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Abstract: The metric function generalizes the concept of distance between two points and hence includes the symmetric property. The aim of this article is to introduce a new and proper extension of Kannan's fixed point theorem to the case of multivalued maps using Wardowski's F-contraction. We show that our result is applicable to a class of mappings where neither the multivalued version of Kannan's theorem nor that of Wardowski's can be applied to determine the existence of fixed points. Application of our result to the solution of integral equations has been provided. A multivalued Reich type generalized version of the result is also established.

Keywords: fixed point; multivalued map; F-contraction; complete metric space; integral equation

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1. Introduction and Preliminaries

Kannan [1] generalized the Banach contraction principle in the following manner which assured that even certain discontinuous functions might possess fixed points.

Theorem 1. *[1] Let $(\Im, \zeta)$ be a complete metric space. The self-map $\Upsilon : \Im \to \Im$ is called a Kannan map if there is a constant $a \in [0, 1)$ such that*

$$\zeta(\Upsilon\theta, \Upsilon\vartheta) \leq \frac{a}{2}[\zeta(\theta, \Upsilon\theta) + \zeta(\vartheta, \Upsilon\vartheta)]$$

for all $\theta, \vartheta \in \Im$. Then Υ has a unique fixed point, where the element $\theta \in \Im$ satisfying $\Upsilon\theta = \theta$ is called a fixed point of Υ.

Subrahmanyam [2] showed that Kannan's theorem could be used to characterize metric completeness. Reich [3] further generalized Banach's Contraction Principle and observed that Kannan's theorem is a particular case of it with suitable choice of the constants.

Theorem 2. *[3] Consider the complete metric space $(\Im, \zeta)$. Suppose the self-map $\Upsilon : \Im \to \Im$ satisfies the following:*

$$\zeta(\Upsilon\theta, \Upsilon\vartheta) \leq l\zeta(\theta, \Upsilon\theta) + m\zeta(\vartheta, \Upsilon\vartheta) + n\zeta(\theta, \vartheta), \text{for all } \theta, \vartheta \in \Im,$$

where $l, m, n \in \mathbb{R}_+$ satisfy $l + m + n < 1$. Then Υ admits a unique fixed point.

$l = m = 0$ provides Banach contraction principle while $l = m, n = 0$ produces Kannan's theorem.

Wardowski [4] defined the concept of F-contraction as given next.

Definition 1. *Let $\mathcal{F}$ denote the class of all such functions $F : (0, +\infty) \to (-\infty, +\infty)$ satisfying the following assumptions:*

(F1) *F is strictly increasing, i.e., for all $u, v \in (0, +\infty)$, $u < v$ implies $F(u) < F(v)$;*

(F2) *For each sequence $\{u_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$, $\lim_{n\to+\infty} u_n = 0$ if and only if $\lim_{n\to+\infty} F(u_n) = -\infty$;*

(F3) *There exists $t \in (0, 1)$ such that $\lim_{u\to0^+} u^t F(u) = 0$.*

If $(\Im, \zeta)$ is a metric space, then a mapping $\Upsilon : \Im \to \Im$ is said to be an $F-$contraction if there exist $\tau > 0$, $F \in \mathcal{F}$, such that for all $\theta, \vartheta \in \Im$,

$$\zeta(\Upsilon\theta, \Upsilon\vartheta) > 0 \Rightarrow \tau + F(\zeta(\Upsilon\theta, \Upsilon\vartheta)) \leq F(\zeta(\theta, \vartheta)).$$

Nadler [5] started the research on fixed points for multivalued maps with the help of Hausdorff concept, i.e., by considering the distance between two arbitrary sets in the following manner.

Let $(\Im, \zeta)$ be a complete metric space (in short, MS) and let $\mathcal{CB}(\Im)$ denote the class of all nonempty closed and bounded subsets of the nonempty set $\Im$. Then for $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\Im)$, define the map $\mathcal{H} : \mathcal{CB}(\Im) \times \mathcal{CB}(\Im) \to [0, \infty)$ by

$$\mathcal{H}(\mathcal{A}, \mathcal{B}) = \max\{\sup_{\xi\in\mathcal{B}} \Delta(\xi, \mathcal{A}), \sup_{\delta\in\mathcal{A}} \Delta(\delta, \mathcal{B})\},$$

where $\Delta(\delta, \mathcal{B}) = \inf_{\xi\in\mathcal{B}} \zeta(\delta, \xi)$. $(\mathcal{CB}(\Im), \mathcal{H})$ is called the Pompeiu-Hausdorff metric space generated by the metric ζ.

Definition 2. *[5] $v \in \Im$ is said to be a fixed point of the multivalued map $\Gamma : \Im \to \mathcal{CB}(\Im)$ if $v \in \Gamma v$. The set of all fixed points of Γ is denoted by $Fix(\Gamma)$.*

Remark 1.
1. *In the MS $(\mathcal{CB}(\Im), \mathcal{H})$, $\theta \in \Im$ is a fixed point of Υ if and only if $\Delta(\theta, \Upsilon\theta) = 0$.*
2. *The metric function $\zeta : \Im \times \Im \to [0, \infty)$ is continuous in the sense that if $\{\theta_n\}, \{\vartheta_n\}$ are two sequences in $\Im$ with $(\theta_n, \vartheta_n) \to (\theta, \vartheta)$ for some $\theta, \vartheta \in \Im$, as $n \to \infty$, then $\zeta(\theta_n, \vartheta_n) \to \zeta(\theta, \vartheta)$ as $n \to \infty$. Similarly, the function Δ is continuous because if $\theta_n \to \theta$ as $n \to \infty$, then $\Delta(\theta_n, \mathcal{A}) \to \Delta(\theta, \mathcal{A})$ as $n \to \infty$ for any $\mathcal{A} \subseteq \Im$.*

We list the following results to be used in the sequel.

Lemma 1. *[6,7] Let $(\Im, \zeta)$ be a MS and $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\Im)$. Then*

1. *$\Delta(\mu, \mathcal{B}) \leq \zeta(\mu, \gamma)$ if $\gamma \in \mathcal{B}$ and $\mu \in \Im$;*
2. *$\Delta(\mu, \mathcal{B}) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$ if $\mu \in \mathcal{A}$.*

Lemma 2. *[5] Suppose that $\mathcal{A}, \mathcal{B} \in \mathcal{CB}(\Im)$ and $v \in \mathcal{A}$. If $p > 0$, then there exists $\xi \in \mathcal{A}$ satisfying*

$$\zeta(v, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}) + p.$$

But there may not exist a point $\xi \in \mathcal{B}$ satisfying

$$\zeta(v, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B}).$$

However, if $\mathcal{B}$ is compact, then a point ξ exists satisfying $\zeta(v, \xi) \leq \mathcal{H}(\mathcal{A}, \mathcal{B})$.

Reich provided a multivalued version of his famous result as follows.

Definition 3. *[8] A multivalued map $\Gamma : \Im \to Cl(\Im)$ (where $Cl(\Im)$ is the family of nonempty closed subsets of $\Im$) is called a Reich-type multivalued (l, m, n)-contraction if there are constants $l, m, n \in \mathbb{R}^+$ satisfying $l + m + n < 1$ such that*

$$\mathcal{H}(\Gamma\theta, \Gamma\vartheta) \leq l\Delta(\theta, \Gamma\theta) + m\Delta(\vartheta, \Gamma\vartheta) + n\zeta(\theta, \vartheta),$$

for each $\theta, \vartheta \in \Im$.

Remark 2. *It was proved in [8] that a Reich-type multivalued (l, m, n)-contraction in a complete MS possesses a fixed point. When $n = 0$ and $l = m$, the above definition reduces to the multivalued version of Kannan-type contraction.*

Multivalued version of Wardowski's theorem was given by Altun et al. [9] as follows.

Definition 4. *[9] Let $(\Im, \zeta)$ be a MS. A multivalued map $\Gamma : \Im \to \mathcal{CB}(\Im)$ is called a multivalued F-contraction (MVFC, in short) if there is a constant $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(\mathcal{H}(\Gamma\mu, \Gamma\nu)) \leq F(\eta(\mu, \nu)) \quad (1)$$

for all $\mu, \nu \in \Im$ with $\Gamma\mu \neq \Gamma\nu$.

Remark 3. *In a complete MS, an MVFC possesses a fixed point.*

Recently, Kannan's and Reich's fixed point theorems have been studied and extended in several directions. Particularly we refer to the research of Aydi et al. [10,11], Bojor [12,13], Choudhury and Kundu [14], Debnath and de La Sen [15,16], Debnath et al. [17,18], Gornicki [19], Karapinar et al. [20], Mohammadi et al. [21]. Some important work on the application of multivalued F-contractions were recently carried out by Sgroi and Vetro [22] and Ali and Kamran [23].

In this article, first we introduce a proper generalization of Kannan's theorem for multivalued maps via F-contraction and further introduce a Reich-type generalization of the same. We present an application of our multivalued Kannan-type F-contraction to the solution of integral equations.

2. Multivalued Kannan Type F-contraction

In this section, we provide a proper extension of Kannan's theorem for multivalued maps using Wardowski's technique.

Definition 5. *Let $(\Im, \zeta)$ be a MS. The map $\Gamma : \Im \to \mathcal{CB}(\Im)$ is called a generalized multivalued Kannan-type F-contraction (GMKFC, in short) if there are constants $a, b \in (0, 1)$ satisfying $a + b < 1$, $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(\mathcal{H}(\Gamma\theta, \Gamma\vartheta)) \leq aF(\Delta(\theta, \Gamma\theta)) + bF(\Delta(\vartheta, \Gamma\vartheta)) \quad (2)$$

for all $\theta, \vartheta \in \Im \setminus Fix(\Gamma)$ with $\Gamma\theta \neq \Gamma\vartheta$, where $Fix(\Gamma)$ is the collection of all fixed points of Γ.

Theorem 3. *Let $(\Im, \zeta)$ be a complete MS. A GMKFC, $\Gamma : \Im \to \mathcal{CB}(\Im)$ such that $\Gamma\theta$ is compact for each $\theta \in \Im$ possesses a fixed point.*

Proof. Fix $\theta_0 \in \Im$ and choose $\theta_1 \in \Gamma\theta_0$. Since $\Gamma\theta_0$ is compact, by Lemma 2, we can select $\theta_2 \in \Gamma\theta_1$ such that $\zeta(\theta_2, \theta_1) \leq \mathcal{H}(\Gamma\theta_1, \Gamma\theta_0)$. Similarly we may consider $\theta_3 \in \Gamma\theta_2$ such that $\zeta(\theta_3, \theta_2) \leq \mathcal{H}(\Gamma\theta_2, \Gamma\theta_1)$ and so on. Continuing this way we generate a sequence $\{\theta_n\}$ satisfying $\theta_{n+1} \in \Gamma\theta_n$ such that $\zeta(\theta_{n+1}, \theta_n) \leq \mathcal{H}(\Gamma\theta_n, \Gamma\theta_{n-1})$.

Assume that $\theta_n \notin \Gamma\theta_n$ for all $n \geq 0$, because otherwise we obtain a fixed point. Thus $\Delta(\theta_n, \Gamma\theta_n) > 0$, for all $n \geq 0$.

Taking $\theta = \theta_n$ and $\vartheta = \vartheta_{n-1}$ in (2), we have

$$\begin{aligned}\tau + F(\zeta(\theta_{n+1}, \theta_n)) &\leq \tau + F(\mathcal{H}(\Gamma\theta_n, \Gamma\theta_{n-1})) \\ &\leq aF(\Delta(\theta_n, \Gamma\theta_n)) + bF(\Delta(\theta_{n-1}, \Gamma\theta_{n-1})) \\ &< aF(\zeta(\theta_n, \theta_{n+1})) + (1-a)F(\zeta(\theta_{n-1}, \theta_n)), (\text{since } b < 1-a). \end{aligned} \quad (3)$$

Let $\zeta(\theta_n, \theta_{n-1}) \leq \zeta(\theta_{n+1}, \theta_n)$. Then from (3), we have

$$\begin{aligned}\tau + F(\zeta(\theta_{n+1}, \theta_n)) &\leq \tau + F(\mathcal{H}(\Gamma\theta_n, \Gamma\theta_{n-1}))) \\ &< aF(\zeta(\theta_n, \theta_{n+1})) + (1-a)F(\zeta(\theta_{n+1}, \theta_n)) \\ &= F(\zeta(\theta_{n+1}, \theta_n)),\end{aligned}$$

which is a contradiction.

Therefore, $\eta(\theta_{n+1}, \theta_n) < \zeta(\theta_n, \theta_{n-1})$ for all $n \geq 1$. Thus from (3), we have

$$\tau + F(\zeta(\theta_{n+1}, \theta_n) < F(\zeta(\theta_n, \theta_{n-1}).$$

Consequently, we obtain

$$F(\zeta(\theta_n, \theta_{n+1})) < F(z\eta(\theta_{n-1}, \theta_n)) - \tau < \ldots < F(\zeta(\theta_0, \theta_1)) - n\tau, \quad (4)$$

for all $n \geq 1$.

Taking limit in (4) as $n \to \infty$, we have that

$$\lim_{n\to\infty} F(\zeta(\theta_n, \theta_{n+1})) = -\infty. \quad (5)$$

Hence by condition $(F2)$, we have $\lim_{n\to\infty} \zeta(\theta_n, \theta_{n+1}) = 0$.

Let $c_n = \zeta(\theta_n, \theta_{n+1})$. So, $\lim_{n\to\infty} c_n = 0$. Thus, for any $n \in \mathbb{N}$, we have

$$c_n^k(F(c_n) - F(c_0)) \leq -c_n^k n\tau < 0. \quad (6)$$

Taking limit in (6) as $n \to \infty$ and using $(F3)$, we have $\lim_{n\to\infty} c_n^k n = 0$. Thus there exists $n_0 \in \mathbb{N}$ such that $c_n^k n \leq 1$ for all $n \geq n_0$, i.e., $c_n \leq \frac{1}{n^{\frac{1}{k}}}$ for all $n \geq n_0$.

Let $m, n \in \mathbb{N}$ with $m > n \geq n_0$. Then

$$\begin{aligned}\zeta(\theta_n, \theta_m) &\leq \sum_{i=n}^{m-1} \zeta(\theta_i, \theta_{i+1}) = \sum_{i=n}^{m-1} c_i \\ &\leq \sum_{i=n}^{\infty} c_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.\end{aligned}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent for $k \in (0,1)$, we have $\zeta(\theta_n, \theta_m) \to 0$ as $m, n \to \infty$. Hence $\{\theta_n\}$ is Cauchy and $(\Im, \zeta)$ being complete, we have $\theta_n \to \theta$ for some $\theta \in \Im$.

We claim that θ is a fixed point of Γ. We consider the two cases.

Case I: There exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\Gamma\theta_{n_k} = \Gamma\theta$ for all $k \in \mathbb{N}$.

Then $\Delta(\theta, \Gamma\theta) = \lim_{k\to\infty} \Delta(\theta_{n_{k+1}}, \Gamma\theta) \leq \lim_{k\to\infty} \mathcal{H}(\Gamma\theta_{n_k}, \Gamma\theta) = 0$.

Case II: There exists $n_1 \in \mathbb{N}$ such that $\Gamma\theta_n \neq \Gamma\theta$ for all $n \geq n_1$. Then

$$\begin{aligned}\tau + F(\Delta(\theta_{n+1}, \Gamma\theta)) &= \tau + F(\mathcal{H}(\Gamma\theta_n, \Gamma\theta)) \\ &\leq aF(\Delta(\theta_n, \Gamma\theta_n)) + bF(\Delta(\theta, \Gamma\theta)) \\ &= aF(\zeta(\theta_n, \theta_{n+1})) + bF(\Delta(\theta, \Gamma\theta)). \end{aligned} \tag{7}$$

Taking limit in (7) as $n \to \infty$, we have $F(\Delta(\theta_{n+1}, \Gamma\theta)) \to -\infty$. Hence $\lim_{n\to\infty} \Delta(\theta_{n+1}, \Gamma\theta) = 0$. Thus $\Delta(\theta, \Gamma\theta) = 0$. □

Remark 4. *In [21], Mohammadi et al. studied interpolative multivalued Ćirić-Reich-Rus type F-contraction which is an extension of Reich's [3] theorem. It is to be noted that our new result, i.e, Theorem 3 is not a particular case of Theorem 2.7 in [21]. Because in [21], the condition $\alpha = 0$ is not permissible.*

Next, we provide an example which shows that Theorem 3 can be used to prove existence of fixed point results for such mappings where neither Kannan's nor Wardowski's theorem is applicable.

Example 1. *Consider $\Lambda = \{0, 1, 2\}$ with the metric*

$$\zeta(\theta, \vartheta) = \begin{cases} 0, \text{ if } \theta = \vartheta, \\ \frac{5}{4}, \text{ if } (\theta, \vartheta) = (1, 2) \text{ or } (\theta, \vartheta) = (2, 1), \\ 1, \text{ otherwise.} \end{cases}$$

Clearly (Λ, ζ) is a complete MS. Define the multivalued map $\Gamma : \Lambda \to \mathcal{CB}(\Lambda)$ by

$$\Gamma\theta = \begin{cases} \{0\}, \text{ if } \theta = 0, 1 \\ \{1, 2\}, \text{ if } \theta = 2. \end{cases}$$

Let $\theta = 0, \vartheta = 2$. Then $F(\mathcal{H}(\Gamma\theta, \Gamma\vartheta)) = F(\mathcal{H}(\{0\}, \{1, 2\})) = F(1)$ and $F(\zeta(\theta, \vartheta)) = F(\zeta(0, 2)) = F(1)$. Thus in this case we can not find any $\tau > 0$ such that $\tau + F(\mathcal{H}(\Gamma\theta, \Gamma\vartheta)) \leq F(\zeta(\theta, \vartheta))$, i.e., the multivalued version of Wardowski's theorem (see Remark 3) is not applicable.

Further, with $\theta = 0, \vartheta = 2$, if the condition $\mathcal{H}(\Gamma\theta, \Gamma\vartheta) \leq \lambda\{\Delta(\theta, \Gamma\theta) + \Delta(\vartheta, \Gamma\vartheta)\}$ is to be satisfied, then we should have $\mathcal{H}(\{0\}, \{1, 2\}) \leq \lambda\{\Delta(0, \Gamma 0) + \Delta(2, \Gamma 2)\}$, i.e., $1 \leq \lambda\{\Delta(0, \{0\}) + \Delta(2, \{1, 2\})\}$, i.e., $1 \leq \lambda \cdot 0 = 0$, which is not satisfied by any $\lambda > 0$. Hence multivalued Kannan's theorem (see, Remark 2) is also not applicable either.

Finally, if we assume that $\theta, \vartheta \in \Lambda \setminus Fix(\Gamma)$ with $\Gamma\theta \neq \Gamma\vartheta$, then it is easy to see that the condition $F(\mathcal{H}(\Gamma\theta, \Gamma\vartheta)) \leq aF(\Delta(\theta, \Gamma\theta)) + bF(\Delta(\vartheta, \Gamma\vartheta))$ is trivially satisfied for any $a, b \in (0, 1)$ with $a + b < 1$, $\tau > 0$ and $F \in \mathcal{F}$. We observe that $Fix(\Gamma) = \{0, 2\}$.

We present another example to illustrate Theorem 3 as follows.

Example 2. *Consider the set $\Im = [0, \infty)$ endowed with the usual metric $\zeta(\theta, \vartheta) = |\theta - \vartheta|$ for all $\theta, \vartheta \in \Im$. Define the multivalued map $\Gamma : \Im \to \mathcal{CB}(\Im)$ by*

$$\Gamma\theta = \begin{cases} \{0\}, \text{ if } \theta \in [0, 5) \\ \{\theta, \theta + 1\}, \text{ if } \theta \geq 5. \end{cases}$$

Let $\theta, \vartheta \notin Fix(\Gamma)$, then clearly $\theta, \vartheta \in (0, 5)$. In that case, $\mathcal{H}(\Gamma\theta, \Gamma\vartheta) = \mathcal{H}(\{0\}, \{0\}) = 0$. Thus, we observe that Γ is a GMKFC with $\tau = \ln 2$, $F(t) = t, t > 0$ and any $a, b \in (0, 1)$ with $a + b < 1$. Therefore, all conditions of Theorem 3 are satisfied and Γ has a fixed point. In fact, Γ has infinitely many fixed points.

3. Multivalued Reich Type F-Contraction

Here we introduce generalized multivalued Reich-type F-contraction (GMRFC, in short) by increasing the degrees of freedom of the constants in GMKFC. We show that GMKFC introduced in the previous section is a particular case of GMRFC for suitable choice of the constants.

Definition 6. *Let $(\Im, \zeta)$ be a complete MS. A map $\Gamma : \Im \to \mathcal{CB}(\Im)$ is said to be a GMRFC if there exist $a, b, c \in (0,1)$ with $a+b+c<1$, $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\tau + F(\mathcal{H}(\Gamma\theta, \Gamma\vartheta)) \leq aF(\Delta(\theta, \Gamma\theta)) + bF(\Delta(\vartheta, \Gamma\vartheta)) + cF(\zeta(\theta, \vartheta)) \tag{8}$$

for all $\theta, \vartheta \in \Im \setminus Fix(\Gamma)$ with $\Gamma\theta \neq \Gamma\vartheta$.

Theorem 4. *Let $(\Im, \zeta)$ be a complete MS. A GMRFC, $\Gamma : \Im \to \Im$ such that $\Gamma\theta$ is compact for each $\theta \in \Im$ admits a fixed point.*

Proof. Similar to the proof of Theorem 3, we construct a sequence $\{\theta_n\}$.

Putting $\theta = \theta_n$ and $\vartheta = \theta_{n-1}$ in (8), we have

$$\begin{aligned}
\tau + F(\zeta(\theta_{n+1}, \theta_n)) &\leq \tau + F(\mathcal{H}(\Gamma\theta_n, \Gamma\theta_{n-1})) \\
&\leq aF(\Delta(\theta_n, \Gamma\theta_n)) + bF(\Delta(\theta_{n-1}, \Gamma\theta_{n-1})) + cF(\zeta(\theta_n, \theta_{n-1})) \\
&\leq aF(\zeta(\theta_n, \theta_{n+1})) + bF(\zeta(\theta_{n-1}, \theta_n)) + cF(\zeta(\theta_n, \theta_{n-1})) \\
&< aF(\zeta(\theta_n, \theta_{n+1})) + bF(\zeta(\theta_{n-1}, \theta_n)) + (1-a-b)F(\zeta(\theta_n, \theta_{n-1})),
\end{aligned} \tag{9}$$

(since $c < 1-a-b$).

Rest of the proof may be obtained in a similar manner as the proof of Theorem 3 and hence omitted. □

Remark 5.
1. *In Theorem 4, if we take $c = 0$, then Theorem 3 is obtained. Thus GMKFC introduced in this paper is a particular case of GMRFC when $c = 0$.*
2. *Theorem 4 is more general than Theorem 2.7 in [21] in terms of relaxation of degrees of freedom of the constants involved.*

4. An Application to Integral Equations

In this section we present an application of Theorem 3 to the solution of a particular Volterra type integral equation.

Let $C([0,\lambda], \mathbb{R})$ be the space of all real valued continuous functions defined on $[0,\lambda]$. For any $\varphi \in C([0,\lambda], \mathbb{R})$ and fixed arbitrary $\tau > 0$, define $\|\varphi\| = \sup_{r\in[0,\lambda]}\{|\varphi(r)|e^{-\tau r}\}$. It is easy to see that the norm $\|\cdot\|$ is equivalent to the supremum norm. The metric ζ on $C([0,\lambda], \mathbb{R})$ is defined by

$$\zeta(\varphi, \psi) = \sup_{r\in[0,\lambda]} \{|\varphi(r) - \psi(r)|e^{-\tau r}\}$$

for all $\varphi, \psi \in C([0,\lambda], \mathbb{R})$.

Consider the following integral equation

$$\varphi(r) = q(r) + \int_0^r K(u, v, \varphi(v))dv, r \in [0,\lambda], \tag{10}$$

where

(A) $q : [0,\lambda] \to \mathbb{R}$ and $K : [0,\lambda] \times [0,\lambda] \times \mathbb{R} \to \mathbb{R}$ are continuous;

(B) $K(u, v, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing for all $u, v \in [0,\lambda]$;

(C) there is $\varphi_0 \in C([0,\lambda],\mathbb{R})$ such that for all $r \in [0,\lambda]$, the following is true:

$$\varphi_0(r) \le q(r) + \int_0^r K(u,v,\varphi_0(v))dv.$$

Theorem 5. *Suppose that conditions $(A)-(C)$ hold. Further, suppose there exist $\tau \in [1,\infty)$ and $a,b \in (0,1)$ with $a+b<1$ satisfying*

$$\begin{aligned}&\Big|K(u,v,\varphi)-K(u,v,\psi)\Big|\\ &\qquad \le \tau e^{-\tau}\left(\left\|\varphi-\int_0^r K(u,v,\varphi)dv\right\|\right)^a \cdot \left(\left\|\psi-\int_0^r K(u,v,\psi)dv\right\|\right)^b \cdot e^{\tau v(1-(a+b))}, \qquad (11)\end{aligned}$$

for all $u,v \in [0,\lambda]$ and $\varphi,\psi \in \mathbb{R}$. Then the integral Equation (10) has a solution.

Proof. Define a map $\Gamma : C([0,\lambda],\mathbb{R}) \to C([0,\lambda],\mathbb{R})$ by

$$\Gamma(\varphi)(r) = q(r) + \int_0^r K(u,v,\varphi(v))dv, r \in [0,\lambda].$$

For each $r \in [0,\lambda]$, we have

$$\begin{aligned}\Big|\Gamma(\varphi)(r)-\Gamma(\psi)(r)\Big| &\le \int_0^r \Big|K(u,v,\varphi(v))-K(u,v,\psi(v))\Big|dv\\ &\le \int_0^r \tau e^{-\tau}\Big|\varphi(v)-\Gamma(\varphi)(v)\Big|^a \cdot \Big|\psi(v)-\Gamma(\psi)(v)\Big|^b e^{\tau v(1-(a+b))}dv\\ &\le \int_0^r e^{\tau v}\tau e^{-\tau}\Big|\varphi(v)-\Gamma(\varphi)(v)\Big|^a e^{-\tau va} \cdot \Big|\psi(v)-\Gamma(\psi)(v)\Big|^b e^{-\tau vb}dv\\ &\le \tau e^{-\tau}\|\varphi-\Gamma\varphi\|^a \cdot \|\psi-\Gamma\psi\|^b \int_0^r e^{\tau v}dv\\ &\le \tau e^{-\tau}\|\varphi-\Gamma\varphi\|^a \cdot \|\psi-\Gamma\psi\|^b \cdot \frac{e^{\tau r}}{\tau}\\ &\le e^{-\tau}\|\varphi-\Gamma\varphi\|^a \cdot \|\psi-\Gamma\psi\|^b \cdot e^{\tau r}\\ \Longrightarrow \Big|\Gamma(\varphi)(r)-\Gamma(\psi)(r)\Big|e^{-\tau r} &\le e^{-\tau}\|\varphi-\Gamma\varphi\|^a \cdot \|\psi-\Gamma\psi\|^b\\ \Longrightarrow \zeta(\Gamma\varphi,\Gamma\psi) &\le e^{-\tau}(\zeta(\varphi,\Gamma\varphi))^a \cdot (\zeta(\psi,\Gamma\psi))^b\\ \Longrightarrow \ln[\zeta(\Gamma\varphi,\Gamma\psi)] &\le \ln[e^{-\tau}(\zeta(\varphi,\Gamma\varphi))^a \cdot (\zeta(\psi,\Gamma\psi))^b]. \qquad (12)\end{aligned}$$

After some routine calculation we have that

$$\tau + \ln[\zeta(\Gamma\varphi,\Gamma\psi)] \le a\ln[\zeta(\varphi,\Gamma\varphi)] + b\ln[\zeta(\psi,\Gamma\psi)]. \qquad (13)$$

Taking $F(\delta) = \ln(\delta)$, $\delta > 0$, we have form (13) that

$$\tau + F(\zeta(\Gamma\varphi,\Gamma\psi)) \le aF(\zeta(\varphi,\Gamma\varphi)) + bF(\zeta(\psi,\Gamma\psi)),$$

for all $\varphi,\psi \in C([0,\lambda],\mathbb{R}) \setminus Fix(\Gamma)$ with $\Gamma\varphi \ne \psi h$.

Hence Theorem 3 is applicable to Γ and we conclude that Γ has a fixed point. Therefore, the integral Equation (10) has a solution. □

5. Conclusions

We have introduced new and proper extensions of multivalued Kannan type F-contraction and found its application to the solution of integral equations. It has been shown that our result is applicable to certain class of mappings where neither the multivalued version of Kannnan nor that of Wardowski can be used. Finding metric completeness characterization in terms of GMKFC is a suggested future work.

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