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Teruo Yaguchi and H.M. Srivastava

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Abstract

The object of the present paper is to derive several properties of the subclasses $\mathscr{P}_b^n(\alpha)$, $\mathscr{S}_b^{\star n}(\alpha)$, and $\mathscr{C}_b^n(\alpha)$ of analytic functions with real part greater than α. The main results of this paper provide interesting generalizations of the corresponding work done recently by S.K. Lee, S. Owa, and H.M. Srivastava [2], and by C.P. McCarty ([3], [4]).

1. Introduction, Notations, and Preliminaries

Let Ω denote the class of functions $\varphi(z)$ which are analytic in the open unit disk

$$\mathscr{U} = \{z : |z| < 1\}$$

and satisfy the inequality:

$$|\varphi(z)| \leqq 1 \quad (z \in \mathcal{U}).$$

Also let $\mathcal{P}^n(\alpha)$ denote the class of functions of the form:

(1.1) $$f(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N} = \{1,2,3,\cdots\}),$$

which are analytic in the disk $\mathcal{U}$ and satisfy the inequality:

(1.2) $$\mathrm{Re}\{f(z)\} > \alpha \quad (z \in \mathcal{U})$$

for some α $(0 \leqq \alpha < 1)$. A function $f(z) \in \mathcal{P}^n(\alpha)$ is said to be in the class $\mathcal{P}^n_b(\alpha)$ if it also satisfies the condition:

(1.3) $$a_n = 2b(1-\alpha) = \frac{f^{(n)}(0)}{n!} \quad (n \in \mathbb{N};\ 0 \leqq \alpha < 1)$$

for some b $(0 \leqq b \leqq 1)$.

The definition of the subclass $\mathcal{P}^n_b(\alpha)$ of the class $\mathcal{P}^n(\alpha)$ follows Carathéodory's lemma (*cf.* [1, p. 41]) which states that the absolute value of any coefficient of a function in the class

$$\mathcal{P}(0) \equiv \mathcal{P}^1(0)$$

is not greater than 2.

Suppose that $0 \leqq \alpha < 1$ and $0 \leqq b \leqq 1$, and that the functions

(1.4) $$F(z) = z + \frac{2b(1-\alpha)}{n} z^{n+1} + a_{n+2} z^{n+2} + \cdots$$

and

$$G(z) = z + \frac{2b(1-\alpha)}{n(n+1)} z^{n+1} + a_{n+2} z^{n+2} + \cdots \tag{1.5}$$

are analytic in the disk $\mathcal{U}$. Then $F(z)$ defined by (1.4) is said to be in the class $\mathcal{S}_b^{\star n}(\alpha)$ if and only if

$$\frac{z\, F'(z)}{F(z)} \in \mathcal{P}_b^n(\alpha) \qquad (z \in \mathcal{U}\,;\ 0 \leqq \alpha < 1;\ 0 \leqq b \leqq 1). \tag{1.6}$$

Furthermore, let $\mathcal{C}_b^n(\alpha)$ denote the class of functions $G(z)$ defined by (1.5) which satisfy the *additional* condition:

$$1 + \frac{z\, G''(z)}{G'(z)} \in \mathcal{P}_b^n(\alpha) \qquad (z \in \mathcal{U}). \tag{1.7}$$

The classes

$$\mathcal{P}_b(\alpha) \equiv \mathcal{P}_b^1(\alpha) \text{ and } \mathcal{S}_b^{\star}(\alpha) \equiv \mathcal{S}_b^{\star 1}(\alpha) \qquad (0 \leqq \alpha < 1;\ 0 \leqq b \leqq 1) \tag{1.8}$$

were introduced and studied systematically by McCarty ([3], [4]). On the other hand, the class

$$\mathcal{C}_b(0) \equiv \mathcal{C}_b^1(0) \qquad (0 \leqq b \leqq 1) \tag{1.9}$$

was introduced by Pashkoulera [6] who also studied the class $\mathcal{P}_b(0)$. More recently, Lee, Owa, and Srivastava [2] gave several interesting properties of the classes $\mathcal{P}_b(\alpha)$ and [*cf*. Equation (1.9)]

(1.10) $$\mathscr{C}_b(\alpha) \equiv \mathscr{C}_b^1(\alpha) \qquad (0 \leqq \alpha < 1;\ 0 \leqq b \leqq 1).$$

In this paper we aim at presenting a systematic study of the general class $\mathscr{P}_b^n(\alpha)$ introduced here. Our main results (Theorems 3, 4, and 5) and the other results (Theorems 1 and 2) provide interesting generalizations of the corresponding observations made earlier by Lee, Owa, and Srivastava [2], and by McCarty ([3], [4]).

In order to introduce the concept of subordination between analytic functions, which we shall require in our present investigation, let $f(z)$ and $g(z)$ be analytic in the open unit disk $\mathscr{U}$. Then the function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $h(z)$ analytic in $\mathscr{U}$, with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1,$$

such that

(1.11) $$f(z) = g(h(z)) \qquad (z \in \mathscr{U}).$$

We denote this subordination by

(1.12) $$f(z) \prec g(z).$$

In particular, if $g(z)$ is univalent in $\mathscr{U}$, the subordination (1.12) is equivalent to (*cf*. [1, p. 190])

(1.13) $$f(0) = g(0) \quad \text{and} \quad f(\mathscr{U}) = g(\mathscr{U}).$$

We now recall the following well-known results which will be needed in the proofs of our theorems.

Lemma A (Nehari [5, p. 167]). *If* $\varphi(z) \in \Omega$, *then*

$$|\varphi'(z)| \leqq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathcal{U}) \tag{1.14}$$

and

$$\frac{|z_0| - |z|}{1 - |z_0\ z|} \leqq |\varphi(z| \leqq \frac{|z_0| + |z|}{1 + |z_0\ z|} \quad (z \in \mathcal{U}), \tag{1.15}$$

provided that $z_0 = \varphi(0)$.

Lemma B (Lindelöf's principle [1, p. 191]). *If*

$$\varphi(z) \prec \Phi(z), \tag{1.16}$$

then the image of each disk

$$\mathbb{D}_r = \{z : |z| \leqq r < 1 \quad (0 \leqq r < 1)\} \tag{1.17}$$

under the function $\varphi(z)$ *is contained in the image of the same disk under the function* $\Phi(z)$, *that is,*

$$\varphi(\mathbb{D}_r) \subset \Phi(\mathbb{D}_r) \quad (0 \leqq r < 1). \tag{1.18}$$

2. General Inequalities for the Class $\mathcal{P}_b^n(\alpha)$

In this section we prove some general inequalities associated with the class $\mathcal{P}_b^n(\alpha)$. Theorem 1 below is a kind of distortion theorem for functions belonging to the class $\mathcal{P}_b^n(\alpha)$.

THEOREM 1. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{P}_b^n(\alpha)$, *then*

$$|f'(z)| \leqq (\mathrm{Re}\{f(z)\}-\alpha)\,\frac{2r^{n-1}\{nb + [n+1 + (n-1)b^2]r + nbr^2\}}{[1 + br + (b+r)r^n][1 + br - (b+r)r^n]} \tag{2.1}$$

$$(r = |z| < 1).$$

The result is sharp for the function $f_1(z)$ *defined by*

$$f_1(z) = \frac{1 + bz + (1-2\alpha)(b+z)z^n}{1 + bz - (b+z)z^n} \tag{2.2}$$

at

$$z = r_1 < 1 \qquad (r_1 = |z| < 1).$$

Proof. Let the function $g(z)$ be defined by

$$g(z) = \frac{f(z) - \alpha}{1 - \alpha} = 1 + 2b\,z^n + \cdots \qquad (n \in \mathbb{N}). \tag{2.3}$$

Then, clearly, $g(z) \in \mathcal{P}_b^n(0)$, that is,

$$\operatorname{Re}\{g(z)\} > 0 \quad (z \in \mathcal{U}) \quad \text{and} \quad g(0) = 1,$$

which shows that the function $g(z)$ is subordinate to the function

$$G(z) = \frac{1 + z}{1 - z} .$$

Since there is a function $w(z) \in \Omega$ such that

$$(2.4) \qquad g(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + 2b\, z^n + \cdots \qquad (n \in \mathbb{N}),$$

we have

$$w(z) = b\, z^n + \cdots \quad \text{and} \quad |w(z)| \leqq 1.$$

Therefore, there exists a function $\varphi(z) \in \Omega$ such that $w(z) = z^n\, \varphi(z)$, and we find from (2.4) that

$$(2.5) \qquad g(z) = \frac{1 + z^n\, \varphi(z)}{1 - z^n\, \varphi(z)} \quad (n \in \mathbb{N}).$$

Now differentiate both sides of (2.5) with respect to z and make use of (2.3). We thus find that

$$(2.6) \qquad |f'(z)| = \frac{2|z^n\varphi'(z) + nz^{n-1}\varphi(z)|}{|1 - z^n\varphi(z)|^2}$$

$$= \frac{2|z^n\varphi'(z) + nz^{n-1}\varphi(z)|}{1 - |z^n\varphi(z)|^2}\,(\mathrm{Re}\{f(z)\} - \alpha).$$

Applying the triangle inequality, and then the assertion (1.14) of Lemma A, to the right-hand side of (2.6), we obtain

$$(2.7)\quad |f'(z)| \leqq \frac{2(\mathrm{Re}\{f(z)\}-\alpha)\,|z|^{n-1}}{1 - |z|^2}\left[\frac{|z| + n(1-|z|^2)\,|\varphi(z)| - |z|\,|\varphi(z)|^2}{1 - |z|^{2n}|\varphi(z)|^2}\right].$$

We can show that the expression in the brackets on the right-hand side of (2.7) is monotonic (increasing) with respect to $|\varphi(z)|$ as follows. Let $r = |z| < 1$, $x = |\varphi(z)| \leqq 1$, and

$$(2.8)\qquad h(x) = \frac{r + n(1-r^2)x - rx^2}{1 - r^{2n}\,x^2}.$$

Then

$$(2.9)\qquad h'(x) = \frac{H(x)}{(1-r^{2n}\,x^2)^2},$$

where

$$(2.10)\qquad H(x) = n(1-r^2) - 2r(1-r^{2n})x + n(1-r^2)r^{2n}\,x^2.$$

Since $y = H(x)$ is a parabola opening upward (in the positive y-direction) and

$$H'(x_0) = 0 \quad \text{when} \quad x_0 = \frac{1 - r^{2n}}{n(1-r^2)\,r^{2n-1}} > 1,$$

we have only to show that $H(1) > 0$ in order to prove that

$$h'(x) > 0 \quad \text{for} \quad 0 \leqq x \leqq 1 \quad \text{and} \quad 0 \leqq r < 1,$$

that is, $h(x)$ is monotonic (increasing) with respect to $x \in [0,1]$ for each fixed $r \in [0,1)$. Indeed it is not difficult to show that $H(1) > 0$, that is, that

$$n\,r^{2n} + n > 2r(1+r^2+\cdots+r^{2(n-1)}) \qquad (0 \leqq r < 1), \tag{2.11}$$

by using the principle of mathematical induction and differentiation.

Finally, we apply the assertion (1.15) of Lemma A with

$$\varphi(z) = \frac{w(z)}{z^n} = b + \cdots \quad \text{and} \quad z_0 = b$$

to the right-hand side of (2.6), and we obtain

$$|f'(z)| \leqq \frac{2(\mathrm{Re}\{f(z)\}-a)\,r^{n-1}}{1 - r^2}\; h\left[\frac{b + r}{1 + br}\right] \tag{2.12}$$

$$= \frac{2(\mathrm{Re}\{f(z)\}-a)\,r^{n-1}\;\{nb + [n + 1 + (n-1)b^2]\,r + nbr^2\}}{[1+br+(b+r)r^n]\;[1+br-(b+r)r^n]},$$

which proves the theorem.

For $n=1$, Theorem 1 leads us at once to

COROLLARY 1 (McCarty [3, p. 213, Corollary 2]). *If* $f(z) \in \mathcal{P}_b(\alpha)$, *then*

$$|f'(z)| \leqq \frac{2(\mathrm{Re}\{f(z)\}-\alpha)}{1-r^2}\left\{\frac{b+2r+br^2}{1+2br+r^2}\right\} \quad (r=|z|<1). \tag{2.13}$$

The result is sharp for the function $f_2(z)$ *defined by*

$$f_2(z) = \frac{1+bz+(1-2\alpha)(b+z)z}{1-z^2} \tag{2.14}$$

at

$$z = r_2 < 1 \quad (r_2 = |z| < 1).$$

Next we state and prove

THEOREM 2. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{P}_b^n(\alpha)$, *then*

$$|f(z) - A_n| \leqq D_n \quad (r=|z|<1;\ n \in \mathbb{N}), \tag{2.15}$$

where

$$A_n = \frac{(1+br)^2 + (1-2\alpha)(b+r)^2\, r^{2n}}{[1+br+(b+r)r^n][1+br-(b+r)r^n]} \quad (r=|z|<1;\ n\in\mathbb{N}) \tag{2.16}$$

and

$$D_n = \frac{2(1-\alpha)(b+r)(1+br)\ r^n}{[1 + br + (b+r)r^n][1 + br - (b+r)r^n]} \quad (r = |z| < 1;\ n \in \mathbb{N}). \tag{2.17}$$

The result is sharp for the function $f_1(z)$ *defined by* (2.2) *at*

$$z = r_1 < 1 \quad (r_1 = |z| < 1).$$

Proof. Since $f(z) \in \mathscr{P}_b^n(\alpha)$, by using (2.3) and (2.4), we can find some functions

$$w(z) = bz^n + \cdots \in \Omega \quad (n \in \mathbb{N}) \text{ and } \varphi(z) \in \Omega$$

such that

$$w(z) = z^n \varphi(z) \quad (n \in \mathbb{N}) \text{ and } f(z) = \frac{1 + (1-2\alpha)w(z)}{1 - w(z)} \quad (0 \leqq \alpha < 1). \tag{2.18}$$

Let

$$\Phi(z) = \frac{z + b}{1 + bz} \quad (0 \leqq b \leqq 1) \text{ and } T(z) = \frac{1 + (1-2\alpha)z}{1 - z} \quad (0 \leqq \alpha < 1). \tag{2.19}$$

Suppose also that [*cf*. Equation (1.16)]

$$\varphi(z) \prec \Phi(z).$$

Then, by virtue of Lemma B, the image of each disk $\mathbb{D}_r$ defined by (2.4) under the function $w(z)$ is contained in the image of the same disk under the function $z^n \Phi(z)$. Furthermore, the image of each disk $\mathbb{D}_r$ under the function $f(z)$ is contained in the image of $\mathbb{D}_r$ under $T(z^n \Phi(z))$.

Finally, using the inequality

$$|\Phi(z)| \leqq \Phi(|z|),$$

we find that the image of each disk $\mathbb{D}_r$ under the function $f(z)$ is contained in the image

$$|\zeta - A_n| \leqq D_n \quad (n \in \mathbb{N})$$

of $\mathbb{D}_r$ under the function $T(|z|^{n-1}\,\Phi(|z|)z)$.

This evidently completes the proof of Theorem 2.

From Theorem 2 with $n = 1$ we can easily deduce

Corollary 2 (McCarty [4, p. 154, Lemma 1]). *If* $f(z) \in \mathcal{P}_b(\alpha)$, *then*

(2.20) $$|f(z) - A| \leqq D \qquad (r = |z| < 1),$$

where

(2.21) $$A = \frac{(1+br)^2 + (1-2\alpha)(b+r)^2\, r^2}{(1+2br+r^2)(1-r^2)} \quad (r = |z| < 1)$$

and

(2.22) $$D = \frac{2(1-\alpha)(b+r)(1+br)r}{(1+2br+r^2)(1-r^2)} \quad (r = |z| < 1).$$

The result is sharp for the function $f_2(z)$ *defined by* (2.14) *at*

$$z = r_2 < 1 \quad (r_2 = |z| < 1).$$

Next we apply Theorems 1 and 2 to prove our first main result contained in

Theorem 3. *If the function* $f(z)$ *defined by* (1.1) *is in the class* $\mathcal{P}_b^n(\alpha)$ *and* $\rho \geqq 0$, *then*

$$\operatorname{Re}\left\{f(z) + \frac{z\,f'(z)}{f(z)+\rho}\right\} \geqq (\operatorname{Re}\{f(z)\}-\alpha)(1-E_n) + \alpha \tag{2.23}$$

and

$$\operatorname{Re}\left\{f(z) + \frac{z\,f'(z)}{f(z)+\rho}\right\} \leqq (\operatorname{Re}\{f(z)\}-\alpha)(1+E_n) + \alpha, \tag{2.24}$$

where

$$E_n = \frac{2r^n\{nb + [n+1+(n-1)b^2]r + nbr^2\}}{[(1+\rho)(1+br) + (2\alpha-1+\rho)(b+r)r^n]\ [1 + br - (b+r)r^n]} \quad (n \in \mathbb{N}) \tag{2.25}$$

and $r = |z| < 1$.

The assertion (2.23) *with* $b = 0$ *and* $n = 2m$ $(m \in \mathbb{N})$ *is sharp for the function* $f_1(z)$ *defined by* (3.2) *at*

$$z = -r_1 > -1 \quad (r_1 = |z| < 1).$$

Proof. By virtue of Theorem 2, we have

$$(2.26)\qquad |f(z)+\rho| \geqq |A_n+\rho| - D_n$$

$$= \frac{(1+\rho)(1+br) + (2\alpha-1+\rho)(b+r)\ r^n}{1 + br + (b+r)r^n}\ .$$

Hence, with the aid of Theorem 1, we obtain

$$(2.27)\qquad \mathrm{Re}\left\{f(z) + \frac{z\ f'(z)}{f(z)\ +\ \rho}\right\} \geqq \mathrm{Re}\{f(z)\} - \left|\frac{z\ f'(z)}{f(z)\ +\ \rho}\right|$$

$$\geqq \mathrm{Re}\{f(z)\} - (\mathrm{Re}\{f(z)\}-\alpha)\ E_n$$

$$= (\mathrm{Re}\{f(z)\}-\alpha)(1-E_n) + \alpha,$$

which proves the assertion (2.23). Furthermore, we have

$$(2.28)\qquad \mathrm{Re}\left\{f(z) + \frac{z\ f'(z)}{f(z)\ +\ \rho}\right\} \leqq \mathrm{Re}\{f(z)\} + \left|\frac{z\ f'(z)}{f(z)\ +\ \rho}\right|$$

$$\leqq \mathrm{Re}\{f(z)\} + (\mathrm{Re}\{f(z)\}-\alpha)\ E_n$$

$$= (\mathrm{Re}\{f(z)\}-\alpha)(1+E_n) + \alpha,$$

which proves the assertion (2.24).

Setting $n = 1$ in Theorem 3, we readily have

Corollary 3 (Lee, Owa, and Srivastava [2, p. 132, Theorem 1]). *If* $f(z) \in \mathcal{P}_b(\alpha)$ *and* $\rho \geqq 0$, *then*

$$\operatorname{Re}\left\{f(z) + \frac{z\,f'(z)}{f(z) + \rho}\right\} \geqq (\operatorname{Re}\{f(z)\}-\alpha)(1-E) + \alpha \tag{2.29}$$

and

$$\operatorname{Re}\left\{f(z) + \frac{z\,f'(z)}{f(z) + \rho}\right\} \leqq (\operatorname{Re}\{f(z)\}-\alpha)(1+E) + \alpha, \tag{2.30}$$

where

$$E = \frac{2r(b+2r+br^2)}{(1-r^2)[1 + \rho + 2(\alpha+\rho)br + (2\alpha-1+\rho)r^2]}$$

and $r = |z| < 1$.

3. The General Classes $\mathcal{S}_b^{\star n}(\alpha)$ and $\mathcal{C}_b^{n}(\alpha)$

In this section we obtain the radius of starlikeness of order α and the radius of convexity of order α of certain functions associated appropriately with functions belonging to the classes $\mathcal{S}_b^{\star n}(\alpha)$ and $\mathcal{C}_b^{n}(\alpha)$.

We begin by proving

Theorem 4. *If the function* $F(z)$ *defined by* (1.4) *is in the class* $\mathcal{S}_b^{\star n}(\alpha)$, *then the function* $f(z)$ *defined by*

$$f(z) = \frac{1}{1 + \rho}\{\rho F(z) + zF'(z)\} \qquad (\rho \geqq 0) \tag{3.1}$$

is starlike of order α *in the disk* $|z| < R_n$, *where* R_n *is the smallest root in the closed interval* $[0,1]$ *of the equation*:

$$(3.2) \qquad [1 + br - (b+r)r^n]\ [(1+\rho)(1+br) + (2\alpha-1+\rho)(b+r)r^n]$$

$$- 2r^n\{nb + [n + 1 + (n-1)b^2]\,r + nbr^2\} = 0.$$

This result when $b = 0$ *and* $n = 2m$ $(m \in \mathbb{N})$ *is sharp for the function* $F_1(z)$ *defined by*

$$(3.3) \qquad F_1(z) = z(1-z^{n+1})^{-2(1-\alpha)/(n+1)}$$

at

$$z = -r_1 > -1 \quad (r_1 = |z| < 1).$$

Proof. Corresponding to the function $F(z) \in \mathcal{S}_b^{\star n}(\alpha)$, let the function $p(z) \in \mathcal{P}_b^n(\alpha)$ be defined by

$$(3.4) \qquad p(z) = \frac{z\ F'(z)}{F(z)}\ .$$

Then, by applying Theorem 3, we have

$$(3.5) \qquad \mathrm{Re}\left\{\frac{z\ f'(z)}{f(z)}\right\} = \mathrm{Re}\left\{p(z) + \frac{z\ p'(z)}{p(z)\ +\ \rho}\right\}$$

$$\geqq (\mathrm{Re}\{p(z)\}-\alpha)(1-E_n) + \alpha$$

$$> \alpha \qquad (|z| < R_n),$$

where R_n is the smallest root in $[0,1]$ of Equation (3.2), and E_n is defined by (2.25). This proves Theorem 4.

For $n = 1$, Theorem 4 immediately yields

Corollary 4 (Lee, Owa, and Srivastava [2, p. 134, Theorem 2]). *If the function* $F(z)$ *defined by* (1.4) *with* $n = 1$, *that is, by*

$$F(z) = z + 2b(1-\alpha)z^2 + a_3z^3 + \cdots \tag{3.6}$$

is in the class $\mathscr{S}^{\star}_b(\alpha)$, *then the function* $f(z)$ *defined by* (3.1) *is starlike of order* α *for* $|z| < R$, *where* R *is the smallest root in the closed interval* $[0,1]$ *of the equation:*

$$1 + \rho + 2(\alpha+\rho-1)br + 2(\alpha-3)r^2 - 2b(\alpha+\rho+1)r^3 - (2\alpha+\rho-1)r^4 = 0, \tag{3.7}$$

which is Equation (3.2) *with* $n = 1$.

The following result is obtained by setting $\rho = 0$ in Theorem 4.

Corollary 5. *If the function* $F(z)$ *defined by* (1.4) *is in the class* $\mathscr{S}^{\star n}_b(\alpha)$, *then the function* $F(z)$ *is convex of order* α *for* $|z| < r_n$, *where* r_n *is the smallest root in the closed interval* $[0,1]$ *of the equation:*

$$(3.8) \qquad [1 + br - (b+r)r^n]\,[1 + br + (2\alpha-1)(b+r)r^n]$$

$$- 2r^n\{nb + [n + 1 + (n-1)b^2]\,r + nbr^2\} = 0,$$

which is Equation (3.2) *with* $\rho = 0$.

This result with $b = 0$ *and* $n = 2m$ $(m \in \mathbb{N})$ *is sharp for the function* $F_1(z)$ *defined by* (3.3) *at*

$$z = -r_1 > -1 \quad (r_1 = |z| < 1).$$

Finally, we prove

Theorem 5. *If the function* $G(z)$ *defined by* (1.5) *is in the class* $\mathscr{C}_b^n(\alpha)$, *then function* $g(z)$ *defined by*

$$(3.9) \qquad g(z) = \frac{1}{\rho + 1}\{\rho G(z) + zG'(z)\} \qquad (\rho \geqq 0)$$

is convex of order α *for* $|z| < R_n$, *where* R_n *is given in Theorem* 4.

This result with $b = 0$ *and* $n = 2m$ $(m \in \mathbb{N})$ *is sharp for the function* $G_1(z)$ *defined by*

$$(3.10) \qquad G_1(z) = \int_0^z (1-\zeta^{n+1})^{-2(1-\alpha)/(n+1)}\, d\zeta$$

at

$$z = -r_2 > -1 \quad (r_2 = |z| < 1).$$

Proof. Let the function $p(z) \in \mathscr{P}_b^n(\alpha)$ be defined by

$$p(z) = 1 + \frac{z\ G''(z)}{G'(z)}\ . \tag{3.11}$$

Then, by applying Theorem 3, we have

$$\begin{aligned} \operatorname{Re}\left\{1 + \frac{z\ g''(z)}{g'(\mathrm{z})}\right\} &= \operatorname{Re}\left\{p(z) + \frac{z\ p(z)}{p(z) + \rho}\right\} \\ &\geqq (\operatorname{Re}\{p(z)\} - \alpha)(1 - E_n) + \alpha \\ &> \alpha \qquad (|z| < R_n), \end{aligned} \tag{3.12}$$

where R_n is given in Theorem 4. This proves Theorem 5.

Letting $n = 1$ in Theorem 5, we readily have

Corollary 6 (Lee, Owa, and Srivastava [2, p. 135, Theorem 3]). *If* $G(z) \in \mathscr{C}_b(\alpha)$, *then* $g(z)$ *defined by* (3.9) *is convex of order* α *for* $|z| < R$, *where* R *is given in Corollary* 4.

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TERUO YAGUCHI:
Department of Mathematics
College of Humanities and Sciences
Nihon University
Setagaya, Tokyo 156
Japan

H.M. SRIVASTAVA:
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 2Y2
Canada