A CLASS OF FINITE q-SERIES. III

by

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SUMMARY

Simple proofs based only upon some rather elementary results are presented for several interesting generalizations (involving series with essentially arbitrary terms) of a number of finite summation formulas for certain classes of hypergeometric functions of one and two variables. Applications of these general summation formulas to various multivariable hypergeometric functions, and their further generalizations (and unifications) and q-extensions, are also considered. The main results (2.3), (4.5), (6.1), (6.2) and (6.12) below, and their special cases including (for example) (2.1), (3.1), (3.5), (3.8), (4.1), (4.3), (5.1), (5.3) to (5.6), (6.14), (6.15) and (6.18), are believed to be new.

1. INTRODUCTION

In the usual notations, let

$$(1.1) \qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, \text{ if } n = 0, \\ \lambda(\lambda+1) \ \dots \ (\lambda+n-1), \ \forall n \in \{1,2,3,\dots\}, \end{cases}$$

and define the binomial (or combinatorial) coefficient $\binom{\lambda}{n}$, for a complex number λ, by

$$(1.2) \qquad \binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{\lambda(\lambda-1) \ \dots \ (\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} .$$

Also let $\{\lambda_n\}$, $\{\mu_n\}$ and $\{\nu_n\}$ be arbitrary complex sequences. In a recent paper [4] we showed, for an arbitrary non-negative integer N, that

(1.3) $$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \lambda_{\ell+m}\, \mu_\ell\, \nu_m\, (-n)_\ell\, (-N+n)_m \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= \lambda_N\, x^N \sum_{n=0}^{N} \binom{N}{n} \mu_{N-n}\, \nu_n \left(-\frac{y}{x}\right)^n,$$

and also gave a q-extension of (1.3). Subsequently, in terms of a bounded double sequence $\{\Omega(\ell,m)\}$, we proved the finite summation formula [5]

(1.4) $$\sum_{n=0}^{N} \binom{\lambda}{n} \binom{\mu}{N-n} \sum_{\ell=0}^{\infty} \sum_{m=0}^{N-n} \binom{N-n}{m} \binom{\mu}{m}^{-1} \Omega(\ell,m) \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= \binom{\lambda+\mu}{N} \sum_{\ell=0}^{\infty} \sum_{m=0}^{N} \binom{N}{m} \binom{\lambda+\mu}{m}^{-1} \Omega(\ell,m) \frac{x^\ell}{\ell!} \frac{y^m}{m!},$$

and also derived a q-extension of a multivariable generalization of (1.4).

Each of our earlier results (1.3) and (1.4) was motivated by, and provides an interesting unification (and generalization) of, a fairly large number of finite summation formulas for hypergeometric functions of one and two variables, which are scattered in the literature (see [4] and [5] for details). The object of the present sequel to our earlier papers [4] and [5] is first to derive a multivariable extension of a mild generalization of (1.3) and show how this extension can be applied to deduce finite summation formulas for various classes of hypergeometric functions of two, three and more variables. We then prove, using some rather elementary results, several similar generalizations (involving series with essentially arbitrary terms) of the finite hypergeometric summation formula of Manocha and Sharma (cf. [1], p. 475, Equation (31)):

(1.5) $$\sum_{n=0}^{N} \binom{N}{n} \frac{(\alpha)_{N-n}\, (\gamma)_n}{(\beta)_{N-n}\, (\delta)_n} \left(-\frac{y}{x}\right)^n$$

$$\cdot\ {}_2F_1\left[\begin{matrix}-n,\ \alpha+N-n;\\ \beta+N-n;\end{matrix}\ x\right]\ {}_2F_1\left[\begin{matrix}-N+n,\ \gamma+n;\\ \delta+n;\end{matrix}\ y\right]$$

$$= \frac{(\alpha)_N}{(\beta)_N}\ {}_3F_2\left[\begin{matrix}-N,\ \gamma,\ 1-\beta-N;\\ \delta,\ 1-\alpha-N;\end{matrix}\ \frac{y}{x}\right],$$

which was proven in [1] and, once again, in a recent paper by Qureshi and Pathan [3] using the same method based upon the fractional derivative operator $\mathcal{D}_z^{\mu}$ defined by

(1.6) $$\mathcal{D}_z^{\mu}\left\{z^{\lambda-1}\right\} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}\ z^{\lambda-\mu-1}$$

$$(\mu \neq \lambda;\ \lambda \neq 0, -1, -2, \ldots).$$

Finally, in Section 6 we present some further generalizations (and unifications) as well as q-extensions of our main results considered in Sections 2 and 4 below. Our multivariable summation formulas (2.3), (4.5), (6.1) and (6.2), the q-summation formula (6.12), and many of their numerous special cases considered in this paper are believed to be new.

2. GENERALIZATIONS OF THE SUMMATION FORMULA (1.3)

By closely examining our proof of the hypergeometric form of the finite summation formula (1.3), detailed in Section 2 of our earlier paper [4], we are led rather immediately to an obvious generalization of (1.3) in the form:

(2.1) $$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \lambda_{\ell+m}\ \Omega(\ell,m)\ (-n)_{\ell}\ (-N+n)_m \frac{x^{\ell}}{\ell!} \frac{y^m}{m!}$$

$$= \lambda_N \, x^N \sum_{n=0}^{N} \binom{N}{n} \, \Omega(N-n,\, n) \left(-\frac{y}{x}\right)^n,$$

which would evidently reduce to (1.3) in the special case when

$$\Omega(\ell,m) = \mu_\ell \, \nu_m, \qquad \ell,\, m = 0,\, 1,\, 2,\, \ldots \; . \tag{2.2}$$

More generally, for every bounded multiple sequence

$$\{\Lambda(k_1,\ldots,k_r;\ell)\}, \quad k_i,\, \ell = 0,\, 1,\, 2,\, \ldots; \quad i = 1,\, \ldots,\, r,$$

we shall show that

$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \Lambda(k_1,\ldots,k_r;\ell+m)\, \Omega(\ell,m) \tag{2.3}$$

$$\cdot\, (-n)_\ell \, (-N+n)_m \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= x^N \, \Phi_N(z_1,\ldots,z_r) \sum_{n=0}^{N} \binom{N}{n} \, \Omega(N-n,\, n) \left(-\frac{y}{x}\right)^n,$$

where, for convenience,

$$\Phi_N(z_1,\ldots,z_r) = \sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r;N) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}, \tag{2.4}$$

provided that the multiple series in (2.4) converges absolutely.

Proof. Let $\mathcal{S}$ denote the left-hand side of the summation formula (2.3). Then, within the r-dimensional region of convergence of the multiple series in (2.4), we find from (2.3), (1.1) and (1.2) that

$$\mathcal{S} = \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{n=0}^{N} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \Lambda(k_1,\ldots,k_r;\ell+m)\ \Omega(\ell,m)$$

$$\cdot \frac{(-1)^{\ell+m+n}\ N!}{(n-\ell)!\ (N-m-n)!} \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell,m\geqq 0} \Lambda(k_1,\ldots,k_r;\ell+m)\ \Omega(\ell,m) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}$$

$$\cdot \frac{(-x)^\ell}{\ell!} \frac{(-y)^m}{m!} \sum_{n=\ell}^{N-m} \frac{(-1)^n\ N!}{(n-\ell)!\ (N-m-n)!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell,m=0}^{\ell+m\leqq N} \binom{N}{\ell+m} (\ell+m)!\ \Lambda(k_1,\ldots,k_r;\ \ell+m)\ \Omega(\ell,m)$$

$$\cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^\ell}{\ell!} \frac{(-y)^m}{m!} \sum_{n=0}^{N-\ell-m} (-1)^n \binom{N-\ell-m}{n}.$$

Since

(2.5) $$\sum_{n=0}^{N} (-1)^n \binom{N}{n} = \delta_{N,0} \qquad N = 0,\ 1,\ 2,\ \ldots,$$

where $\delta_{m,n}$ is the familiar Kronecker delta, the innermost sum in the last expression for $\mathcal{S}$ is nil unless $\ell + m = N$ (in which case the sum is 1), and we have

$$\mathcal{S} = \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell+m=N} \binom{N}{\ell+m} (\ell+m)!\ \Lambda(k_1,\ldots,k_r;\ell+m)\ \Omega(\ell,m)$$

$$\cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^\ell}{\ell!} \frac{(-y)^m}{m!},$$

or, equivalently,

$$(2.6) \qquad \mathcal{S} = x^N \sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r;\, N) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}$$

$$\cdot \sum_{m=0}^{N} \binom{N}{m} \Omega(N-m,\, m) \left(-\frac{y}{x}\right)^m,$$

which, in view of the definition (2.4), immediately yields the right-hand side of the summation formula (2.3).

For $z_1 = \ldots = z_r = 0$, our general result (2.3) would reduce at once to the finite summation formula (2.1).

3. FINITE SUMMATION FORMULAS INVOLVING MULTIVARIABLE HYPERGEOMETRIC FUNCTIONS

By suitably specializing the multiple sequences

$\{\Omega(\ell,m)\}$ and $\{\Lambda(k_1,\ldots,k_r;\ell)\}$, k_i, ℓ, $m = 0, 1, 2, \ldots; i = 1, \ldots, r$,

the general result (2.3) can be applied to derive various finite summation formulas involving certain classes of hypergeometric functions of several variables, such as the (Srivastava-Daoust) generalized Lauricella function of $r + 2$ variables (cf. [6], p. 37, Equation (21) et seq.). In particular, if we take $r = 1$, and let $\{\Delta(\ell,m)\}$ be a bounded double sequence, then this special case of (2.3) may be stated in the form:

$$(3.1) \qquad \sum_{n=0}^{N} (-1)^n \binom{N}{n} \sum_{\kappa=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \Delta(\kappa,\, \ell+m)\, \Omega(\ell,m)$$

$$\cdot (-n)_\ell \, (-N+n)_m \frac{z^\kappa}{\kappa!} \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= x^N \Psi_N(z) \sum_{n=0}^{N} \binom{N}{n} \Omega(N-n,\ n) \left(-\frac{y}{x}\right)^n,$$

where, for convenience,

(3.2) $$\Psi_N(z) = \sum_{\kappa=0}^{\infty} \Delta(\kappa,N) \frac{z^\kappa}{\kappa!},$$

provided that the series in (3.2) is absolutely convergent.

In (3.1) we now set

(3.3) $$\Delta(\ell,m) = \frac{\prod_{j=1}^{p} (a_j)_{\ell+m} \prod_{j=1}^{r} (c_j)_\ell \prod_{j=1}^{u} (g_j)_m}{\prod_{j=1}^{k} (b_j)_{\ell+m} \prod_{j=1}^{s} (d_j)_\ell \prod_{j=1}^{v} (h_j)_m}$$

and

(3.4) $$\Omega(\ell,m) = \frac{\prod_{j=1}^{\lambda} (\alpha_j)_{\ell+m} \prod_{j=1}^{\rho} (\gamma_j)_\ell \prod_{j=1}^{\tau} (\xi_j)_m}{\prod_{j=1}^{\mu} (\beta_j)_{\ell+m} \prod_{j=1}^{\sigma} (\delta_j)_\ell \prod_{j=1}^{\omega} (\eta_j)_m},$$

and interpret the inner triple series occurring on the left-hand side of (3.1) as Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ (cf., e.g., [6], p. 44, Equation (14) et seq.), and we shall obtain the finite summation formula:

(3.5) $$\sum_{n=0}^{N} (-1)^n \binom{N}{n} F^{(3)} \left[\begin{matrix} (a_p)::\ (g_u),\ (\alpha_\lambda);\ \text{—};\ \text{—}: \\ \\ (b_k)::\ (h_v),\ (\beta_\mu);\ \text{—};\ \text{—}: \end{matrix} \right.$$

$$\left. \begin{matrix} -n,\ (\gamma_\rho);\ -N+n,\ (\xi_\tau);\ (c_r); \\ \\ (\delta_\sigma);\qquad (\eta_\omega);\ (d_s); \end{matrix} \right| x,y,z \right]$$

$$= \frac{\prod_{j=1}^{p} (a_j)_N \prod_{j=1}^{u} (g_j)_N \prod_{j=1}^{\lambda} (\alpha_j)_N \prod_{j=1}^{\rho} (\gamma_j)_N}{\prod_{j=1}^{k} (b_j)_N \prod_{j=1}^{v} (h_j)_N \prod_{j=1}^{\mu} (\beta_j)_N \prod_{j=1}^{\sigma} (\delta_j)_N}$$

$$\cdot\ x^N\ {}_{p+r}F_{k+s}\left[\begin{matrix} (a_p)+N,\ (c_r); \\ \\ (b_k)+N,\ (d_s); \end{matrix} \ z \right]$$

$$\cdot\ {}_{1+\tau+\sigma}F_{\omega+\rho}\left[\begin{matrix} -N,\ (\xi_\tau),\ 1-(\delta_\sigma)-N; \\ \\ (\eta_\omega),\ 1-(\gamma_\rho)-N; \end{matrix} \ (-1)^{\rho-\sigma} \frac{y}{x} \right],$$

where (a_p) abbreviates the array of p parameters $a_1, \ldots, a_p$, with similar interpretations for (b_k), (c_r), (d_s), (α_λ), (β_μ), *et cetera*, an empty product being understood as 1.

An interesting special case of (3.5) occurs when we set

(3.6) $$\rho = \sigma - 1 = \tau = \omega - 1 = 0,\quad \delta_1 = \alpha + 1,\quad \eta_1 = \beta + 1,$$

and identify the resulting hypergeometric function ${}_2F_1$ as a Jacobi polynomial $P_N^{(\alpha,\beta)}(\zeta)$ defined by

(3.7) $$P_N^{(\alpha,\beta)}(\zeta) = \sum_{n=0}^{N} \binom{N+\alpha}{N-n} \binom{N+\beta}{n} \left(\frac{\zeta-1}{2}\right)^n \left(\frac{\zeta+1}{2}\right)^{N-n}$$

$$= \binom{N+\beta}{N} \left(\frac{\zeta-1}{2}\right)^N {}_2F_1\left[\begin{matrix} -N, -\alpha-N; \\ \\ \beta+1; \end{matrix} \quad \frac{\zeta+1}{\zeta-1}\right].$$

We thus obtain the summation formula:

(3.8)
$$\sum_{n=0}^{N} (-1)^n \binom{N}{n} F^{(3)}\left[\begin{matrix} (a_p) :: (g_u), (\alpha_\lambda); —; —: \\ \\ (b_k) :: (h_v), (\beta_\mu); —; —: \end{matrix}\right.$$

$$\left.\begin{matrix} -n; -N+n; (c_r); \\ \\ \alpha+1; \beta+1; (d_s); \end{matrix} \quad x,y,z \right]$$

$$= \frac{\prod_{j=1}^{p} (a_j)_N \prod_{j=1}^{u} (g_j)_N \prod_{j=1}^{\lambda} (\alpha_j)_N}{\prod_{j=1}^{k} (b_j)_N \prod_{j=1}^{v} (h_j)_N \prod_{j=1}^{\mu} (\beta_j)_N}$$

$$\cdot \frac{(-1)^N (x+y)^N}{(\alpha+1)_N (\beta+1)_N} {}_{p+r}F_{k+s}\left[\begin{matrix} (a_p)+N, (c_r); \\ \\ (b_k)+N, (d_s); \end{matrix} \quad z\right] P_N^{(\alpha,\beta)}\left(\frac{y-x}{y+x}\right).$$

A very specialized form of our summation formula (3.8) when

(3.9) $\qquad p = k = u = r = 1 \quad \text{and} \quad v = s = \lambda = \mu = 0$

happens to be the main result in an earlier paper by Pathan ([2], p. 59, Equation (2.3)) who indeed used a markedly different method to prove this special case of (3.8).

We remark in passing that the special cases of both (3.5) and (3.8), when $z = 0$, were derived in our paper [4] as obvious consequences of the general result (1.3).

4. GENERALIZATIONS OF THE SUMMATION FORMULA (1.5)

Let $\{\lambda_n\}$ and $\{\mu_n\}$ be arbitrary complex sequences. Then a closer look at (1.5) would suggest the existence of a straightforward generalization of (1.5) in the form:

$$\sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \lambda_{N-n+\ell}\, \mu_{m+n}\, (-n)_\ell\, (-N+n)_m \frac{x^\ell}{\ell!} \frac{y^m}{m!} = \sum_{n=0}^{N} \binom{N}{n} \lambda_{N-n}\, \mu_n \left(-\frac{y}{x}\right)^n, \tag{4.1}$$

which evidently reduces to (1.5) in the special case when

$$\lambda_n = \frac{(\alpha)_n}{(\beta)_n} \quad \text{and} \quad \mu_n = \frac{(\gamma)_n}{(\delta)_n}, \qquad n = 0,\ 1,\ 2,\ \ldots\ . \tag{4.2}$$

In terms of a triple sequence $\{A(\ell,m,n)\}$, the finite summation formula (4.1) admits itself of a further generalization given by

$$\sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} A(\ell+m,\ N-n+\ell,\ m+n) \cdot (-n)_\ell\, (-N+n)_m \frac{x^\ell}{\ell!} \frac{y^m}{m!} = \sum_{n=0}^{N} \binom{N}{n} A(0,\ N-n,\ n) \left(-\frac{y}{x}\right)^n, \tag{4.3}$$

which corresponds to (4.1) when we set

$$A(\ell,m,n) = \lambda_m\, \mu_n, \quad \ell,\ m,\ n = 0,\ 1,\ 2,\ \ldots\ . \tag{4.4}$$

More generally, for every bounded multiple sequence

$$\{B(k_1,\ldots,k_r;\ell,m,n)\},\ k_i,\ \ell,\ m,\ n = 0,\ 1,\ 2,\ \ldots;\ i = 1,\ \ldots,\ r,$$

we now show that

$$(4.5)\qquad \sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{\ell=0}^{n}\sum_{m=0}^{N-n} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n) \cdot (-n)_\ell\,(-N+n)_m \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^\ell}{\ell!}\frac{y^m}{m!}$$

$$= \sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} B(k_1,\ldots,k_r;0,N-n,n)\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!},$$

provided that the multiple series in (4.5) converges absolutely.

Proof. Denoting, for convenience, the left-hand side of the summation formula (4.5) by $\mathcal{S}^*$, we find from (4.5), (1.1) and (1.2) that

$$\mathcal{S}^* = \sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{n=0}^{N}\sum_{\ell=0}^{n}\sum_{m=0}^{N-n} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n) \cdot \frac{(-1)^{\ell+m+n}\,N!}{(n-\ell)!\,(N-m-n)!}\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^{-n+\ell}}{\ell!}\frac{y^{m+n}}{m!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{\ell,m\geqq 0}\sum_{n=\ell}^{N-m} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n) \cdot \frac{(-1)^{\ell+m+n}\,N!}{(n-\ell)!\,(N-m-n)!}\frac{z_1^{k_1}}{k_1!}\cdots\frac{z_r^{k_r}}{k_r!}\frac{x^{-n+\ell}}{\ell!}\frac{y^{m+n}}{m!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty}\sum_{\ell,m=0}^{\ell+m\leqq N}\binom{N}{\ell+m}(\ell+m)!\,\frac{y^\ell}{\ell!}\frac{(-y)^m}{m!}$$

$$\cdot \sum_{n=0}^{N-\ell-m} \binom{N-\ell-m}{n} \left(-\frac{y}{x}\right)^n B(k_1,\ldots,k_r;\ell+m,N-n,\ell+m+n)$$

$$\cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},$$

it being assumed that the multiple series in (4.5) converges absolutely.

Now apply the elementary identity

$$\sum_{\ell,m=0}^{\infty} f(\ell+m) \frac{x^\ell}{\ell!} \frac{y^m}{m!} = \sum_{\ell=0}^{\infty} f(\ell) \frac{(x+y)^\ell}{\ell!}, \tag{4.6}$$

which immediately yields the reduction formula

$$\sum_{\ell,m=0}^{\infty} f(\ell+m) \frac{x^\ell}{\ell!} \frac{(-x)^m}{m!} = f(0) \tag{4.7}$$

for any bounded sequence $\{f(n)\}$, and we readily obtain

$$\mathcal{S}^* = \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n B(k_1,\ldots,k_r;0,N-n,n)$$

$$\cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},$$

which, upon interchanging the order of summation appropriately, gives us

$$\mathcal{S}^* = \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} B(k_1,\ldots,k_r;0,N-n,n)$$

$$\cdot \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}.$$

This evidently completes the proof of our summation formula (4.5) which would reduce at once to (4.3) in the special case when

(4.8) $$z_1 = \ldots = z_r = 0$$

and

(4.9) $$B(0,\ldots,0;\ell,m,n) = A(\ell,m,n), \quad \ell,\ m,\ n = 0,\ 1,\ 2,\ \ldots\ .$$

5. APPLICATIONS OF THE GENERAL RESULT (4.5)

Just as in the case of the summation formula (2.3), we can suitably specialize the multiple sequence

$$\{B(k_1,\ldots,k_r;\ell,m,n)\},\ k_i,\ \ell,\ m,\ n = 0,\ 1,\ 2,\ \ldots;\ i = 1,\ \ldots,\ r$$

with a view to applying the general result (4.5) in order to deduce various finite summation formulas involving such classes of multivariable hypergeometric functions as the (Srivastava-Daoust) generalized Lauricella function of $r + 2$ variables (cf. [6], p. 37, Equation (21) et seq.). In the particular case when $r = 1$, if $\{C(\kappa,\ell,m,n)\}$ is a bounded quadruple sequence, (4.5) assumes the form:

(5.1) $$\sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \sum_{\kappa=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} C(\kappa,\ell+m,N-n+\ell,m+n)$$

$$\cdot\ (-n)_\ell\ (-N+n)_m \frac{z^\kappa}{\kappa!} \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \sum_{\kappa=0}^{\infty} C(\kappa,0,N-n,n) \frac{z^\kappa}{\kappa!}\ ,$$

provided that each side of (5.1) exists.

In (5.1) we now put

$$(5.2)\qquad C(\kappa,\ell,m,n) = \frac{\prod_{j=1}^{p}(a_j)_{\kappa+\ell}\prod_{j=1}^{r}(c_j)_{\kappa+m}\prod_{j=1}^{u}(g_j)_{\kappa+n}\prod_{j=1}^{\lambda}(\alpha_j)_{m+n}}{\prod_{j=1}^{k}(b_j)_{\kappa+\ell}\prod_{j=1}^{s}(d_j)_{\kappa+m}\prod_{j=1}^{v}(h_j)_{\kappa+n}\prod_{j=1}^{\mu}(\beta_j)_{m+n}}$$

$$\cdot \frac{\prod_{j=1}^{\rho}(\gamma_j)_{\kappa}\prod_{j=1}^{\tau}(\xi_j)_{\ell}\prod_{j=1}^{\theta}(e_j)_{m}\prod_{j=1}^{t}(\varepsilon_j)_{n}}{\prod_{j=1}^{\sigma}(\delta_j)_{\kappa}\prod_{j=1}^{\omega}(\eta_j)_{\ell}\prod_{j=1}^{\phi}(f_j)_{m}\prod_{j=1}^{w}(\zeta_j)_{n}},$$

and interpret the inner triple series occurring on the left-hand side of (5.1) as Srivastava's triple hypergeometric function $F^{(3)}[x,y,z]$ (cf., e.g., [6], p. 44, Equation (14) et seq.). Upon interpreting the inner series on the right-hand side of (5.1) as a generalized hypergeometric function, we thus obtain the finite summation formula:

$$(5.3)\qquad \sum_{n=0}^{N}\binom{N}{n}\left(-\frac{y}{x}\right)^{n}\frac{\prod_{j=1}^{r}(c_j)_{N-n}\prod_{j=1}^{\theta}(e_j)_{N-n}\prod_{j=1}^{u}(g_j)_{n}\prod_{j=1}^{t}(\varepsilon_j)_{n}}{\prod_{j=1}^{s}(d_j)_{N-n}\prod_{j=1}^{\phi}(f_j)_{N-n}\prod_{j=1}^{v}(h_j)_{n}\prod_{j=1}^{w}(\zeta_j)_{n}}$$

$$\cdot\, F^{(3)}\left[\begin{matrix}(a_p)::\ (\alpha_\lambda)+N,\ (\xi_\tau);\ (g_u)+n;\ (c_r)+N-n: \\ (b_k)::\ (\beta_\mu)+N,\ (\eta_\omega);\ (h_v)+n;\ (d_s)+N-n:\end{matrix}\right.$$

$$\left.\begin{matrix}-n,\ (e_\theta)+N-n;\ -N+n,\ (\varepsilon_t)+n;\ (\gamma_\rho); \\ (f_\phi)+N-n;\qquad (\zeta_w)+n;\ (\delta_\sigma);\end{matrix}\ x,y,z\right]$$

$$= \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \frac{\prod_{j=1}^{r} (c_j)_{N-n} \prod_{j=1}^{\theta} (e_j)_{N-n} \prod_{j=1}^{u} (g_j)_n \prod_{j=1}^{t} (\varepsilon_j)_n}{\prod_{j=1}^{s} (d_j)_{N-n} \prod_{j=1}^{\phi} (f_j)_{N-n} \prod_{j=1}^{v} (h_j)_n \prod_{j=1}^{w} (\zeta_j)_n}$$

$$\cdot \; {}_{p+\rho+u+r}F_{k+\sigma+v+s}\left[\begin{matrix} (a_p),\ (\gamma_\rho),\ (g_u)+n,\ (c_r)+N-n; \\ \\ (b_k),\ (\delta_\sigma),\ (h_v)+n,\ (d_s)+N-n; \end{matrix} \; z\right],$$

where, for convergence of the generalized hypergeometric series,

$$p + \rho + u + r \leqq k + \sigma + v + s + 1 \quad (\text{equality when } |z| < 1).$$

For $z = 0$, the finite summation formula (5.3) evidently yields

$$(5.4) \qquad \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \frac{\prod_{j=1}^{r} (c_j)_{N-n} \prod_{j=1}^{u} (g_j)_n}{\prod_{j=1}^{s} (d_j)_{N-n} \prod_{j=1}^{v} (h_j)_n}$$

$$\cdot \; F^{p:1+r;1+u}_{k:\ s;\ v}\left[\begin{matrix} (a_p):\ -n,\ (c_r)+N-n;\ -N+n,\ (g_u)+n; \\ \\ (b_k):\qquad (d_s)+N-n;\qquad (h_v)+n; \end{matrix} \; x,y\right]$$

$$= \frac{\prod_{j=1}^{r} (c_j)_N}{\prod_{j=1}^{s} (d_j)_N} \; {}_{1+u+s}F_{v+r}\left[\begin{matrix} -N,\ (g_u),\ 1-(d_s)-N; \\ \\ (h_v),\ 1-(c_r)-N; \end{matrix} \; (-1)^{r-s} \frac{y}{x}\right],$$

which, for $p = k = 0$, reduces immediately to the elegant form:

(5.5)
$$\sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \frac{\prod_{j=1}^{r} (c_j)_{N-n} \prod_{j=1}^{u} (g_j)_n}{\prod_{j=1}^{s} (d_j)_{N-n} \prod_{j=1}^{v} (h_j)_n}$$

$$\cdot\ {}_{1+r}F_s\left[\begin{matrix} -n,\ (c_r)+N-n; \\ \\ (d_s)+N-n; \end{matrix}\ x\right] {}_{1+u}F_v\left[\begin{matrix} -N+n,\ (g_u)+n; \\ \\ (h_v)+n; \end{matrix}\ y\right]$$

$$= \frac{\prod_{j=1}^{r} (c_j)_N}{\prod_{j=1}^{s} (d_j)_N}\ {}_{1+u+s}F_{v+r}\left[\begin{matrix} -N,\ (g_u),\ 1-(d_s)-N; \\ \\ (h_v),\ 1-(c_r)-N; \end{matrix}\ (-1)^{r-s}\frac{y}{x}\right].$$

Formula (1.5) is an obvious special case of (5.5) when

$$r = s = u = v = 1,\quad c_1 = \alpha,\quad d_1 = \beta,\quad g_1 = \gamma \text{ and } h_1 = \delta.$$

Yet another special case of the general hypergeometric summation formula (5.3) occurs when we set $r = s = u = v = 0$. Upon simplifying the right-hand side of (5.3) in this special case, we obtain the summation formula

(5.6)
$$\sum_{n=0}^{N} \binom{N}{n} \left(-\frac{y}{x}\right)^n \frac{\prod_{j=1}^{\theta} (e_j)_{N-n} \prod_{j=1}^{t} (\varepsilon_j)_n}{\prod_{j=1}^{\phi} (f_j)_{N-n} \prod_{j=1}^{w} (\zeta_j)_n}$$

$$\cdot\ F^{(3)}\left[\begin{matrix} (a_p) :: (\alpha_\lambda)+N,\ (\xi_\tau);\ \text{—};\ \text{—}: \\ \\ (b_k) :: (\beta_\mu)+N,\ (\eta_\omega);\ \text{—};\ \text{—}: \end{matrix}\right.$$

$$\left.\begin{matrix} -n,\ (e_\theta)+N-n;\ -N+n,\ (\varepsilon_t)+n;\ (\gamma_\rho); \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad x,y,z \\ (f_\phi)+N-n;\qquad (\zeta_w)+n;\ (\delta_\sigma); \end{matrix}\right]$$

$$= \frac{\prod_{j=1}^{\theta} (e_j)_N}{\prod_{j=1}^{\phi} (f_j)_N}\ {}_{p+\rho}F_{k+\sigma}\left[\begin{matrix} (a_p),\ (\gamma_\rho); \\ \qquad\qquad z \\ (b_k),\ (\delta_\sigma); \end{matrix}\right]$$

$$\cdot\ {}_{1+t+\phi}F_{w+\theta}\left[\begin{matrix} -N,\ (\varepsilon_t),\ 1-(f_\phi)-N; \\ \qquad\qquad (-1)^{\theta-\phi}\frac{y}{x} \\ (\zeta_w),\ 1-(e_\theta)-N; \end{matrix}\right],$$

which incidentally may be regarded as a straightforward three-variable extension of the hypergeometric summation formula (5.4).

6. FURTHER GENERALIZATIONS AND q-EXTENSIONS

Our proof of the general multivariable summation formula (4.5) depends heavily upon the reduction formula (4.7). If, instead of (4.7), we make use of the series identity (4.6), we shall similarly obtain an interesting further generalization of (4.5) in the form:

$$(6.1)\qquad \sum_{n=0}^{N} \binom{N}{n}\left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n)$$

$$\cdot\ (-n)_\ell\ (-N+n)_m \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^\ell}{\ell!} \frac{y^m}{m!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{N} \binom{N}{\ell} \ell! \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{(Y-y)^{\ell}}{\ell!}$$

$$\cdot \sum_{n=0}^{N-\ell} \binom{N-\ell}{n} \left(-\frac{Y}{x}\right)^n B(k_1,\ldots,k_r;\ell,N-n,n+\ell),$$

which would immediately yield (4.5) in the special case when $Y = y$.

More generally, in terms of a bounded multiple sequence

$$\{C(k_1,\ldots,k_r;\ell,m,n,\nu)\},\ k_i,\ \ell,\ m,\ n,\ \nu = 0,\ 1,\ 2,\ \ldots\ ;$$

$$i = 1,\ \ldots,\ r,$$

we can apply the proofs of (2.3) and (4.5) <u>mutatis mutandis</u> in order to derive the following unification (and generalization) of the multivariable summation formulas (2.3) and (4.5):

$$(6.2)\qquad \sum_{n=0}^{N} \binom{N}{n} \left(-\frac{Y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} C(k_1,\ldots,k_r;\ell,m,N-n+\ell,m+n)$$

$$\cdot\ (-n)_{\ell}\ (-N+n)_m \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{x^{\ell}}{\ell!} \frac{y^m}{m!}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell,m=0}^{\ell+m\leqq N} \binom{N}{\ell+m} (\ell+m)! \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!} \frac{Y^{\ell}}{\ell!} \frac{(-y)^m}{m!}$$

$$\cdot \sum_{n=0}^{N-\ell-m} \binom{N-\ell-m}{n} \left(-\frac{Y}{x}\right)^n C(k_1,\ldots,k_r;\ell,m,N-n,\ell+m+n),$$

which, in view of the series identity (4.6), reduces to (6.1) in the special case when

$$(6.3)\qquad C(k_1,\ldots,k_r;\ell,m,n,\nu) = B(k_1,\ldots,k_r;\ell+m,n,\nu),$$

$$k_i,\ \ell,\ m,\ n,\ \nu = 0,\ 1,\ 2,\ \ldots;\quad i = 1,\ \ldots,\ r.$$

In order to deduce the multivariable summation formula (2.3) as a special case of (6.2), we set $Y = x$ and

$$(6.4)\qquad C(k_1,\ldots,k_r;\ell,m,n,\nu) = \Lambda(k_1,\ldots,k_r;\ell+m)\ \Omega(\ell,m),$$

$$k_i,\ \ell,\ m,\ n,\ \nu = 0,\ 1,\ 2,\ \ldots;\quad i = 1,\ \ldots,\ r,$$

and make use of the definition (2.4) and the elementary combinatorial series identity (2.5).

With a view to presenting the q-extensions of the various finite summation formulas considered in this paper, we begin by recalling the definition (cf. [6], p. 346 et seq.)

$$(6.5)\qquad (\lambda;q)_\mu = \prod_{j=0}^{\infty} \left\{\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right\}$$

for arbitrary q, λ and μ, $|q| < 1$, so that

$$(6.6)\qquad (\lambda;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1}), & \forall\ n \in \{1,2,3,\ldots\}, \end{cases}$$

and [cf. Equation (1.1)]

$$(6.7)\qquad \lim_{q\to 1} \left\{\frac{(q^\lambda;q)_n}{(q^\mu;q)_n}\right\} = \frac{(\lambda)_n}{(\mu)_n},\qquad n = 0,\ 1,\ 2,\ \ldots,$$

for arbitrary λ and μ, $\mu \neq 0, -1, -2, \ldots$.

We shall also need the q-binomial coefficient defined, for arbitrary λ, by [cf. Equation (1.2)]

$$\begin{bmatrix}\lambda\\ n\end{bmatrix} = (-1)^n q^{\frac{1}{2}n(2\lambda-n+1)} \frac{(q^{-\lambda};q)_n}{(q;q)_n}, \qquad n = 0, 1, 2, \ldots, \tag{6.8}$$

so that, if N is an integer,

$$\begin{bmatrix}N\\ n\end{bmatrix} = \frac{(q;q)_N}{(q;q)_n (q;q)_{N-n}} = \begin{bmatrix}N\\ N-n\end{bmatrix}, \qquad 0 \leqq n \leqq N. \tag{6.9}$$

Since

$$(q^{-n};q)_\ell = (-1)^\ell q^{\frac{1}{2}\ell(\ell-2n-1)} \frac{(q;q)_n}{(q;q)_{n-\ell}}, \qquad (0 \leqq \ell \leqq n), \tag{6.10}$$

and

$$(q^{-N+n};q)_m = (-1)^m q^{\frac{1}{2}m(m-2N+2n-1)} \frac{(q;q)_{N-n}}{(q;q)_{N-m-n}}, \tag{6.11}$$

$$(0 \leqq m \leqq N-n;\ 0 \leqq n \leqq N),$$

which indeed follow readily from (6.8) and (6.9), it is not difficult to prove (along the lines detailed in Sections 2 and 4) the following q-extension of the general result (6.2):

$$\sum_{n=0}^{N} \begin{bmatrix}N\\ n\end{bmatrix} \left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} C(k_1,\ldots,k_r;\ell,m,N-n+\ell,m+n) \tag{6.12}$$

$$\cdot\, q^{\frac{1}{2}n(n-2m-1)} (q^{-n};q)_\ell (q^{-N+n};q)_m \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{(qx)^\ell}{(q;q)_\ell} \frac{(q^N y)^m}{(q;q)_m}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell,m=0}^{\ell+m\leq N} \begin{bmatrix} N \\ \ell+m \end{bmatrix} (q;q)_{\ell+m}\, q^{\frac{1}{2}m(m-1)} \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{Y^{\ell}}{(q;q)_{\ell}} \frac{(-y)^m}{(q;q)_m}$$

$$\cdot \sum_{n=0}^{N-\ell-m} \begin{bmatrix} N-\ell-m \\ n \end{bmatrix} \left(-\frac{Y}{x}\right)^n q^{\frac{1}{2}n(n-1)}\, C(k_1,\ldots,k_r;\ell,m,N-n,\ell+m+n),$$

which holds true whenever both sides exist.

In the special case of the q-summation formula (6.12) when the constraint (6.3) is satisfied, if we apply the q-series identity [cf. Equation (4.6)]

$$(6.13) \qquad \sum_{\ell,m=0}^{\infty} f(\ell+m)\, q^{\frac{1}{2}m(m-1)} \frac{x^{\ell}}{(q;q)_{\ell}} \frac{y^m}{(q;q)_m}$$

$$= \sum_{\ell=0}^{\infty} f(\ell) \left(-\frac{y}{x};q\right)_{\ell} \frac{x^{\ell}}{(q;q)_{\ell}},$$

which is an immediate consequence of the q-binomial expansion (cf. [6], p. 348, Equation (274)), we obtain the result

$$(6.14) \qquad \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \left(-\frac{Y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n)$$

$$\cdot\, q^{\frac{1}{2}n(n-2m-1)} (q^{-n};q)_{\ell} (q^{-N+n};q)_m \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{(qx)^{\ell}}{(q;q)_{\ell}} \frac{(q^N y)^m}{(q;q)_m}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (q;q)_{\ell} \left(\frac{Y}{y};q\right)_{\ell} \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{Y^{\ell}}{(q;q)_{\ell}}$$

$$\cdot \sum_{n=0}^{N-\ell} \begin{bmatrix} N-\ell \\ n \end{bmatrix} \left(-\frac{Y}{x}\right)^n q^{\frac{1}{2}n(n-1)}\, B(k_1,\ldots,k_r;\ell,N-n,n+\ell),$$

which is a q-extension of the multivariable summation formula (6.1).

For $Y = y$, the ℓ-series occurring on the right-hand side of (6.14) reduces to its first term given by $\ell = 0$, and we thus find from (6.14) that

$$(6.15)\qquad \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \left(-\frac{y}{x}\right)^n \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} B(k_1,\ldots,k_r;\ell+m,N-n+\ell,m+n)$$

$$\cdot\; q^{\frac{1}{2}n(n-2m-1)} (q^{-n};q)_\ell (q^{-N+n};q)_m \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{(qx)^\ell}{(q;q)_\ell} \frac{(q^N y)^m}{(q;q)_m}$$

$$= \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \left(-\frac{y}{x}\right)^n q^{\frac{1}{2}n(n-1)}$$

$$\cdot \sum_{k_1,\ldots,k_r=0}^{\infty} B(k_1,\ldots,k_r;0,N-n,n) \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}},$$

which is a q-extension of the multivariable summation formula (4.5).

The q-summation formula (6.15) can indeed be deduced directly from (6.12) by setting $Y = y$, and making use of the constraint (6.3) and the identity [cf. Equation (4.7)]

$$(6.16)\qquad \sum_{\ell,m=0}^{\infty} f(\ell+m)\, q^{\frac{1}{2}m(m-1)} \frac{x^\ell}{(q;q)_\ell} \frac{(-x)^m}{(q;q)_m} = f(0),$$

which is an immediate consequence of (6.13).

Finally, we deduce a q-extension of the multivariable summation formula (2.3) as a special case of (6.12). Setting $Y = x$ in (6.12) and using the constraint (6.4), if we appeal to the elementary q-identity:

$$(6.17)\qquad \sum_{n=0}^{N} (-1)^n \begin{bmatrix} N \\ n \end{bmatrix} q^{\frac{1}{2}n(n-1)} = \delta_{N,0}, \qquad N = 0, 1, 2, \ldots,$$

which is a q-extension of the combinatorial series identity (2.5), we obtain from (6.12) the multivariable q-summation formula:

(6.18)
$$\sum_{n=0}^{N} (-1)^n \begin{bmatrix} N \\ n \end{bmatrix} \sum_{k_1,\ldots,k_r=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=0}^{N-n} \Lambda(k_1,\ldots,k_r;\ell+m)\, \Omega(\ell,m)$$
$$\cdot\, q^{\frac{1}{2}n(n-2m-1)} (q^{-n};q)_\ell (q^{-N+n};q)_m \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}} \frac{(qx)^\ell}{(q;q)_\ell} \frac{(q^N y)^m}{(q;q)_m}$$
$$= x^N\, \Xi_{N,q}(z_1,\ldots,z_r) \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} q^{\frac{1}{2}n(n-1)}\, \Omega(N-n,n) \left(-\frac{y}{x}\right)^n,$$

where

(6.19)
$$\Xi_{N,q}(z_1,\ldots,z_r) = \sum_{k_1,\ldots,k_r=0}^{\infty} \Lambda(k_1,\ldots,k_r;N) \frac{z_1^{k_1}}{(q;q)_{k_1}} \cdots \frac{z_r^{k_r}}{(q;q)_{k_r}},$$

provided that the multiple series in (6.19) converges absolutely.

Formula (6.18) provides a q-extension of the multivariable summation formula (2.3). Its special case when (4.8) holds true in conjunction with (2.2) and

(6.20)
$$\Lambda(0,\ldots,0;n) = \lambda_n, \qquad n = 0, 1, 2, \ldots,$$

leads to a q-extension of (1.3), which indeed was given earlier by us [4]. For $r = 1$, (6.18) would immediately yield a q-extension of the summation formula (3.1). Furthermore, in view of the limit relationship (6.7), and since

(6.21)
$$\lim_{q \to 1} \frac{\{(1-q)z\}^n}{(q;q)_n} = \frac{z^n}{n!}, \qquad n = 0, 1, 2, \ldots,$$

each of the multivariable q-summation formulas (6.12), (6.14), (6.15) and (6.18) would naturally yield the corresponding results (discussed in this paper) in the limit when $q \to 1$ in an appropriate manner.

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