Existence and Nonlinear Stability of Dynamic Solutions to the Vlasov Equation under a $\frac{1}{r^2}$ Potential

by

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Abstract

This dissertation analyzes the existence and nonlinear stability of spherically symmetric dynamic solutions to the Vlasov equation under an inverse-square potential, known as the Vlasov-Manev system. This is an interesting mathematical problem because compared to a potential of the form $-\frac{1}{r^\alpha}$, where $1 < \alpha < 2$, the singularities which are encountered are much stronger and the analytical problems encountered are much more difficult. The first two Chapters give a brief historical background and necessary introductory material, as well as a summary of what is to follow. In the subsequent Chapters, several formulae for the potential and the force term which would apply in a spherically symmetric dynamic solution under an inverse-square potential are derived. Some of these are particularly well suited to solutions which have compact support. With these formulae in hand, some examples of anisotropic steady state solutions which are compactly supported are developed. The two examples differ markedly in that the force term in the first one becomes unboundedly large, while it is bounded in the second example. It is then shown how any such solution can be rescaled to produce infinitely many solutions with support on a sphere of any positive radius. Next, the existence and nonlinear stability of solutions under the Newtonian potential is investigated. The energy-Casimir method is introduced and used to establish the nonlinear stability. The existence of a large class of such nonlinearly stable solutions is proved. An example of an isotropic steady-state solution under the Newtonian potential is constructed. After this, the existence and nonlinear stability of solutions under the inverse-square potential is investigated, and an example of an isotropic steady-state solution under this potential is constructed. A comparison of the density profiles of the isotropic steady-state solutions

constructed under the Newtonian and the inverse-square potential show a remarkable similarity, in spite of the more serious singularities in the latter case.

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To

Nancy, Erin and Sean.

Chapter 1

Introduction

One of the first and best-known confirmations of Einstein's theory of general relativity was the correct prediction of the previously unexplained portion of the advance of the perihelion of the planet Mercury by 43 arc seconds per century (the total precession is 574 seconds of arc per century, of which all but 43 seconds can be explained by Newton's laws and classical mechanics, taking into account all the effects of the other planets).

However, an alternative way to produce this effect was proposed in a series of papers published by Manev [38, 39, 40, 41] between 1924 and 1930, involving a correction to the attractive Newtonian potential, $U(r) = -\frac{\gamma}{r}$, of the type

$$U(r) = -\frac{\gamma}{r} - \frac{\epsilon}{r^2} \tag{1.1}$$

We shall refer to (1.1) as a "Manev" potential, with $U_n(r) := -\frac{\gamma}{r}$ as the "Newtonian" part and $U_m(r) := -\frac{\epsilon}{r^2}$ as the "Manev correction" or "Pure Stellar Manev Potential".

In the last of his four papers, Manev [41] proposed the values $\gamma = \mu$ and $\epsilon = \frac{3\mu^2}{2c^2}$ ($\mu = GM$ and c being the gravitational parameter of the two-body system and the speed of light, respectively), presenting physical arguments in favor of this choice of the constants.

Diacu, Mioc and Stoica [13] describe how in 1908, the physicist Max Planck stated a more general action-reaction principle, verified by special relativity, and from which Newton's third law followed as a theorem. Making use of those results, Manev showed that by applying the more general action-reaction principle to classical mechanics, he was naturally led to a law given by a potential of the type given by equation (1.1). Thus, Manev considered this model as a substitute to general relativity. The advantage of Manev's model is that it explains solar-system phenomena with the same accuracy as relativity, but without leaving the framework of classical mechanics.

The Manev correction is insignificant on Galactic scales due to the large distances between stars. In globular clusters, which are agglomerations of roughly one million stars, typically seventy light-years in diameter, the average distance between stars is relatively much smaller. Hence, close encounters between stars are relatively more frequent, and corrections such as the Manev correction may matter over large periods of time.

There are also reasons for studying a $-\frac{\epsilon}{r^2}$ potential which are strictly mathematical. Compared to a potential of the form $-\frac{\epsilon}{r^\alpha}$, where $1 < \alpha < 2$, the singularities which are encountered with the Pure Stellar Manev potential are much stronger and the analytical problems encountered are much more difficult. For example, the force term under such a potential, which is the integral given in Chapter 2 as equation (2.7), is defined only as a Cauchy principal value.

In this dissertation we investigate whether spherically symmetric, nonlinearly stable steady-state solutions can be obtained in solutions depending only on the Manev correction. This provides assurance that steady-state solutions under the Newtonian potential with the Manev correction will also exist. For simplicity, we shall be interested in finding spherically symmetric steady-state solutions to the Vlasov equation, which we now introduce, given the Pure Stellar Manev potential.

1.1 The Vlasov-Poisson System

In the Newtonian case the full Vlasov-Poisson system is

$$\frac{\partial \Psi}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \Psi - \nabla_{\mathbf{x}} U(t, \mathbf{x}) \cdot \nabla_{\mathbf{v}} \Psi = 0 \quad \text{(Vlasov's equation)} \tag{1.2}$$

$$\rho(t, \mathbf{x}) := \int_{\mathbf{R}^3} \Psi(t, \mathbf{x}, \mathbf{v}) \; d\mathbf{v} \tag{1.3}$$

$$\triangle U(t, \mathbf{x}) = 4\pi\rho(t, \mathbf{x}) \quad \text{(Poisson's equation)} \tag{1.4}$$

such that $\Psi(0) = \Psi_0$, where Ψ_0 is a given density function of $(\mathbf{x}, \mathbf{v})$, and $\mathbf{x} \in \mathbf{R}^3$, $\mathbf{v} \in \mathbf{R}^3$.

Vlasov's equation (1.2) is also known as the collisionless Boltzmann equation. Astrophysicists refer to it in the stellar dynamics context as "Jeans' equation".

We now present a brief derivation of the Poisson's equation. Starting from $U(r) = -\frac{\gamma}{r}$, Newton derived the following result: that the gravitational force acting on any point at a distance r from the origin in a spherically symmetric distribution of particles is equal to the force which would result if all the mass inside radius r were concentrated at the origin, and all the mass outside of radius r is neglected. Today this is given as a challenging exercise for second-year students of multivariable calculus, see for example Edwards and Penney [15]. Many authors, for example Guo and Rein [25] use the following symbol for the mass inside a sphere of radius r centred at the origin:

$$m_\rho(r) = \int_0^r 4\pi s^2 \rho(s) \; ds \tag{1.5}$$

Newton's classical result can then be expressed as

$$U'(r) = \frac{m_\rho(r)}{r^2} \tag{1.6}$$

which can be rewritten as

$$r^2 U'(r) = \int_0^r 4\pi s^2 \rho(s) \; ds \tag{1.7}$$

Now take the derivative of both sides with respect to r and solve for ρ to get

$$\frac{1}{r^2}\left(r^2U'(r)\right)' = 4\pi\rho(r) \tag{1.8}$$

which is just Poisson's equation (1.4) in the spherically symmetric case.

Under the two assumptions of spherical symmetry of the initial condition Ψ_0 and no time dependence, Batt, Faltenbacher and Horst [2] write $\Psi(\mathbf{x},\mathbf{v}) = \Phi(r,w,F)$ for the choice of coordinates

$$r := |\mathbf{x}|, \quad w := \frac{1}{|\mathbf{x}|}\mathbf{x}\cdot\mathbf{v}, \quad F := |\mathbf{x}|^2|\mathbf{v}|^2 - (\mathbf{x}\cdot\mathbf{v})^2 = |\mathbf{x}\times\mathbf{v}|^2 \tag{1.9}$$

Definition 1.1 *A function $f(\mathbf{x},\mathbf{v})$ is called spherically symmetric if it depends only on $|\mathbf{x}|$, $|\mathbf{v}|$ and $\mathbf{x}\cdot\mathbf{v}$.*

We shall restrict our attention to steady spherically symmetric solutions. The new coordinate w is the component of velocity $\mathbf{v}$ in the direction of $\mathbf{x}$, and F is the square of the length of the angular momentum vector. Using the square prevents unnecessary involvement of square root signs in calculations. In these new coordinates the Vlasov-Poisson system (1.2 - 1.4) takes the form

$$w\frac{\partial\Phi}{\partial r}(r,w,F) + \left(\frac{F}{r^3} - U'(r)\right)\frac{\partial\Phi}{\partial w}(r,w,F) = 0 \quad \text{(Vlasov's equation)} \tag{1.10}$$

$$\rho(r) = \frac{\pi}{r^2}\int_{F>0}\int_{w\in\mathbf{R}}\Phi(r,w,F)\ dw\ dF \tag{1.11}$$

$$\frac{1}{r^2}(r^2U'(r))' = 4\pi\rho(r) \quad \text{(Poisson's equation)} \tag{1.12}$$

The first two equations, Vlasov's equation and the density, do not depend on the particular nature of the potential and remain valid for the Pure Stellar Manev potential.

The potential $U(r)$ is only determined up to a constant, and we note that there are two main conventions regarding the potential used in the literature. We begin with the definition as used in Rein [46].

$$U_n(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \tag{1.13}$$

In this form, the potential has the feature that when applied to a spherically symmetric gravitating particle distribution, there is a negative value of $U_n(r)$ at the origin (where $r = |\mathbf{x}| = 0$), and $\lim_{r\to\infty} U_n(r) = 0$. Later, we will move to the version of potential used in BFH [2]

$$U_n(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} + \int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{y}|} \tag{1.14}$$

This potential function has been translated so that $U_m(0) = 0$ and $U_m(r) > 0$ for $r > 0$.

1.2 Jeans' Theorem

Throughout this dissertation, we will make extensive use of steady-state distribution functions $\Psi(\mathbf{x}, \mathbf{v})$ which are of the form

$$\Psi(\mathbf{x}, \mathbf{v}) = \varphi(E, F) \tag{1.15}$$

where F is as defined above in the new coordinate system (1.9), and

$$E(\mathbf{x}, \mathbf{v}) = \frac{1}{2}\mathbf{v}^2 + U(\mathbf{x}) \tag{1.16}$$

denotes the particle energy. Note that our distribution function $\Psi(\mathbf{x}, \mathbf{v})$ is implicitly defined in terms of φ, $\mathbf{x}$ and $\mathbf{v}$ by

$$\begin{aligned} \Psi(\mathbf{x}, \mathbf{v}) - \varphi(E, F) &= \\ \Psi(\mathbf{x}, \mathbf{v}) - \varphi\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} - \int\int \frac{\Psi(\mathbf{y}, \mathbf{v})}{|\mathbf{x}-\mathbf{y}|^n}\, d\mathbf{v}\, d\mathbf{y}, |\mathbf{x}\times\mathbf{v}|^2\right) &= 0 \end{aligned} \tag{1.17}$$

where $n = 1$ in the Newtonian case and $n = 2$ in the Pure Stellar Manev case.

The justification for the use of these distribution functions is Jeans' Theorem, named after the astronomer Sir James Jeans. Jeans went to Trinity College Cambridge in 1896 having won a mathematics scholarship. There, he was a fellow student with G. H. Hardy. They were both students of Alfred North Whitehead. Jeans was awarded an Isaac Newton Studentship in astronomy and optics, and in 1901 was elected a Fellow of Trinity. His publications include *The Dynamical Theory of Gasses* (1903), *Theoretical Mechanics* (1906), *The Mathematical Theory of Electricity and Magnetism* (1908), *Radiation and Quantum Theory* (1914), *Problems of Cosmogony and Stellar Dynamics* (1919) and *The Nebular Hypothesis and Modern Cosmogony* (1922). Jeans died in 1946, having received numerous honours and awards, including a knighthood in 1928.

For a concise statement of Jeans' Theorem, we shall refer to the standard Astronomy textbook by Binney and Tremaine [8].

Definition 1.2 *A constant of motion in a given force field is any function* $C(\mathbf{x}, \mathbf{v}, t)$ *of the coordinates, velocities and time that is constant along any stellar orbit; that is, if the position and velocity along an orbit are given by* $\mathbf{x}(t)$ *and* $\mathbf{v}(t) = \frac{d\mathbf{x}}{dt}$*, then*

$$C\left[\mathbf{x}(t_1), \mathbf{v}(t_1), t_1\right] = C\left[\mathbf{x}(t_2), \mathbf{v}(t_2), t_2\right] \tag{1.18}$$

for any t_1 *and* t_2.

Definition 1.3 *An integral of motion* $I(\mathbf{x}, \mathbf{v})$ *is any function of the phase-space coordinates* $(\mathbf{x}, \mathbf{v})$ *alone that is constant along any orbit:*

$$I\left[\mathbf{x}(t_1), \mathbf{v}(t_1)\right] = I\left[\mathbf{x}(t_2), \mathbf{v}(t_2)\right]. \tag{1.19}$$

According to equation (1.19), a function of the phase-space coordinates $I(\mathbf{x}, \mathbf{v})$ is an integral if and only if

$$\frac{d}{dt} I\left[\mathbf{x}(t), \mathbf{v}(t)\right] = 0 \tag{1.20}$$

along all orbits. With the equations of motion this becomes

$$\frac{dI}{dt} = \nabla_x I \cdot \frac{d\mathbf{x}}{dt} + \nabla_{\mathbf{v}} I \cdot \frac{d\mathbf{v}}{dt} = 0 \tag{1.21}$$

which can be rewritten

$$\mathbf{v} \cdot \nabla_x I - \nabla_x U \cdot \nabla_{\mathbf{v}} I = 0 \tag{1.22}$$

Comparing this with equation (1.2), we see that the condition for I to be an integral of motion is identical with the condition for I to be a steady-state solution of the Vlasov's equation, also known as the collisionless Boltzmann equation. This leads to the following theorem.

Jeans' Theorem [8] Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the galactic potential U, and any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation. (the reference to galactic potential is due to Binney and Tremaine [8] being intended for an astronomy audience)

Proof. Suppose f is a steady-state solution of the collisionless Boltzmann equation. Then, as we have just seen, f is an integral of motion, and the first part of the theorem is proved. Conversely, if I_1 to I_n are n integrals, and if f is any function of n variables, then

$$\frac{d}{dt} f\left[I_1(\mathbf{x}, \mathbf{v}), \ldots, I_n(\mathbf{x}, \mathbf{v})\right] = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m} \frac{dI_m}{dt} = 0 \tag{1.23}$$

and f is seen to satisfy the collisionless Boltzmann equation. □

An astronomer viewing a distant galaxy knows that the galaxy has finite mass and is bounded in extent. If we simply write down a function f of n variables, each variable being an integral of motion, we will usually find that the associated steady-state solution is nonphysical - it will have infinite mass, or be noncompactly supported, or both. The issue of which functions f produce physically meaningful solutions, in particular, solutions with finite mass and compact support, is addressed by Batt, Faltenbacher and Horst [2]. They also address the issue of self-consistency; that the three equations in the Vlasov-Poisson system must be mutually compatible. Their necessarily much more complex proof of Jeans' Theorem is discussed below, but before that let us see Sir James Jeans' version of the theorem which now bears his name.

We review exactly what Jeans stated in his paper [31], and then proceed to the proof of Jeans' Theorem presented in BFH [2]. In section II, *On the most General Law of Distribution possible for a Cluster or Universe in a Steady State* we find:

"Hence we see that for a universe in steady motion the law of distribution f must be such that $f = constant$ is a first integral, independent of the time, of the three equations of motion of a star (which Jeans states as Newton's Second Law, $Force = m\frac{d\mathbf{v}}{dt}$, in each of the x, y and z directions).

Whatever the nature of the motion, or whatever the arrangement of the universe, one first integral of these equations (of motion) is always known, namely the energy-integral

$$E := \frac{1}{2}(u^2 + v^2 + w^2) - \Omega = cons. \tag{1.24}$$

(where u, v and w are the components of velocity and Ω has the opposite sign to our U) E being the energy per unit mass.

If, as will usually be the case, this is the only first integral of the equations of motion, f must be a function of E, and the law of distribution must be of the form

$$f = \phi(E) = \phi\left[\frac{1}{2}(u^2 + v^2 + w^2 - 2\Omega)\right] \tag{1.25}$$

In the case now under consideration, in which no integral exists other than the energy-equation, this value of f will give the most general law of distribution possible. The formula $f = Ce^{-q(u^2+v^2+w^2-2\Omega)}$ (where $q > 0$) is a special case. It will be noticed that Schwarzschild's ellipsoidal law cannot be included in formula (1.25), nor indeed can any law which is consistent with the existence of star-streaming; the law of distribution (1.25) requires that at every point the proper motions of the stars shall favour all directions in space equally.

We must next consider cases in which there is more than one first integral of the equations of motion. (Here Jeans defines $\langle\tilde{w_1}, \tilde{w_2}, \tilde{w_3}\rangle = \langle u, v, w\rangle \times \langle x, y, z\rangle$, i.e. $\tilde{w_1} \ldots \tilde{w_3}$ are the three components of the angular momentum vector.)

Thus $\tilde{w_1} = cons.$ will be an integral if at every point of the universe we have that the resultant force passes through the x-axis. This requires that the universe shall be arranged so that the equipotentials shall all be surfaces of revolution. Thus the universe in general must be symmetrical about an axis. For such a universe the most general law of distribution consistent with the existence of a steady state will be

$$f = \phi(E, \tilde{w_1}) = \phi\left[\frac{1}{2}(u^2 + v^2 + w^2 - 2\Omega), zv - yw\right] \tag{1.26}$$

of which the law of distribution (1.25) previously found is a special case.

It is possible to have the three integrals of motion

$$\tilde{w_1} = cons., \quad \tilde{w_2} = cons., \quad \tilde{w_3} = cons., \tag{1.27}$$

if the universe is such that at every point the resultant force on every star passes through the centre of gravity of the whole universe. This requires that the universe shall be "centrobaric", so that it must be either a spherical universe or a universe in which all the stars remain for ever in one plane. In the former case the most general law of distribution is

$$f = \phi(E, \tilde{w_1}, \tilde{w_2}, \tilde{w_3}) \tag{1.28}$$

In the latter case, if the stars are all in the plane $x = 0$, the most general law of distribution possible is obtained by putting $x = 0$ and $u = 0$ in formula (1.28), and the law of distribution then reduces to that given by (1.26)."

This is the original statement of Jeans' Theorem. We turn to BFH [2], who proceed far beyond just a proof of Jeans' Theorem. They also address the issue of self-consistency of the three equations (1.10-1.12) of the Vlasov-Poisson system, and seek solutions with the physically meaningful features of finite mass and compact support. There is no concise statement of Jeans' Theorem, followed by a single proof. Instead, the theorem is built up bit by bit over several pages, culminating in their Theorem 3.9. The authors define the mapping J, $J : G \to \mathbf{R}^2$, as that which maps (r, w, F) to $(E(r, w, F), F)$. It is then proved, in the lengthy proof of Theorem 2.2, that if $\Phi : G \to \mathbf{R}$ is an integral of the system then there exists a unique function $\varphi : J(G) \to \mathbf{R}$ such that $\Phi = \varphi \circ J$.

The second requirement of such a solution is the consistency condition

$$\rho(r) = \frac{\pi}{r^2} \int_{F>0} \int_{w \in \mathbf{R}} \Phi(r, w, F)\, dw\, dF \qquad \text{a.e. on } (0, \infty) \tag{1.29}$$

and the third is the Poisson equation

$$\frac{1}{r^2}\left(r^2 U'(r)\right)' = 4\pi\rho(r) \qquad \text{a.e. on } (0, \infty). \tag{1.30}$$

If (Φ, ρ, U) is such a solution, then for r, $r_0 > 0$ we have

$$U'(r) = \frac{1}{r^2}\left[r_0^2 U'(r_0) + 4\pi \int_{r_0}^{r} s^2 \rho(s)\, ds\right] \tag{1.31}$$

It then follows from Theorem 2.2 [2] that there exists a unique non-negative measurable function φ on $J(G)$ associated with Φ — that is $\Phi = \varphi \circ J$ — such that

$$\begin{aligned}
\rho(r) &= \frac{\pi}{r^2} \int_{F>0} \int_{w \in \mathbf{R}} \varphi\left(\frac{1}{2}w^2 + \frac{1}{2}\frac{F}{r^2} + U(r), F\right) dw\, dF & (1.32)\\
&= 2\pi \int_{\tilde{F}>0} \int_{w>0} \varphi\left(\frac{1}{2}w^2 + \frac{1}{2}\tilde{F} + U(r), r^2\tilde{F}\right) dw\, d\tilde{F} & (1.33)\\
&= \frac{1}{4\pi} h_\varphi\left(r, U(r)\right) \qquad \text{a.e. on } (0, \infty) & (1.34)
\end{aligned}$$

with

$$h_\varphi(r,u) := 8\pi^2 \int_{\tilde{F}>0} \int_{w>0} \varphi\left(\frac{1}{2}w^2 + \frac{1}{2}\tilde{F} + u, r^2\tilde{F}\right) \, dw \, d\tilde{F} \tag{1.35}$$

and

$$\frac{1}{r^2}\left(r^2 U'(r)\right)' = h_\varphi(r, U(r)) \qquad \text{a.e. on } (0,\infty). \tag{1.36}$$

Conversely, if a non-negative measurable function φ is given and h_φ is defined by (1.35), then we obtain a solution (Φ, ρ, U) associated with φ if there exists a solution $U : (0,\infty) \to \mathbf{R}$ of (1.36) in the sense of Carathéodory (that is, U is differentiable with absolutely continuous U', and satisfies (1.36) a.e.). In fact, if we define

$$\Phi(r,w,F) := \varphi\left(\frac{1}{2}w^2 + \frac{1}{2}\frac{F}{r^2} + U(r), F\right) \qquad \text{on } G \tag{1.37}$$

$$\rho(r) := \frac{1}{4\pi}\, h_\varphi(r, U(r)) \qquad \text{on } (0,\infty), \tag{1.38}$$

then (Φ, ρ, U) is such a solution. Hence, to prove the existence of stationary, spherically symmetric, stellar dynamic solutions we need to solve (1.36).

This is the approach which we will take in constructing isotropic, stationary, spherically symmetric stellar dynamic solutions with finite mass and compact support in $\mathbf{R}^6$.

BFH [2] now investigate the equation

$$\frac{1}{r^2}\left(r^2 U'\right)' = h(r, U) \tag{1.39}$$

for a general right-hand side h.

They define for $r_0 > 0$ and $\alpha, \beta \in \mathbf{R}$ a solution $U : (0,\infty) \to \mathbf{R}$ of (1.39) in the sense of Carathéodory (defined above), said to be of type $\langle r_0, \alpha, \beta \rangle$ if

$$U(r_0) = \alpha, \qquad U'(r_0) = \beta \tag{1.40}$$

and for $\alpha \in \mathbf{R}$, a solution U is said to be of type $\langle 0, \alpha \rangle$ if

$$\lim_{r \to 0} U(r) = \alpha. \tag{1.41}$$

In this case, if $h = h_\varphi$ for some φ, the corresponding solution (Φ, ρ, U) is said to be of type $\langle r_0, \alpha, \beta \rangle$ or of type $\langle 0, \alpha \rangle$, respectively.

BFH [2] then prove the following theorems, prefaced by the comment that they have made a particular attempt to keep the assumptions of Theorems 3.6 and 3.8 very general in order to cover applications to examples with discontinuous and unbounded functions φ.

3.6. Theorem. [2] Let $\alpha \in \mathbf{R}$ and let $h : (0, \infty) \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions:

(i) For all $u \in \mathbf{R}$ the function $h(\cdot, u)$ is measurable;

(ii) $sh(\cdot, \alpha) \in L^1_{loc}[0, \infty)$; i.e. $s \mapsto sh(s, \alpha) \in L^1_{loc}[0, \infty)$.

(iii) for all pairs $(r_1, u) \in \{(0, \alpha)\} \cup \{(0, \infty) \times \mathbf{R}\}$, there exists a number $\delta > 0$ and a function

$$L_{r_1 u} : (r_1, r_1 + \delta) \to [0, \infty] \ \text{ with } \ (r - r_1) L_{r_1 u} \in L^1[r_1, r_1 + \delta] \tag{1.42}$$

such that for all $r \in (r_1, r_1 + \delta)$ and $u_1, u_2 \in [u - \delta, u + \delta]$ we have

$$|h(r, u_1) - h(r, u_2)| \leq L_{r_1 u}(r) |u_1 - u_2|; \tag{1.43}$$

(iv) there exists $H \in L^1_{loc}(0, \infty)$, $H \geq 0$, such that for all $r > 0$ and $u \in \mathbf{R}$ we have

$$|h(r, u)| \leq H(r)(1 + |u|). \tag{1.44}$$

Then (1.39) has a unique solution U of type $\langle 0, \alpha \rangle$.

The proof of Theorem 3.6 [2] is lengthy and makes use of the contraction mapping principle and Gronwall's Lemma.

Theorem 3.8 [2] is very similar to Theorem 3.6, and deals with the existence and uniqueness of solutions U of type $\langle r_0, \alpha, \beta \rangle$. We shall be primarily concerned with solutions U of type $\langle 0, \alpha \rangle$ covered by Theorem 3.6.

Now BFH [2] reach

3.9. Theorem. [2] Let φ be a non-negative measurable function such that, for given α or given r_0, α, β, the function h_φ defined by (1.35) satisfies the assumptions of Theorem 3.6 or 3.8, respectively. Then there exists a unique stationary, spherically symmetric, stellar dynamic solution (Φ, ρ, U) associated with φ of type $\langle 0, \alpha \rangle$ or $\langle r_0, \alpha, \beta \rangle$, respectively.

This concludes the discussion of Jeans' Theorem and the related issues of self-consistency of the three equations (1.10-1.12) of the Vlasov-Poisson system in the context of solutions with finite mass and compact support in BFH [2].

To summarize up to this point, Jeans states (but does not rigorously prove) that if E is the only first integral of motion which we wish to consider, then a steady state must be of the form $f = \phi(E) = \phi(\frac{1}{2}\mathbf{v}^2 + U)$. If there exist more than one first integral of motion, then a steady state must be of the form $f = \phi($ some or all of the integrals of motion $)$. Theorem 3.9 of BFH [2], gives precise conditions under which a non-negative measurable function $\varphi(E, F)$ will produce a unique stationary, spherically symmetric, stellar dynamic solution (Φ, ρ, U) associated with φ.

In Chapter 4 we will construct some anisotropic steady-state solutions under the Pure Stellar Manev potential in which the distribution function $f(\mathbf{x}, \mathbf{v}) = \phi(E, F)$. We will not construct anisotropic steady-state solutions in the Newtonian case, as there are already several such examples in BFH [2]. In the last two chapters we move to the more difficult problem of constructing isotropic steady-state solutions under both potentials, involving functions φ which depend on E only, $\varphi(E)$.

1.3 The Vlasov-Manev System

The Pure Stellar Manev potential, applied to a spherically symmetric attracting particle distribution, is given by

$$U_m(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\ d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} \tag{1.45}$$

In the Pure Stellar Manev case, the Poisson equation does not hold. We are left with (1.45) and

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} U_m(t,\mathbf{x})\cdot\nabla_{\mathbf{v}} f = 0 \quad \text{(Vlasov's equation)} \tag{1.46}$$

$$\rho(t,\mathbf{x}) := \int_{\mathbf{R}^3} f(t,\mathbf{x},\mathbf{v})\ \ d\mathbf{v} \tag{1.47}$$

In Chapter 3 we derive expressions for spherically symmetric distributions of particles which will facilitate the computations in the later chapters. Some of them are particularly well suited for particle distributions which have compact support. These expressions for the potential $U_m(r)$ and the force $-U_m'(r)$ under the Pure Stellar Manev potential are novelties which to our knowledge are not yet available in the literature.

Our first illustration of the use of the new expressions is to compute the Pure Stellar Manev potential for a Gaussian distribution of total mass 1.

1.4 Rescaling of Time-Dependent and Steady-State Solutions

In Chapter 4 anisotropic steady-state solutions under the Pure Stellar Manev potential are constructed. We begin with an example in which the force on a particle could become unboundedly large, even though all energies are bounded. Then, we proceed to a solution in which both energies and forces are bounded.

Now any steady-state solution can be expanded into a family of infinitely many steady-state solutions by an appropriate rescaling. We will take a result established in [9] and use it in a rescaling proposed in [46]. We start with a general interaction potential

$$U(r) = \frac{\alpha}{r^n}, \qquad n > 1 \tag{1.48}$$

and rescale the Vlasov equation. Let x_0, v_0 and t_0 be typical length, velocity and time scales related by $x_0 = v_0\, t_0$, and further that ρ_0 is a typical value of the spatial density. We pass to a dimensionless form of the Vlasov equation by setting

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{x_0}, \;\; \tilde{\mathbf{v}} = \frac{\mathbf{v}}{v_0}, \;\; \tilde{t} = \frac{t}{t_0} \tag{1.49}$$

and then

$$f(\mathbf{x}, \mathbf{v}, t) = \rho_0\, v_0^{-3}\, \tilde{f}(\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, \tilde{t}). \tag{1.50}$$

In [9] it is then established that preserving solutions to the Vlasov equation requires that in any rescaling

$$\frac{\rho\, x^{3-n}}{v^2} \tag{1.51}$$

is kept constant. Since $n = 2$ for the Pure Stellar Manev potential, this means that

$$\frac{\rho\, x}{v^2} \tag{1.52}$$

is conserved. This result will be used in the following rescaling function suggested in [46].

$$\overline{f}(\mathbf{x}, \mathbf{v}) = a f(b\mathbf{x}, c\mathbf{v}), \qquad a, b, c > 0 \tag{1.53}$$

In (r, w, F) coordinates this becomes

$$\overline{\Phi}(r, w, F) = a\Phi(br, cw, b^2c^2F), \qquad a, b, c > 0 \tag{1.54}$$

In either coordinate system, we can start from a known steady-state solution and by rescaling in both the three dimensions of position and the three dimensions of velocity, obtain infinitely many other steady-state solutions, with any radius of support desired.

Another type of rescaling known as projective invariance, originally due to Sophus Lie, is presented in Bobylev, Dukes, Illner and Victory [9] and Illner [29]. Projective invariance is not valid in the Newtonian case; it only holds for a pure $\pm\frac{\epsilon}{r^2}$ potential. The following Theorem is found in [29]:

Theorem 3.1 Let $f(t, \mathbf{x}, \mathbf{v})$ be a solution of (1.2) where the force is given by

$$\begin{aligned} -\nabla_x U[\rho](t, \mathbf{x}) &= -\epsilon \int \frac{1}{|\mathbf{x}-\mathbf{y}|^2} \nabla_{\mathbf{y}} \rho(t, \mathbf{y}) \, d\mathbf{y} \\ &= -2\epsilon \int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4} \rho(t, \mathbf{y}) \, d\mathbf{y} \end{aligned} \tag{1.55}$$

Note that the last integral on the right must be interpreted as a Cauchy principal value; the integration domain is always all space, so the integral is defined, e.g., if ρ is at least Hölder continuous. Suppose that f exists on a time interval $[0, t_0)$. Let $a > 0$ and set

$$\tau = \frac{t}{1+at} \qquad \mathbf{y} = \frac{\mathbf{x}}{1+at} \qquad \mathbf{w} = \mathbf{v}(1+at) - a\mathbf{x}.$$

Then $F(t, \mathbf{y}, \mathbf{w}) := f(t, \mathbf{x}, \mathbf{v})$ solves equation (1.2) with respect to $\tau, \mathbf{y}, \mathbf{w}$ on an interval

$$\tau \in \left[0, \frac{t_0}{1+at_0}\right]$$

and for the initial value $F(0, \mathbf{y}, \mathbf{w}) = f_0(\mathbf{y}, \mathbf{w} + a\mathbf{y})$.

1.5 Isotropic Steady-State Solutions

Chapter 5 begins with the energy-Casimir method, which arose in the plasma physics literature in the late 1950's. One of the earliest citations is Kruskal and Oberman [36], which appeared in 1958. This method is used to establish the existence and nonlinear stability of isotropic, stationary, spherically symmetric stellar dynamic solutions with finite mass and compact support in $\mathbf{R}^6$.

Batt, Morrison and Rein [3] investigated the existence and linear stability of stationary solutions of the Vlasov-Poisson system in three dimensions in the context of both plasma physics and stellar dynamics. We will establish nonlinear stability in both the Newtonian and the Pure Stellar Manev cases by means of the energy-Casimir method used in Rein [46], which was there limited to the plasma physics case. We adapted Rein's method to our purpose.

Chapter 5 concludes with the production of steady-state solutions under the Newtonian potential whose distribution function is dependent only on particle energy E. This means that these solutions are isotropic: at any given point, a given speed will be equiprobable in all directions.

Chapter 6 begins with a nonlinear stability theorem in the Pure Stellar Manev case. It concludes with the production of an isotropic steady-state solution in the Pure Stellar Manev case. To produce this solution, we make use of established existence and uniqueness results for a solution to a certain class of integral equation. We then proceed to solve the problem using the method of Picard iteration, which is carried out by hand for the first iteration, and then numerically after that.

1.6 Appendices

Appendix A investigates several Fourier transform results on the Pure Stellar Manev potential. In the Newtonian case, we calculate the potential from the density $\rho(r)$.

We can also solve the inverse problem: given a Newtonian potential at every radius r, can we recover the density distribution which produced it? We can, using Poisson's equation, rewritten in the form

$$\rho = \frac{1}{4\pi} \triangle U_n \tag{1.56}$$

Nowhere in the literature could we find a corresponding result for a $\frac{1}{r^2}$ potential. Accordingly, in Appendix A we develop formulae which permit the recovery of the density from the potential in this case.

Appendix B gives the details of some calculations which were made in Section 6.2 establishing that a certain kernel is an L^2-*kernel.*

Chapter 2

A Comparative Study of Newtonian vs. Manev Stellar Dynamics

The first two of the three equations in the Vlasov-Poisson system presented in Section 1.1 in Chapter 1, the Vlasov equation (1.2) and the density (1.3), do not depend on the particular potential and are valid in both Newtonian and Pure Stellar Manev dynamics.

However, serious difficulties arise due to the lack of the Poisson equation (1.4)

$$\triangle U(t, \mathbf{x}) = 4\pi\rho(t, \mathbf{x}) \tag{2.1}$$

in the Pure Stellar Manev case.

The Poisson equation permits convenient simplifications in the Newtonian case. We begin with Green's First Theorem

$$\int_V \left[\psi_1 \nabla^2 \psi_2 + \nabla\psi_1 \cdot \nabla\psi_2\right] \, dV = \int_S \psi_1 \nabla\psi_2 \cdot dS \tag{2.2}$$

Now if both functions vanish sufficiently rapidly at infinity, or if either of them is compactly supported, then the right hand side will be zero. This permits the familiar vector version of integration by parts, where the integration is carried out over all of $\mathbf{R}^3$:

$$\int_{\mathbf{R}^3} \nabla\psi_1 \cdot \nabla\psi_2 \, dV = -\int_{\mathbf{R}^3} \psi_1 \nabla^2 \psi_2 \, dV \tag{2.3}$$

This in turn means that instead of having to calculate the total potential energy in the Newtonian case using the expression

$$\frac{1}{2}\int_{\mathbf{R}^3} U(\mathbf{x})\ \rho(\mathbf{x})\ d\mathbf{x} = -\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\ d\mathbf{y}\ d\mathbf{x} \tag{2.4}$$

one can simply use

$$\begin{aligned} & \frac{1}{2}\int_{\mathbf{R}^3} U(\mathbf{x})\ \rho(\mathbf{x})\ d\mathbf{x} \\ = \ & \frac{1}{8\pi}\int_{\mathbf{R}^3} U(\mathbf{x})\ 4\pi\rho(\mathbf{x})\ d\mathbf{x} \\ = \ & \frac{1}{8\pi}\int_{\mathbf{R}^3} U(\mathbf{x})\ \nabla^2 U(\mathbf{x})\ d\mathbf{x} \\ = \ & -\frac{1}{8\pi}\int_{\mathbf{R}^3} \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{x}} U\ d\mathbf{x} \end{aligned} \tag{2.5}$$

which is customarily written

$$-\frac{1}{8\pi}\int_{\mathbf{R}^3} |\nabla_{\mathbf{x}} U|^2\ d\mathbf{x} \tag{2.6}$$

Throughout the remaining chapters, there will be several instances in which an equation which holds under the Pure Stellar Manev potential will be considerably more complicated than the corresponding equation under the Newtonian potential, due to the inapplicability of such simplifications.

Bobylev, Dukes, Illner and Victory [9] point out that the Pure Stellar Manev force term

$$E_2[\rho](\mathbf{x},t) = -\nabla_{\mathbf{x}} U_m[\rho](\mathbf{x},t) = -2\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4}\ \rho(\mathbf{y},t)\ d\mathbf{y} \tag{2.7}$$

is well defined if ρ is Hölder continuous with exponent $0 < \alpha < 1$ and $\rho \in L^1$, so we make this a requirement of our density, ρ, in problems involving the Pure Stellar

Manev potential. Illner [29] points out that in the Newtonian case, the sufficient requirement for the force term to be well defined is that density ρ be both bounded and integrable.

We proceed with a result from BDIV [9], that conservation of energy holds in the Pure Stellar Manev case just as it does in the Newtonian case. We denote by

$$E_{psm}(t) = \frac{1}{2}\left[\int\int \mathbf{v}^2 f(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}\ d\mathbf{x} - \int\int \frac{1}{|\mathbf{x}-\mathbf{y}|^2}\,\rho(\mathbf{y},t)\,\rho(\mathbf{x},t)\ d\mathbf{x}\ d\mathbf{y}\right] \quad (2.8)$$

the total energy for the Pure Stellar Manev case.

To prove conservation of E_{psm}, differentiate $\int\int \mathbf{v}^2 f\ d\mathbf{x}\ d\mathbf{v}$ with respect to t and use the Vlasov equation:

$$\frac{d}{dt}\int\int \mathbf{v}^2 f\ d\mathbf{x}\ d\mathbf{v} = \int\int \mathbf{v}^2(-\mathbf{v}\cdot\nabla_{\mathbf{x}} f)\ d\mathbf{x}\ d\mathbf{v} - \int\int \mathbf{v}^2 E_2\cdot\nabla_{\mathbf{v}} f\ d\mathbf{x}\ d\mathbf{v} \quad (2.9)$$

The first term on the right hand side is zero if f has compact support in $\mathbf{x}$ or vanishes sufficiently fast at infinity. By construction, our steady states have compact support in both $\mathbf{x}$ and $\mathbf{v}$. We will use the widely understood symbol $j(\mathbf{x},t) = \int \mathbf{v} f\ d\mathbf{v}$, see for example Glassey [20]. After an integration by parts, the second term on the right hand side becomes

$$\begin{aligned}
\int\int 2\mathbf{v}\cdot E_2\ f\ d\mathbf{v}\ d\mathbf{x} &= 2\int E_2\cdot j\ d\mathbf{x} \\
&= -2\int \nabla U_m\cdot j\ d\mathbf{x} \\
&= 2\int U_m\ \mathrm{div} j\ d\mathbf{x} \\
&= -2\int U_m\ \partial_t\rho\ d\mathbf{x} \\
&= -\frac{d}{dt}\int U_m\ \rho\ d\mathbf{x} \qquad (2.10)
\end{aligned}$$

This calculation uses the Continuity Equation $\rho_t + \mathrm{div}_{\mathbf{x}} j = 0$. Collecting terms gives the desired conservation of total energy in the Pure Stellar Manev case.

It should be mentioned that this calculation can be generalized to the case of any potential with a sufficiently weak singularity that satisfies Newton's third law. With the potential

$$U_V(\mathbf{x}) = -\int \rho(\mathbf{y})V(|\mathbf{x}-\mathbf{y}|)\;d\mathbf{y}, \tag{2.11}$$

the energy becomes

$$E_V(t) = \frac{1}{2}\left[\int\int \mathbf{v}^2 f(\mathbf{x},\mathbf{v},t)\,d\mathbf{v}\;d\mathbf{x} - \int\int \rho(\mathbf{y})\;\rho(\mathbf{x})\;V(|\mathbf{x}-\mathbf{y}|)\;d\mathbf{x}\;d\mathbf{y}\right] \tag{2.12}$$

and the steps on the previous page follow (modulo integrations by parts).

BDIV [9], in their Section 2.2, use an argument first introduced by Horst [27, 28] to compute the second derivative of the moment of inertia

$$\frac{d^2}{dt^2}\int \mathbf{x}^2\rho(\mathbf{x},t)\;d\mathbf{x} = -\frac{d}{dt}\left(\int\int \mathbf{x}^2\mathbf{v}\cdot\nabla_{\mathbf{x}}f\;d\mathbf{v}\;d\mathbf{x} + \int\int \mathbf{x}^2 E_2\cdot\nabla_{\mathbf{v}}f\;d\mathbf{v}\;d\mathbf{x}\right) \tag{2.13}$$

where E_2 is defined by (2.7). The second integral on the right hand side is zero if f vanishes rapidly enough with respect to velocity. In the first integral, we integrate by parts, and use the equation again, to obtain

$$\begin{aligned}\frac{d^2}{dt^2}\int \mathbf{x}^2\rho(\mathbf{x},t)\;d\mathbf{x} &= -\int\int 2(\mathbf{x}\cdot\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{x}}f + E_2\cdot\nabla_{\mathbf{v}}f)\;d\mathbf{v}\;d\mathbf{x} \qquad (2.14)\\ &= \int\int 2\mathbf{v}^2 f(\mathbf{x},\mathbf{v},t)\;d\mathbf{v}\;d\mathbf{x} + \int\int 2(\mathbf{x}\cdot E_2)f(\mathbf{x},\mathbf{v},t)\;d\mathbf{v}\;d\mathbf{x}\end{aligned}$$

where we used integration by parts in both terms. The first term on the right is 4 times the kinetic energy. For the second term, we use the structure of the term E_2 (2.7) to compute formally

$$\begin{aligned}-\int 4\mathbf{x}\cdot\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4}\;\rho(\mathbf{y},t)\;\rho(\mathbf{x},t)\;d\mathbf{y}\;d\mathbf{x} &=\\ -4\int\int \frac{1}{|\mathbf{x}-\mathbf{y}|^2}\;\rho(\mathbf{y},t)\;\rho(\mathbf{x},t)\;d\mathbf{x}\;d\mathbf{y}&\\ -\int 4\mathbf{y}\cdot\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4}\;\rho(\mathbf{y},t)\;\rho(\mathbf{x},t)\;d\mathbf{y}\;d\mathbf{x}&\end{aligned} \tag{2.15}$$

and by interchanging $\mathbf{x}$ and $\mathbf{y}$ in the last term we see that

$$\int\int 2(\mathbf{x}\cdot E_2)f(\mathbf{x},\mathbf{v},t)\;d\mathbf{v}\;d\mathbf{x} = -2\int\int \frac{1}{|\mathbf{x}-\mathbf{y}|^2}\;\rho(\mathbf{x},t)\;\rho(\mathbf{y},t)\;d\mathbf{x}\;d\mathbf{y} \tag{2.16}$$

which BDIV point out is just 4 times the potential energy. Hence they have proved that

$$\frac{d^2}{dt^2}\int \mathbf{x}^2\rho(\mathbf{x},t)\;d\mathbf{x} = 4E_{psm}(t) = 4E_{psm}(0) \tag{2.17}$$

where we have used conservation of energy in the last equality.

The identity (2.17) is remarkable and revealing. First, observe that the quantity $\int \mathbf{x}^2\rho(\mathbf{x},t)\;d\mathbf{x}$ is by definition nonnegative. If the total energy $E_{psm}(0)$ is negative, the time evolution of the moment of inertia is given by a downward parabola which must become negative for $t \geq t_0$, where t_0 can be explicitly computed in terms of the initial energy and the initial values of the moment of inertia and the quantity $\int\int \mathbf{x}\cdot\mathbf{v}\;f\;d\mathbf{x}\;d\mathbf{v}$. It follows that the solution of the Pure Stellar Manev system will not exist globally if $E_{psm}(0) < 0$, and the breakdown (i.e. the formation of a singularity) will happen at some time before t_0.

BDIV [9] do not carry out similar calculations for total energy $E_{psm}(0) > 0$, but that would mean that the time evolution of the moment of inertia is given by an upward parabola, which could result in either a moment of inertia which becomes zero, i.e. the formation of a singularity at some finite time t_0, or a moment of inertia which never becomes zero but after some finite time is strictly increasing.

We can conclude that $E_{psm} = 0$ is a necessary, but not necessarily sufficient, condition for a steady state in the Pure Stellar Manev case. This means that the total kinetic energy and total potential energy of the steady state solution are of equal magnitude and of opposite sign.

Compare this with the Virial Theorem, first proved by R. Clausius in 1870, for the Newtonian case, which can be found in Goldstein [21]. The following statement of the theorem is reproduced from Binney and Tremaine [8]. If the system is in a steady state, and K is the total kinetic energy and W is the total potential energy, then these two quantities are related by

$$2K + W = 0 \tag{2.18}$$

A corollary to this theorem is that the total energy of the system in the Newtonian case is given by

$$E_{tot} = K + W = -K = \frac{1}{2}W \tag{2.19}$$

Chapter 3

The Manev Potential

3.1 Overview

The objective of this Chapter is to derive expressions for spherically symmetric distributions of particles which will facilitate the computations in the later Chapters.

In the next section, we define new variables in $\mathbf{R}^3$ which are then used to evaluate the Pure Stellar Manev potential, given by

$$U_m(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} \tag{3.1}$$

Since we are in the spherically symmetric case, this expression depends on $r = |\mathbf{x}|$.

Our first illustration of the use of the new expressions is to compute the Pure Stellar Manev potential for a Gaussian distribution of total mass 1.

We then develop several alternative expressions for the Pure Stellar Manev potential and force, some of which are especially well suited for particle distributions which have compact support.

3.2 The Manev Potential

We shall be interested in finding spherically symmetric steady-state solutions to the Vlasov equation under the Pure Stellar Manev Potential.

We proceed to develop several methods of calculating the potential $U_m(r)$ under the Pure Stellar Manev potential in the spherically symmetric case. Recall from Section 1.1 that the potential is only determined up to a constant. We begin with the definition as used in [9].

$$U_m(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} \tag{3.2}$$

Later, we will move to the version of potential used in [2]

$$U_m(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{y}|^2} \tag{3.3}$$

For $\rho = \rho(r)$ given ($r \geq 0$), let

$$F(x) = \int_0^x \tau\rho(\tau)\, d\tau \tag{3.4}$$

Provided that $\tau\rho(\tau)$ is integrable on some closed interval, then $F(x)$ is a continuous function of bounded variation on that closed interval, and we have, for almost all x in that closed interval

$$F'(x) = x\rho(x) \tag{3.5}$$

Because of the spherical symmetry, we assume without loss of generality that the point under consideration is located at $\mathbf{x} = \langle 0, 0, r\rangle$. We translate every point in $\mathbf{R}^3$ by $\langle 0, 0, -r\rangle$, which moves each point down by a distance r so that $\mathbf{x}$ is now at the origin, and the center of the spherically symmetric distribution is now at $\langle 0, 0, -r\rangle$. We define variables $s = |\mathbf{x}-\mathbf{y}|$ and $\tilde{s} = |\langle 0, 0, -r\rangle - \mathbf{y}|$.

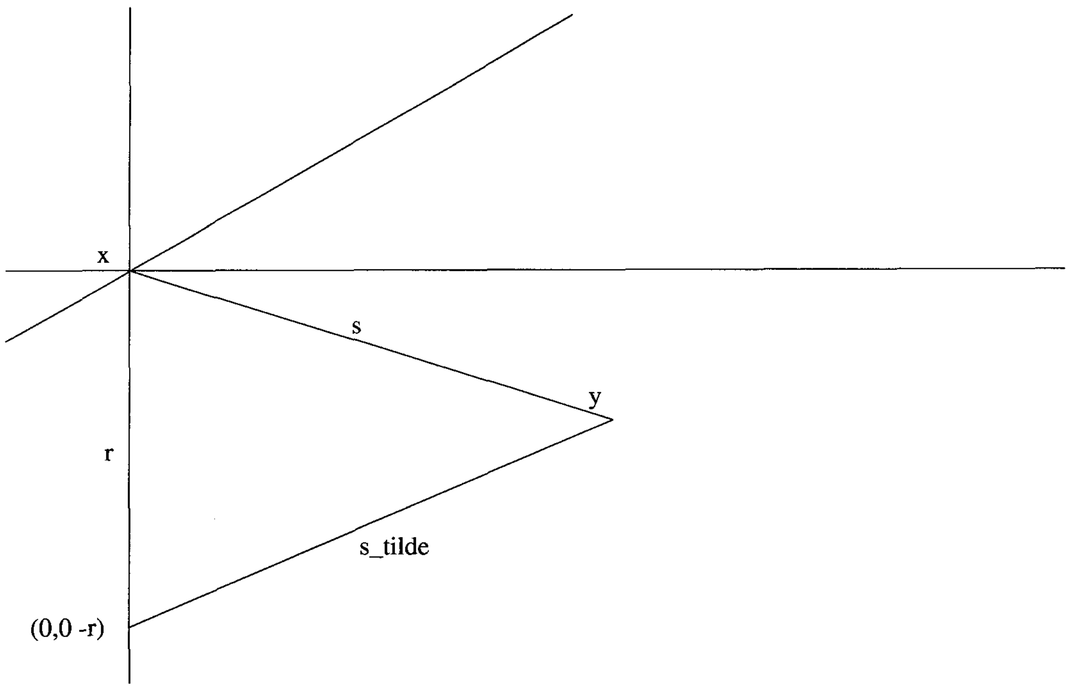

Figure 3.1: Variables used in the integrations to compute $U_m(r)$

The point $\mathbf{x}$ *which was at* $\langle 0,0,r\rangle$ *has been translated to the origin. The origin has been translated to the point* $\langle 0,0,-r\rangle$. *The point* $\mathbf{y}$ *is at a distance of* s *from the point* $\mathbf{x}$ *and at a distance of* $\tilde{s}$ *from the point* $\langle 0,0,-r\rangle$. *The angle* ϕ *is the angle between the position vector* $\mathbf{y}$ *and the positive* z*-axis.*

From the cosine law we have

$$\begin{aligned} \tilde{s}^2 &= r^2 + s^2 - 2rs\cos(\pi - \phi) \\ &= r^2 + s^2 + 2rs\cos(\phi) \end{aligned} \tag{3.6}$$

from which we obtain

$$\sin\phi \, d\phi = -\frac{\tilde{s}\, d\tilde{s}}{rs} \tag{3.7}$$

which will be used in computing the next integral. We now present some representations of $U_m(r)$ needed later.

Proposition 3.1

$$U_m(r) = -2\pi \int_0^r \int_{r-s}^{r+s} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds - 2\pi \int_r^\infty \int_{s-r}^{s+r} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds \tag{3.8}$$

Proof.

$$\begin{aligned} U_m(r) &= -\int_0^{2\pi} \int_0^\infty \int_0^\pi \frac{\rho(\tilde{s})}{s^2}\, s^2\, \sin\phi\, d\phi\, ds\, d\theta \\ &= -2\pi \int_0^\infty \int_0^\pi \rho(\tilde{s})\, \sin\phi\, d\phi\, ds \\ &= 2\pi \int_0^r \int_{r+s}^{r-s} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds + 2\pi \int_r^\infty \int_{s+r}^{s-r} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds \\ &= -2\pi \int_0^r \int_{r-s}^{r+s} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds - 2\pi \int_r^\infty \int_{s-r}^{s+r} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds \quad \square \end{aligned} \tag{3.9}$$

This computes $U_m(r)$ as in [9], or in [46]. The version used in [2] would then be

$$\begin{aligned} U_m(r) &= -2\pi \int_0^r \int_{r-s}^{r+s} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds - 2\pi \int_r^\infty \int_{s-r}^{s+r} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds \\ &\quad + \int_0^{2\pi} \int_0^\infty \int_0^\pi \frac{\rho(\tilde{s})}{\tilde{s}^2}\, \tilde{s}^2\, \sin\phi\, d\phi\, d\tilde{s}\, d\theta \\ &= -2\pi \int_0^r \int_{r-s}^{r+s} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds - 2\pi \int_r^\infty \int_{s-r}^{s+r} \frac{\tilde{s}\,\rho(\tilde{s})}{rs}\,d\tilde{s}\,ds \\ &\quad +4\pi \int_0^\infty \rho(\tilde{s})\,d\tilde{s} \end{aligned} \tag{3.10}$$

We proceed to compute alternative formulae for the Pure Stellar Manev potential $U_m(r)$. One outcome of these alternatives will be the expression for the Laplacian of $U_m(r)$. First we note that under the conditions of spherical symmetry, $\rho(-r) := \rho(r)$. Using our previously defined function (3.4) for $F(x)$, we have

Proposition 3.2

$$\begin{aligned} U_m(r) &= -2\pi \int_0^\infty \frac{1}{rs}\,[F(r+s) - F(r-s)]\,ds \\ &\quad +4\pi \int_0^\infty \rho(\tilde{s})\,d\tilde{s} \end{aligned} \tag{3.11}$$

Proof.

$$\begin{aligned}\int_{r-s}^{r+s} \tilde{s}\,\rho(\tilde{s})\,d\tilde{s} &= \int_0^{r+s} \tilde{s}\,\rho(\tilde{s})\,d\tilde{s} - \int_0^{r-s} \tilde{s}\,\rho(\tilde{s})\,d\tilde{s} \\ &= F(r+s) - F(r-s) \end{aligned} \tag{3.12}$$

and consequently we can rewrite equation (3.10) as

$$\begin{aligned} U_m(r) &= -2\pi \int_0^r \frac{1}{rs}\,[F(r+s) - F(r-s)]\,ds \\ & \quad -2\pi \int_r^\infty \frac{1}{rs}\,[F(s+r) - F(s-r)]\,ds\ + 4\pi \int_0^\infty \rho(\tilde{s})\,d\tilde{s} \end{aligned} \tag{3.13}$$

Now we use that r is an odd function and $\rho(r)$ is an even function to conclude that $F(x)$ is an even function and so $F(s-r) = F(r-s)$. This permits us to finally write

$$\begin{aligned} U_m(r) &= -2\pi \int_0^\infty \frac{1}{rs}\,[F(r+s) - F(r-s)]\,ds \\ & \quad +4\pi \int_0^\infty \rho(\tilde{s})\,d\tilde{s} \quad \square \end{aligned} \tag{3.14}$$

Corollary 3.3 *This permits an alternative representation of the force,*

$$\begin{aligned} U_m'(r) &= -2\pi \int_0^\infty -\frac{1}{r^2 s}\,[F(r+s) - F(r-s)] + \frac{1}{rs}\,[F'(r+s) - F'(r-s)]\,ds \\ &= -2\pi \int_0^\infty \left(\frac{F(r-s) - F(r+s)}{r^2 s}\right) + \frac{(r+s)\rho(r+s) - (r-s)\rho(r-s)}{rs}\,ds \end{aligned} \tag{3.15}$$

Remark. As mentioned at the beginning of this section, the Poisson's equation does not give the Laplacian for the Pure Stellar Manev potential. This can be computed from the above represenation of the force term. We will not be using $\triangle U_m(r)$ in our analysis of Vlasov-Manev problems, but

$$\begin{aligned} \triangle U_m(r) &= \frac{1}{r^2}\,\left(r^2 U_m'(r)\right)' \\ &= -2\pi \int_0^\infty \frac{1}{rs}\,[\rho(r+s) + (r+s)\rho'(r+s) \end{aligned}$$

$$
\begin{aligned}
& -\rho(r-s)-(r-s)\rho'(r-s)]\,ds \\
= \quad & -2\pi \int_0^\infty \frac{1}{rs}\frac{\partial}{\partial r}\left[(r+s)\rho(r+s)-(r-s)\rho(r-s)\right] ds
\end{aligned}
\tag{3.16}
$$

The preceding calculations permit a third representation of the potential.

Proposition 3.4

$$
\begin{aligned}
U_m(r) \quad = \quad & -\frac{2\pi}{r}\int_0^r \int_0^\infty \frac{(\tilde r+s)\rho(\tilde r+s)-(\tilde r-s)\rho(\tilde r-s)}{s}\,ds\,d\tilde r \\
& +4\pi \int_0^\infty \rho(\tilde s)\,d\tilde s
\end{aligned}
\tag{3.17}
$$

Proof. Using (3.14) and (3.15),

$$
\begin{aligned}
& rU_m'(r)+U_m(r) \\
= \quad & -2\pi \int_0^\infty \left(\frac{F(r-s)-F(r+s)}{rs}\right)+\frac{(r+s)\rho(r+s)-(r-s)\rho(r-s)}{s}\,ds \\
& -2\pi \int_0^\infty \frac{1}{rs}\,[F(r+s)-F(r-s)]\,ds\; +4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s \\
= \quad & -2\pi \int_0^\infty \frac{(r+s)\rho(r+s)-(r-s)\rho(r-s)}{s}\,ds+4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s
\end{aligned}
\tag{3.18}
$$

which we integrate to get

$$
\begin{aligned}
rU_m(r) \quad = \quad & -2\pi \int_0^r \int_0^\infty \frac{(\tilde r+s)\rho(\tilde r+s)-(\tilde r-s)\rho(\tilde r-s)}{s}\,ds\,d\tilde r \\
& +4\pi\, r\int_0^\infty \rho(\tilde s)\,d\tilde s
\end{aligned}
\tag{3.19}
$$

Dividing both sides by r,

$$\begin{aligned} U_m(r) &= -\frac{2\pi}{r}\int_0^r\int_0^\infty \frac{(\tilde r+s)\rho(\tilde r+s)-(\tilde r-s)\rho(\tilde r-s)}{s}\,ds\,d\tilde r \\ &\quad +4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s \quad \square \end{aligned} \tag{3.20}$$

In the next chapter, we will construct a steady-state solution under the Pure Stellar Manev law in which forces are bounded. This will require a fourth version of the potential which will facilitate this computation, one which is particularly suited to distributions with compact support. We will also use this fourth version in Chapter 6, in which our distributions have compact support.

Proposition 3.5

$$\begin{aligned} U_m(r) &= -\frac{2\pi}{r}\int_0^r \tilde s\rho(\tilde s)\,\ln\left|\frac{r+\tilde s}{r-\tilde s}\right|\,d\tilde s-\frac{2\pi}{r}\int_r^\infty \tilde s\rho(\tilde s)\,\ln\left|\frac{\tilde s+r}{\tilde s-r}\right|\,d\tilde s \\ &\quad +4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s \end{aligned} \tag{3.21}$$

Proof. We reverse the order of integration in (3.10) to obtain

$$\begin{aligned} U_m(r) &= -2\pi\int_0^r\int_{r-\tilde s}^r \frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s-2\pi\int_r^{2r}\int_{\tilde s-r}^r \frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s \\ &\quad -2\pi\int_0^{2r}\int_r^{\tilde s+r}\frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s-2\pi\int_{2r}^\infty\int_{\tilde s-r}^{\tilde s+r}\frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s \\ &\quad +4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s \\ &= -2\pi\int_0^r\int_{r-\tilde s}^{r+\tilde s}\frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s\;-2\pi\int_r^\infty\int_{\tilde s-r}^{\tilde s+r}\frac{\tilde s\,\rho(\tilde s)}{rs}\,ds\,d\tilde s \\ &\quad +4\pi\int_0^\infty \rho(\tilde s)\,d\tilde s \end{aligned} \tag{3.22}$$

Note that (3.22) is (3.10) with the order of integration reversed and with each occurrence of s in the limits of integration replaced by $\tilde s$. This is due to the fact that the region of integration in the $s\tilde{s}$-plane is symmetric about the line $s = \tilde{s}$. Finally, we perform the inner integrals to obtain

$$\begin{aligned} U_m(r) &= -\frac{2\pi}{r}\int_0^r \tilde{s}\rho(\tilde{s})\,\ln\left|\frac{r+\tilde{s}}{r-\tilde{s}}\right|\,d\tilde{s} - \frac{2\pi}{r}\int_r^\infty \tilde{s}\rho(\tilde{s})\,\ln\left|\frac{\tilde{s}+r}{\tilde{s}-r}\right|\,d\tilde{s} \\ &\quad +4\pi\int_0^\infty \rho(\tilde{s})\,d\tilde{s} \quad \square \end{aligned} \tag{3.23}$$

Note that if the density $\rho(r)$ has compact support, $r \in [0, R]$, where R is the maximum radius, then the upper limits of integration in the last two integrals of (3.23) become R.

Care must be taken in using this formula; close attention must be paid to the limits as the value r is approached both from above and below. In a case such as this one, use is made of

$$\lim_{x\to 0} x\,\ln x = 0 \tag{3.24}$$

to carefully deal with singularities. We have already observed that force under the Pure Stellar Manev law exists as a Cauchy principal value. Here we have a version of potential $U_m(r)$ which is also defined in such a way.

Example. As an example, let us compute the Pure Stellar Manev potential for a Gaussian distribution of total mass of 1 unit given by the density distribution

$$\rho(r) = \frac{1}{(2\pi)^{\frac{3}{2}}}e^{-\frac{r^2}{2}} \tag{3.25}$$

We proceed

$$\begin{aligned} U_m(r) &= -\frac{2\pi}{(2\pi)^{\frac{3}{2}}}\int_0^r\int_{r-s}^{r+s}\frac{\tilde{s}\,e^{-\frac{\tilde{s}^2}{2}}}{rs}\,d\tilde{s}\,ds - \frac{2\pi}{(2\pi)^{\frac{3}{2}}}\int_r^\infty\int_{s-r}^{s+r}\frac{\tilde{s}\,e^{-\frac{\tilde{s}^2}{2}}}{rs}\,d\tilde{s}\,ds \\ &\quad +\frac{4\pi}{(2\pi)^{\frac{3}{2}}}\int_0^\infty e^{-\frac{\tilde{s}^2}{2}}\,d\tilde{s} \\ &= 1 - \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{r^2}{2}}}{r}\int_0^\infty \frac{e^{-\frac{s^2}{2}}\sinh(rs)}{s}\,ds \end{aligned} \tag{3.26}$$

Note that $\lim_{r\to 0} U_m(r) = 0$, and $\lim_{r\to\infty} U_m(r) = 1$. It will be interesting to investigate the force at any point under the preceding mass distribution. Details of an integration by parts are omitted.

$$\begin{aligned}
-U_m'(r) &= \sqrt{\frac{2}{\pi}}\left(-\frac{(r^2+1)}{r^2}\, e^{-\frac{r^2}{2}} \int_0^\infty \frac{e^{-\frac{s^2}{2}} \sinh(rs)}{s}\, ds\right) \\
&\quad + \sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{r^2}{2}}}{r} \int_0^\infty e^{-\frac{s^2}{2}} \cosh(rs) ds \\
&= -2\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{r^2}{2}}}{r^2} \int_0^\infty \frac{e^{-\frac{s^2}{2}} (rs\cosh(rs) - \sinh(rs))}{s^3} ds
\end{aligned} \tag{3.27}$$

Chapter 4

Some Anisotropic Steady-State Solutions

4.1 Overview

The objective of this chapter is to derive two examples of steady state solutions under the Pure Stellar Manev potential which are compactly supported with a maximum radius of $r = 1$. Both of these steady state solutions will be anisotropic, that is, at any point a given speed will not be equiprobable in all directions. Isotropic solutions, which are more difficult to construct, will be presented in the last two chapters.

The first solution will feature a density distribution which is not continuous at $r = 1$, which results in the force on a particle becoming unboundedly large as it approaches $r = 1$. However, the particle energies, even arbitrarily close to $r = 1$, are bounded.

The second solution involves a density distribution with $\rho'(0) = 0$ and $\rho'(1) = 0$ and bounded force acting on all particles.

The last Section implements the nonlinear rescaling of steady state solutions, introduced in Section 1.4, to obtain infinitely many solutions of any desired support radius.

4.2 A Steady-State Solution with Unbounded Force

The first example of a steady-state solution under the Pure Stellar Manev law with nontrivial values in the w-coordinate is suggested by Example 4.3 of Batt, Faltenbacher and Horst [2]. Starting from a density given by

$$\rho(r) = \begin{cases} \frac{3}{4\pi} \;, & 0 < r < 1 \\ 0 \;, r \geq 1. \end{cases} \tag{4.1}$$

which results in a total mass of 1, we proceed to a $\phi(E, F)$ which satisfies the time-independent Vlasov equation and hence determines a steady-state solution. In both of the examples presented in this section, ϕ is dependent on both E and F, which were introduced in Section 1.2, and hence ϕ is anisotropic. Please note that here F means the square of the modulus of angular momentum as defined in equation (1.9), not the function F which was defined in equation (3.4). The dependence of ϕ on E and F is also in accord with Jeans' Theorem.

We first compute the potential energy function.

$$U(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{y}|^2} \tag{4.2}$$

Without loss of generality, we will assume that the point $\mathbf{x}$ has coordinates $(0, 0, r)$. For the first integral, the entire region of the sphere (region R) will be broken into two regions; the interior of the sphere with radius $(1 - r)$ centred at $\mathbf{x}$ (region R_1), and the complement of the interior (region R_2). Thus

$$U(\mathbf{x}) = -\int_{R_1} \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} - \int_{R_2} \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \int_{R} \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{y}|^2} \tag{4.3}$$

We now evaluate each of the three integrals.

$$-\int_{R_1} \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} = -\int_0^{2\pi} \int_0^{\pi} \int_0^{1-r} \frac{3}{4\pi} \frac{1}{s^2}\, s^2 \sin\phi \, ds\, d\phi\, d\theta$$

$$\begin{aligned} &= -4\pi \int_0^{1-r} \frac{3}{4\pi}\, ds \\ &= -3(1-r) \end{aligned} \tag{4.4}$$

$$\begin{aligned} -\int_{R_2} \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} &= -\int_0^{2\pi}\int_0^{\pi}\int_{1-r}^{-r\cos\phi+\sqrt{1-r^2\sin^2\phi}} \frac{3}{4\pi}\frac{1}{s^2}\, s^2\, \sin\phi\, ds\, d\phi\, d\theta \\ &= -2\pi\,\frac{3}{4\pi}\int_0^{\pi}\int_{1-r}^{-r\cos\phi+\sqrt{1-r^2\sin^2\phi}} ds\, d\phi \\ &= -\frac{3}{2}\int_0^{\pi}\left(-r\cos\phi+\sqrt{1-r^2\sin^2\phi}-1+r\right)\sin\phi\, d\phi \\ &= -\frac{3}{2}\left[-1+2r+\frac{(1-r^2)}{2r}\,\ln\left|\frac{1+r}{1-r}\right|\right] \end{aligned} \tag{4.5}$$

$$\begin{aligned} \int_R \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{y}|^2} &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{3}{4\pi}\frac{1}{s^2}\, s^2\, \sin\phi\, ds\, d\phi\, d\theta \\ &= 3 \end{aligned} \tag{4.6}$$

Adding these results together gives

$$U(r) = \frac{3}{2} - \frac{3(1-r^2)}{4r}\,\ln\left|\frac{1+r}{1-r}\right| \tag{4.7}$$

It is easily verified that

$$\lim_{r\to 0} U(r) = 0 \tag{4.8}$$

$$\lim_{r\to 1} U(r) = \frac{3}{2} \tag{4.9}$$

Remark. This potential possesses the elegant power series expansion

$$U(r) = 3(\frac{1}{1\cdot 3}r^2 + \frac{1}{3\cdot 5}r^4 + \frac{1}{5\cdot 7}r^6 + \frac{1}{7\cdot 9}r^8 \ldots) \tag{4.10}$$

We can immediately check that this series converges for $0 \leq r \leq 1$ and use it to confirm the limits just given.

Now the force term is easily derived to be

$$-U'(r) = \frac{3}{2r} - \frac{3}{4}\left(\frac{1+r^2}{r^2}\right) \ln\left|\frac{1+r}{1-r}\right| \tag{4.11}$$

As we would expect, the force term satisfies

$$\lim_{r\to 0} -U'(r) = 0 \tag{4.12}$$

however the limit at the other endpoint, $r = 1$, becomes unboundedly large and negative, and hence fails to exist. The physical meaning of this is quite clear. With finite kinetic energy, a particle can approach arbitrarily close to $r = 1$, but as the force becomes unboundedly large and negative, it will be pulled back into the interior of the sphere.

Remark. It is interesting to compare the force at a point inside the sphere in the Newtonian case to the Pure Stellar Manev case. In the former case,

$$\frac{-U'(r)}{r} = -1 \tag{4.13}$$

while in the latter we have the power series expansion

$$\frac{-U'(r)}{r} = -3\left(\frac{2}{1\cdot 3} + \frac{4}{3\cdot 5}r^2 + \frac{6}{5\cdot 7}r^6 \ldots\right) \tag{4.14}$$

In constructing our $\phi(E, F)$ we shall find it convenient if

$$\lim_{r\to 1} U(r) = \frac{1}{2} \tag{4.15}$$

This is specified so that a particle at the origin with a speed of 1 will have precisely the kinetic energy which it requires to reach, but not exceed, the maximum radius of 1. Since a total particle mass of 1 unit results in a limit as $r \to 1$ of exactly $\frac{3}{2}$, three times the proposed maximum potential energy, we must use a density which is exactly one-third of the original density. So, we will use a density of $\rho(r) = \frac{1}{4\pi}$ in the interior of the sphere. This has potential exactly one-third of (4.7)

$$U(r) = \frac{1}{2} - \frac{(1-r^2)}{4r} \ln\left|\frac{1+r}{1-r}\right| \tag{4.16}$$

Our first example is inspired by Example 4.3 of [2]. It is based on the simple definite integral, in which $k > 0$

$$\begin{aligned} \int_{-k}^{k} \frac{dw}{\sqrt{k^2 - w^2}} &= \qquad \text{let } w = k\sin\theta \\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{k\cos\theta d\theta}{k\cos\theta} &= \theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi \end{aligned} \tag{4.17}$$

Our first example is

$$\phi(E, F) = \begin{cases} \frac{1}{4\pi^3}(1 - 2E + F)^{-\frac{1}{2}} & \text{if } \; E < \frac{1}{2}(1+F), \quad F < r^2 \le 1 \\ 0 \;, & \text{elsewhere.} \end{cases} \tag{4.18}$$

Note how, in accordance with Jeans' Theorem, ϕ is a function of E and F only, which guarantees that the Vlasov equation is automatically satisfied. Non-negativity, integrability and self-consistency are the relevant points. We proceed:

The solution (4.18) becomes, in (r, w, F) coordinates

$$\Phi(r, w, F) = \begin{cases} \frac{1}{4\pi^3}\left(1 - w^2 - \frac{F}{r^2} - 2U(r) + F\right)^{-\frac{1}{2}} \\ \qquad \text{if } \; 1 - w^2 - \frac{F}{r^2} - 2U(r) + F > 0, \quad F < r^2 \le 1 \\ 0 \;, \quad \text{elsewhere.} \end{cases} \tag{4.19}$$

In this coordinate system $E = \frac{1}{2}\mathbf{v}^2 + U(\mathbf{x}) = \frac{1}{2}w^2 + \frac{F}{2r^2} + U(r)$. We will carry out the integral specified in equation (1.11) in order to confirm that this solution correctly produces a density of $\frac{1}{4\pi}$ at every value of $r \in (0,1)$. If we integrate first with respect to w and secondly with respect to F, then the requirement that $1 - w^2 - \frac{F}{r^2} - 2U(r) + F > 0$ means that $w^2 < 1 + F - \frac{F}{r^2} - 2U(r)$ and so the limits of integration for the inner integral are $-\sqrt{1 + F - \frac{F}{r^2} - 2U(r)} < w < \sqrt{1 + F - \frac{F}{r^2} - 2U(r)}$. In the outer integral, the minimum value of F is 0 and the maximum value is r^2. So now we proceed, using (4.17)

$$\begin{aligned} \rho(r) &= \frac{\pi}{r^2}\int_{F>0}\int_{w\in\mathbf{R}} \Phi(r,w,F)\ dw\ dF \\ &= \frac{\pi}{r^2}\int_0^{r^2}\int_{-\sqrt{1+F-\frac{F}{r^2}-2U(r)}}^{\sqrt{1+F-\frac{F}{r^2}-2U(r)}} \frac{1}{4\pi^3\sqrt{1-w^2-\frac{F}{r^2}-2U(r)+F}}\ dw\ dF \\ &= \frac{\pi}{r^2}\int_0^{r^2}\frac{\pi}{4\pi^3}\ dF \\ &= \frac{1}{4\pi} \end{aligned} \tag{4.20}$$

Remark. It is now possible to examine the potential function at points outside of the unit sphere. We will again assume without loss of generality that the point is located at $\mathbf{x} = (0,0,r)$. When outside the sphere, it is now more complicated to integrate about the point $(0,0,r)$, so we will integrate about the origin instead. As before, s is the distance from a point in the interior of the sphere to the origin, and we will use $d = |\mathbf{x} - \mathbf{y}|$ to represent the distance between two points.

$$\begin{aligned} U(\mathbf{x}) &= -\int \frac{\rho(\mathbf{y})\ d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \int \frac{\rho(\mathbf{y})\ d\mathbf{y}}{|\mathbf{y}|^2} \\ &= -\int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{1}{4\pi d^2}\ s^2\sin\phi\ ds\ d\phi\ d\theta + 1 \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^{\pi}\int_0^{1}\frac{s^2\sin\phi}{r^2+s^2-2rs\cos\phi}ds\,d\phi\ +1 \\
&= \frac{1}{2}+\frac{(r^2-1)}{4r}\ \ln\left|\frac{r+1}{r-1}\right| \qquad (4.21)
\end{aligned}
$$

Note that this is the same expression which gives the potential at any point in the interior of the sphere. It is easily checked that

$$\lim_{r\to\infty} U(r) = 1 \qquad (4.22)$$

4.3 Steady-State Solution with Bounded Force

The second example of a steady-state solution under the Pure Stellar Manev law with nontrivial values in the w-coordinate is obtained by modifying the density in the first example. Obtaining the potential $U(r)$ by a similar integration to that used in the first example would be computationally very difficult, so instead we will use Proposition 3.5 from Chapter 3. In using this formula, close attention must be paid to the limits as the value r is approached both from above and below. In a case such as this one, use is made of

$$\lim_{x\to 0} x\ \ln x = 0 \qquad (4.23)$$

to carefully deal with singularities. Recall (3.23),

$$U(r) = -\frac{2\pi}{r}\int_0^{r} s\rho(s)\ \ln\left|\frac{r+s}{r-s}\right|\ ds - \frac{2\pi}{r}\int_r^{R} s\rho(s)\ \ln\left|\frac{s+r}{s-r}\right|\ ds + 4\pi\int_0^{R}\rho(s)\ ds \qquad (4.24)$$

We will start with the density

$$\rho(r) = \begin{cases} (1-r^2)^2 \quad , \quad 0<r<1 \\ 0 \quad , \quad r\geq 1. \end{cases} \qquad (4.25)$$

and compute the resulting $U(r)$. We will then adjust the density by a constant factor so that $U(1) = \frac{1}{2}$ as in the preceding example. The motivation for this choice of density is that $\rho'(0) = 0$, $\rho'(1) = 0$ and there is a change of concavity from concave down to concave up at $r = \frac{1}{\sqrt{3}}$, which gives the density profile a similar shape to the isotropic solutions which we will construct in the last two Chapters. The extra degree of freedom in having a distribution function which depends on both E and F, as opposed to the isotropic case where there is dependence on E alone, essentially permits us freedom to "sculpt" the density profile.

$$\begin{aligned}
U(r) &= -\frac{2\pi}{r}\int_0^r s(1-s^2)^2 \ln\left|\frac{r+s}{r-s}\right| ds - \frac{2\pi}{r}\int_r^1 s(1-s^2)^2 \ln\left|\frac{s+r}{s-r}\right| ds \\
&\quad + 4\pi\int_0^1 (1-s^2)^2\, ds \\
&= -\frac{2\pi}{r}\left(\frac{1}{2}s^2 - \frac{1}{2}s^4 + \frac{1}{6}s^6\right)\ln\left|\frac{r+s}{r-s}\right| \Bigg|_0^r \\
&\quad +2\pi\int_0^r \frac{s^2 - s^4 + \frac{1}{3}s^6}{r^2 - s^2}\, ds \\
&\quad -\frac{2\pi}{r}\left(\frac{1}{2}s^2 - \frac{1}{2}s^4 + \frac{1}{6}s^6\right)\ln\left|\frac{s+r}{s-r}\right| \Bigg|_r^1 \\
&\quad -2\pi\int_r^1 \frac{s^2 - s^4 + \frac{1}{3}s^6}{s^2 - r^2}\, ds + \frac{32\pi}{15} \\
&= -\frac{\pi}{3r}\left(1-r^2\right)^3 \ln\left|\frac{1+r}{1-r}\right| - \frac{2\pi}{3}\left(r^4 - \frac{8}{3}r^2 + \frac{11}{5}\right) + \frac{32\pi}{15} \qquad (4.26)
\end{aligned}$$

We now have

$$\lim_{r\to 0} U(r) = -\frac{2\pi}{3} - \frac{22\pi}{15} + \frac{32\pi}{15} = 0 \qquad (4.27)$$

as it should, and

$$\begin{aligned}
\lim_{r\to 1} U(r) &= 0 - \frac{2\pi}{3}\left(1 - \frac{8}{3} + \frac{11}{15}\right) + \frac{32\pi}{15} \\
&= \frac{16\pi}{9} \qquad (4.28)
\end{aligned}$$

So to produce the desired potential $\lim_{r\to 1} U(1) = \frac{1}{2}$, we will multiply this density by a factor of $\frac{9}{32\pi}$.

Our second example is then

$$\phi(E,F) = \begin{cases} \frac{9}{32\pi^3}\,(1-2E+F)^{-\frac{1}{2}} & \text{if} \quad E < \frac{1}{2}(1+F), \quad F < r^2(1-r^2)^2 \leq 1 \\ 0 \ , & \text{elsewhere.} \end{cases} \tag{4.29}$$

Note the different restriction on F from the first example. This becomes

$$\Phi(r,w,F) = \begin{cases} \frac{9}{32\pi^3}\left(1 - w^2 - \frac{F}{r^2} - 2U(r) + F\right)^{-\frac{1}{2}} \\ \qquad \text{if} \quad 1 - w^2 - \frac{F}{r^2} - 2U(r) + F > 0, \quad F < r^2(1-r^2)^2 \leq 1 \\ 0 \ , \quad \text{elsewhere.} \end{cases} \tag{4.30}$$

Checking that this Φ gives back the correct density ρ is carried out in a similar manner to the calculation performed in equation (4.20).

It is interesting to examine the force produced at points in the interior of the sphere, because the magnitude of the force is not a strictly increasing function of r over $0 < r < 1$ as it was in the first example in Section 4.2.

$$\begin{aligned} & -U'(r) \\ = \ & \frac{3}{32}\left(-\frac{(1-r^2)^3}{r^2}\ \ln\left|\frac{1+r}{1-r}\right| - 6(1-r^2)^2\ \ln\left|\frac{1+r}{1-r}\right| + \frac{2(1-r^2)^2}{r} + 8r^3 - \frac{32}{3}r\right) \\ = \ & \frac{3}{32}\left(-\frac{(1+5r^2)}{r^2}(1-r^2)^2\ \ln\left|\frac{1+r}{1-r}\right| + \frac{30r^4-44r^2+6}{3r}\right) \end{aligned} \tag{4.31}$$

Now we have

$$\lim_{r\to 0} -U'(r) = 0 \tag{4.32}$$

and

$$\lim_{r\to 1} -U'(r) = -\frac{1}{4} \tag{4.33}$$

with the force reaching a maximum magnitude of approximately -0.7283428 at $r = 0.5623467$.

4.4 Rescaling of Steady-State Models

Recall from Section 1.4 that any steady-state solution can be expanded into a family of infinitely many steady-state solutions by an appropriate rescaling. In the case of the Pure Stellar Manev potential, the quantity

$$\frac{\rho\,\mathbf{x}}{\mathbf{v}^2} \tag{4.34}$$

must be conserved in the process. This will be used now in the following rescaling function suggested in [46].

$$\overline{f}(\mathbf{x},\mathbf{v}) = af(b\mathbf{x}, c\mathbf{v}), \qquad a,b,c>0 \tag{4.35}$$

In (r,w,F) coordinates this becomes

$$\overline{\Phi}(r,w,F) = a\Phi(br, cw, b^2c^2F), \qquad a,b,c>0 \tag{4.36}$$

So our first example can be rescaled

$$\overline{\Phi}(r,w,F) = \begin{cases} \frac{a}{4\pi^3}\left(1-c^2w^2-\frac{c^2F}{r^2}-2U(br)+b^2c^2F\right)^{-\frac{1}{2}} \\ \quad \text{if } \ 1-c^2w^2-\frac{c^2F}{r^2}-2U(br)+b^2c^2F>0, \quad F<\frac{r^2}{c^2}\le 1 \\ 0 \ , \quad \text{elsewhere.} \end{cases} \tag{4.37}$$

Note that the condition $F<r^2\le 1$ has been replaced by $F<\frac{r^2}{c^2}\le 1$. Now in order to check that the Vlasov equation is satisfied, it is necessary to compute a rescaled potential $\overline{U}(r)$ and then a rescaled force $-\overline{U'}(r)$. We must first compute the rescaled $\overline{\rho}(r)$.

$$\begin{aligned} \overline{\rho}(r) &= \frac{\pi}{r^2}\int_0^{\frac{r^2}{c^2}}\int_{-\frac{1}{c}\sqrt{1-\frac{c^2F}{r^2}-2U(br)+b^2c^2F}}^{\frac{1}{c}\sqrt{1-\frac{c^2F}{r^2}-2U(br)+b^2c^2F}} \\ &\quad \frac{a}{4\pi^3}\left(1-c^2w^2-\frac{c^2F}{r^2}-2U(br)+b^2c^2F\right)^{-\frac{1}{2}}\,dw\,dF \\ &= \frac{a}{4\pi c^3} \end{aligned} \tag{4.38}$$

Note that since $\overline{\rho}(r)$ is only nonzero in the interior of a sphere of radius $\frac{1}{b}$, we have

$$\overline{\rho}(r) = \frac{a}{c^3}\,\rho(br) \tag{4.39}$$

Now we use the earlier result in [9] to place a restriction on the constant a. Combining (4.34) with (4.39), we obtain the restriction $a = bc$ in order to preserve a steady state. This permits us to rewrite (4.39) as

$$\overline{\rho}(r) = \frac{b}{c^2}\,\rho(br) \tag{4.40}$$

Note that even with the restriction on a we have two degrees of freedom in any rescaling of a steady-state solution. We can change our space and our velocity scale. Now we can compute

$$\begin{aligned}
\overline{U}(r) &= -\int \frac{\rho(\mathbf{y})\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \int \frac{\rho(\mathbf{y})\,d\mathbf{y}}{|\mathbf{y}|^2} \\
&= -\frac{b}{c^2}\int \frac{\rho(b\mathbf{y})\,d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} + \frac{b}{c^2}\int \frac{\rho(b\mathbf{y})\,d\mathbf{y}}{|\mathbf{y}|^2} \\
&= \frac{1}{c^2}U(br)
\end{aligned} \tag{4.41}$$

from which comes the force term needed to check the Vlasov equation

$$-\overline{U}'(r) = -\frac{b}{c^2}U'(br) \tag{4.42}$$

Now we have

$$\begin{aligned}
& w\frac{\partial\overline{\Phi}}{\partial r} + \left(\frac{F}{r^3} - \overline{U}'(r)\right)\frac{\partial\overline{\Phi}}{\partial w} \\
= & \frac{-\frac{a}{8\pi^3}\left(\frac{2c^2wF}{r^3} - 2bwU'(br) - \frac{2c^2wF}{r^3} + 2c^2w\overline{U}'(r)\right)}{\left(1 - c^2w^2 - \frac{c^2F}{r^2} - 2U(br) + b^2c^2F\right)^{\frac{3}{2}}} \\
= & \ 0
\end{aligned} \tag{4.43}$$

Chapter 5

Existence and Nonlinear Stability of Solutions under Newtonian Potential

5.1 Overview

In this Chapter, we establish the existence and nonlinear stability of isotropic, stationary, spherically symmetric solutions with finite mass and compact support in $\mathbf{R}^6$ under the Newtonian potential. The same topic under the Pure Stellar Manev potential will be the subject of the next Chapter. Batt, Morrison and Rein [3] investigated the existence and linear stability of stationary solutions of the Vlasov-Poisson system in three dimensions in the context of both plasma physics and stellar dynamics. We will establish nonlinear stability by means of the energy-Casimir method used in Rein [46], which there was limited to the plasma physics case.

5.2 The Energy-Casimir Method

The energy-Casimir method is presented in Holm, Marsden, Ratiu and Weinstein [26]. We shall follow the brief review of the method which is set out in Rein [46]. Let the system under consideration be described by the conservative equation of motion

$$\dot{u} = A(u) \tag{5.1}$$

on some state space X, $A : D(A) \to X$ a (non-linear) operator, and let u_0 be the stationary solution whose stability we want to investigate. The following steps lead to a stability result for u_0:

1. Find the energy (Hamiltonian) $H : X \to \mathbf{R}$ of the system; $\frac{d}{dt}H(u(t)) = 0$ along solutions.
2. Relate u_0 to a further conserved quantity $C : X \to \mathbf{R}$ such that u_0 is a critical point of $H_C := H + C$, i.e. $DH_C(u_0) = 0$.
3. Show that the quadratic part in the expansion of H_C at u_0

$$H_C(u) = H_C(u_0) + DH_C(u_0)(u - u_0) + D^2H_C(u_0)(u - u_0, u - u_0) + \cdots \tag{5.2}$$

is either positive definite or negative definite. (For our example it will turn out to be negative definite.) More precisely, find a norm $||\cdot||$ on X such that

$$H_C(u) - H_C(u_0) - DH_C(u_0)(u - u_0) \leq c||u - u_0||^2, u \in X, \tag{5.3}$$

for some $c < 0$. Note that $H_C(u)$ will be maximized at $u = u_0$.
4. Find a norm $|||\cdot|||$ on X with respect to which H_C is continuous at u_0.

If steps (1)-(3) can be carried through, then for any solution

$$||u(t) - u_0||^2 \leq \left|\frac{1}{c}\left(H_C(u(0)) - H_C(u_0)\right)\right| \tag{5.4}$$

and with step (4) we conclude that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|||u(0) - u_0||| < \delta$ implies $||u(t) - u_0|| < \epsilon, \quad t \geq 0$, i.e. u_0 is (non-linearly) stable.

The conserved quantity C is usually called a Casimir functional.

In the following sections we first apply the energy-Casimir method to the Newtonian case, and then in the next Chapter to the Pure Stellar Manev case. The construction of the steady states in both cases is essentially the same. As introduced in Chapter 3, in both cases the potential $U(r)$ is only determined up to a constant. The definition used in Bobylev, Dukes, Illner and Victory [9] and in Rein [46] is that $U(r) < 0$ and $\lim_{r\to\infty} U(r) = 0$. We will use this definition, so that the graph of the potential energy $U_0(r)$ of the steady state f_0 will have the features in Figure 5.1 which appears at the top of the following page.

There is no loss of generality in setting the maximum radius for the steady state solution at $R = 1$, as the methods of Section 4.4 permit the solution to be rescaled so that its support in the $\mathbf{x}$ variable has any desired maximum radius.

We are looking for distribution functions $f_0(z)$ of the steady state solution of the type

$$f_0(z) = \varphi(E_0(z)), \quad z \in \mathbf{R}^6, \tag{5.5}$$

where

$$E_0(z) := \frac{1}{2}\mathbf{v}^2 + U_0(\mathbf{x}) \tag{5.6}$$

denotes the particle energy.

We restrict $\varphi(E_0) \geq 0$ to

$$\varphi(E_0) = 0 \ \text{ for } -\infty < E_0 \leq E_{min} \text{ and } E_{max} < E_0 \tag{5.7}$$

with $\varphi(E_0)$ strictly increasing and C^1 for $E_{min} < E_0 < E_{max}$, where E_{min} and E_{max} are as defined for the graph of $U_0(r)$. Consequently, all the particles in the steady state have energies which prevent them from escaping the sphere of radius 1 centred at the origin.

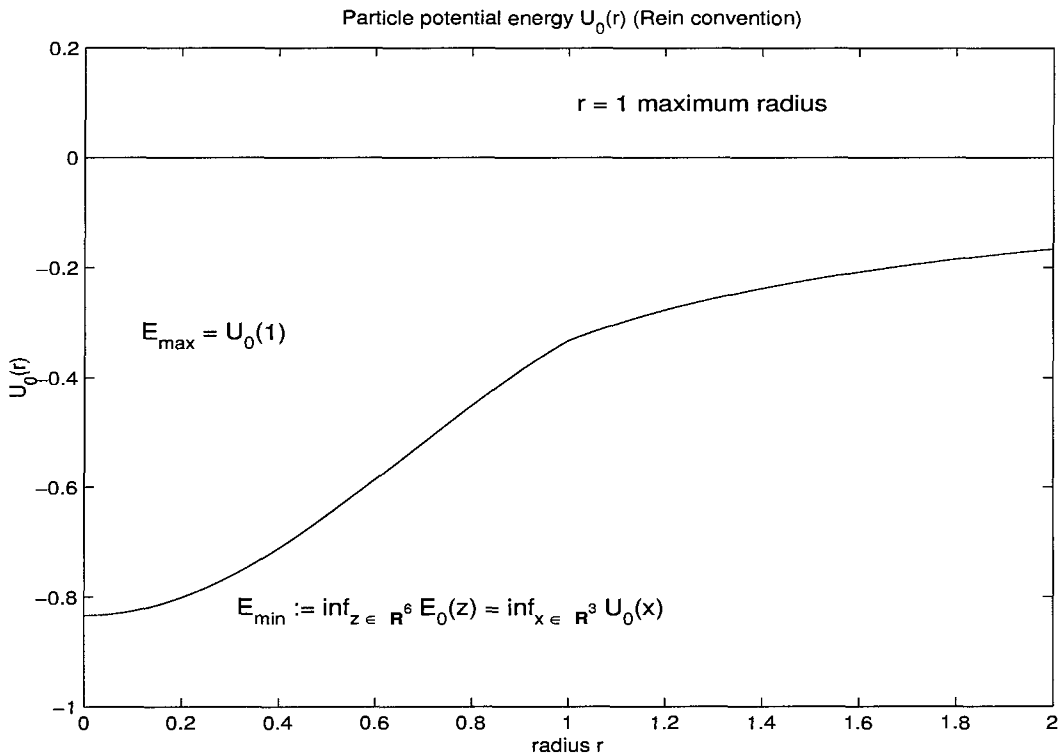

Figure 5.1: Potential Energy U_0 of the steady state f_0

This is the potential energy function which appears as (5.79) in Section 5.4. Density $\rho(r)$ is supported on $0 < r < 1$. Potential energy $U_0(r)$ is supported on $0 < r < \infty$. In this diagram, $E_{max} = U_0(1) = -\frac{1}{3}$ and $E_{min} := \inf_{z \in \mathbf{R}^6} E_0(z) = \inf_{\mathbf{x} \in \mathbf{R}^3} U_0(\mathbf{x}) = -\frac{5}{6}$.

Rein [46] mentions that Holm, Marsden, Ratiu and Weinstein [26] point out that the appearance of unboundedly large velocities could cause the energy-Casimir method to run into trouble if it is applied to the Vlasov-Poisson system. We avoid this difficulty because by construction of the steady state no particle has more velocity than that required to reach the radius $r = 1$ in the spherical solution.

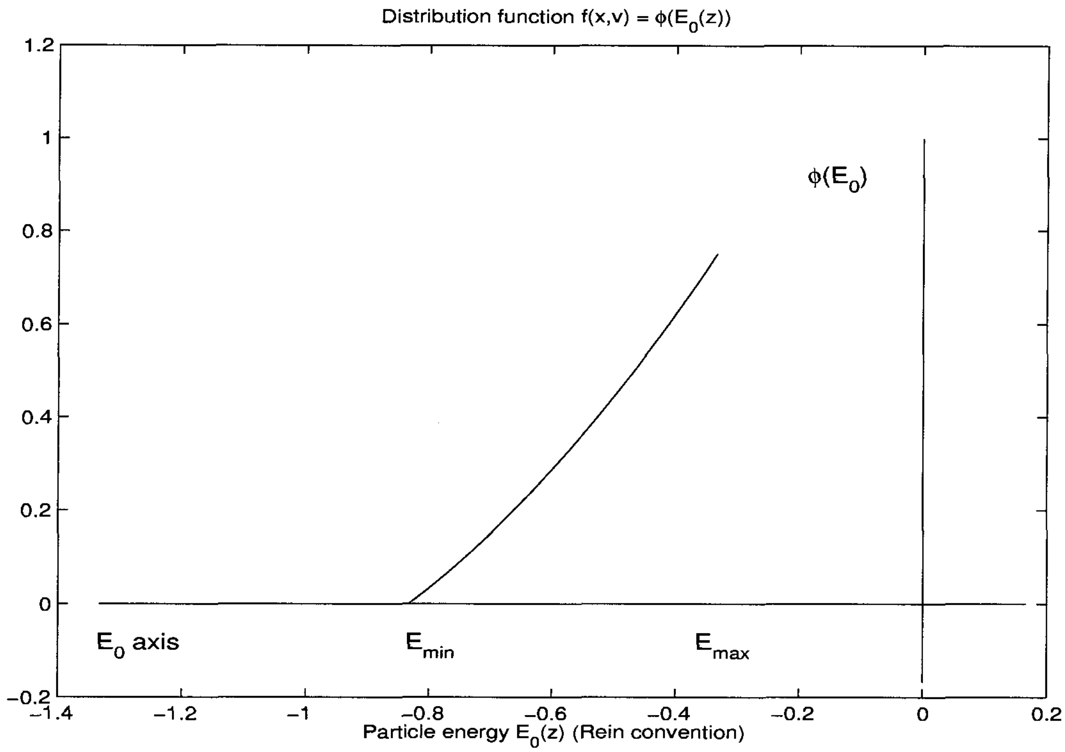

Figure 5.2: A possible distribution function $f(\mathbf{x}, \mathbf{v}) = \varphi(E_0(z))$
Distribution function $\varphi(E_0(z))$ *is supported on* $E_{min} < E_0 < E_{max}$*, where we are following the Rein convention on* E_0*.*

The stability condition which we derive in both cases is that the distribution function $f_0(z)$ is equal to $\varphi(E_0(z))$, which is a strictly increasing function of the particle energy on the support of the distribution function, with a jump discontinuity at the maximum value of the particle energy. Please see Figure 5.2 above. Rein [46] found that the plasma physics case required that the function φ was a strictly decreasing function of the particle energy, with a jump discontinuity at the minimum value of the particle energy. Batt, Morrison and Rein [3] investigated stellar dynamics distribution functions with a jump discontinuity of φ at the maximum value of the particle energy, but only their linear stability.

The construction of φ is motivated to meet the requirements of our proof of nonlinear stability. Then, our distribution function $f(\mathbf{x}, \mathbf{v})$ is implicitly defined in terms of φ, $\mathbf{x}$ and $\mathbf{v}$ by

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}) &= \varphi(E) \\ &= \varphi\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v} - \int\int \frac{f(\mathbf{y}, \mathbf{v})}{|\mathbf{x}-\mathbf{y}|^n}\ d\mathbf{v}\ d\mathbf{y}\right) \end{aligned} \tag{5.8}$$

where $n = 1$ in the Newtonian case and $n = 2$ in the Pure Stellar Manev case. We restrict our φ to those for which $\varphi'(E) > 0$ so that φ^{-1} exists. If the condition that $\varphi'(E) > 0$ is not met, then our nonlinear stability argument will not hold. So, we could also proceed on the basis that our distribution function $f(\mathbf{x}, \mathbf{v})$ is implicitly defined in terms of φ^{-1}, $\mathbf{x}$ and v by

$$\varphi^{-1}\left(f(\mathbf{x}, \mathbf{v})\right) = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} - \int\int \frac{f(\mathbf{y}, \mathbf{v})}{|\mathbf{x}-\mathbf{y}|^n}\ d\mathbf{v}\ d\mathbf{y} \tag{5.9}$$

5.3 The Newtonian Case

Recall that in the Newtonian case the time-dependent Vlasov-Poisson system is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} U(t, \mathbf{x})\cdot\nabla_{\mathbf{v}} f = 0 \quad \text{(Vlasov's equation)} \tag{5.10}$$

$$\rho(t, \mathbf{x}) := \int_{\mathbf{R}^3} f(t, \mathbf{x}, \mathbf{v})\ d\mathbf{v} \tag{5.11}$$

$$\triangle U(t, \mathbf{x}) = 4\pi\rho(t, \mathbf{x}) \quad \text{(Poisson's equation)} \tag{5.12}$$

Define the state space

$$X := \{f \in C_c^1(\mathbf{R}^6) | f \geq 0\}, \tag{5.13}$$

where the index c means that the functions have compact support. The first proof of global existence of a smooth solution with large (unrestricted size) data was produced in 1989 by Pfaffelmoser [42], although it was not published until 1992.

Schaeffer [50] proved then that for any $\overset{\circ}{f} \in X$ there exists a unique global classical solution f of the above time-dependent system with $f(0) = \overset{\circ}{f}$.

We will prove existence and non-linear stability of non-trivial steady solutions (f_0, U_0) of the system (5.10 - 5.12) having the following properties:

$$f_0 \in C_c(\mathbf{R}^6), \qquad U_0 \in C_b(\mathbf{R}^3) \cap C^2(\mathbf{R}^3) \tag{5.14}$$

and

$$f_0(z) = \varphi(E_0(z)), \quad z \in \mathbf{R}^6, \tag{5.15}$$

where

$$E_0(z) := \frac{1}{2}\mathbf{v}^2 + U_0(\mathbf{x}) \tag{5.16}$$

denotes the particle energy.

We will make the same assumptions on φ and U_0 as in Batt, Morrison and Rein [3]

$$\varphi \in L^\infty_{loc}(\mathbf{R}), \quad \varphi \geq 0, \tag{5.17}$$

$$E_{max} := \inf\{E \in \mathbf{R} | \varphi(\tilde{E}) = 0 \;\; a.e. \text{ for } \tilde{E} > E\}, \tag{5.18}$$

$$\varphi \in C^1\left(-\infty, E_{max}\right) \text{ with } \varphi' \in L^1_{loc}\left((-\infty, E_{max}]\right), \tag{5.19}$$

Here $\varphi \in L^\infty_{loc}(\mathbf{R})$ means that $\varphi|_K \in L^\infty(K)$ for every compact interval $K \subset \mathbf{R}$, and $L^1_{loc}\left((-\infty, E_{max}]\right)$ is defined analogously.

For U_0

$$U_0 \in C^2(\mathbf{R}^3), \tag{5.20}$$

$$U_0 \text{ is bounded; } U_{min} := \inf_{\mathbf{x}\in\mathbf{R}^3} U_0(\mathbf{x}) < E_{max}, \tag{5.21}$$

$$\text{the set } B := \{(\mathbf{x}, \mathbf{v}) \in \mathbf{R}^6 | \frac{1}{2}\mathbf{v}^2 + U_0(\mathbf{x}) \leq E_{max}\} \text{ is bounded and } \partial B \text{ has measure zero.} \tag{5.22}$$

There is one relaxation to these assumptions which we may make, and that is that in (5.19), we may allow φ to be continuous but have a discontinuous derivative at an E value which will be $E_{min} < E_{max}$.

Conditions (5.18) and (5.21) imply that the energy levels of the distribution function f_0 vary between the values $E_{min} = U_{min}$ and E_{max}. Together with (5.22) this means that f_0 has phase-space support in the bounded set B, with $\overset{\circ}{f}(z) = 0$ for $z \notin B$. Note that $\varphi(E_0) = 0$ for $E \leq E_{min}$. In particular, the steady state has finite radius, i.e., there exists a radius $R_0 > 0$ such that $f_0(\mathbf{x}, \mathbf{v}) = 0$ for $|\mathbf{x}| > R_0$, and by (5.17) it has finite mass:

$$\int\int_{\mathbf{R}^6} f_0(\mathbf{x}, \mathbf{v})\, d\mathbf{v}\, d\mathbf{x} \leq \mathrm{vol}(B) \sup_{E \in [U_{min}, E_{max}]} \varphi(E) < \infty \tag{5.23}$$

We write $(f_0, U_0) \in \mathbf{S}$ if (f_0, U_0) is a stationary solution of the Vlasov-Poisson system satisfying the above assumptions.

Also, as in BMR [3] we make some additional assumptions:

$$f \in L^1(\mathbf{R}^6) \tag{5.24}$$

$$f \in C\left([0,\infty), L^1(\mathbf{R}^6)\right) \tag{5.25}$$

$$\rho_f \in C\left([0,\infty), L^\infty(\mathbf{R}^3)\right) \tag{5.26}$$

$$U_{\rho_f} \in C\left([0,\infty), C^1_b(\mathbf{R}^3)\right) \tag{5.27}$$

$$\tag{5.28}$$

The existence of a large class of stationary solutions $(f_0, U_0) \in \mathbf{S}$ is established in [3], section 6. Among these are steady states satisfying $\varphi'(E) > 0$ in the stellar dynamics case. In section 5 of that paper it was established that such steady states are stable, but only that they are linearly stable.

In this Chapter and the next, we will restrict our attention to functions φ which depend on E only, $\varphi(E)$, i.e. isotropic solutions.

We now proceed to establish the existence and nonlinear stability of isotropic, stationary, spherically symmetric stellar dynamic solutions with finite mass and compact support in $\mathbf{R}^6$. We will continue using the energy-Casimir method used in [46], which was there limited to the plasma physics case.

First, to establish existence, we begin by pointing out that this can be done in more than one way. In Guo and Rein [25], existence is established by first selecting a Casimir functional which is subject to several criteria which are necessary in order that it possess a minimizer with all the desired properties. A minimizer $f_0(\mathbf{x}, \mathbf{v}) = q(E_{max} - E, L)$ is then found, where their L has precisely the same meaning as our F. It is then shown that since q depends only on E and L, that $f_0(\mathbf{x}, \mathbf{v})$ solves the steady-state Vlasov equation. Finally it is shown that

$$\triangle U_0 = \frac{1}{r^2}\left(r^2 U_0'\right) = 4\pi\rho_0 \tag{5.29}$$

establishing the existence of (f_0, ρ_0, u_0) as a solution of the Vlasov-Poisson system.

The approach to establish existence which we are following is that taken in Rein [46], in which a distribution function $f_0(\mathbf{x}, \mathbf{v}) = f_0(z) = \varphi(E_0(z)), \quad z \in \mathbf{R}^6$ is selected, subject to several criteria in order that it be an extremum of a Casimir functional with all the desired properties. Since $f_0(\mathbf{x}, \mathbf{v})$ depends only on E_0, it solves the steady-state Vlasov equation. Rein refers to the existence proof in Section 3 of Batt, Morrison and Rein [3].

The following Theorem establishes the nonlinear stability.

Theorem 5.1 *Let (f_0, U_0) be a stationary solution of the system (5.10 - 5.12) with the above properties. Then (f_0, U_0) is non-linearly stable in the following sense:*

For every $C_1 > 0$ there exists $C_2 > 0$ such that for $\overset{\circ}{f} \in X$ with $\overset{\circ}{f} \leq C_1$ (that is, $\sup_z |\overset{\circ}{f}(z)| \leq C_1$), the time-dependent solution with initial value $f(0) = \overset{\circ}{f}$ satisfies the estimate

$$||f(t) - f_0||_2^2 \leq C_2 \left(\int (1 + \mathbf{v}^2) | \overset{\circ}{f} - f_0 | (z) \, dz + || \overset{\circ}{\rho} - \rho_0 ||_{\frac{6}{5}}^2 \right), t \geq 0.$$

Proof. We follow the steps of the energy-Casimir method as outlined in Section 5.2.

Conservation of Energy: For $f \in X$ define

$$\begin{aligned} E_{kin}(f) &:= \frac{1}{2} \int \mathbf{v}^2 f(z) \, dz, & (5.30) \\ E_{pot}(f) &:= \frac{1}{2} \int U(\mathbf{x}) \rho(\mathbf{x}) \, d\mathbf{x} & (5.31) \\ &= \frac{1}{8\pi} \int U \triangle U \, d\mathbf{x} & (5.32) \\ &= -\frac{1}{8\pi} \int \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{x}} U \, d\mathbf{x} & (5.33) \\ &= -\frac{1}{8\pi} \int |\nabla_{\mathbf{x}} U|^2 \, d\mathbf{x} & (5.34) \end{aligned}$$

where we have used (5.12) and the integration by parts which we derived as (2.3) and (2.5) in Chapter 2 in the computation of an alternative version of E_{pot} and where

$$U(\mathbf{x}) = -\int \frac{\rho(\mathbf{y}) \, d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|} \tag{5.35}$$

denotes the Newtonian gravitational potential generated by the density ρ and vanishing at infinity. The total energy

$$H(f) := E_{kin}(f) + E_{pot}(f), \qquad f \in X \tag{5.36}$$

is conserved along classical solutions of the Vlasov-Poisson system (5.10- 5.12).

Construction of a Casimir functional: First note that

$$\begin{aligned} E_{kin}(f) &= E_{kin}(f_0) + \frac{1}{2}\int \mathbf{v}^2(f-f_0)(z)\,dz, &(5.37)\\ E_{pot}(f) &= E_{pot}(f_0) + \frac{1}{2}\int U(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int U_0(\mathbf{x})\rho_0(\mathbf{x})d\mathbf{x} &(5.38)\\ &= E_{pot}(f_0) - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} &(5.39)\end{aligned}$$

so that

$$\begin{aligned} &H(f)\\ =\;& H(f_0) + \frac{1}{2}\int \mathbf{v}^2(f-f_0)(z)\,dz - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz - \int U_0(\mathbf{x})(f-f_0)(z)\,dz\\ &-\frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz + \iint \frac{\rho_0(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\rho(\mathbf{x})\,d\mathbf{x} - \iint \frac{\rho_0(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\rho_0(\mathbf{x})\,d\mathbf{x}\\ &-\frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ &-\frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} \end{aligned} \tag{5.40}$$

Note: we now replace the third integral on the last line of the preceding calculation, by an integral of equal value obtained by switching the variables $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned} =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} + \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ &-\frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x} - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz - \frac{1}{2}\iint \frac{(\rho(\mathbf{y})-\rho_0(\mathbf{y}))\,(\rho(\mathbf{x})-\rho_0(\mathbf{x}))}{|\mathbf{x}-\mathbf{y}|}d\mathbf{y}\,d\mathbf{x}\\ =\;& H(f_0) + \int E_0(z)(f-f_0)(z)\,dz - \frac{1}{8\pi}\int |\nabla_{\mathbf{x}} U_{(f-f_0)}(\mathbf{x})|^2\,d\mathbf{x} \end{aligned} \tag{5.41}$$

for all $f \in X$. $E_0(z)$ is as defined in (5.6).

Take any C^2 function $\Phi : [0, \infty) \to \mathbf{R}$ with $\Phi(0) = 0$ and define

$$C(f) := \int \Phi\left(f(z)\right)\, dz, \qquad f \in X. \tag{5.42}$$

Since the flow Z is measure preserving, C is a conserved quantity, and, at least formally,

$$C(f) = C(f_0) + \int \Phi'\left(f_0(z)\right)(f - f_0)(z)\, dz + \cdots \tag{5.43}$$

Thus, if we define

$$H_C := H + C \tag{5.44}$$

then $DH_C(f_0)$ vanishes if we choose Φ such that $\Phi'\left(f_0(z)\right) = -E_0(z), \ \ z \in \mathbf{R}^6$. To make this construction precise, define

$$E_{min} := \inf_{z \in \mathbf{R}^6} E_0(z) = \inf_{\mathbf{x} \in \mathbf{R}^3} U_0(\mathbf{x}) \tag{5.45}$$

$E_{min} < E_{max}$ since otherwise the stationary solution is trivial in the sense that the total mass would be zero. Now

$$\varphi : [E_{min}, E_{max}] \to [0, \varphi_{max}] \tag{5.46}$$

is one-to-one and onto, where $\varphi_{max} := \varphi(E_{max}) = ||f_0||_\infty$, and for the inverse function

$$\varphi^{-1} : [0, \varphi_{max}] \to [E_{min}, E_{max}], \tag{5.47}$$

we have that

$$\varphi^{-1} \in C\left([0, \varphi_{max}]\right) \cap C^1\left((0, \varphi_{max}]\right). \tag{5.48}$$

Define

$$\Phi(\zeta) := -\int_0^\zeta \varphi^{-1}(\xi)\, d\xi, \qquad \zeta \in [0, \varphi_{max}]. \tag{5.49}$$

Then $\Phi \in C^1([0,\varphi_{max}])$, $\Phi(0)=0$, $\Phi'(\zeta) = -\varphi^{-1}(\zeta)$, $\zeta \in [0,\varphi_{max}]$, and $\Phi \in C^2((0,\varphi_{max}])$ with

$$\begin{aligned}
\Phi''(\zeta) &= -\frac{1}{\varphi'\left(\varphi^{-1}(\zeta)\right)} \\
&\leq -\left(\sup\left\{\varphi'(E) | E_{min} \leq E \leq E_{max}\right\}\right)^{-1} \\
&=: c_\varphi \in (-\infty, 0), \qquad \zeta \in (0,\varphi_{max}].
\end{aligned} \tag{5.50}$$

We extend Φ to a function $\Phi \in C^2((0,\infty))$ in such a way that $\Phi''(\zeta) = \Phi''(\varphi_{max})$, $\zeta \geq \varphi_{max}$ so that the estimate $\Phi''(\zeta) \leq c_\varphi < 0$ holds for all $\zeta > 0$.

Negative-definiteness of the quadratic part of H_C:

$$\begin{aligned}
& H_C(f) - H_C(f_0) \\
= & \int \left[\Phi(f(z)) - \Phi(f_0(z)) + E_0(z)(f-f_0)(z)\right] dz - \frac{1}{8\pi}\int \left|\nabla_{\mathbf{x}} U_{(f-f_0)}(\mathbf{x})\right|^2 d\mathbf{x} \\
\leq & \int \left[\Phi(f(z)) - \Phi(f_0(z)) + E_0(z)(f-f_0)(z)\right] dz.
\end{aligned} \tag{5.51}$$

Let $z \in \mathbf{R}^6$ be such that $f_0(z) > 0$. Then $E_0(z) \in (E_{min}, E_{max}]$, i.e. $E_0(z)$ is in the interval where φ is invertible, and by construction of Φ,

$$E_0(z) = \varphi^{-1}\left(\varphi(E_0(z))\right) = -\Phi'\left(f_0(z)\right). \tag{5.52}$$

Therefore,

$$\begin{aligned}
[\cdots] &= \Phi(f(z)) - \Phi(f_0(z)) - \Phi'(f_0(z))(f-f_0)(z) \\
&= \lim_{\epsilon \searrow 0}\left(\Phi(f(z)+\epsilon) - \Phi(f_0(z)) - \Phi'(f_0(z))(f+\epsilon-f_0)(z)\right).
\end{aligned} \tag{5.53}$$

By Taylor's theorem

$$(\cdots) = \frac{1}{2}\Phi''(\zeta_\epsilon)(f+\epsilon-f_0)^2(z) \leq \frac{1}{2}c_\varphi|f+\epsilon-f_0|^2(z) \tag{5.54}$$

for some ζ_ϵ between $f_0(z)$ and $f(z)+\epsilon$, and thus

$$[\cdots] \leq \frac{1}{2} c_\varphi |f - f_0|^2(z) \tag{5.55}$$

for all $z \in \mathbf{R}^6$ with $f_0(z) > 0$.

Now assume that $f_0(z) = 0$. Then

$$E_0(z) \leq E_{min} = \varphi^{-1}(0) = -\Phi'(0) \tag{5.56}$$

and we can estimate the above integrand as follows:

$$\begin{aligned} [\cdots] &= \Phi(f(z)) - \Phi(0) + E_0(z) f(z) \\ &\leq \Phi(f(z)) - \Phi(0) - \Phi'(0) f(z) \\ &= \lim_{\epsilon \searrow 0} \left(\Phi(f(z)+\epsilon) - \Phi(\epsilon) - \Phi'(\epsilon) f(z) \right). \end{aligned} \tag{5.57}$$

Again, Taylor's theorem yields

$$(\cdots) = \frac{1}{2} \Phi''(\zeta_\epsilon)(f(z) + \epsilon - \epsilon)^2 \leq \frac{1}{2} c_\varphi |f(z)|^2 \tag{5.58}$$

and thus

$$[\cdots] \leq \frac{1}{2} c_\varphi |f - f_0|^2(z) \tag{5.59}$$

also if $f_0(z) = 0$. Note that the above limiting arguments were necessary since Φ is C^2 only on the open interval $(0, \infty)$. We have shown that

$$H_C(f) - H_C(f_0) \leq \frac{1}{2} c_\varphi ||f - f_0||_2^2, \qquad f \in X, \tag{5.60}$$

and the third step from Section 5.2 is complete.

Continuity of H_C at f_0: There are various norms on X with respect to which H_C is continuous. We present one possible choice. From (5.40) and (5.42) we have

$$|H_C(f) - H_C(f_0)| \;=\; \left| \int E_0(z)(f - f_0)(z)\, dz - \frac{1}{8\pi} \int |\nabla_{\mathbf{x}} U_{(f-f_0)}(\mathbf{x})|^2 \, d\mathbf{x} \right.$$

$$
\begin{aligned}
&+\int \Phi\left(f(z)\right)-\Phi\left(f_0(z)\right)\,dz\Bigg| && (5.61)\\
\leq\ & \left|\int E_0(z)(f-f_0)(z)\,dz\right|+\left|\frac{1}{8\pi}\int |\nabla_{\mathbf{x}}U_{(f-f_0)}(\mathbf{x})|^2\,d\mathbf{x}\right| \\
&+\left|\int \Phi\left(f(z)\right)-\Phi\left(f_0(z)\right)\,dz\right| && (5.62)\\
\leq\ & \int |E_0(z)|\,|f-f_0|\,(z)\,dz+\frac{1}{8\pi}\int |\nabla_{\mathbf{x}}U_{(f-f_0)}(\mathbf{x})|^2\,d\mathbf{x} \\
&+\int \left|\Phi\left(f(z)\right)-\Phi\left(f_0(z)\right)\right|\,dz && (5.63)
\end{aligned}
$$

Let $C_1 > 0$ and $f \in X$ with $f \leq C_1$. Define

$$L := \max\left\{\left|\Phi'(\zeta)\right| : \zeta \in [0, \max\{C_1, ||f_0||_\infty\}]\right\}. \qquad (5.64)$$

Then by Sobolev's inequality (see equation (4.17) on p. 121 of Glassey [20])

$$\int |\nabla_{\mathbf{x}}U_{(f-f_0)}(\mathbf{x})|^2\,d\mathbf{x} \leq c||\rho-\rho_0||^2_{\frac{6}{5}} \qquad (5.65)$$

where c is some constant and therefore

$$|H_C(f)-H_C(f_0)| \leq C\left(\int \left(|E_{min}|+L\right)|f-f_0|(z)\,dz+||\rho-\rho_0||^2_{\frac{6}{5}}\right), \qquad (5.66)$$

where the constant C depends on the stationary solution and — via L — on the constant C_1.

This completes the fourth step from Section 5.2. Putting the estimates of steps three and four together as indicated in Section 5.2 proves the theorem. □

5.4 Steady-State Solution under Newtonian Potential

We now produce a steady-state solution under the Newtonian potential whose distribution function $f(\mathbf{x},\mathbf{v}) = \varphi(E)$. As before, this means that the distribution function $f(\mathbf{x},\mathbf{v})$ is implicitly defined in terms of $\varphi(E), \mathbf{x}$ and $\mathbf{v}$. The lack of dependence of

φ on F make this an isotropic steady-state solution. That is, at any point a given speed will be equiprobable in all directions.

Isotropic solutions here appear after anisotropic solutions as they are more difficult to construct. Allowing φ to be dependent on both E and F gives an additional degree of freedom, which literally permits one to "sculpt" the density profile $\rho(r)$. In both of the anisotropic steady state examples in Chapter 4 we were able to start from a desired density profile $\rho(r)$ and use the dependence of φ on F to produce the desired distribution function f.

When constructing isotropic solutions, two of the equations in the Vlasov-Poisson or Vlasov-Manev system (the excluded one being the Vlasov equation) give a relation between the two functions $\rho(r)$ and $U(r)$ which must then be solved simultaneously to give a self-consistent solution. We cannot select $\rho(r)$ arbitrarily because it may not produce the appropriate $U(r)$.

We encounter similar difficulties to those in Batt, Faltenbacher and Horst [2] where it is stated, "Explicit examples of stationary stellar dynamic models (their term for solutions) are not too numerous because, in most cases, the integration for h_φ is too involved to be carried out explicitly." The following example, and the one constructed under the Pure Stellar Manev potential at the end of Chapter 6, are isotropic steady states, but they do not meet all of the requirements from Chapter 5 for nonlinear stability. Specifically, we do not have that φ is a bounded function of E, nor is $\varphi(0) = 0$ (using the Batt, Faltenbacher and Horst [2] convention for E). We can make φ a bounded function of E by restricting the domain of the function from $E \in [0, \frac{1}{2}]$ to $E \in [0, \frac{1}{2} - \epsilon]$ for some arbitrarily small $\epsilon > 0$. We can make $\varphi(0) = 0$ by, for example, changing to $\varphi(E) = k\left((1-2E)^{-\frac{1}{2}} - (1-E)^{-\frac{1}{2}}\right)$. However, these changes could be made only at the cost of raising the difficulty level of obtaining an explicit solution far beyond the level of any solution which is presented in BFH [2].

We seek an isotropic steady-state solution of the form $\varphi(E) = k\,(1-2E)^{-\frac{1}{2}}$

when $0 \leq E < \frac{1}{2}$, where we must solve for the value of k. The inspiration for this is that it is an isotropic version of the anisotropic steady state solution presented in Section 4.3 of BFH [2]. This means that it is also an isotropic version of the first of our anisotropic steady-state solutions, presented in Section 4.2 of Chapter 4. Here we are using the BFH [2] convention for the particle energy $E = \frac{1}{2}\mathbf{v}^2 + U(\mathbf{x})$, which is that a particle which is at rest and which is located at the origin has kinetic energy of zero, potential energy of zero and hence total energy $E = 0$. As in our examples in Chapter 4, we seek a density $\rho(r)$ which is a decreasing function of r, decreasing from a maximum value at $r = 0$ to a minimum value of 0 at $r = 1$.

We begin with the steady-state version of the second equation in the Vlasov-Poisson system

$$\begin{aligned}
\rho(\mathbf{x}) &= \int_{\mathbf{R}^3} f(\mathbf{x}, \mathbf{v}) \; d\mathbf{v} \\
&= \int_{\mathbf{R}^3} \varphi(E) \; d\mathbf{v} \\
&= 4\pi \int_0^\infty \varphi(E) \; v^2 \; dv \\
&= 4\pi \int_0^\infty \sqrt{2}\, \varphi(E) \; \sqrt{E-U} \; v \, dv \\
&= 4\pi\sqrt{2} \int_u^\infty \varphi(E) \; \sqrt{E-U} \; dE \qquad (5.67)
\end{aligned}$$

This equation appears as Lemma 6.2 in Batt, Morrison and Rein [3], where it appears as

$$h_\varphi(u) = 4\pi\sqrt{2} \int_u^\infty \varphi(E) \; \sqrt{E-U} \; dE \qquad (5.68)$$

Here the fact that the function $U(r)$ is strictly increasing and hence invertible has been used to produce a density function of particle potential energy U instead of radius r.

Now we substitute $\varphi(E) = k\,(1-2E)^{-\frac{1}{2}}$ when $0 \leq E < \frac{1}{2}$ into this expression to obtain

$$\begin{aligned}
\rho(r) &= h_{\varphi(U(r))} \\
&= 4\pi\sqrt{2}\int_{U(r)}^{\frac{1}{2}} \varphi(E)\ \sqrt{E-U}\ dE \\
&= 4\pi k\sqrt{2}\int_{U(r)}^{\frac{1}{2}} \sqrt{\frac{E-U}{1-2E}}\ dE \\
&= 2\pi k\left(-\sqrt{1-2E}\sqrt{2E-2U}-(1-2U)\arctan\left(\frac{\sqrt{1-2E}}{\sqrt{2E-2U}}\right)\right)\Bigg|_{U(r)}^{\frac{1}{2}} \\
&= k\pi^2(1-2U(r))
\end{aligned} \tag{5.69}$$

establishing a relation between the decreasing function $\rho(r)$ and the increasing function $U(r)$.

So now we must solve the following system for $0 < r < 1$:

$$\begin{aligned}
\rho(r) &= k\pi^2(1-2U(r)) \\
\frac{1}{r^2}\left(r^2U'(r)\right)' &= 4\pi\rho(r)
\end{aligned} \tag{5.70}$$

subject to the initial conditions (using the Batt, Faltenbacher and Horst [2] convention on $U(r)$) that $U(0) = 0$ and $U'(0) = 0$. These conditions will be met by any positive value of k. We will fix a value for k by requiring that $U(1) = \frac{1}{2}$ in order that $\rho(1) = 0$, i.e. that our density decreases until it reaches the value 0 at $r = 1$. It is important to remember that this system is only valid for $0 < r < 1$. For $r > 1$, $\rho(r) = 0$ while $U(r)$ is positive and an increasing function of r.

We rewrite the second equation in the system as

$$U''(r) + \frac{2}{r}U'(r) = 4\pi\rho(r) \tag{5.71}$$

and substitute in the first equation to obtain

$$U''(r) + \frac{2}{r}U'(r) = 4k\pi^3\left(1-2U(r)\right) \tag{5.72}$$

which we rewrite as

$$rU''(r) + 2U'(r) + 8k\pi^3 rU(r) = 4k\pi^3 r \tag{5.73}$$

This is a linear, nonhomogeneous second-order ordinary differential equation. It is easily checked that $r = 0$ is a regular singular point. The initial conditions $U(0) = 0$ and $U'(0) = 0$ suggest the use of Laplace transforms as a solution method, and so we proceed

$$\begin{aligned}
-(-r)U''(r) + 2U'(r) - 8k\pi^3(-r)U(r) &= 4k\pi^3 r \\
-\frac{d}{ds}\left(s^2U(s) - sU(0) - U'(0)\right) + 2\left(sU(s) - U(0)\right) - 8k\pi^3\frac{d}{ds}\left(U(s)\right) &= \frac{4k\pi^3}{s^2} \\
-2sU(s) - s^2U'(s) + U(0) + 2sU(s) - 2U(0) - 8k\pi^3U'(s) &= \frac{4k\pi^3}{s^2} \\
-\left(s^2 + 8k\pi^3\right)U'(s) = \frac{4k\pi^3}{s^2}
\end{aligned}$$

Using partial fractions, this can be rewritten as

$$U'(s) = -\frac{1}{2s^2} + \frac{\frac{1}{2}}{s^2 + 8k\pi^3} \tag{5.74}$$

which is readily integrated to give

$$U(s) = \frac{1}{2s} - \frac{1}{2}\left(\frac{1}{\sqrt{8k\pi^3}}\arctan\frac{\sqrt{8k\pi^3}}{s}\right) \tag{5.75}$$

Now from any standard table of Laplace transforms, the inverse Laplace transform of $\arctan\frac{a}{s}$ is $\frac{\sin at}{t}$. This gives the following solution to our second-order ordinary differential equation with its two initial conditions

$$U(r) = \frac{1}{2}\left(1 - \frac{\sin\sqrt{8k\pi^3}\,r}{\sqrt{8k\pi^3}\,r}\right) \tag{5.76}$$

There are infinitely many values of k which satisfy $U(1) = \frac{1}{2}$, however only the smallest such positive value, $k = \frac{1}{8\pi}$ is acceptable, because all the others result in a potential function $U(r)$ which is oscillatory and not strictly increasing. So our solution is

$$U(r) = \frac{1}{2}\left(1 - \frac{\sin \pi r}{\pi r}\right) \tag{5.77}$$

Substituting this into the first equation in our system (6.40) gives

$$\rho(r) = \frac{\pi}{8}\frac{\sin \pi r}{\pi r} = \frac{\sin \pi r}{8r} \tag{5.78}$$

which has a maximum value of $\lim_{r\to 0} \rho(r) = \frac{\pi}{8}$ at the origin and decreases monotonically to $\rho(1) = 0$.

If we follow Rein's [46] convention for $U(r)$ we obtain

$$U(r) = \begin{cases} -\frac{1}{3} - \frac{\sin \pi r}{2\pi r} & \text{if} \quad 0 < r \leq 1 \\ -\frac{1}{3r} & \text{if} \quad 1 < r. \end{cases} \tag{5.79}$$

This potential $U(r)$ is plotted and appears as Figure 5.1 in section 5.2.

Now that we have the value of k, our isotropic steady-state solution in the Newtonian case is given by

$$\varphi(E) = \begin{cases} \frac{1}{8\pi}(1-2E)^{-\frac{1}{2}} & \text{if} \quad E < \frac{1}{2} \\ 0 & \text{elsewhere.} \end{cases} \tag{5.80}$$

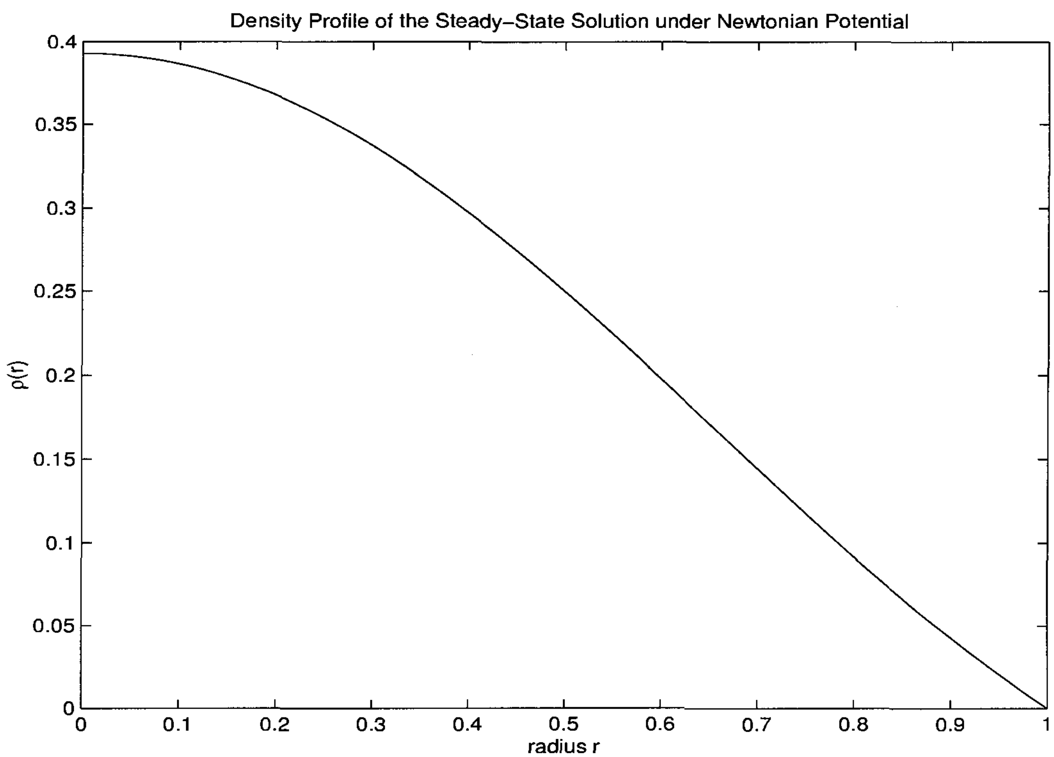

Figure 5.3: Density Profile of the Steady-State Solution under Newtonian Potential
This is the plot of the density function $\rho(r)$ *which appears as (5.78) in this Section. Density* $\rho(r)$ *is supported on* $0 < r < 1$.

Chapter 6

Existence and Nonlinear Stability of Solutions under Pure Stellar Manev Potential

6.1 The Pure Stellar Manev Case

Recall from Section 5.3 that we placed restrictions on f, f_0, $\overset{\circ}{f}$, φ, ρ and U.

In addition to all the restrictions which we placed on f, f_0, $\overset{\circ}{f}$, φ, ρ and U we must place some further restrictions in the Pure Stellar Manev case.

Recall from Chapter 2 that in the Pure Stellar Manev case, we require that our density ρ be Hölder continuous with exponent $0 < \alpha < 1$ and $\rho \in L^1$, whereas in the Newtonian case we only require that it be both bounded and integrable.

Recall also from Chapter 2 that $E_{psm} = 0$ is a necessary, but not necessarily sufficient, condition for a steady state in the Pure Stellar Manev case. We conclude that in the Pure Stellar Manev case, the initial value $f(0) = \overset{\circ}{f}$ of the time-dependent perturbed solution must be restricted to be one which results in a total energy of zero, otherwise a singularity could result, which would of course not be nonlinearly stable. Stability analysis for the Pure Stellar Manev case must be restricted to the manifold of solutions for which $E \equiv 0$. Because the system is isolated and conserves energy, this is a reasonable restriction. Our definition of the state space X must be modified from (5.13) to

$$X := \{f \in C_c^1(\mathbf{R}^6) | f \geq 0 \quad \text{and} \quad E_{psm}(f) = 0\} \tag{6.1}$$

With these additional restrictions on our density, ρ, and on our initial value $f(0) = \overset{\circ}{f}$, we are now ready to establish the existence and nonlinear stability of isotropic, stationary, spherically symmetric stellar dynamic solutions with finite mass and compact support in $\mathbf{R}^6$ in the Pure Stellar Manev case.

The approach to establish existence which we are following is that taken in Rein [46], in which a distribution function $f_0(\mathbf{x}, \mathbf{v}) = f_0(z) = \varphi(E_0(z)), \quad z \in \mathbf{R}^6$ is selected, subject to several criteria in order that it be an extremum of a Casimir functional with all the desired properties. Since $f_0(\mathbf{x}, \mathbf{v})$ depends only on E_0, it will solve the steady-state Vlasov equation. Rein refers to the existence proof in Section 3 of Batt, Morrison and Rein [3].

The following Theorem establishes the nonlinear stability.

Theorem 6.1 *Let (f_0, U_0) be a stationary solution of the system (1.45 - 1.47) with all of the properties required in Section 5.3, together with the further properties required in the present section. Then (f_0, U_0) is non-linearly stable in the following sense: For every $C_1 > 0$ there exists $C_2 > 0$ such that for $\overset{\circ}{f} \in X$ with $\overset{\circ}{f} \leq C_1$ (that is, $\sup_z | \overset{\circ}{f}(z)| \leq C_1$), the time-dependent solution with initial value $f(0) = \overset{\circ}{f}$ satisfies the estimate*

$$||f(t) - f_0||_2^2 \leq C_2 \left(\int (1 + \mathbf{v}^2) | \overset{\circ}{f} - f_0|(z)\, dz + || \overset{\circ}{\rho} - \rho_0||_{\frac{3}{2}}^2 \right), t \geq 0.$$

Proof. We follow the steps of the energy-Casimir method as outlined in Section 5.2, similar to the proof of Theorem 5.1 but with important differences.

Conservation of Energy: For $f \in X$ define

$$E_{kin}(f) := \frac{1}{2}\int \mathbf{v}^2 f(z)\, dz, \tag{6.2}$$
$$E_{pot}(f) := \frac{1}{2}\int U(\mathbf{x})\rho(\mathbf{x})\, d\mathbf{x} \tag{6.3}$$

Note that we no longer have equations (5.32- 5.34) as in the Newtonian case because we no longer have the Poisson equation (5.12).

For the conservation of total energy

$$H(f) := E_{kin}(f) + E_{pot}(f), \qquad f \in X \tag{6.4}$$

we have the result of Bobylev, Dukes, Illner and Victory [9] which was produced in Chapter 2.

Construction of a Casimir functional: First note that

$$\begin{aligned}
E_{kin}(f) &= E_{kin}(f_0) + \frac{1}{2}\int \mathbf{v}^2(f - f_0)(z)\, dz, & (6.5)\\
E_{pot}(f) &= E_{pot}(f_0) + \frac{1}{2}\int U(\mathbf{x})\rho(\mathbf{x})d\mathbf{x} - \frac{1}{2}\int U_0(\mathbf{x})\rho_0(\mathbf{x})d\mathbf{x} & (6.6)\\
&= E_{pot}(f_0) - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x} & (6.7)
\end{aligned}$$

so that

$$\begin{aligned}
&H(f)\\
=\;& H(f_0) + \frac{1}{2}\int \mathbf{v}^2(f - f_0)(z)\, dz - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x}\\
=\;& H(f_0) + \int E_0(z)(f - f_0)(z)\, dz - \int U_0(\mathbf{x})(f - f_0)(z)\, dz\\
&-\frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x}\\
=\;& H(f_0) + \int E_0(z)(f - f_0)(z)\, dz + \iint \frac{\rho_0(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\rho(\mathbf{x})\, d\mathbf{x} - \iint \frac{\rho_0(\mathbf{y})d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\rho_0(\mathbf{x})\, d\mathbf{x}\\
&-\frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x} + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2}d\mathbf{y}\, d\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= H(f_0) + \int E_0(z)(f - f_0)(z)\, dz + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \\
&+\frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} - \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x}
\end{aligned} \tag{6.8}
$$

Note: we now replace the third integral on the last line of the preceding calculation, by an integral of equal value obtained by switching the variables $\mathbf{x}$ and $\mathbf{y}$.

$$
\begin{aligned}
&= H(f_0) + \int E_0(z)(f - f_0)(z)\, dz + \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \\
&+\frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} - \frac{1}{2}\iint \frac{\rho_0(\mathbf{y})\rho_0(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} - \frac{1}{2}\iint \frac{\rho(\mathbf{y})\rho(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \\
&= H(f_0) + \int E_0(z)(f - f_0)(z)\, dz \\
&-\frac{1}{2}\iint \frac{(\rho(\mathbf{y}) - \rho_0(\mathbf{y}))\,(\rho(\mathbf{x}) - \rho_0(\mathbf{x}))}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x}
\end{aligned} \tag{6.9}
$$

for all $f \in X$.

Take any function $\Phi : [0, \infty) \to \mathbf{R}$ with $\Phi(0) = 0$ which is C^2 and define

$$C(f) := \int \Phi\left(f(z)\right)\, dz, \qquad f \in X. \tag{6.10}$$

Since the flow Z is measure preserving, C is a conserved quantity, and at least formally,

$$C(f) = C(f_0) + \int \Phi'\left(f_0(z)\right)(f - f_0)(z)\, dz + \cdots \tag{6.11}$$

Thus, if we define

$$H_C := H + C \tag{6.12}$$

then $DH_C(f_0)$ vanishes if we choose Φ such that $\Phi'\left(f_0(z)\right) = -E_0(z),\ \ z \in \mathbf{R}^6$. To make this construction precise, define

$$E_{min} := \inf_{z\in\mathbf{R}^6} E_0(z) = \inf_{\mathbf{x}\in\mathbf{R}^3} U_0(\mathbf{x}) \tag{6.13}$$

$E_{min} < E_{max}$ since otherwise the stationary solution is trivial. Now

$$\varphi : [E_{min}, E_{max}] \to [0, \varphi_{max}] \tag{6.14}$$

is one-to-one and onto, where $\varphi_{max} := \varphi(E_{max}) = ||f_0||_\infty$, and for the inverse function

$$\varphi^{-1} : [0, \varphi_{max}] \to [E_{min}, E_{max}], \tag{6.15}$$

we have that

$$\varphi^{-1} \in C\left([0, \varphi_{max}]\right) \cap C^1\left((0, \varphi_{max}]\right). \tag{6.16}$$

Define

$$\Phi(\zeta) := -\int_0^\zeta \varphi^{-1}(\xi)\, d\xi, \qquad \zeta \in [0, \varphi_{max}]. \tag{6.17}$$

Then $\Phi \in C^1\left([0, \varphi_{max}]\right)$, $\Phi(0) = 0$, $\Phi'(\zeta) = -\varphi^{-1}(\zeta)$, $\zeta \in [0, \varphi_{max}]$, and $\Phi \in C^2\left((0, \varphi_{max}]\right)$ with

$$\begin{aligned}\Phi''(\zeta) &= -\frac{1}{\varphi'\left(\varphi^{-1}(\zeta)\right)} \\ &\leq -\left(\sup\left\{\varphi'(E) | E_{min} \leq E \leq E_{max}\right\}\right)^{-1} \\ &=: c_\varphi \in (-\infty, 0), \qquad \zeta \in (0, \varphi_{max}].\end{aligned} \tag{6.18}$$

We extend Φ to a function $\Phi \in C^2\left((0, \infty)\right)$ in such a way that $\Phi''(\zeta) = \Phi''(\varphi_{max})$, $\zeta \geq \varphi_{max}$ so that the estimate $\Phi''(\zeta) \leq c_\varphi < 0$ holds for all $\zeta > 0$.

Negative-definiteness of the quadratic part of H_C:

$$\begin{aligned}& H_C(f) - H_C(f_0) \\ = & \int \left[\Phi\left(f(z)\right) - \Phi\left(f_0(z)\right) + E_0(z)(f - f_0)(z)\right]\, dz \qquad (6.19) \\ & -\frac{1}{2}\iint \frac{\left(\rho(\mathbf{y}) - \rho_0(\mathbf{y})\right)\left(\rho(\mathbf{x}) - \rho_0(\mathbf{x})\right)}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \\ \leq & \int \left[\Phi\left(f(z)\right) - \Phi\left(f_0(z)\right) + E_0(z)(f - f_0)(z)\right]\, dz. \qquad (6.20)\end{aligned}$$

Let $z \in \mathbf{R}^6$ be such that $f_0(z) > 0$. Then $E_0(z) \in (E_{min}, E_{max}]$, i.e. $E_0(z)$ is in the interval where φ is invertible, and by construction of Φ,

$$E_0(z) = \varphi^{-1}\left(\varphi(E_0(z))\right) = -\Phi'\left(f_0(z)\right). \tag{6.21}$$

Therefore,

$$\begin{aligned}[\cdots] &= \Phi\left(f(z)\right) - \Phi\left(f_0(z)\right) - \Phi'\left(f_0(z)\right)(f - f_0)(z) \\ &= \lim_{\epsilon \searrow 0} \left(\Phi(f(z)+\epsilon) - \Phi(f_0(z)) - \Phi'(f_0(z))(f+\epsilon-f_0)(z)\right). \end{aligned} \tag{6.22}$$

By Taylor's theorem

$$(\cdots) = \frac{1}{2}\Phi''(\zeta_\epsilon)(f+\epsilon-f_0)^2(z) \le \frac{1}{2}c_\varphi|f+\epsilon-f_0|^2(z) \tag{6.23}$$

for some ζ_ϵ between $f_0(z)$ and $f(z)+\epsilon$, and thus

$$[\cdots] \le \frac{1}{2}c_\varphi|f-f_0|^2(z) \tag{6.24}$$

for all $z \in \mathbf{R}^6$ with $f_0(z) > 0$. Now assume that $f_0(z) = 0$. Then

$$E_0(z) \le E_{min} = \varphi^{-1}(0) = -\Phi'(0) \tag{6.25}$$

and we can estimate the above integrand as follows:

$$\begin{aligned}[\cdots] &= \Phi\left(f(z)\right) - \Phi(0) + E_0(z)f(z) \\ &\le \Phi\left(f(z)\right) - \Phi(0) - \Phi'(0)f(z) \\ &= \lim_{\epsilon \searrow 0} \left(\Phi(f(z)+\epsilon) - \Phi(\epsilon) - \Phi'(\epsilon)f(z)\right). \end{aligned} \tag{6.26}$$

Again, Taylor's theorem yields

$$(\cdots) = \frac{1}{2}\Phi''(\zeta_\epsilon)(f(z)+\epsilon-\epsilon)^2 \le \frac{1}{2}c_\varphi|f(z)|^2 \tag{6.27}$$

and thus

$$[\cdots] \le \frac{1}{2}c_\varphi|f-f_0|^2(z) \tag{6.28}$$

also if $f_0(z) = 0$. Note that the above limiting arguments were necessary since Φ is C^2 only on the open interval $(0, \infty)$. We have shown that

$$H_C(f) - H_C(f_0) \leq \frac{1}{2} c_\varphi ||f - f_0||_2^2, \qquad f \in X, \tag{6.29}$$

and the third step from Section 5.2 is complete.

Continuity of H_C at f_0: There are various norms on X with respect to which H_C is continuous. We present one possible choice. From (6.8) and (6.10) we have

$$\begin{aligned}
& |H_C(f) - H_C(f_0)| \\
= & \left| \int E_0(z)(f - f_0)(z)\, dz - \frac{1}{2} \iint \frac{(\rho(\mathbf{y}) - \rho_0(\mathbf{y}))\,(\rho(\mathbf{x}) - \rho_0(\mathbf{x}))}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \right. \\
& \left. + \int \Phi\left(f(z)\right) - \Phi\left(f_0(z)\right)\, dz \right| \qquad (6.30) \\
\leq & \left| \int E_0(z)(f - f_0)(z)\, dz \right| + \left| \frac{1}{2} \iint \frac{(\rho(\mathbf{y}) - \rho_0(\mathbf{y}))\,(\rho(\mathbf{x}) - \rho_0(\mathbf{x}))}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \right| \\
& + \left| \int \Phi\left(f(z)\right) - \Phi\left(f_0(z)\right)\, dz \right| \qquad (6.31) \\
\leq & \int |E_0(z)|\, |f - f_0|\,(z)\, dz + \frac{1}{2} \iint \frac{|\rho(\mathbf{y}) - \rho_0(\mathbf{y})|\ |\rho(\mathbf{x}) - \rho_0(\mathbf{x})|}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \\
& + \int |\Phi\left(f(z)\right) - \Phi\left(f_0(z)\right)|\, dz \qquad (6.32)
\end{aligned}$$

Let $C_1 > 0$ and $f \in X$ with $f \leq C_1$. Define

$$L := \max\left\{|\Phi'(\zeta)| : \zeta \in [0, \max\{C_1, ||f_0||_\infty\}]\right\}. \tag{6.33}$$

Now we require a different Sobolev's inequality from the one which was used to establish (5.65) in the Newtonian case. From p. 31 of Reed and Simon [45] we have

Example 3 (Sobolev's inequality) Let $0 < \lambda < n$ and suppose that $f \in L^p(\mathbf{R}^n)$, $h \in L^r(\mathbf{R}^n)$, with $p^{-1} + r^{-1} + \lambda n^{-1} = 2$ and $1 < p, r < \infty$. Then

$$\int\int \frac{|f(x)|\ |h(y)|}{|x - y|^\lambda}\, d^n x\, d^n y \ \leq C_{p,r,\lambda,n}\, ||f||_p\, ||h||_r \tag{6.34}$$

So now we take $p = r = \frac{3}{2}$ together with $\lambda = 2$ and $n = 3$ to obtain

$$\iint \frac{|\rho(\mathbf{y}) - \rho_0(\mathbf{y})| \; |\rho(\mathbf{x}) - \rho_0(\mathbf{x})|}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y}\, d\mathbf{x} \;\leq c||\rho - \rho_0||^2_{\frac{3}{2}} \tag{6.35}$$

where c is some constant and therefore

$$|H_C(f) - H_C(f_0)| \;\leq\; C\left(\int (|E_{min}| + L)\,|f - f_0|(z)\, dz + ||\rho - \rho_0||^2_{\frac{3}{2}}\right) \tag{6.36}$$

where the constant C depends on the stationary solution and — via L — on the constant C_1.

This completes the fourth step from Section 5.2. Putting the estimates of steps three and four together as indicated in Section 5.2 proves the theorem. □

6.2 Steady-State Solution under Pure Stellar Manev Potential

As in the previous Chapter, we seek an isotropic steady-state solution of the form $\varphi(E) = c\,(1 - 2E)^{-\frac{1}{2}}$ when $0 \leq E < \frac{1}{2}$, where we must solve for the value of the constant c. By precisely the same steps as in the Newtonian solution, we reach

$$\rho(r) = c\pi^2\,(1 - 2U(r)) \qquad \text{for } 0 < r < 1. \tag{6.37}$$

Now our task is to couple this equation with the third equation in the Pure Stellar Manev case, which is the potential (3.2):

$$U_m(\mathbf{x}) = -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2} \tag{6.38}$$

or with any of the expressions which were derived from it in Chapter 3. As in the previous Chapter, it is important to remember that this system of the two equations (6.37) and (6.38) is only to be solved for $0 < r < 1$. For $r > 1$, $\rho(r) = 0$ while $U(r)$ is positive and an increasing function of r. If we use (3.23), we obtain

$$\rho(r) \;=\; c\pi^2(1 - 2U(r))$$

$$= c\pi^2 \left(1 + \frac{4\pi}{r} \int_0^r \tilde{s}\rho(\tilde{s}) \ln\left|\frac{r+\tilde{s}}{r-\tilde{s}}\right| d\tilde{s} + \frac{4\pi}{r} \int_r^1 \tilde{s}\rho(\tilde{s}) \ln\left|\frac{\tilde{s}+r}{\tilde{s}-r}\right| d\tilde{s} \right.$$
$$\left. -8\pi \int_0^1 \rho(\tilde{s})\, d\tilde{s}\right) \tag{6.39}$$

We recognize this as a Fredholm integral equation of the second type, which has the general form

$$\phi(x) = f(x) + \lambda \int_a^b k(x,t)\, \phi(t)\, dt \qquad (a \le x \le b) \tag{6.40}$$

The background material on Fredholm integral equations of the second type which follows is reproduced from Porter and Sterling [43]. Rather than work with three integrals, Porter and Sterling treat integral equations of our type (6.39) in the following concise format:

$$\rho(r) \;=\; c\pi^2 + 4c\pi^3 \int_0^1 \left(\frac{\tilde{s}}{r} \ln\left|\frac{r+\tilde{s}}{r-\tilde{s}}\right| - 2\right) \rho(\tilde{s})\, d\tilde{s} \tag{6.41}$$

which clearly identifies $\phi(x)$, $f(x)$, λ, a, b and the kernel $k(x,t)$.

We now prove existence and uniqueness of the solution to the integral equation (6.39). First we define an L^2-*kernel* and show that our kernel is an L^2-*kernel.* Sections 3.3 and 3.4 of [43] show that if k is an L^2-*kernel*, then the integral operator it generates, that is, the linear map $K : L^2(0,1) \to L^2(0,1)$ defined by

$$(K\phi)(x) = \int_0^1 k(x,t)\, \phi(t)\, dt \qquad (0 \le x \le 1) \tag{6.42}$$

is a bounded operator and

$$||K|| \le \left\{\int_0^1 \int_0^1 |k(x,t)|^2\, dx\, dt\right\}^{\frac{1}{2}} \tag{6.43}$$

Porter and Sterling [43] then establish that if k is an L^2-*kernel* on $[0,1]\times[0,1]$ and K is the bounded operator from $L^2(0,1)$ to itself generated by k, then K is compact.

Next, we use a theorem concerning the spectrum of a compact operator to establish that we can find infinitely many values of λ which are not eigenvalues of

K. Lastly, we use the Fredholm Alternative theorem to establish that for any such choice of λ that our equation (6.39) has a unique square-integrable solution. The following statement of the Fredholm Alternative theorem is reproduced from [43].

Theorem 3.3[43] Let H be the Hilbert space $L^2(a,b)$ and $k(x,t)$ be a kernel on $[a,b] \times [a,b]$ such that $(K\phi)(x) = \int_a^b k(x,t)\ \phi(t)\ dt$ defines a compact operator on $L^2(a,b)$. Then

either for every square-integrable function f on $[a,b]$ the integral equation

$$\phi(x) - \lambda \int_a^b k(x,t)\ \phi(t)\ dt = f(x) \qquad (a \leq x \leq b) \tag{6.44}$$

has a unique square-integrable solution ϕ,

or the homogeneous integral equation

$$\psi(x) - \lambda \int_a^b k(x,t)\ \psi(t)\ dt = 0 \qquad (a \leq x \leq b) \tag{6.45}$$

has a non-trivial square-integrable solution ψ.

We now proceed, starting with the definition of an L^2-*kernel.*

Definition 6.2 *We say that a complex-valued measurable function k defined on $[a,b] \times [a,b]$ is an L^2-kernel if it has the property that*

$$\int_a^b \int_a^b |k(x,t)|^2\ dx\ dt\ <\ \infty \tag{6.46}$$

To show that our kernel is an L^2-*kernel,* we only need to show that

$$\frac{\tilde{s}}{r} \ln \left| \frac{r+\tilde{s}}{r-\tilde{s}} \right| \tag{6.47}$$

is an L^2-*kernel,* since Minkowski's inequality tells us that the L^2 norm of the sum of two measurable functions is less than or equal to the sum of their L^2 norms. The details of this calculation are presented in Appendix B.

Now we introduce the following theorem on the spectrum of a compact operator.

Theorem 4.7[43] Let H be a Hilbert space of infinite dimension and let $K \in B(H)$ be compact. Then the spectrum of K consists of 0 together with a finite or countably infinite set of eigenvalues. Each non-zero point of $\sigma(K)$ (the spectrum of K) is an eigenvalue with a finite-dimensional set of corresponding eigenvectors and if there are infinitely many eigenvalues, $\{\mu_n : n \in \mathbf{N}\}$, then $\mu_n \to 0$ as $n \to \infty$.

This means that there are uncountably infinitely many positive real numbers λ which are not eigenvalues of K. For any choice among these possible values for λ, by the Fredholm Alternative theorem, equation (6.41) has a unique square-integrable solution $\rho(r)$. We will produce the solution using Picard iteration.

To solve equation (6.40) by Picard iteration, we define a sequence of approximations

$$\begin{aligned} \phi_0(x) &= f(x) \\ \forall n \geq 1, \quad \phi_n(x) &= f(x) + \lambda \int_a^b k(x,t)\, \phi_{n-1}(t)\, dt \qquad (a \leq x \leq b) \end{aligned} \tag{6.48}$$

If we let $a_0 = c\pi^2$, the first two functions in our sequence of approximations are

$$\begin{aligned} \rho_0(r) &= a_0 \\ \rho_1(r) &= a_0 + 4\pi a_0 \int_0^1 \left(\frac{\tilde{s}}{r} \ln \left| \frac{r+\tilde{s}}{r-\tilde{s}} \right| - 2 \right) a_0 \, d\tilde{s} \end{aligned} \tag{6.49}$$

Computing the first iteration $\rho_1(r)$ by hand using equation (4.10) from Chapter 4 we obtain

$$\rho_1(r) = a_0 \left[1 - 8\pi a_0 \left(\frac{r^2}{1 \cdot 3} + \frac{r^4}{3 \cdot 5} + \frac{r^6}{5 \cdot 7} \cdots \right) \right] \tag{6.50}$$

Note that the first iteration is a power series consisting only of even powers of r. Hand calculation shows that if $\rho(r)$ is given by any even power of r, then $U(r)$ will be a power series consisting only of even powers of r. This means that every iteration starting with the first will be of this form. Further iterations were performed using

MATLAB. MATLAB computes a new column vector containing new coefficients of the power series

$$\rho(r) = a_0 + a_2 r^2 + a_4 r^4 + a_6 r^6 \cdots \tag{6.51}$$

by performing the following matrix multiplication

$$8\pi a_0 \begin{bmatrix} \frac{1}{8\pi a_0} & 0 & 0 & \cdots & \cdots \\ \frac{1}{3(-1)} & \frac{1}{3(1)} & \frac{1}{3(3)} & \cdots & \cdots \\ \frac{1}{5(-3)} & \frac{1}{5(-1)} & \frac{1}{5(1)} & \cdots & \cdots \\ \frac{1}{7(-5)} & \frac{1}{7(-3)} & \frac{1}{7(-1)} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \\ a_4 \\ a_6 \\ \cdots \end{bmatrix}$$

where the column vector on the right contains the coefficients from the previous iteration step. One hundred iterations were performed, by which point each iteration was producing a difference of much less than 10^{-14} in every coefficient. Finally, we solved for the value of a_0 which would satisfy the requirement that $U(1) = \frac{1}{2}$. The approximate numerical solution obtained is

$$\begin{aligned} \rho(r) &= 0.0874739879 - 0.1631047432\, r^2 + 0.0986157202\, r^4 - 0.0275708576\, r^6 \\ &+0.0047567683\, r^8 - 0.0004150071\, r^{10} + 0.0000914971\, r^{12} + 0.0000332010\, r^{14} \\ &+0.0000246832\, r^{16} + 0.0000177421\, r^{18} \cdots\cdots\cdots \end{aligned} \tag{6.52}$$

The graph of this density profile appears on the next page.

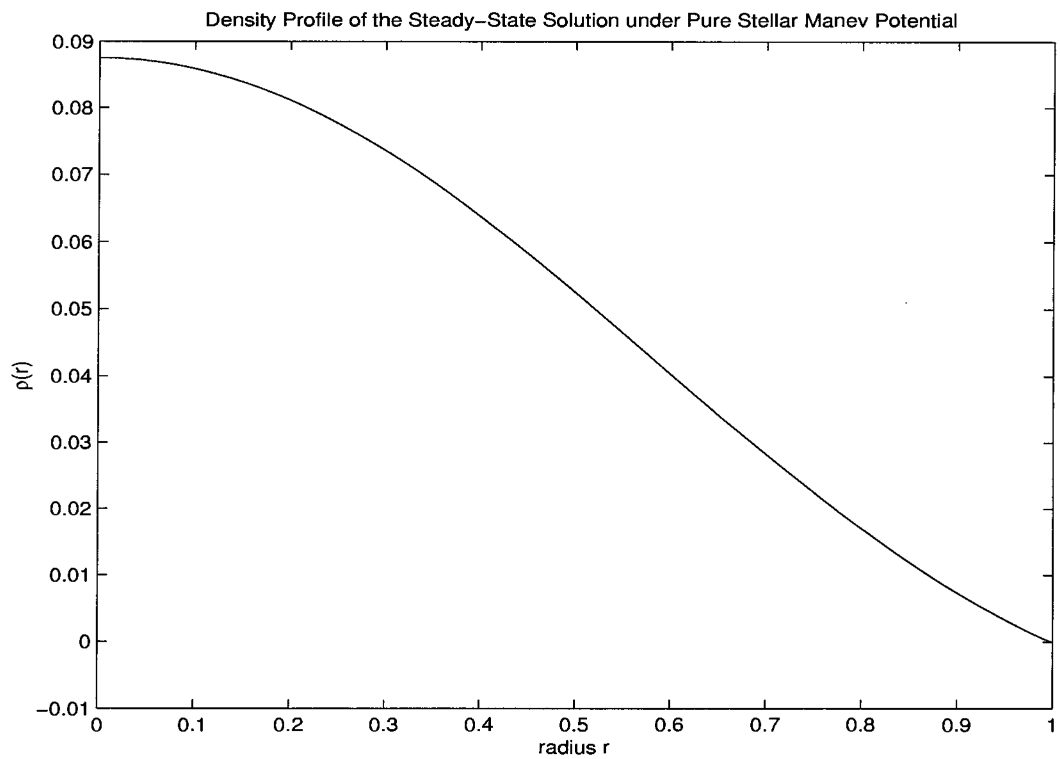

Figure 6.1: Density Profile of the Steady-State Solution under Pure Stellar Manev Potential

This is the plot of the density function $\rho(r)$ *which appears as (6.52) in this Section. Density* $\rho(r)$ *is supported on* $0 < r < 1$. *Note the similarity to Figure 5.3 at the end of Chapter 5.*

Chapter 7

Discussion and Conclusions

The demonstration of the existence and nonlinear stability of dynamic solutions under a $\frac{1}{r^2}$ potential was broken down into several steps. After a brief historical introduction, which included the Vlasov-Poisson system and Jeans' Theorem, the Vlasov-Manev system was presented. This was followed by a comparative study of Newtonian vs. Manev stellar dynamics, with emphasis on the more severe singularities which are encountered with the Pure Stellar Manev (PSM) potential. Of these, the most severe is that the integral which gives the force term is defined only as a Cauchy principal value. Calculations in the PSM potential are considerably more difficult due to the lack of the Poisson equation. Another consideration is that a steady state solution under the PSM potential must have a total energy of zero.

In this dissertation, several expressions for the potential energy and the force term in a spherically symmetric attracting particle distribution under PSM were derived. Many of these are particularly well suited to distributions which have compact support. Distributions with finite mass and compact support were the goal as they are the most physically meaningful.

Two anisotropic steady-state solutions under PSM, both with finite mass and compact support, were constructed. In the first of these, the force term becomes unboundedly large due to a discontinuity in the density, $\rho(r)$, at $r = 1$. In the second, the density is such that the force term is bounded. Using a rescaling method suggested in Rein [46], it was shown how the first solution could be rescaled to any radius of support and any finite maximum particle speed desired.

Existence and nonlinear stability of an infinitude of isotropic, stationary, spherically symmetric solutions with finite mass and compact support in $\mathbf{R}^6$ under the Newtonian potential was established, using a method found in Rein [46], which there was limited to the plasma physics case. This was accompanied by an introduction to the energy-Casimir method, which provides a plan for the construction of a suitable Casimir functional. An example of an isotropic steady-state solution under the Newtonian potential was then constructed, which involved explicitly solving the Vlasov-Poisson system.

In this dissertation, the same constructions were performed for the Pure Stellar Manev potential and the Vlasov-Manev system. The Sobolev inequality for the continuity of the energy-Casimir functional H_C at the steady-state solution f_0 in the Newtonian case involved the square of the same $L^{\frac{6}{5}}$ norm of $\rho - \rho_0$ as that found by Rein [46] in the plasma physics case. However, in this dissertation it is established that in the Manev case the same expression involves the $L^{\frac{3}{2}}$ norm. Another interesting result is the similarity between the density profiles of the isotropic steady states which appear as Figure 5.3 at the end of Chapter 5 and Figure 6.1 at the end of Chapter 6.

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Appendix A

Some Properties and Applications of the Fourier Transform

A.1 The Fourier Transform

In Chapter 3 we developed several representations of the Pure Stellar Manev Potential, $U_m(r)$. In each of these, $U_m(r)$ is given in terms of density, $\rho(r)$. In the Newtonian potential, we have Poisson's equation (1.4)

$$\triangle U = 4\pi\rho(r) \tag{A.1}$$

so that

$$\rho(r) = \frac{1}{4\pi}\triangle U \tag{A.2}$$

This gives density, $\rho(r)$, as a functional of the Newtonian potential, $U_n(r)$. At present, there is no corresponding expression which gives density, $\rho(r)$, as a functional of the Pure Stellar Manev potential, $U_m(r)$. We will make use of Fourier transforms to establish such formulas. Our task will be complicated by the fact that many of the Fourier transforms we encounter will not exist in the usual sense, because the required integrals will not exist. However, these Fourier transforms will exist in the sense of distributions.

There is more than one version of the Fourier transform. All of the background material in this Chapter on the Fourier transform is reproduced from Taylor [52]. The Fourier transform is defined by

Definition A.1 *The Fourier transform $\hat{f}(\xi) = (\mathcal{F}f)(\xi)$ of a function $f(x)$ is*

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int f(x) e^{-ix\cdot\xi}\, dx \tag{A.3}$$

where ξ is the variable in the frequency domain and $f \in L^1(\mathbf{R}^n)$ (this requirement will be relaxed later). We have that

$$\mathcal{F} : L^1(\mathbf{R}^n) \longrightarrow L^\infty(\mathbf{R}^n) \tag{A.4}$$

We define $\mathcal{F}^*$ by

$$\mathcal{F}^* f(\xi) = \tilde{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int f(x) e^{ix\cdot\xi}\, dx \tag{A.5}$$

Definition A.2 *The Schwartz space of rapidly decreasing functions,* S

$$\mathsf{S}(\mathbf{R}^n) = \{f \in C^\infty(\mathbf{R}^n) |\ x^\beta D^\alpha f \in L^\infty(\mathbf{R}^n)\ \ \forall \alpha, \beta \geq 0\} \tag{A.6}$$

where $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, with $D_j = -i\partial/\partial x_j$. We have that

$$\mathcal{F} : \mathsf{S}(\mathbf{R}^n) \longrightarrow \mathsf{S}(\mathbf{R}^n) \tag{A.7}$$

and

$$(\mathcal{F}u, v) = (u, \mathcal{F}^* v), \tag{A.8}$$

for $u, v \in \mathsf{S}(\mathbf{R}^n)$, where (u, v) denotes the L^2 inner product, $(u, v) = \int u(x)\overline{v(x)}dx$. We have the inversion formula

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I \quad \text{on} \quad \mathsf{S}(\mathbf{R}^n) \tag{A.9}$$

This means that if we restrict functions to the Schwartz space S, then we are guaranteed that not only will the Fourier transform $\hat{f}(\xi)$ exist, but that the Inverse Fourier transform will give us back $f(x)$.

In light of (A.8) and the Fourier inversion formula (A.9), we have that, for $u, v \in \mathsf{S}(\mathbf{R}^n)$

Theorem A.3 *Parseval Theorem*

$$(\mathcal{F}u, \mathcal{F}v) = (u, v) = (\mathcal{F}^*u, \mathcal{F}^*v) \tag{A.10}$$

Thus $\mathcal{F}$ and $\mathcal{F}^*$ extend uniquely from $\mathsf{S}(\mathbf{R}^n)$ to isometries on $L^2(\mathbf{R}^n)$ and are inverses to each other. This gives us the Plancherel theorem

Theorem A.4 *Plancherel Theorem*

The Fourier transform

$$\mathcal{F} : L^2(\mathbf{R}^n) \longrightarrow L^2(\mathbf{R}^n) \tag{A.11}$$

is unitary, with inverse $\mathcal{F}^*$.

The inversion formulas of (A.4) and (A.9) do not provide for the inversion of $\mathcal{F}$ in (A.4). We will obtain this later in this chapter.

A.2 Distributions

We continue to follow the treatment in Taylor [52]. We begin with the concept of a tempered distribution. This is a continuous linear functional

$$w : \mathsf{S}(\mathbf{R}^n) \longrightarrow \mathbf{C} \tag{A.12}$$

where $\mathsf{S}(\mathbf{R}^n)$ is the Schwartz space defined in Section A.1. The space $\mathsf{S}(\mathbf{R}^n)$ has a topology, determined by the seminorms

$$p_k(u) = \sum_{|\alpha|\le k} sup_{x\in\mathbf{R}^n} \langle x\rangle^k |D^\alpha u(x)|. \tag{A.13}$$

The distance function

$$d(u,v) = \sum_{k=0}^{\infty} 2^{-k} \frac{p_k(u-v)}{1+p_k(u-v)} \tag{A.14}$$

makes $\mathsf{S}(\mathbf{R}^n)$ a complete metric space; with such a topology it is a Fréchet space. For a linear map w as in (A.12) to be continuous, it is necessary and sufficient that, for some k, C,

$$|w(u)| \le C\, p_k(u),\ \forall u \in \mathsf{S}(\mathbf{R}^n). \tag{A.15}$$

The action of w is often written as follows:

$$w(u) = \langle u, w\rangle. \tag{A.16}$$

The set of all continuous linear functionals on $\mathsf{S}(\mathbf{R}^n)$ is denoted

$$\mathsf{S}'(\mathbf{R}^n) \tag{A.17}$$

and is called the space of tempered distributions.

The space $\mathsf{S}'(\mathbf{R}^n)$ has a topology, called the weak* topology, or sometimes simply the weak topology, in terms of which a directed family w_γ converges to w weakly in $\mathsf{S}'(\mathbf{R}^n)$ if and only if, for each $u \in \mathsf{S}(\mathbf{R}^n)$, $\langle u, w_\gamma\rangle \longrightarrow \langle u, w\rangle$.

There is a natural injection

$$L^p(\mathbf{R}^n) \hookrightarrow \mathsf{S}'(\mathbf{R}^n) \tag{A.18}$$

for any $p \in [1, \infty]$, given by

$$\langle u, f \rangle = \int u(x)\, f(x)\, dx, \quad u \in \mathsf{S}(\mathbf{R}^n), \quad f \in L^p(\mathbf{R}^n). \tag{A.19}$$

Similarly, any finite measure on $(\mathbf{R}^n)$ gives an element of $\mathsf{S}'(\mathbf{R}^n)$. The basic example is the Dirac "delta function" δ, defined by

$$\langle u, \delta \rangle = u(0). \tag{A.20}$$

Also, each differential operator $D_j = -i\partial/\partial x_j$ acts on $\mathsf{S}'(\mathbf{R}^n)$, by the definition

$$\langle u, D_j w \rangle = -\langle D_j u, w \rangle, \quad u \in \mathsf{S}, \quad w \in \mathsf{S}'. \tag{A.21}$$

Iterating, we see that each $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ acts on S':

$$D^\alpha : \mathsf{S}'(\mathbf{R}^n) \longrightarrow \mathsf{S}'(\mathbf{R}^n) \tag{A.22}$$

and we have

$$\langle u, D^\alpha w \rangle = (-1)^{|\alpha|} \langle D^\alpha u, w \rangle \tag{A.23}$$

for $u \in \mathsf{S}$, $w \in \mathsf{S}'$.

The Fourier transform $\mathcal{F} : \mathsf{S}(\mathbf{R}^n) \to \mathsf{S}(\mathbf{R}^n)$, from Section A.1, extends to S' by the formula

$$\langle u, \mathcal{F} w \rangle = \langle \mathcal{F} u, w \rangle; \tag{A.24}$$

we can also set

$$\langle u, \mathcal{F}^* w \rangle = \langle \mathcal{F}^* u, w \rangle; \tag{A.25}$$

to get

$$\mathcal{F}, \mathcal{F}^* : \mathsf{S}'(\mathbf{R}^n) \longrightarrow \mathsf{S}'(\mathbf{R}^n). \tag{A.26}$$

The maps (A.26) are continuous when $\mathsf{S}'(\mathbf{R}^n)$ is given the weak* topology, as follows from the definitions.

The Fourier inversion formula (A.9) yields

Proposition A.5 *We have*

$$\mathcal{F}^*\mathcal{F} = \mathcal{F}\mathcal{F}^* = I \quad \text{on} \quad \mathsf{S}'(\mathbf{R}^n) \tag{A.27}$$

Proof. Using (A.24) and (A.25), if $u \in \mathsf{S}$, $w \in \mathsf{S}'$,

$$\langle u, \mathcal{F}^*\mathcal{F}w\rangle = \langle \mathcal{F}^*u, \mathcal{F}w\rangle = \langle \mathcal{F}\mathcal{F}^*u, w\rangle = \langle u, w\rangle, \tag{A.28}$$

and a similar analysis works for $\mathcal{F}\mathcal{F}^*w$.

So now we have a usable Fourier transform of a function f which is not in $L^1(\mathbf{R}^n)$, but which is only locally integrable. We will consider $\hat{f}(\xi) = (\mathcal{F}f)(\xi)$ to be the Fourier transform of such a function f provided that for any test function $\phi \in \mathsf{S}$ we have

$$\langle \mathcal{F}f, \phi\rangle = \langle f, \mathcal{F}\phi\rangle \tag{A.29}$$

A.3 Preliminary computations

Continuing with the material as presented in Taylor [52], we have convolution

$$u * v(x) = \int u(y)v(x-y)dy \tag{A.30}$$

which gives a bilinear map

$$\mathsf{S}(\mathbf{R}^n) \times \mathsf{S}(\mathbf{R}^n) \longrightarrow \mathsf{S}(\mathbf{R}^n). \tag{A.31}$$

The convolution theorem

$$\mathcal{F}(u * v) = (2\pi)^{\frac{n}{2}}\hat{u}(\xi)\hat{v}(\xi) \tag{A.32}$$

holds.

Since we are interested only in spherically symmetric solutions, we use the rotational invariance of the Laplace operator on $\mathbf{R}^n$; one has

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_S \tag{A.33}$$

where $\triangle_S$ is the Laplace operator on the unit sphere S^{n-1}. In the case of a spherically symmetric function, or even a rotationally invariant tempered distribution, the Laplace operator simplifies to

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} \tag{A.34}$$

We now consider the Fourier transform of a spherically symmetric steady-state function, $F(x) = f(r)$, $r = |x|$. Details of the derivation are in Taylor [52] and are omitted here.

$$\hat{F}(\xi) = |\xi|^{1-\frac{n}{2}} \int_0^\infty f(r) J_{\frac{n}{2}-1}(r|\xi|) r^{\frac{n}{2}}\, dr \tag{A.35}$$

where for general $\nu \in \mathbf{C}$ satisfying $\Re(\nu) > -\frac{1}{2}$, where $\Re(\nu)$ is the real part of the complex number ν, the Bessel function $J_\nu(z)$ is defined to be

$$J_\nu(z) = \left[\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\right]^{-1}\left(\frac{z}{2}\right)^\nu \int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}} e^{izt}\, dt \tag{A.36}$$

Since $n = 3$, (A.35) becomes

$$\hat{F}(\xi) = |\xi|^{-\frac{1}{2}} \int_0^\infty f(r)\sqrt{\frac{2}{\pi}}\,\frac{\sin(r|\xi|)}{\sqrt{r|\xi|}} r^{\frac{3}{2}}\, dr \tag{A.37}$$

For example, let us take the Fourier transforms of $f(r) = \frac{1}{r^2}$ and $f(r) = \frac{1}{r}$ in $\mathbf{R}^3$. In the first case we have

$$\begin{aligned}
\hat{F}(\xi) &= |\xi|^{-\frac{1}{2}} \int_0^\infty \frac{1}{r^2}\sqrt{\frac{2}{\pi}}\,\frac{\sin(r|\xi|)}{\sqrt{r|\xi|}} r^{\frac{3}{2}}\, dr \\
&= |\xi|^{-\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{|\xi|}} \int_0^\infty \frac{\sin(r|\xi|)}{r}\, dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\xi|}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin u}{u}\,du \\
&= \frac{1}{|\xi|}\sqrt{\frac{2}{\pi}}\,\frac{\pi}{2} \\
&= \sqrt{\frac{\pi}{2}}\,\frac{1}{|\xi|} \qquad \text{(A.38)}
\end{aligned}
$$

In the second case, $f(r) = \frac{1}{r}$, when we attempt to take the Fourier transform using the integral (A.37), the integral

$$
\begin{aligned}
\hat{F}(\xi) &= |\xi|^{-\frac{1}{2}}\int_0^\infty \frac{1}{r}\sqrt{\frac{2}{\pi}}\,\frac{\sin(r|\xi|)}{\sqrt{r|\xi|}}r^{\frac{3}{2}}\,dr \\
&= \frac{1}{|\xi|}\sqrt{\frac{2}{\pi}}\int_0^\infty \sin(r|\xi|)\,dr \\
&= \frac{1}{|\xi|^2}\sqrt{\frac{2}{\pi}}\int_0^\infty \sin u\,du \qquad \text{(A.39)}
\end{aligned}
$$

is undefined. Fortunately, Taylor [52] establishes, using a detailed derivation from analytic function theory, the result that for values (possibly complex-valued) of m whose real part is strictly between $-n$ and 0, i.e. $m \in \mathbf{C} : -n < \Re(m) < 0$, the Fourier transform of r^m is

$$
2^{m+\frac{n}{2}}\frac{\Gamma(\frac{m+n}{2})}{\Gamma(-\frac{1}{2}m)}|\xi|^{-m-n} \qquad \text{(A.40)}
$$

There is a more concise derivation of the same result in Constantinescu [12] which we shall summarize briefly, using notation consistent with that used in Taylor [52]. We note that

$$
\mathcal{F}(r^m)(\xi) = (2\pi)^{-\frac{n}{2}}\int r^m e^{-ix\cdot\xi}\,dx \qquad \text{(A.41)}
$$

where $r = |x|$ and the integral converges for $-n < \Re(m) < 0$. Now for any real $t > 0$, with the substitution $y = tx$, $|y| = \sqrt{y_1^2 + y_2^2 + y_3^2}$,

$$
\begin{aligned}
\mathcal{F}(r^m)(t\xi) &= (2\pi)^{-\frac{n}{2}}\int r^m e^{-itx\cdot\xi}\,dx \\
&= (2\pi)^{-\frac{n}{2}}\int |y|^m t^{-m-n} e^{-iy\cdot\xi}\,dy \\
&= t^{-m-n}\mathcal{F}(r^m)(\xi) \qquad \text{(A.42)}
\end{aligned}
$$

It follows that

$$\mathcal{F}(r^m)(\xi) = c(m)\rho^{-m-n} \tag{A.43}$$

where $\rho = |\xi|$, with $c(m)$ a constant depending only on m. We now solve for $c(m)$, using (A.3) and the test function $\phi(x) = e^{-\frac{r^2}{2}}$, where $r = |x|$.

$$(\mathcal{F}(r^m), \mathcal{F}(\phi)) = (r^m, \phi) \tag{A.44}$$

First we compute the left hand side, making the substitution $s = \frac{\rho^2}{2}$, where $\rho = |\xi|$

$$\begin{aligned}
&\int \mathcal{F}(r^m)(\xi) e^{-\frac{|\xi|^2}{2}}\, d\xi \\
= \quad & c(m) \int |\xi|^{-m-n} e^{-\frac{|\xi|^2}{2}}\, d\xi \\
= \quad & c(m) A_{n-1} \int_0^\infty \rho^{-m-n} \rho^{n-1} e^{-\frac{\rho^2}{2}}\, d\rho \\
= \quad & c(m) A_{n-1} \int_0^\infty \rho^{-m-2} e^{-\frac{\rho^2}{2}} \rho\, d\rho \\
= \quad & c(m) 2^{-\frac{(m+1)}{2}} A_{n-1} \int_0^\infty s^{-\frac{m}{2}-1} e^{-s}\, ds \\
= \quad & c(m) 2^{-\frac{(m+1)}{2}} A_{n-1} \Gamma(-\frac{m}{2})
\end{aligned} \tag{A.45}$$

where $A_{n-1} = vol(S^{n-1})$, the volume of the unit sphere in $\mathbf{R}^{n-1}$. Similarly, we compute the right hand side, making the substitution $t = \frac{r^2}{2}$

$$\begin{aligned}
&\int r^m \phi\, dx \\
= \quad & \int r^m e^{-\frac{r^2}{2}}\, dx \\
= \quad & A_{n-1} \int_0^\infty r^m e^{-\frac{r^2}{2}} r^{n-1}\, dr \\
= \quad & A_{n-1} \int_0^\infty r^{m+n-2} e^{-\frac{r^2}{2}} r\, dr \\
= \quad & 2^{\frac{m+n}{2}-1} A_{n-1} \int_0^\infty t^{\frac{m+n}{2}-1} e^{-t}\, dt \\
= \quad & 2^{\frac{m+n}{2}-1} A_{n-1} \Gamma(\frac{m+n}{2})
\end{aligned} \tag{A.46}$$

Now, equating the two sides and solving for $c(m)$ yields

$$c(m) = \frac{2^{m+\frac{n}{2}}\Gamma(\frac{m+n}{2})}{\Gamma(-\frac{m}{2})} \tag{A.47}$$

and (A.40) is established.

So now, for example, the Fourier transform of $f(r) = \frac{1}{r}$ in $\mathbf{R}^3$ can be computed readily as

$$\begin{aligned} \mathcal{F}(\frac{1}{r})(\xi) &= 2^{-1+\frac{3}{2}}\frac{\Gamma(\frac{-1+3}{2})}{\Gamma(\frac{1}{2})}|\xi|^{1-3} \\ &= \frac{1}{|\xi|^2}\sqrt{\frac{2}{\pi}} \end{aligned} \tag{A.48}$$

A.4 Potential computations

As the first demonstration of the power of the Fourier transform, we will verify (1.4) in the Newtonian potential case, i.e. that

$$\begin{aligned} U_n(\mathbf{x}) &= -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \\ &= -\rho * \frac{1}{|x|} \end{aligned} \tag{A.49}$$

solves the Poisson equation.

Taking the Fourier transform of both sides of (A.49), using (A.30) and (A.48)

$$\begin{aligned} \widehat{U_n} &= -(2\pi)^{\frac{3}{2}} \widehat{\left(\frac{1}{|x|}\right)} \hat{\rho} \\ &= -\frac{4\pi}{|\xi|^2} \hat{\rho} \end{aligned} \tag{A.50}$$

gives us

$$-|\xi|^2 \widehat{U_n} = 4\pi \hat{\rho} \tag{A.51}$$

Now take the inverse Fourier transform of both sides to get (1.4), Poisson's equation.

$$\triangle U_n = 4\pi\rho \tag{A.52}$$

In particular, (A.52) shows how ρ can be reconstructed from u_n. We now carry out the analogous computation for the Pure Stellar Manev potential.

$$\begin{aligned} U_m(\mathbf{x}) &= -\int \frac{\rho(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} \\ &= -\rho * \frac{1}{|x|^2} \end{aligned} \tag{A.53}$$

Taking the Fourier transform of both sides of (A.53), using (A.30) and (A.38)

$$\begin{aligned} \widehat{U_m} &= -(2\pi)^{\frac{3}{2}} \left(\widehat{\frac{1}{|x|^2}}\right) \hat{\rho} \\ &= -\frac{2\pi^2}{|\xi|}\, \hat{\rho} \end{aligned} \tag{A.54}$$

gives us

$$-|\xi|^2\, \widehat{U_m} = 2\pi^2 |\xi|\, \hat{\rho} \tag{A.55}$$

that is,

$$(\widehat{\triangle U_m}) = 2\pi^2 |\xi|\, \hat{\rho} \tag{A.56}$$

which can be rewritten

$$\begin{aligned} \hat{\rho} &= \frac{1}{2\pi^2} (\widehat{\triangle U_m}) \frac{1}{|\xi|} \\ &= (2\pi)^{\frac{3}{2}} \frac{1}{4\pi^4} (\widehat{\triangle U_m}) \left(\widehat{\frac{1}{|x|^2}}\right) \end{aligned} \tag{A.57}$$

Now take the inverse Fourier transform of both sides to get the density ρ as a functional of the Pure Stellar Manev potential U_m

$$\begin{aligned} \rho &= \frac{1}{4\pi^4} \triangle U_m * \frac{1}{|x|^2} \\ &= \frac{1}{4\pi^4} \int \frac{\triangle U_m(\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2} \end{aligned} \tag{A.58}$$

Now we can use

$$\begin{aligned} \nabla_{\mathbf{y}} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|^2} \right) &= \nabla_{\mathbf{y}} \frac{1}{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} \\ &= 2\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4} \end{aligned} \tag{A.59}$$

and integrate (A.58) by parts to obtain

$$\rho = -\frac{1}{2\pi^4} \int \frac{\nabla U_m(\mathbf{y}) \cdot (\mathbf{x}-\mathbf{y})\, d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^4} \tag{A.60}$$

As a demonstration of (A.58), let us compute the density at the origin of the Steady-State Solution with Unbounded Force of Section 4.2. Using (4.16) and (4.21) together with (A.34), we obtain

$$\triangle U_m = \begin{cases} \frac{1}{1-r^2} + \frac{1}{2r} \ln\left|\frac{1+r}{1-r}\right| & , \quad 0 < r < 1 \\ -\frac{1}{r^2-1} + \frac{1}{2r} \ln\left|\frac{r+1}{r-1}\right| & , r > 1. \end{cases} \tag{A.61}$$

As we noted after computing (4.21), the Laplacian of the potential is actually given by the same expression at any point interior or exterior to the sphere of radius 1. Now we compute

$$\begin{aligned} \rho(0) &= \frac{1}{4\pi^4} \int \frac{\triangle U_m(\mathbf{y})\, d\mathbf{y}}{|\mathbf{0}-\mathbf{y}|^2} \\ &= \frac{1}{\pi^3} \int_0^1 \frac{1}{1-r^2} + \frac{1}{2r} \ln\left|\frac{1+r}{1-r}\right| \; dr + \frac{1}{\pi^3} \int_1^\infty -\frac{1}{r^2-1} + \frac{1}{2r} \ln\left|\frac{r+1}{r-1}\right| \; dr \\ &= \frac{1}{\pi^3} \int_0^\infty \frac{1}{1-r^2} + \frac{1}{2r} \ln\left|\frac{1+r}{1-r}\right| \; dr \\ &= \frac{1}{4\pi} \end{aligned} \tag{A.62}$$

consistent with the Steady-State Solution with Unbounded Force. Note that, due to the singularity at $r = 1$, the integral exists only as a Cauchy principal value. The singularity was investigated two different ways, first by taking epsilon limits on either side of $r = 1$ and taking the limit as epsilon goes to zero, and secondly repeating the epsilon limits numerically using MATLAB.

Appendix B

Details of Calculations from Section 6.2

Here we present the details of the calculations needed in Section 6.2 which establish that the kernel

$$k(r,s) = \frac{s}{r}\ln\left|\frac{r+s}{r-s}\right| \tag{B.1}$$

is an L^2-*kernel*, i.e. that

$$\int_0^1\int_0^1 |k(r,s)|^2\,dr\,ds \tag{B.2}$$

exists. Note that the tildes have been dropped.

The square $(r,s)\in[0,1]\times[0,1]$ in the r,s-plane must be divided into two regions to carry out the integration. They are the upper half of the square located above the diagonal $s=r$, and then the lower half of the square, located below the diagonal $s=r$. We will use polar coordinates and appropriate substitutions to carry out both integrals. Since we are already using the variable r, we will use R for the variable used in polar coordinates. We will also use the numerical integration capabilities of MAPLE.

Above the diagonal, we have

$$\int_0^1\int_r^1\left(\frac{s}{r}\ln\left|\frac{s+r}{s-r}\right|\right)^2 ds\,dr \qquad \text{let } \frac{s}{r}=\tan\theta$$

$$
\begin{aligned}
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\csc\theta} \tan^2\theta \left(\ln\left|\frac{\tan\theta+1}{\tan\theta-1}\right|\right)^2 R\,dR\,d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\csc^2\theta\,\tan^2\theta \left(\ln\left|\frac{\tan\theta+1}{\tan\theta-1}\right|\right)^2 d\theta \qquad \text{let } w=\tan\theta \\
&= \int_1^{\infty} \frac{1}{2}\left(\ln\left|\frac{w+1}{w-1}\right|\right)^2 dw \\
&= 3.289868134 \qquad \text{by MAPLE} \qquad\qquad \text{(B.3)}
\end{aligned}
$$

Below the diagonal,

$$
\begin{aligned}
&\int_0^1 \int_0^r \left(\frac{s}{r}\ln\left|\frac{r+s}{r-s}\right|\right)^2 ds\,dr \qquad \text{let } \frac{s}{r}=\tan\theta \\
&= \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \tan^2\theta \left(\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right|\right)^2 R\,dR\,d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{2}\sec^2\theta\,\tan^2\theta \left(\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right|\right)^2 d\theta \qquad \text{let } w=\tan\theta \\
&= \int_0^1 \frac{1}{2}w^2\left(\ln\left|\frac{1+w}{1-w}\right|\right)^2 dw \\
&= 1.214978022 \qquad \text{by MAPLE} \qquad\qquad \text{(B.4)}
\end{aligned}
$$

This concludes the calculation that our kernel is an L^2-*kernel.*