CONTINUITY PROPERTIES OF HYPERLAT AND REDUCING SUBSPACES

by

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DMS–500–IR April 1989

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ABSTRACT

We determine points of continuity, upper semicontinuity and lower semicontinuity of the following two maps from the space of all linear operators on a finite–dimensional complex vector space $\mathscr{V}$ into sets of subspaces of $\mathscr{V}$: **Hyperlat**, which associates with every operator A the set of subspaces invariant under the commutant of A, and **Red**, which associates with A the set of reducing subspaces of A.

1. INTRODUCTION

Let $\mathscr{V}$ be a vector space over the complex field, having a finite dimension $n \geq 2$ and let $A \in L(\mathscr{V})$, the space of all linear operators on $\mathscr{V}$. A subspace $\mathscr{M}$ of $\mathscr{V}$ is called *invariant* under A if $A\mathscr{M} \subset \mathscr{M}$. It is called *hyperinvariant* under A if it is invariant under every linear operator commuting with A, and it is called *reducing* for A if $\mathscr{M}$ has a complement $\mathscr{N}$ such that both $\mathscr{M}$ and $\mathscr{N}$ are invariant under A. The set of all invariant, respectively hyperinvariant, subspaces under an operator A is a lattice under the operations of "intersection" and "vector sum" and are denoted by **Lat** A, respectively **Hyperlat** A. The set of reducing subspaces of A is denoted by **Red** A. (Incidentally, **Red** A is not always a lattice, see §5.)

To discuss the concept of continuity for the functions **Lat**, **Hyperlat** and **Red**, we describe a notion of convergence in $2^{\mathscr{S}(\mathscr{V})}$, the set of all subsets of $\mathscr{S}(\mathscr{V})$ where $\mathscr{S}(\mathscr{V})$ denotes the set of all subspaces of $\mathscr{V}$ (*Cf.* [4], [8, Ch. 13,14]). First we introduce an inner product on $\mathscr{V}$ and define the distance $d(\mathscr{M},\mathscr{N})$ between two subspaces $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{V}$ by

$$d(\mathscr{M},\mathscr{N}) = \|P_{\mathscr{M}} - P_{\mathscr{N}}\| \tag{1.1}$$

where $P_{\mathscr{M}}$ denotes the orthogonal projection of $\mathscr{V}$ onto $\mathscr{M}$ and $\|\cdot\|$ denotes the operator norm on $L(\mathscr{V})$. This defines a metric on the set $\mathscr{S}(\mathscr{V})$ of subspaces of $\mathscr{V}$ under which $\mathscr{S}(\mathscr{V})$ is a compact metric space. The following alternative formula [8, Theorem 13.1.1] for $d(\mathscr{M},\mathscr{N})$ is quite useful,

$$d(\mathscr{M},\mathscr{N}) = \max\left\{ \sup_{x \in \mathscr{M}_1} d(x,\mathscr{N}),\ \sup_{y \in \mathscr{N}_1} d(y,\mathscr{M})\right\} \tag{1.2}$$

where $\mathscr{M}_1$ and $\mathscr{N}_1$ are the unit spheres of $\mathscr{M}$ and $\mathscr{N}$ respectively. In addition, if P_1 and P_2 are any projections with ranges $\mathscr{M}$ and $\mathscr{N}$ respectively, then

$$d(\mathscr{M},\mathscr{N}) \le \|P_1 - P_2\|, \tag{1.3}$$

(see [8, Theorem 13.1.1]).

Now that we have a metric on $\mathscr{S}(\mathscr{V})$, we define $\liminf \mathscr{S}_k$ and $\limsup \mathscr{S}_k$ for a sequence $\{\mathscr{S}_k\}$ of sets of subspaces of $\mathscr{V}$ as follows:

$\mathscr{M} \in \liminf \mathscr{S}_k$ if there exists $\mathscr{M}_k \in \mathscr{S}_k$ such that $\mathscr{M}_k \to \mathscr{M}$,

$\mathscr{M} \in \limsup \mathscr{S}_k$ if there exists a subsequence $\mathscr{M}_{k_i} \in \mathscr{S}_{k_i}$ such that $\mathscr{M}_{k_i} \to \mathscr{M}$.

It is obvious that $\liminf \mathscr{S}_k \subset \limsup \mathscr{S}_k$. If equality holds, the common value will be denoted by $\lim \mathscr{S}_k$ and we say that $\{\mathscr{S}_k\}$ *converges.*

A function $\varphi : L(\mathscr{V}) \to 2^{\mathscr{S}(\mathscr{V})}$ is called *lower semicontinuous* (l.s.c.) at A if

$$\varphi(A) \subset \liminf \varphi(A_k)$$

whenever $A_k \to A$. Similarly φ is called *upper semicontinuous* (u.s.c.) at A if $A_k \to A$ implies that

$$\varphi(A) \supset \limsup \varphi(A_k).$$

Since every member of $\limsup \varphi(A_k)$ belongs to $\liminf \varphi(A_{m_k})$ for a subsequence $\{A_{m_k}\}$, we have that φ is u.s.c. at A if and only if

$$\varphi(A) \supset \liminf \varphi(A_k)$$

for every sequence $\{A_k\}$ converging to A. Finally, the function φ is called *continuous* at A if it is both u.s.c. and l.s.c. at A.

Conway and Halmos [4] showed that **Lat** is upper semicontinuous and that it is continuous at A if and only if the minimal polynomial of A coincides with its characteristic polynomial (see also [1], [3] for related results). In this paper we determine points of continuity, upper semicontinuity and lower semicontinuity of the functions **Hyperlat** and **Red** as well as the related function $\mathbf{Lat}_\perp$ (see §6).

If φ is any of the functions **Lat**, **Hyperlat**, or **Red** and if S is an invertible linear operator from a vector space $\mathscr{W}$ onto $\mathscr{V}$, then $\varphi(S^{-1}AS) = S^{-1}\varphi(A)$. Consequently, continuity properties of φ are invariant under similarity. This fact will be exploited repeatedly, by replacing $\mathscr{V}$ by C^n and A by its Jordan matrix representation.

2. PRELIMINARIES

A vector v is called a *cyclic vector* for the operator A if $\{v, Av, \cdots, A^{n-1}v\}$ is a basis for $\mathscr{V}$. If A has a cyclic vector we say that A is a *cyclic* operator. Cyclic operators are also called *non–derogatory*. Each of the following is known to be equivalent to A being cyclic (see [10, Chapter 7]): (i) The characteristic polynomial of A coincides with its minimal polynomial. (ii) Each operator on $\mathscr{V}$ commuting with A is a polynomial in A. (iii) For each eigenvalue of A, the corresponding eigenspace has dimension one. (iv) For each eigenvalue of A, there corresponds only one Jordan cell in the Jordan canonical form.

The first assertion of the following lemma is proved in [4]. Our proof is different.

Lemma 1. *The set of cyclic operators is an open subset of* $L(\mathscr{V})$. *Indeed, if* v *is a cyclic vector of an operator* A, *then* v *is a cyclic vector for all operators that are sufficiently near* A.

Proof. Let $\mathscr{B}$ be a basis for $\mathscr{V}$ and denote the column of coefficients of a vector v with respect to $\mathscr{B}$ by [v]. If v is a cyclic vector for A, then $v, Av, \cdots, A^{n-1}v$ are linearly independent and so

$$\det\Big[[v],[Av],\cdots,[A^{n-1}v]\Big] \neq 0.$$

By continuity of the determinant, there exist $\epsilon > 0$ such that if $\|B - A\| < \epsilon$, then

$$\det\Big[[v],[Bv],\cdots,[B^{n-1}v]\Big] \neq 0$$

and so v is a cyclic vector for B. ■

We now introduce some notation and terminology. The standard basis of $\mathbb{C}^n$ will be denoted by $\{e_1,\cdots,e_n\}$. The linear space generated by a subset S of a vector space $\mathscr{V}$ is denoted by $\langle S\rangle$. The spectrum of an operator A, *i.e.* the set of eigenvalues of A, will be denoted by spec A. We will always write the Jordan form as an upper triangular matrix, *i.e.* by a Jordan cell we mean a matrix of the form

$$J = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & \lambda & 1 \\ & & & & & \lambda \end{bmatrix}.$$

3. A DECOMPOSITION THEOREM

We prove a theorem which allows us to reduce the subsequent proofs to the case where A has only one eigenvalue. First we start with an auxilliary lemma.

Lemma 2. *If* $\{\mathscr{M}_k\}$ *is a sequence of subspaces and* $\{T_k\}$ *is a sequence of operators such that* $\mathscr{M} = \lim \mathscr{M}_k$ *and* $T = \lim T_k$ *with* T *invertible, then*

$$T\mathscr{M} = \lim T_k \mathscr{M}_k .$$

Proof. By omitting a finite number of elements, if necessary, we may assume that T_k is invertible for every k and that $\|T_k\|$ and $\|T_k^{-1}\|$ are uniformly bounded by a positive number b. Using (1.2), it is immediate that

$$d(T\mathcal{M}, T\mathcal{M}_k) \leq b^2\, d(\mathcal{M}, \mathcal{M}_k),$$

$$d(T\mathcal{M}_k, T_k\mathcal{M}_k) \leq b\, \|T_k - T\|.$$

The result follows. ∎

We now state the decomposition theorem, but first we give some notation. By an equation such as $A = B \oplus C$, we mean that $\mathcal{V}$ is a direct sum of two subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, each invariant under A with $B = A|\mathcal{V}_1$ and $C = A|\mathcal{V}_2$.

Theorem 1. *Let* $A = B \oplus C$ *where* $\operatorname{spec} B \cap \operatorname{spec} C = \emptyset$ *and let* φ *be any of the functions* **Lat**, **Hyperlat** *or* **Red**. *Then* A *is a point of upper semicontinuity (respectively lower semicontinuity) of* φ *if and only if* B *and* C *are.*

Proof. Denote the indicated decomposition of $\mathcal{V}$ by $\mathcal{V}_1 \oplus \mathcal{V}_2$. It is well–known ([2],[7]) that the function φ splits, *i.e.* $\varphi(A) = \varphi(B) \oplus \varphi(C)$. (For results on splitting of invariant subspaces in infinite dimensions, see [5] and [6].) If A is a point of lower semicontinuity of φ and if one of B and C (say B) is not, then there exists a sequence $B_k \to B$ such that $\varphi(B) \not\subset \liminf \varphi(B_k)$. Let $A_k = B_k \oplus C$. By continuity of spec [11, p. 118], we have $\operatorname{spec} B_k \cap \operatorname{spec} C = \emptyset$ and so $\varphi(A_k) = \varphi(B_k) \oplus \varphi(C)$ for k large enough. Therefore

$$\liminf \varphi(A_k) = \liminf \varphi(B_k) \oplus \varphi(C)$$

and so $\varphi(A) \not\subset \liminf \varphi(A_k)$ contradicting the assumption of lower semicontinuity. A similar proof holds for upper semicontinuity. This proves the "only if" part of the theorem.

To prove the converse, let $\sigma_1 = \text{spec } B$ and $\sigma_2 = \text{spec } C$. If P is the projection on $\mathscr{V}_1$ along $\mathscr{V}_2$, then

$$P = \frac{1}{2\pi i} \int_\Gamma (\zeta I - A)^{-1}\, d\zeta$$

where Γ is a simple closed path with σ_1 contained in the interior of Γ and σ_2 in the exterior of Γ (see, *e.g.* [13, §5.7]).

If $A_k \to A$, then by continuity of spec, we have that for k large enough, $\text{spec } A_k = \sigma_1^{(k)} \cup \sigma_2^{(k)}$ such that $\sigma_1^{(k)} \subset \text{int } \Gamma$ and $\sigma_2^{(k)} \subset \text{ext } \Gamma$. Let

$$P_k = \frac{1}{2\pi i} \int_\Gamma (\zeta 1 - A_k)^{-1}\, d\zeta .$$

Therefore $P_k \to P$. Let $S_k = P_k P + (1-P_k)(1-P)$, then $S_k P = P_k P$ and $S_k(1-P) = (1-P_k)(1-P)$ and so

$$S_k(\mathscr{V}_1) \subset \text{range } P_k \quad \text{and} \quad S_k(\mathscr{V}_2) \subset \text{range}(I-P_k).$$

Furthermore $S_k \to I$, so for k large enough, we have that S_k are invertible, thus

$$S_k(\mathscr{V}_1) = \text{range } P_k\,, \qquad S_k(\mathscr{V}_2) = \text{range } (I-P_k).$$

Therefore $S_k^{-1} A_k S_k$ leaves $\mathscr{V}_1$ and $\mathscr{V}_2$ invariant and $S_k^{-1} A_k S_k \to A$. So

$$S_k^{-1} A_k S_k = B_k \oplus C_k$$

with $B_k \to B$ and $C_k \to C$.

Now assume that φ is l.s.c. at B and C, and let $\mathscr{L} \in \varphi(A)$. So $\mathscr{L} = \mathscr{M} \oplus \mathscr{N}$ with $\mathscr{M} \in \varphi(B)$ and $\mathscr{N} \in \varphi(C)$. It follows that $\mathscr{M} = \lim \mathscr{M}_k$ and $\mathscr{N} = \lim \mathscr{N}_k$ where $\mathscr{M}_k \in \varphi(B_k)$ and $\mathscr{N}_k \in \varphi(C_k)$. Thus $\mathscr{L}_k := \mathscr{M}_k \oplus \mathscr{N}_k \in \varphi(S_k^{-1} A_k S_k)$ and $\mathscr{L}_k \to \mathscr{L}$ so $S_k \mathscr{L}_k \in \varphi(A_k)$ and $S_k \mathscr{L}_k \to \mathscr{L}$ by Lemma 2. Therefore φ is l.s.c. at A.

Next assume that φ is u.s.c. at B and C. Let $\mathscr{L}_k \in \varphi(A_k)$, and let $\mathscr{L}_k \to \mathscr{L}$. To prove that φ is u.s.c. at A, we must show that $\mathscr{L} \in \varphi(A)$. Now $S_k^{-1} \mathscr{L}_k \in \varphi(S_k^{-1} A_k S_k)$ and so

$$S_k^{-1} \mathscr{L}_k = \mathscr{M}_k \oplus \mathscr{N}_k$$

with $\mathscr{M}_k \in \varphi(B_k)$ and $\mathscr{N}_k \in \varphi(C_k)$. By Lemma 1, we have that $S_k^{-1} \mathscr{L}_k \to \mathscr{L}$. It follows that $\lim \mathscr{M}_k$ and $\lim \mathscr{N}_k$ also exist. (This is easy if $\mathscr{V}_1$ is orthogonal to $\mathscr{V}_2$. The general case can be reduced to the case of orthogonal decomposition by a similarity transformation.) Denoting $\lim \mathscr{M}_k$ and $\lim \mathscr{N}_k$ by $\mathscr{M}$ and $\mathscr{N}$ respectively, we have $\mathscr{M} \in \liminf \varphi(B_k) \subset \varphi(B)$ and similarly $\mathscr{N} \in \varphi(C)$. Thus $\mathscr{L} = \mathscr{M} \oplus \mathscr{N} \in \varphi(A)$. ■

4. HYPERLAT

A detailed description of **Hyperlat** A is given in [7] but we will make no use of it in this paper. We start by giving an example showing that, unlike **Lat**, the function **Hyperlat** is not always upper semicontinuous.

EXAMPLE. Suppose $\mathscr{V} = \mathscr{W}_1 \oplus \mathscr{W}_2$, $A_k = \frac{1}{k} I \oplus 0$ and $A = 0$. Then $A_k \to A$, and

$$\textbf{Hyperlat}\ A_k = \left\{\{0\}, \mathscr{W}_1, \mathscr{W}_2, \mathscr{V}\right\}$$

$$\textbf{Hyperlat}\ 0 = \left\{\{0\}, \mathscr{V}\right\}.$$

Therefore **Hyperlat** is not u.s.c. at 0.

The next example shows that **Hyperlat** is not l.s.c. if $n \geq 3$. (It is l.s.c. for $n = 2$; this follows from Theorem 2 below.)

Example. First let $\mathcal{V} = \mathbf{C}^3$ and

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{k} \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that

$$\textbf{Hyperlat } A_k = \Big\{ \{0\}, \langle e_1 \rangle, \langle e_1, e_2 \rangle, \mathcal{V} \Big\}.$$

Also $\langle e_1, e_3 \rangle = \text{null}(A)$ and so $\langle e_1, e_3 \rangle \in$ **Hyperlat** A. Thus **Hyperlat** is not l.s.c. at A. A similar example can be constructed on every finite dimensional space with dimension greater than 3.

For an eigenvalue λ of A recall that the *spectral subspace* (or the root space) corresponding to λ is defined by $\mathcal{M}(\lambda) := \text{null}(A - \lambda I)^n$. We call the decomposition of $\mathcal{V}$ into a direct sum of spectral subspaces and the corresponding decomposition of A the *spectral decomposition* of A. (This is also called the primary decomposition.)

Theorem 2. *The function* **Hyperlat** *is:*

(i) *upper semicontinuous at* A *if and only if* A *is cyclic;*

(ii) *lower semicontinuous at* A *if and only if the restriction of* A *to each spectral subspace (corresponding to an eigenvalue) is either cyclic or is a scalar multiple of the identity;*

(iii) *continuous at* A *if and only if* A *is cyclic.*

Proof. In view of Theorem 1 and the spectral decomposition, we need only prove the theorem for an operator A having only one eigenvalue. Furthermore, by translation, we may assume that the eigenvalue is 0, *i.e.* A is nilpotent.

We start by proving the "if" parts of all three assertions of the theorem. First, we observe that every scalar multiple of I is a point of lower semicontinuity of **Hyperlat** since **Hyperlat**(λI) is the trivial lattice $\left\{\{0\}, \mathscr{V}\right\}$.

Next, we consider a cyclic operator A. We will show that **Hyperlat** is both u.s.c. and l.s.c. at A. Toward that end let $A_k \to A$. By Lemma 1, we may assume that A_k is cyclic for every k. Since the commutant of a cyclic operator equals the algebra generated by it, we have

$$\textbf{Hyperlat}\ A_k = \textbf{Lat}\ A_k \quad \text{and} \quad \textbf{Hyperlat}\ A = \textbf{Lat}\ A.$$

Thus, the upper semicontinuity of **Hyperlat** at A follows from the upper semicontinuity of **Lat** [3]. To prove lower semicontinuity, let v be a cyclic vector of A and assume that A is nilpotent. By Lemma 1, the vector v is also a cyclic vector for A_k for k large enough. If $\mathscr{M} \in$ **Hyperlat** A, then

$$\mathscr{M} = \langle A^{\ell}v, A^{\ell+1}v, \cdots, A^{n-1}v\rangle$$

for some $\ell \in \{0,1,\cdots,n-1\}$. Let S_k be the operator determined by

$$S_k A^j v = A_k^j v, \qquad j = 0,1,\cdots,n-1.$$

Then $S_k \to I$ and so $S_k \mathcal{M} \to \mathcal{M}$ by Lemma 2. But $S_k \mathcal{M} \in$ **Lat** $A_k =$ **Hyperlat** A_k, and so $\mathcal{M} \in \liminf$ **Hyperlat** A_k. This proves the lower semicontinuity.

We now prove the "only if" parts. Assume that A is nilpotent and not cyclic. We identify A with its Jordan matrix representation,

$$A = J_1 \oplus \cdots \oplus J_m, \qquad m \geq 2,$$

where each J_j is a nilpotent Jordan cell. Denote the corresponding decomposition of $\mathcal{V}$ by

$$\mathcal{V} = \mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_m .$$

Let A_k be the matrix obtained from A by replacing each zero on the superdiagonal by $\frac{1}{k}$. Thus $A_k \to A$. Also A_k is cyclic for every k and so **Hyperlat** $A_k =$ **Lat** $A_k = \mathcal{C}$ where $\mathcal{C}$ is the chain

$$\mathcal{C} = \left\{ \{0\}, \langle e_1 \rangle, \langle e_1,e_2 \rangle, \cdots, C^n \right\} .$$

But $\text{null}(A) \in$ **Hyperlat** A and $\text{null}(A) \notin \mathcal{C}$ unless $A = 0$. This proves that **Hyperlat** is not l.s.c. at a nilpotent A unless A is cyclic or $A = 0$.

To show that **Hyperlat** is not u.s.c. at A, it suffices to show that $\mathcal{W}_1 \notin$ **Hyperlat** A. This follows from the fact that the matrix

$$\begin{bmatrix} 0 & \cdots & 0 \\ B & & \\ \vdots & & \\ 0 & & 0 \end{bmatrix}$$

where

$$B = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & & \\ 0 & & & 0 \end{bmatrix}$$

commutes with A. ■

Remark. The proof above contains a proof (different from that of [4]) of the fact that **Lat** is l.s.c. at every cyclic operator.

5. THE FUNCTION RED

We start by showing that **Red**(A) is not always a lattice.

Example. Let $\mathscr{V} = \mathbb{C}^3$ and

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $\mathscr{M}_1 = \langle e_1 \rangle$ and $\mathscr{M}_2 = \langle e_1 + e_2 \rangle$, then $\mathscr{M}_1$ and $\mathscr{M}_2 \in$ **Red** A since both are invariant and have $\langle e_2, e_3 \rangle$ as an invariant complement. But $\mathscr{M}_1 + \mathscr{M}_2 \notin$ **Red** A since an invariant complement, if it existed, must be spanned by an eigenvector, and every eigenvector belongs to $\mathscr{M}_1 + \mathscr{M}_2$.

Next we show that **Red** is neither l.s.c. nor u.s.c. everywhere. In each example $\mathscr{V} = \mathbb{C}^2$.

Example. Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $A_k = \begin{bmatrix} 0 & 1/k \\ 0 & 0 \end{bmatrix}$. Then **Red** $A_k = \{\{0\}, \mathscr{V}\}$, but **Red** A is the set of all subspaces of $\mathscr{V}$. So **Red** A $\not\subset$ lim inf **Red** A_k.

Example. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_k = \begin{bmatrix} 1/k & 1 \\ 0 & -1/k \end{bmatrix}$. Then **Red** $A = \left\{\{0\}, \mathscr{V}\right\}$ and $\langle e_1 \rangle \in$ **Red**(A_k) for all k. So **Red** $A \not\supseteq \limsup$ **Red** A_k.

Theorem 3. *The function* **Red** *is*

(i) *upper semicontinuous at* A *if and only if* A *is diagonalizable*;

(ii) *lower semicontinuous at* A *if and only if* A *is cyclic*;

(iii) *continuous at* A *if and only if* A *has* n *distinct eigenvalues.*

Proof. Again by Theorem 1 we may assume that A is nilpotent.

(i) If A is diagonalizable and nilpotent, then $A = 0$ and **Red** $A = \mathscr{S}(\mathscr{V})$, the set of all subspaces of $\mathscr{V}$. Thus **Red** is u.s.c. at A.

If A is not diagonalizable, denote the matrix Jordan representation of A by $A = J_1 \oplus \cdots \oplus J_\ell$. Let D_k be a sequence of $n \times n$ diagonal matrices, each having distinct diagonal elements with $D_k \to 0$ and let $A_k = A + D_k$. Since A_k has n distinct eigenvalues, we have **Red** $A_k =$ **Lat** A_k and each consists of all spans of sets of eigenvectors. In particular if $\mathscr{N} = \text{null } A$, then $\mathscr{N} \in$ **Red** A_k for every k. To show that **Red** is not u.s.c. at A, it suffices to show that $\mathscr{N} \notin$ **Red** A. However if $\{\mathscr{N}, \mathscr{M}\}$ reduce A, then $A = 0 \oplus B$ on $\mathscr{N} \oplus \mathscr{M}$ and $\text{rank } B = \text{rank } A = \dim \mathscr{M}$. But B is nilpotent, so $\mathscr{M} = 0$ and $A = 0$, a contradiction.

(ii) If A is cyclic and nilpotent, then **Red** A is the trivial lattice and so **Red** is l.s.c. at A. Conversely, if A is nilpotent and not cyclic, then the Jordan representation of A is

$$A = J_1 \oplus J_2 \oplus \cdots \oplus J_m, \qquad m \geq 2.$$

Denote the corresponding decomposition of $\mathscr{V}$ by

$$\mathscr{V} = \mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_m .$$

Let A_k be the matrix obtained from A by replacing every 0 on the superdiagonal by $\frac{1}{k}$. Then $A_k \to A$, A_k is nilpotent cyclic, and so **Red** $A_k = \left\{\{0\}, \mathscr{V}\right\}$. But $\mathscr{W}_1 \in$ **Red** A and $\mathscr{W}_1 \notin \liminf$ **Red** A_k. Therefore **Red** is not l.s.c. at A.

Finally, part (iii) follows from (i) and (ii). ∎

6. ORTHOGONALLY REDUCING SUBSPACES

Let $\mathscr{V}$ be an inner product space, $A \in L(\mathscr{V})$. A subspace $\mathscr{W}$ is called *orthogonally reducing* for A if $\mathscr{W}$ and $\mathscr{W}^\perp \in$ **Lat** A. The set of all orthogonally reducing subspaces of A is a Boolean algebra which we will denote by $\mathbf{Lat}_\perp(A)$.

The set of all orthogonally irreducible operators is dense (see [9] or [12]). Therefore if $\mathbf{Lat}_\perp$ is l.s.c. at A, then A must be orthogonally irreducible. The converse is clearly true.

On the other hand, $\mathbf{Lat}_\perp$ is u.s.c. everywhere. To prove this, assume that $A_k \to A$ and let $\mathscr{M}_k$ be a sequence of subspaces in $\mathbf{Lat}_\perp(A_k)$ converging to $\mathscr{M}$. Denote by P_k (respectively, P) the orthogonal projection on $\mathscr{M}_k$ (respectively, $\mathscr{M}$). Then $A_kP_k = P_kA_k$. Upon taking limits, we obtain $AP = PA$, *i.e.* $\mathscr{M} \in \mathbf{Lat}_\perp(A)$.

We summarize these observations in the following theorem.

THEOREM 4. *The function* $\mathbf{Lat}_\perp$ *is upper semicontinuous at every point and is continuous at* A *if and only if* A *is orthogonally irreducible.*

REMARK. Theorem 4 is valid for the function $\mathrm{Lat}_\perp$ on $B(\mathscr{H})$, the space of all bounded operators on an infinite–dimensional Hilbert space $\mathscr{H}$. The proof is the same as the one above.

ACKNOWLEDGEMENT

This work was done while the first author was visiting the University of Victoria. He would like to express his thanks to Bob Miers and John Phillips for their hospitality.

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