}\binom{\ell-\alpha-n-1}{\ell} P_n^{(\alpha+k-\ell,\,\beta-k+\ell)}(x)\, t^k\, \tau^{\ell}$$
$$= (1-t)^{\beta}\,\{1-(1-t)\tau\}^{\alpha}\; P_n^{(\alpha,\beta)}(X) \quad (n \in \mathbb{N}_0;\, |t|<1;\, |\tau|<|1-t|^{-1}), \tag{5.6}$$

where, for convenience,

$$X = X(x;t,\tau) := x-(1+x)t+(1-t)\tau\,\{1-x+(1+x)t\}. \tag{5.7}$$

In view of the identity (4.2), the generating functions (5.4) and (5.5) are substantially the same result (with the rôles of α and β interchanged). More importantly, the generating function (5.6) is a rather straightforward consequence of the generating functions (5.4) and (5.5). Denoting, for convenience, the first member of (5.6) by $\mathcal{S}(x;t,\tau)$, and applying the generating function (5.5) with α and β replaced by $\alpha-\ell$ and $\beta+\ell$, respectively, we readily find that

$$\mathcal{S}(x;t,\tau) = \sum_{\ell=0}^{\infty} \binom{\ell-\alpha-n-1}{\ell} \tau^{\ell} \cdot \sum_{k=0}^{\infty} \binom{k-\beta-\ell-n-1}{k} P_n^{(\alpha-\ell+k,\, \beta+\ell-k)}(x)\, t^k$$
$$= (1-t)^{\beta} \sum_{\ell=0}^{\infty} \binom{\ell-\alpha-n-1}{\ell} P_n^{(\alpha-\ell,\, \beta+\ell)}(-t+(1-t)\,x) \cdot \{(1-t)\tau\}^{\ell} \quad (|t|<1). \tag{5.8}$$

We now make use of the generating function (5.4) with x and t replaced by

$$-t+(1-t)\,x \qquad \text{and} \qquad (1-t)\tau\,,$$

respectively, and (5.8) would lead us immediately to the second member of (5.6).

Next we give a simple derivation of the *parent* result (5.4) *without* using the group-theoretic method employed by the earlier authors. Indeed, if we make use of the hypergeometric representation (1.4) on the left-hand side of the generating function (5.4), we obtain

$$\mathcal{S}_1 := \sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\, \beta+k)}(x)\, t^k$$
$$= \binom{\alpha+n}{n} \sum_{k=0}^{\infty} \binom{k-\alpha-1}{k} t^k \cdot \sum_{r=0}^{n} \frac{(-n)_r\, (\alpha+\beta+n+1)_r}{(\alpha-k+1)_r} \frac{\{\frac{1}{2}(1-x)\}^r}{r!}, \tag{5.9}$$

where we have also used the relationship:

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n} \qquad (n \in \mathbb{N}_0; \quad \lambda \in \mathbb{C}).$$

Since

$$(\alpha-k+1)_r = (\alpha+1)_r \binom{k-\alpha-1}{k} \binom{k-\alpha-r-1}{k}^{-1} \qquad (k, r \in \mathbb{N}_0),$$

upon inverting the order of summation in (5.9), we have

$$\mathcal{S}_1 = \binom{\alpha+n}{n} \sum_{r=0}^{n} \frac{(-n)_r\,(\alpha+\beta+n+1)_r}{(\alpha+1)_r}\,\frac{\left\{\frac{1}{2}(1-x)\right\}^r}{r!} \cdot \sum_{k=0}^{\infty} \binom{k-\alpha-r-1}{k} t^k. \tag{5.10}$$

The inner sum in (5.10) can be evaluated, when $|t|<1$, by means of the binomial expansion:

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k} z^k = (1-z)^{-\lambda} \quad (|z|<1), \tag{5.11}$$

and we thus find from (5.10) that

$$\mathcal{S}_1 = (1-t)^{\alpha} \binom{\alpha+n}{n}\, {}_2F_1\left[\begin{matrix} -n,\ \alpha+\beta+n+1; \\ \alpha+1; \end{matrix}\ \frac{1}{2}(1-x)(1-t)\right] \quad (|t|<1),$$

which, in view of the hypergeometric representation (1.4) once again, immediately yields the right-hand side of the generating function (5.4).

In terms of the generalized hypergeometric function defined by (1.5), it is not difficult to prove similarly that

$$\sum_{k=0}^{\infty} \binom{\lambda+k-1}{k}\, {}_uF_{v+1}\left[\begin{matrix} \alpha_1,\cdots,\alpha_u; \\ 1-\lambda-k,\ \beta_1,\cdots,\beta_v; \end{matrix}\ z\right] t^k = (1-t)^{-\lambda}\, {}_uF_{v+1}\left[\begin{matrix} \alpha_1,\cdots,\alpha_u; \\ 1-\lambda,\ \beta_1,\cdots,\beta_v; \end{matrix}\ z(1-t)\right] \quad (|t|<1), \tag{5.12}$$

which would immediately yield the generating function (5.4) in the special case when

$$u-2=v=0,\ \ \alpha_1=-n,\ \ \alpha_2=\alpha+\beta+n+1\ \ (n\in\mathbb{N}_0),$$
$$\lambda=-\alpha,\ \ \text{and}\ \ z=\frac{1}{2}(1-x).$$

Furthermore, in view of the confluent hypergeometric representation (1.15) for the classical Laguerre polynomials, (5.12) with

$$u-1=v=0,\quad \alpha_1=-n\ \ (n\in\mathbb{N}_0),\quad \lambda=-\alpha,\quad \text{and}\quad z=x$$

gives us the generating function:

$$\sum_{k=0}^{\infty}\binom{k-\alpha-n-1}{k} L_n^{(\alpha-k)}(x)\,t^n \\ = (1-t)^{\alpha}\, L_n^{(\alpha)}\left(x(1-t)\right) \quad \left(|t|<1\right), \tag{5.13}$$

which can, of course, be deduced directly from (5.4) by appealing to the limit relationship [43, p. 103, Equation (5.3.4)]:

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty}\left\{P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}.$$

For a further generalization of the hypergeometric generating function (5.12), we introduce a sequence $\{\zeta_k^{(\lambda,\rho)}(z)\}_{k=0}^{\infty}$ defined by

$$\zeta_k^{(\lambda,\rho)}(z) = \zeta_k^{(\lambda,\rho)}\left[\alpha_1,\cdots,\alpha_u;\ \beta_1,\cdots,\beta_v\ :\ z\right] \\ := {}_uF_{\rho+v}\left(\alpha_1,\cdots,\alpha_u;\ \Delta(\rho;\,1-\lambda-k),\ \beta_1,\cdots,\beta_v;\ z\right), \tag{5.14}$$

where, for convenience, $\Delta(\rho;\lambda)$ abbreviates the array of ρ parameters

$$\frac{\lambda}{\rho},\ \frac{\lambda+1}{\rho},\cdots,\ \frac{\lambda+\rho-1}{\rho} \qquad (\rho\in\mathbb{N}).$$

We thus obtain the generating function:

$$\sum_{k=0}^{\infty}\binom{\lambda+m+k-1}{k}\zeta_{m+k}^{(\lambda,\rho)}(z)\,t^k \\ = (1-t)^{-\lambda-m}\ \zeta_m^{(\lambda,\rho)}\left(z(1-t)^{\rho}\right) \qquad (m\in\mathbb{N}_0;\ |t|<1), \tag{5.15}$$

which, for $\rho=1$ (and λ replaced by $\lambda-m$), yields (5.12).

The generating function (5.15) is analogous to the known result [38, p. 312, Equation (62)]:

$$\sum_{k=0}^{\infty}\binom{\lambda+m+k-1}{k}\omega_{m+k}^{(\lambda,\rho)}(z)\,t^k \\ = (1-t)^{-\lambda-m}\ \omega_m^{(\lambda,\rho)}\left(\frac{z}{(1-t)^{\rho}}\right) \qquad (m\in\mathbb{N}_0;\quad |t|<1), \tag{5.16}$$

where [*cf.* Definition (5.14)]

$$\begin{aligned}\omega_k^{(\lambda,\rho)}(z) &= \omega_k^{(\lambda,\rho)}\left[\alpha_1,\cdots,\alpha_u;\quad \beta_1,\cdots,\beta_v:\ z\right]\\ &:= {}_{\rho+u}F_v\left(\Delta(\rho;\lambda+k),\ \alpha_1,\cdots,\alpha_u;\ \beta_1,\cdots,\beta_v;\ z\right).\end{aligned} \tag{5.17}$$

A multivariable extension of the generating function (5.16) can be derived easily from another known result (*cf.*, *e.g.*, Srivastava and Manocha [39, p. 490, Problem 2]). As a matter of fact, if we set

$$\Omega_k^{(\lambda)}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r) := \sum_{k_1,\cdots,k_r=0}^{\infty} (\lambda+k)_K\ A(k_1,\cdots,k_r)\, z_1^{k_1}\cdots z_r^{k_r} \tag{5.18}$$

$$\left(K := k_1\sigma_1+\cdots+k_r\sigma_r;\quad k_j\in\mathbb{N}_0;\quad \lambda,\sigma_j\in\mathbb{C};\quad j=1,\cdots,r\right),$$

where $\{A(k_1,\cdots,k_r)\}$ is a suitably bounded multiple sequence of complex numbers, we thus find that

$$\begin{aligned}&\sum_{k=0}^{\infty}\binom{\lambda+m+k-1}{k}\Omega_{m+k}^{(\lambda)}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r)\,t^k\\ &\quad= (1-t)^{-\lambda-m}\ \Omega_m^{(\lambda)}\left(\sigma_1,\cdots,\sigma_r;\ \frac{z_1}{(1-t)^{\sigma_1}},\cdots,\frac{z_r}{(1-t)^{\sigma_r}}\right)\\ &\qquad\left(m\in\mathbb{N}_0;\quad |t|<1\right).\end{aligned} \tag{5.19}$$

An analogous multivariable extension of the generating function (5.15) can be proven *directly*, and we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\binom{\lambda+m+k-1}{k} Z_{m+k}^{(\lambda)}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r)\,t^k\\ &\quad= (1-t)^{-\lambda-m}\ Z_m^{(\lambda)}\left(\sigma_1,\cdots,\sigma_r;\ z_1(1-t)^{\sigma_1},\cdots,z_r(1-t)^{\sigma_r}\right)\\ &\qquad\left(m\in\mathbb{N}_0;\quad |t|<1\right),\end{aligned} \tag{5.20}$$

where

$$Z_k^{(\lambda)}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r) := \sum_{k_1,\cdots,k_r=0}^{\infty}\frac{A(k_1,\cdots,k_r)}{(1-\lambda-k)_K}\, z_1^{k_1}\cdots z_r^{k_r} \tag{5.21}$$

$$\left(K := k_1\sigma_1+\cdots+k_r\sigma_r;\quad k_j\in\mathbb{N}_0;\quad \lambda,\sigma_j\in\mathbb{C};\quad j=1,\cdots,r\right),$$

$\{A(k_1, \cdots, k_r)\}$ being (as before) a suitably bounded multiple sequence of complex numbers.

It should be remarked in passing that the multivariable generating functions (5.19) and (5.20) correspond *essentially* to the *special* cases $\beta = 0$ and $\beta = -1$ of the following result due to Srivastava [33] who indeed gave much more general generating functions of several variables (see also Srivastava and Manocha [39, p. 491, Problem 3]):

$$\sum_{k=0}^{\infty} \binom{\alpha + (\beta+1)k}{k} \Lambda_k^{(\alpha,\beta)}(\sigma_1, \cdots, \sigma_r;\ z_1, \cdots, z_r)\, t^k$$
$$= \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\, G\left(z_1(1+\zeta)^{\sigma_1}, \cdots, z_r(1+\zeta)^{\sigma_r}\right), \tag{5.22}$$

where ζ is a function of t defined implicitly by

$$\zeta = t(1+\zeta)^{\beta+1} \qquad (\zeta(0) := 0), \tag{5.23}$$

$$\Lambda_k^{(\alpha,\beta)}(\sigma_1, \cdots, \sigma_r;\ z_1, \cdots, z_r)$$
$$:= \sum_{k_1, \cdots, k_r = 0}^{\infty} \frac{(\alpha + (\beta+1)k + 1)_K}{(\alpha + \beta k + 1)_K}\, A(k_1, \cdots, k_r)\, z_1^{k_1} \cdots z_r^{k_r} \tag{5.24}$$

$$(K := k_1\sigma_1 + \cdots + k_r\sigma_r; \quad k_j \in \mathbb{N}_0; \quad \alpha, \beta, \sigma_j \in \mathbb{C}; \quad j = 1, \cdots, r),$$

and (for convenience)

$$G(z_1, \cdots, z_r) := \sum_{k_1, \cdots, k_r = 0}^{\infty} A(k_1, \cdots, k_r)\, z_1^{k_1} \cdots z_r^{k_r}. \tag{5.25}$$

Returning now to generating functions for the Jacobi and related hypergeometric polynomials, of the types illustrated by (5.4), (5.5), and (5.6), we should like to point out that many such results have been derived (and rederived in a seemingly different form) by using essentially the same group-theoretic method referred to above (see, for example, Chakrabarty [8]; Chongdar [12]; Ghosh [17], [18], [20], [21]; Mukherjee and Chongdar [25]; and Thakurta [44]). In some of these papers (as well as elsewhere in the literature on the subject) these and other authors have derived (and have quite frequently claimed as a *new* result) one or the other version of the following *equivalent* generating functions for the

Jacobi polynomials:

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-n,\beta)}(x)\,t^n = (1+t)^{\alpha} \left\{1-\frac{1}{2}(x-1)t\right\}^{-\alpha-\beta-m-1} \cdot P_m^{(\alpha,\beta)}\left(\frac{x+\frac{1}{2}(x-1)t}{1-\frac{1}{2}(x-1)t}\right) \tag{5.26}$$

$$\left(m \in \mathbb{N}_0; \quad |t| < \min\{1,\, 2|x-1|^{-1}\}\right);$$

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha,\beta-n)}(x)\,t^n = (1-t)^{\beta} \left\{1-\frac{1}{2}(x+1)\,t\right\}^{-\alpha-\beta-m-1} \cdot P_m^{(\alpha,\beta)}\left(\frac{x-\frac{1}{2}(x+1)\,t}{1-\frac{1}{2}(x+1)\,t}\right) \tag{5.27}$$

$$\left(m \in \mathbb{N}_0; \quad |t| < \min\{1,\, 2|x+1|^{-1}\}\right);$$

$$\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-n,\beta-n)}(x)\,t^n = \left\{1+\frac{1}{2}(x+1)\,t\right\}^{\alpha} \left\{1+\frac{1}{2}(x-1)\,t\right\}^{\beta} \cdot P_m^{(\alpha,\beta)}\left(x+\frac{1}{2}(x^2-1)\,t\right) \tag{5.28}$$

$$\left(m \in \mathbb{N}_0; \quad |t| < \min\{2|x+1|^{-1},\, 2|x-1|^{-1}\}\right),$$

each of which is a fairly well-known result (*cf.*, *e.g.*, Manocha and Sharma [22], Singhal and Srivastava [31], Srivastava and Lavoie [37], and Srivastava *et al.* [38]; see also Srivastava and Manocha [39, pp. 164-165, Problems 8 and 9; p. 420, Equations 8.4(14), 8.4(16), and 8.4(18)]).

Weisner's group-theoretic method has also been applied in order to derive generating functions for the Jacobi polynomials involving triple and quadruple series. However, just as the double-series generating function (5.6) was shown above to follow by simply iterating the single-series generating functions (5.4) and (5.5), each of these triple-series and quadruple-series generating functions can be deduced, by straightforward iteration, from some suitable (usually known) single- and double-series generating functions for the Jacobi polynomials. We leave this demonstration as an exercise for the interested reader and recall instead some further single- and double-series generating functions which were derived recently by the aforementioned group-theoretic method.

First of all, Ghosh [17] proved the well-known result (5.28), together with its relatively more familiar special case when $m = 0$, and the following results for the Jacobi polynomials (in corrected *or* modified forms):

$$P_n^{(\alpha,\beta)}(x+2t) = \sum_{\ell=0}^{n} \binom{\alpha+\beta+n+\ell}{\ell} P_{n-\ell}^{(\alpha+\ell,\,\beta+\ell)}(x)\, t^{\ell}; \tag{5.29}$$

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} t^k \sum_{\ell=0}^{n+k} \binom{\alpha+\beta+n-k+\ell}{\ell} P_{n+k-\ell}^{(\alpha-k+\ell,\,\beta-k+\ell)}(x)\, \tau^{\ell} \\ = \left\{1+\frac{1}{2}(x+2\tau+1)\,t\right\}^{\alpha} \left\{1+\frac{1}{2}(x+2\tau-1)\,t\right\}^{\beta} \\ \cdot P_n^{(\alpha,\beta)}\left(x+2\tau+\frac{1}{2}(x^2-1)\,t+2(x+\tau)\,t\tau\right) \\ \left(n\in\mathbb{N}_0;\quad |t|<\min\left\{2|x+2\tau+1|^{-1},\ 2|x+2\tau-1|^{-1}\right\}\right).\end{aligned} \tag{5.30}$$

Since (*cf.*, *e.g.*, Erdélyi *et al.* [13, p. 170, Equation 10.8(17)])

$$\frac{\partial^{\ell}}{\partial t^{\ell}}\left\{P_n^{(\alpha,\beta)}(x+2t)\right\} = \begin{cases} \binom{\alpha+\beta+n+\ell}{\ell}\,\ell!\; P_{n-\ell}^{(\alpha+\ell,\,\beta+\ell)}(x+2t) & (\ell=0,1,\cdots,n) \\ 0 \quad (\ell=n+1,n+2,n+3,\cdots), \end{cases} \tag{5.31}$$

the finite summation formula (5.29) is an immediate consequence of the Taylor expansion of

$$P_n^{(\alpha,\beta)}(x+2t)$$

in powers of t. On the other hand, the (*corrected*) double-series generating function (5.30) follows readily by successively iterating (5.29) (with α and β replaced by $\alpha-k$ and $\beta-k$, respectively) *and* the well-known result (5.28).

Next we recall the *main* result of Chakrabarty [8] in the following *corrected* form (see also Thakurta [44, p. 79, Equation (4.1)]):

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{n} \binom{n+k-\ell}{k} \binom{\ell-\alpha-n-1}{\ell} P_{n+k-\ell}^{(\alpha,\,\beta-k+\ell)}(x)\, t^k\, \tau^\ell$$

$$= (1-t)^\beta \, \{1-\tau(1-t)\}^n \left\{1-\frac{1}{2}(x+1)\,t\right\}^{-\alpha-\beta-n-1} \tag{5.32}$$

$$\cdot\, P_n^{(\alpha,\beta)} \left(\frac{x - \frac{1}{2}(x+1)\,t - \tau(1-t)\, \{1-\frac{1}{2}(x+1)\,t\}}{\{1-\tau(1-t)\}\, \{1-\frac{1}{2}(x+1)\,t\}} \right)$$

$$\left(n \in \mathbb{N}_0; \quad |t| < \min\left\{1,\, 2|x+1|^{-1}\right\}\right),$$

where the inner sum obviously terminates at $\ell = n$ (and not at $\ell = n+k$, as claimed by Chakrabarty [8] as well as Thakurta [44]), since

$$\binom{n+k-\ell}{k} = \binom{n+k-\ell}{n-\ell} = 0 \qquad (\ell = n+1, n+2, n+3, \cdots). \tag{5.33}$$

The double-series generating function (5.32) is a straightforward consequence of the well-known result (5.27) *and* the following analogue of the finite summation formula (5.29):

$$\sum_{\ell=0}^{n} \binom{\ell-\alpha-n-1}{\ell} P_{n-\ell}^{(\alpha,\beta+\ell)}(x)\, t^\ell = (1-t)^n \, P_n^{(\alpha,\beta)} \left(\frac{x-t}{1-t}\right), \tag{5.34}$$

which can be proven easily by applying the hypergeometric representation (1.4) and the binomial expansion (5.11), just as in our derivation of the generating function (5.4). (See also Ghosh [21, p. 155, Equation (4.2)] for (5.34) with t replaced by $-t$, and α interchanged with β.) In fact, upon reversing the order of summation in (5.34), if we replace β by $\beta - n$ and t by t^{-1}, we find from (5.34) that

$$\sum_{\ell=0}^{n} \frac{(-n)_\ell}{(\alpha+1)_\ell} P_\ell^{(\alpha,\,\beta-\ell)}(x)\, t^\ell = \binom{\alpha+n}{n}^{-1} (1-t)^n \, P_n^{(\alpha,\,\beta-n)} \left(\frac{1-xt}{1-t}\right), \tag{5.35}$$

which is an obvious *further* special case ($\lambda = -n$; $n \in \mathbb{N}_0$) of the following very specialized member of several known families of generating functions for the Jacobi and related hypergeometric polynomials (*cf.* Srivastava and Manocha [39, p. 108, Equation 2.3(15)]):

$$\sum_{\ell=0}^{\infty} \frac{(\lambda)_\ell}{(\alpha+1)_\ell} P_\ell^{(\alpha,\,\beta-\ell)}(x)\, t^\ell = (1-t)^{-\lambda} \; {}_2F_1 \left[\begin{matrix} \lambda, \; \alpha+\beta+1; \\ \alpha+1; \end{matrix} \; \frac{(1-x)\,t}{2(t-1)} \right] \tag{5.36}$$

$$\left(|t| < 1\right).$$

Observe also that (5.36) with $\lambda = \alpha + 1$ corresponds to the special case of the well-known generating function (5.27) when $m = 0$.

The corrected (and modified) version of a quadruple-series generating function, which happens to be the main result of Thakurta [44], would follow upon further iterating the double-series generating function (5.32) with single-series generating functions of the aforementioned classes. In an earlier paper, Chongdar [11] had derived the well-known result (5.28), the finite summation formula (5.29), and a double-series generating function analogous to (5.32). This double-series generating function in Chongdar's paper [11] is precisely the same as the main result of Ghosh [17] which we recalled above in a corrected form (5.30). Just as we remarked about (5.32), *another* corrected version of the double-series generating function proven independently by Chosh [17] and Chongdar [11] can be given as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{\ell=0}^{n} \binom{n+k-\ell}{k} \binom{\alpha+\beta+n+\ell}{\ell} P_{n+k-\ell}^{(\alpha-k+\ell,\ \beta-k+\ell)}(x)\, t^k\, \tau^\ell \\
&\quad = \left\{1 + \frac{1}{2}(x+1)\,t\right\}^{\alpha} \left\{1 + \frac{1}{2}(x-1)\,t\right\}^{\beta} \\
&\qquad \cdot P_n^{(\alpha,\beta)}\left(\left(1 + \frac{1}{2}xt\right)\{x(1+t\tau)+2\tau\} - \frac{1}{2}t(1+t\tau)\right)
\end{aligned} \tag{5.37}$$

$$\left(n \in \mathbb{N}_0; \quad |t| < \min\left\{2|x+1|^{-1},\ 2|x-1|^{-1}\right\}; \quad \tau \in \mathbb{C}\right),$$

which is derivable easily by simply iterating the well-known result (5.28) with (5.29). Similarly, another corrected version of the main result of Chakrabarty [8] (see also Thakurta [44, p. 79, Equation (4.1)]), analogous to (5.30), can be derived by iterating the known result (5.27) with (5.34), and we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \binom{n+k}{k} t^k \sum_{\ell=0}^{n+k} \binom{\ell-\alpha-n-k-1}{\ell} P_{n+k-\ell}^{(\alpha,\ \beta-k+\ell)}(x)\,\tau^\ell \\
&\quad = (1-\tau)^n \{1-t(1-\tau)\}^{\beta} \left\{1 - \frac{1}{2}(x-2\tau+1)\,t\right\}^{-\alpha-\beta-n-1} \\
&\qquad \cdot P_n^{(\alpha,\beta)}\left(\frac{x-\tau-\frac{1}{2}(x-2\tau+1)\,t(1-\tau)}{(1-\tau)\left\{1-\frac{1}{2}(x-2\tau+1)\,t\right\}}\right)
\end{aligned} \tag{5.38}$$

$$\left(n \in \mathbb{N}_0; \quad |t| < \min\left\{|1-\tau|^{-1},\ 2|x-2\tau+1|^{-1}\right\}\right),$$

which may be compared also with (5.32).

Finally, we recall the *main* results of Ghosh [18] in the following (*slightly modified* or *corrected*) forms:

$$\sum_{k=0}^{\infty} \binom{k-\alpha-n-1}{k} P_n^{(\alpha-k,\,\beta)}(x)\,t^k$$
$$= (1-t)^{\alpha} \left\{1+\frac{1}{2}(x-1)\,t\right\}^n P_n^{(\alpha,\beta)}\left(\frac{x-\frac{1}{2}(x-1)\,t}{1+\frac{1}{2}(x-1)\,t}\right) \tag{5.39}$$
$$(n \in \mathbb{N}_0; \quad |t|<1);$$

$$\sum_{k=0}^{\infty} \binom{\alpha+\beta+n+k}{k} P_n^{(\alpha+k,\,\beta)}(x)\,t^k$$
$$= (1-t)^{-\alpha-\beta-n-1} P_n^{(\alpha,\beta)}\left(\frac{x+t}{1-t}\right) \qquad (n \in \mathbb{N}_0; \quad |t|<1); \tag{5.40}$$

$$\sum_{k,\ell=0}^{\infty} \binom{\alpha+\beta+n+k-\ell}{k}\binom{\ell-\alpha-n-1}{\ell} P_n^{(\alpha+k-\ell,\,\beta)}(x)\,t^k\,\tau^\ell$$
$$= (1-t)^{-\alpha-\beta-n-1}\{1-(1-t)\,\tau\}^{\alpha}\left\{1+\frac{1}{2}(x+2t-1)\,\tau\right\}^n \tag{5.41}$$
$$\cdot\, P_n^{(\alpha,\beta)}\left(\frac{x+t-\frac{1}{2}(x+2t-1)\,\tau(1-t)}{(1-t)\,\{1+\frac{1}{2}(x+2t-1)\,\tau\}}\right)$$
$$\left(n \in \mathbb{N}_0; \quad |t| < \min\{1,\ |1-\tau^{-1}|\}\right).$$

Formula (5.41) provides the corrected (and slightly modified) form of two (essentially identical) double-series generating functions in Ghosh's paper [18, p. 139, Equation (1.4); p. 140, Equation (1.5)]. In fact, (5.41) follows readily by iterating the generating functions (5.39) and (5.40): First evaluate the k-series by applying (5.40) with α replaced by $\alpha-\ell$, and then make use of (5.39) with x and t replaced by

$$\frac{x+t}{1-t} \qquad \text{and} \qquad (1-t)\,\tau\,,$$

respectively. Thus the basic results in Ghosh's paper [18] are the single-series generating functions (5.39) and (5.40).

Since [*cf.* Equations (4.1) and (4.2)]

$$P_n^{(\alpha-k,\,\beta)}(x) = \left(\frac{x+1}{2}\right)^n P_n^{(\alpha-k,\,-(\alpha+\beta+2n+1)+k)}\left(\frac{3-x}{x+1}\right), \tag{5.42}$$

the generating function (5.39) is equivalent to the already known result (5.4). Furthermore, since [*cf.* Equations (4.1) and (4.2)]

$$P_n^{(\alpha+k,\,\beta)}(x) = \left(\frac{x+1}{2}\right)^n P_n^{(\alpha+k,\,-(\alpha+\beta+2n+1)-k)}\left(\frac{3-x}{x+1}\right), \tag{5.43}$$

the generating function (5.40) is equivalent to the already known result (5.5). Consequently, since the generating functions (5.4) and (5.5) are observed above to be equivalent, the four generating functions (5.4), (5.5), (5.39), and (5.40) are all equivalent to one another.

In view of the well-known hypergeometric representations (*cf.*, *e.g.*, Szegö [43, p. 68, Equation (4.3.2)]; see also Rainville [28, p. 254, Equation (2)], and Equation (3.11) above)

$$P_n^{(\alpha-k,\,\beta)} = \binom{\alpha-k+n}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,\ -\beta-n; \\ \alpha-k+1;\end{matrix}\ \frac{x-1}{x+1}\right] \tag{5.44}$$

and (*cf.* Rainville [28, p. 255, Equation (9)]; see also Srivastava and Manocha [39, p. 91, Problem 16(iv)])

$$P_n^{(\alpha,\beta)}(x) = \binom{\alpha+\beta+2n}{n}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n,\ -\beta-n; \\ -\alpha-\beta-2n;\end{matrix}\ \frac{2}{x+1}\right], \tag{5.45}$$

the generating functions (5.39) and (5.40) would naturally follow also as special cases of the hypergeometric generating function (5.12) when

$$u-2=v=0, \quad \alpha_1=-n, \quad \alpha_2=-\beta-n\ (n\in\mathbb{N}_0), \quad \lambda=-\alpha, \quad \text{and} \quad z=\frac{x-1}{x+1}$$

and

$$u-2=v=0, \quad \alpha_1=-n, \quad \alpha_2=-\beta-n, \quad \lambda=\alpha+\beta+2n+1\ (n\in\mathbb{N}_0), \quad \text{and} \quad z=\frac{2}{x+1},$$

respectively. Thus all the four generating functions (5.4), (5.5), (5.39), and (5.40), and their double-series consequences (5.6) and (5.41) are derivable from a single hypergeometric generating function (5.12).

We remark in passing that the various quadruple sums involving the Jacobi polynomials, which were obtained by Chongdar [12] (and, more recently, by Mukherjee and Chongdar [25]) by using Weisner's group-theoretic method, are simple consequences of the aforementioned known (or easily derivable) single-series generating functions for the Jacobi polynomials. The details involved in these simple derivations may be left as an exercise for the interested reader.

6. Bilinear and Bilateral Generating Functions

Each of the generating functions (5.4), (5.5), (5.12), (5.15), (5.16), (5.26), (5.27), (5.28), (5.39), and (5.40) can be seen to fit easily into the Singhal-Srivastava definition [31, p. 755, Equation (1)]:

$$\sum_{k=0}^{\infty} A_{m,k}\; S_{m+k}(x)\, t^k = f(x,t)\, \{g(x,t)\}^{-m}\; S_m\big(h(x,t)\big) \tag{6.1}$$

$$(m \in \mathbb{N}_0),$$

where the coefficients $A_{m,k}$ are independent of x and t, and f, g, and h are suitable functions of x and t. For example, upon replacing α and β by $\alpha-m$ and $\beta+m$, respectively ($m \in \mathbb{N}_0$), the generating function (5.4) can be rewritten in its equivalent form:

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{k-\alpha+m-n-1}{k}\; P_n^{(\alpha-m-k,\ \beta+m+k)}(x)\, t^k \\ = (1-t)^{\alpha-m}\; P_n^{(\alpha-m,\ \beta+m)}\big(t+(1-t)x\big)\end{aligned} \tag{6.2}$$

$$\big(m,n \in \mathbb{N}_0; \quad |t|<1\big),$$

which is of the type (6.1) with

$$A_{m,k} = \binom{k-\alpha+m-n-1}{k}, \quad f=(1-t)^{\alpha}, \quad g = 1-t,$$

$$h = t+(1-t)x, \quad \text{and} \quad S_k(x) = P_n^{(\alpha-k,\ \beta+k)}(x)$$

$$(k,m,n \in \mathbb{N}_0).$$

Thus, just as in the earlier works of Srivastava *et al.* (*cf.* [31], [35], [37], [39, Chapter 8], and [40]), the generating function (5.4) can be applied to derive various new families of bilinear, bilateral, or mixed multilateral generating functions for the Jacobi polynomials.

Recalling that the generating functions (5.4), (5.5), (5.39), and (5.40) are equivalent and that each of these results is a special case of the hypergeometric generating function (5.12) which, in turn, is contained in (5.15), it would suffice to apply the aforementioned analysis to the general result (5.15). We are thus led to the following class of bilateral generating functions for the sequence $\{\zeta_k^{(\lambda,\rho)}(z)\}_{k=0}^{\infty}$ defined by (5.14):

Theorem 1. *Corresponding to a non-vanishing function* $\Pi_\mu(\xi_1,\cdots,\xi_s)$ *of* s *complex variables* $\xi_1,\cdots,\xi_s$ $(s \in \mathbb{N})$ *and involving a complex parameter* μ, *called the order, let*

$$\Lambda_{m,p,q}^{(1)}[z;\,\xi_1,\cdots,\xi_s;\,t] := \sum_{k=0}^{\infty} a_k\,\zeta_{m+qk}^{(\lambda+\sigma qk,\,\rho)}(z) \cdot \Pi_{\mu+pk}(\xi_1,\cdots,\xi_s)\,t^k \tag{6.3}$$

$$\left(a_k \neq 0;\quad m \in \mathbb{N}_0;\quad \rho,p,q \in \mathbb{N};\quad \sigma \in \mathbb{C}\right)$$

and

$$\Theta_{k,m,q}^{\lambda,p,\mu,\sigma}(z;\,\xi_1,\cdots,\xi_s;\,\eta) := \sum_{\ell=0}^{[k/q]} \binom{\lambda+m+\sigma q\ell+k-1}{k-q\ell} a_\ell\,\zeta_{m+k}^{(\lambda+\sigma q\ell,\,\rho)}(z) \cdot \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\,\eta^\ell, \tag{6.4}$$

where the sequence $\{\zeta_k^{(\lambda,\rho)}(z)\}_{k=0}^{\infty}$ *is defined by* (5.14).

Then

$$\sum_{k=0}^{\infty} \Theta_{k,m,q}^{\lambda,p,\mu,\sigma}(z;\,\xi_1,\cdots,\xi_s;\,\eta)\,t^k = (1-t)^{-\lambda-m} \cdot \Lambda_{m,p,q}^{(1)}\left[z(1-t)^\rho;\quad \xi_1,\cdots,\xi_s;\quad \frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right] \qquad (|t|<1), \tag{6.5}$$

provided that each member of (6.5) *exists.*

Proof. For the sake of convenience, let $\mathcal{S}_2$ denote the first member of the assertion (6.5) of Theorem 1. Then, upon substituting for the polynomials

$$\Theta_{k,m,q}^{\lambda,p,\mu,\sigma}(z;\,\xi_1,\cdots,\xi_s;\,\eta)$$

from (6.4) into the left-hand side of (6.5), we obtain

$$\begin{aligned}\mathcal{S}_2 &= \sum_{k=0}^{\infty} t^k \sum_{\ell=0}^{[k/q]} \binom{\lambda+m+\sigma q\ell+k-1}{k-q\ell} a_\ell\, \zeta_{m+k}^{(\lambda+\sigma q\ell,\,\rho)}(z) \\ &\qquad\qquad \cdot \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\,\eta^\ell \\ &= \sum_{\ell=0}^{\infty} a_\ell\, \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)(\eta t^q)^\ell \\ &\qquad \cdot \sum_{k=0}^{\infty} \binom{\lambda+m+(\sigma+1)q\ell+k-1}{k} \zeta_{m+q\ell+k}^{(\lambda+\sigma q\ell,\,\rho)}(z)\,t^k \\ &\qquad\qquad \left(\Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\neq 0\right),\end{aligned} \tag{6.6}$$

where we have inverted the order of the double summation involved.

The inner k-series in (6.6) can be summed by appealing to the generating function (5.15) with m and λ replaced by $m+q\ell$ and $\lambda+\sigma q\ell$, respectively ($\ell \in \mathbb{N}_0$; $q \in \mathbb{N}$; $\sigma \in \mathbb{C}$), and we thus find from (6.6) that

$$\begin{aligned}\mathcal{S}_2 &= (1-t)^{-\lambda-m} \sum_{\ell=0}^{\infty} a_\ell\, \zeta_{m+q\ell}^{(\lambda+\sigma q\ell,\,\rho)}\left(z(1-t)^\rho\right) \\ &\qquad \cdot \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s) \left\{\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right\}^\ell \qquad (|t|<1),\end{aligned}$$

which, in view of the definition (6.3), is precisely the second member of the assertion (6.5) of Theorem 1.

This evidently completes the proof of Theorem 1 under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, Theorem 1 holds true (at least as a relation between formal power series) for those values of the various parameters and variables involved for which each member of the assertion (6.5) exists.

Various special cases of Theorem 1 will include all those bilateral or mixed multilateral generating functions which would stem in this manner from any of the equivalent generating functions (5.4), (5.5), (5.39), and (5.40) involving the Jacobi polynomials, simply because each of these generating functions was shown above to be contained in the hypergeometric generating function (5.12) which, in turn, is the special case $\rho = 1$ of the generating function (5.15) used in Theorem 1.

Some of the most general families of bilateral or mixed multilateral generating functions stemming from the well-known generating function (5.28) can be found in the work of Srivastava and Popov [40], and each of the generating functions (5.26) and (5.27) is indeed equivalent to (5.28). Furthermore, Chen *et al.* [10, p. 362, Theorem 4] have recently given a result analogous to Theorem 1 that stems from the generating function (5.16).

Having exhausted all of the generating functions belonging to the family (6.1), which we have considered in this paper, we turn to the multivariable generating functions (5.19) and (5.20), each of which fits easily into the following definition with $\mu = m$ $(m \in \mathbb{N}_0)$ (*cf.* Srivastava and Lavoie [37, p. 316, Equation (90)] and Panda [26, p. 28, Equation (3)]; see also Srivastava and Manocha [39, p. 437, Equation 8.5(1)]):

$$\begin{aligned}
&\sum_{k=0}^{\infty} \gamma_{\mu,k}\, \Xi_{\mu+k}(z_1,\cdots,z_r)\, t^k \\
&\quad = \theta(z_1,\cdots,z_r;\ t)\left\{\phi(z_1,\cdots,z_r;\ t)\right\}^{-\mu} \\
&\qquad \cdot \Xi_\mu\big(\psi_1(z_1,\cdots,z_r;\ t),\ \cdots,\ \psi_r(z_1,\cdots,z_r;\ t)\big) \qquad (\mu \in \mathbb{C}),
\end{aligned} \tag{6.7}$$

where the coefficients $\gamma_{\mu,k}$ $(k \in \mathbb{N}_0)$ are independent of $z_1,\cdots,z_r$ and t, and θ, ϕ, $\psi_1,\cdots,\psi_r$ are suitable functions of $z_1,\cdots,z_r$ and t.

The method of proof of Theorem 1 can be applied *mutatis mutandis* in order to derive Theorem 2 and Theorem 3 below, which would evidently yield bilateral or mixed multilateral generating relations for a remarkably wide variety of sequences of functions of several variables.

Theorem 2. *Corresponding to a non-vanishing function* $\Pi_\mu\ (\xi_1,\cdots,\xi_s)$ *of* s *complex variables* $\xi_1,\cdots,\xi_s$ $(s \in \mathbb{N})$ *and involving a complex parameter* μ*, called the order,* *let*

$$\Lambda^{(2)}_{m,p,q}\left[z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ t\right]$$

$$:= \sum_{k=0}^{\infty} a_k\ Z^{(\lambda+\sigma qk)}_{m+qk}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r) \cdot \Pi_{\mu+pk}(\xi_1,\cdots,\xi_s)\, t^k \tag{6.8}$$

$$\left(a_k \neq 0; \quad m \in \mathbb{N}_0; \quad p,q \in \mathbb{N}; \quad \sigma \in \mathbb{C}\right)$$

and

$$\Phi^{\lambda,p,\mu,\sigma}_{k,m,q}\left(z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ \eta\right)$$

$$:= \sum_{\ell=0}^{[k/q]} \binom{\lambda+m+\sigma q\ell+k-1}{k-q\ell} a_\ell\ Z^{(\lambda+\sigma q\ell)}_{m+k}(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r) \cdot \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\, \eta^\ell, \tag{6.9}$$

where the (multivariable and multiparameter) sequence

$$\left\{Z^{(\lambda)}_k\left(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r\right)\right\}_{k=0}^{\infty}$$

is defined by (5.21).

Then

$$\sum_{k=0}^{\infty} \Phi^{\lambda,p,\mu,\sigma}_{k,m,q}\left(z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ \eta\right) t^k$$

$$= (1-t)^{-\lambda-m}\ \Lambda^{(2)}_{m,p,q}\left[z_1(1-t)^{\sigma_1},\cdots,z_r(1-t)^{\sigma_r};\ \xi_1,\cdots,\xi_s;\ \frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right] \qquad (|t|<1), \tag{6.10}$$

provided that each member of (6.10) *exists.*

Theorem 3. *Corresponding to a non-vanishing function* $\Pi_\mu(\xi_1,\cdots,\xi_s)$ *of* s *complex variables* $\xi_1,\cdots,\xi_s$ $(s \in \mathbb{N})$ *and involving a complex parameter* μ, *called the order,* *let*

$$\Lambda^{(3)}_{m,p,q}\left[z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ t\right]$$
$$:=\sum_{k=0}^{\infty} a_k\ \Omega^{(\lambda+\sigma qk)}_{m+qk}\left(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r\right) \tag{6.11}$$
$$\cdot\,\Pi_{\mu+pk}\left(\xi_1,\cdots,\xi_s\right)t^k$$

$$\left(a_k\neq 0;\quad m\in\mathbb{N}_0;\quad p,q\in\mathbb{N};\quad \sigma\in\mathbb{C}\right)$$

and

$$\Psi^{\lambda,p,\mu,\sigma}_{k,m,q}\left(z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ \eta\right)$$
$$:=\sum_{\ell=0}^{[k/q]}\binom{\lambda+m+\sigma q\ell+k-1}{k-q\ell}a_\ell\cdot\Omega^{(\lambda+\sigma q\ell)}_{m+k}\left(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r\right) \tag{6.12}$$
$$\cdot\,\Pi_{\mu+p\ell}\left(\xi_1,\cdots,\xi_s\right)\eta^\ell,$$

where the (multivariable and multiparameter) sequence

$$\left\{\Omega^{(\lambda)}_k\left(\sigma_1,\cdots,\sigma_r;\ z_1,\cdots,z_r\right)\right\}_{k=0}^{\infty}$$

is defined by (5.18).

Then

$$\sum_{k=0}^{\infty}\Psi^{\lambda,p,\mu,\sigma}_{k,m,q}\left(z_1,\cdots,z_r;\ \xi_1,\cdots,\xi_s;\ \eta\right)t^k$$
$$=(1-t)^{-\lambda-m}\ \Lambda^{(3)}_{m,p,q}\left[\frac{z_1}{(1-t)^{\sigma_1}},\cdots,\frac{z_r}{(1-t)^{\sigma_r}};\right. \tag{6.13}$$
$$\left.\xi_1,\cdots,\xi_s;\ \frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right]\qquad(|t|<1),$$

provided that each member of (6.13) *exists.*

In its special case when

$$r=1,\ \sigma_1=\rho\ (\rho\in\mathbb{N}),\quad\text{and}\quad A(k)=\frac{(\alpha_1)_k\cdots(\alpha_u)_k}{(\beta_1)_k\cdots(\beta_v)_k}\cdot\frac{1}{k!}\quad(k\in\mathbb{N}_0),$$

Theorem 2 corresponds essentially to Theorem 1. Theorem 3, on the other hand, provides a similar (multivariable and multiparameter) generalization of a result due to Chen *et al.* [10, p. 362, Theorem 4].

We conclude this paper by remarking that, for each suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function

$$\Pi_\mu(\xi_1, \cdots, \xi_s) \qquad (s \in \mathbb{N}^* := \mathbb{N} \setminus \{1\})$$

is expressed as an appropriate product of simpler functions, Theorems 1, 2, and 3 can be shown to yield various classes of mixed multilateral generating relations for numerous sequences of functions of one and more variables, which belong to the families involved in the definitions (5.14), (5.18), and (5.21).

Acknowledgements

The present investigation was initiated during the second-named author's visit to the Institute of Mathematics (Academia Sinica) at Taipei in July 1993. This work was supported, in part, by the *National Science Council of the Republic of China* under Grant NSC–82–0208–M–1–145 and, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

1. N. Abdul-Halim and W.A. Al-Salam, A characterization of the Laguerre polynomials, *Rend. Sem. Mat. Univ. Padova* 34(1964), 176-179.
2. R. Askey, Jacobi's generating function for Jacobi polynomials, *Proc. Amer. Math. Soc.* **71**(1978), 243-246.
3. R. Askey, An integral of Ramanujan and orthogonal polynomials, *J. Indian Math. Soc.* (*N.S.*) **51**(1987), 27-36.
4. W.N. Bailey, Products of generalized hypergeometric series, *Proc. London Math. Soc.* (2) **28**(1928), 242-254.
5. S.D. Bajpai, A new orthogonality property for the Jacobi polynomials and its applications, *Nat. Acad. Sci. Lett.* **15**(1992), 221-226.
6. S.D. Bajpai, Generating function and orthogonality relations of a class of hypergeometric polynomials, *Pan Amer. Math. J.* 3(1993), no. 1, 95-102.
7. L. Carlitz, The generating function for the Jacobi polynomial, *Rend. Sem. Mat. Univ. Padova* **38**(1967), 86-88.
8. A.B. Chakrabarty, Group theoretic study of certain generalised functions of Jacobi polynomial, *J. Indian Inst. Sci.* **64**(1983), no. 5, 97-103.
9. L.-C. Chen and M.E.H. Ismail, On asymptotics of Jacobi polynomials, *SIAM J. Math. Anal.* **22**(1991), 1442-1449.
10. M.-P. Chen, C.-C. Feng, and H.M. Srivastava, Some generating functions for the generalized Bessel polynomials, *Studies in Appl. Math.* **87**(1992), 351-366.
11. A.K. Chongdar, Group theoretic study for certain generating functions, *Bull. Calcutta Math. Soc.* **77**(1985), 151-157.
12. A.K. Chongdar, On Jacobi polynomials, *Bull. Calcutta Math. Soc.* **78**(1986), 363-374.
13. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. **II**, McGraw-Hill, New York, Toronto, and London, 1953.
14. A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of Integral Transforms*, Vol. **II**, McGraw-Hill, New York, Toronto, and London, 1954.

15. E. Feldheim, Une propriété caractéristique des polynômes de Laguerre, *Comment. Math. Helv.* **13**(1940), 6-10.

16. D. Foata and P. Leroux. Polynômes de Jacobi, interprétation combinatoire et fonction génératrice, *Proc. Amer. Math. Soc.* **87**(1983), 47-53.

17. B. Ghosh, Group-theoretic study of certain generating functions of Jacobi polynomials, *Bull. Calcutta Math. Soc.* **75**(1983), 227-231.

18. B. Ghosh, Group theoretic origins of certain generating functions of Jacobi polynomials. I, *Pure Math. Manuscript* 3(1984), 139-144.

19. B. Ghosh, Group-theoretic origins of certain generating functions of Jacobi polynomials. III, *Pure Math. Manuscript* 4(1985), 5-10.

20. B. Ghosh, Group-theoretic study of certain generating functions involving modified Jacobi polynomials, *Bull. Calcutta Math. Soc.* **78**(1986), 187-192.

21. B. Ghosh, Some generating functions involving hypergeometric polynomials by Lie algebraic method, *Bull. Inst. Math. Acad. Sinica* **16**(1988), 149-155.

22. H.L. Manocha and B.L. Sharma, Some formulae for Jacobi polynomials, *Proc. Cambridge Philos. Soc.* **62**(1966), 459-462.

23. E.B. McBride, *Obtaining Generating Functions*, Springer Tracts in Natural Philosophy, Vol. **21**, Springer-Verlag, New York, Heidelberg, and Berlin, 1971.

24. W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York and London, 1968.

25. M.C. Mukherjee and A.K. Chongdar, Some generating functions of Jacobi polynomials, *Bull. Calcutta Math. Soc.* **85**(1993), 269-274.

26. R. Panda, A theorem on bilateral generating functions, *Glasnik Mat. Ser. III* **11**(**31**)(1976), 27-30.

27. S. Parnes and S.B. Ekhad, A WZ-style proof of Jacobi polynomials' generating function, *Discrete Math.* **110**(1992), 263-264.

28. E.D. Rainville, *Special Functions*, Macmillan, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

29. V. Romanovski, Sur quelques classes nouvelles de polynomes orthogonaux, *C.R. Acad. Sci. Paris* **188**(1929), 1023-1025.

30. R. Sharma and A.K. Chongdar, On generating functions of Jacobi polynomials by group-theoretic method, *Bull. Inst. Math. Acad. Sinica* **20**(1992), 253-259.

31. J.P. Singhal and H.M. Srivastava, A class of bilateral generating functions for certain classical polynomials, *Pacific J. Math.* 42(1972), 755-762.

32. L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, London, and New York, 1966.

33. H.M. Srivastava, Certain generating functions of several variables, *Z. Angew. Math. Mech.* **57**(1977), 339-340.

34. H.M. Srivastava, A note on Jacobi's generating function for the Jacobi polynomials, *Proc. Japan Acad. Ser. A Math. Sci.* **61**(1985), 201-202.

35. H.M. Srivastava, Some bilateral generating functions for a class of special functions. I and II, *Nederl. Akad. Wetensch. Indag. Math.* **42**(1980), 221-233 and 234-246.

36. H.M. Srivastava, A certain family of generating functions for classical orthogonal polynomials, in *Topics in Polynomials of One and Several Variables and Their Applications: A Volume Dedicated to the Memory of P.L. Chebyshev* (1821-1894) (Th. M. Rassias, H.M. Srivastava, and A. Yanushauskas, Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1993, 535-561.

37. H.M. Srivastava and J.-L. Lavoie, A certain method of obtaining bilateral generating functions, *Nederl. Akad. Wetensch. Indag. Math.* **37**(1975), 304-320.

38. H.M. Srivastava, J.-L. Lavoie, and R. Tremblay, The Rodrigues type representations for a certain class of special functions, *Ann. Mat. Pura Appl.* (4) **119**(1979), 9-24.

39. H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

40. H.M. Srivastava and B.S. Popov, A class of generating functions for the Jacobi and related polynomials, *Studies in Appl. Math.* 78(1988), 177-182.

41. H.M. Srivastava and J.P. Singhal, New generating functions for Jacobi and related

polynomials, *J. Math. Anal. Appl.* **41**(1973), 748-752.

42. V. Strehl, Combinatorics of Jacobi-configurations. II: The Srivastava-Singhal generating function revisited, *Discrete Math.* **73**(1988), 221-232.

43. G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. **23**, Fourth edition, American Mathematical Society, Providence, Rhode Island, 1975.

44. B.G. Thakurta, Some generating functions of Jacobi polynomials, *Caribbean J. Math.* **5**(1986), no. 2, 75-81.

45. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions*, Fourth edition, Cambridge University Press, Cambridge, London, and New York, 1927.