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ANALYTIC FUNCTIONS**

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Abstract

The object of this paper is to improve some sufficient conditions for p -valently starlikeness and for starlikeness of order α . Some new starlikeness conditions are also presented.

1. Introduction

Let \mathcal{H} denote the class of functions that are analytic in the open unit disk \mathcal{U} and let

$$\mathcal{A} := \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}.$$

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Denote by $\mathcal{A}(p)$ the subclass of \mathcal{H} consisting of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}).$$

If

$$f(z) \in \mathcal{A}(p) \quad (0 \leq \alpha < p)$$

and

$$\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{U}),$$

then $f(z)$ is said to be p -valently starlike of order α . We represent the class of such functions by $\mathcal{S}_p^*(\alpha)$. Obviously, $\mathcal{S}_1^*(\alpha) = \mathcal{S}^*(\alpha)$ is the familiar class of *starlike functions of order α* .

Let $f(z) \in \mathcal{H}$ and $g(z) \in \mathcal{H}$. We say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a Schwarz function $w(z) \in \mathcal{H}$, $w(0) = 0$, and $|w(z)| < 1$ in \mathcal{U} , such that $f(z) = g(w(z))$. It is known that, if $g(z)$ is univalent in \mathcal{U} , then $f \prec g$ is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

In this note, we improve some sufficient conditions for p -valently starlikeness and for starlikeness of order α , and also obtain some new starlikeness conditions in terms of

$$\frac{z f'(z)}{f(z)} \quad \text{and} \quad \frac{z f''(z)}{f'(z)}.$$

2. Preliminaries

We need the following lemmas to prove our results.

Lemma 1 (Miller *et al.* [5]). *Let $g(z) \in \mathcal{H}$ be a convex function in \mathcal{U} (i.e., $g(z)$ is univalent and $g(\mathcal{U})$ is a convex domain). If $\gamma \neq 0$, $\Re(\gamma) \geq 0$, and $f(z) \in \mathcal{H}$, then*

$$f \prec g \Rightarrow \frac{1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \prec \frac{1}{z^\gamma} \int_0^z g(t) t^{\gamma-1} dt. \quad (1)$$

If $f(z) \in \mathcal{H}$ with $f(0) = 0$, then (1) holds true in the case $\gamma = 0$.

Lemma 2 (Miller and Mocanu [4]). Let $s > 0$, $A, B \in [-1, 1]$, and $A \neq B$. If $p(z) \in \mathcal{H}$ satisfies

$$p(z) + \frac{zp'(z)}{sp(z)} \prec \frac{1 + Az}{1 + Bz},$$

then

$$p(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz},$$

where $q(z)$ given by

$$q(z) = \begin{cases} \frac{z^s(1+Bz)^{s(A-B)/B}}{s \int_0^z t^{s-1}(1+Bt)^{s(A-B)/B} dt} & (B \neq 0) \\ \frac{z^s e^{sAz}}{s \int_0^z t^{s-1} e^{sAt} dt} & (B = 0) \end{cases}$$

is the best dominant.

Lemma 3 (Jack [2]). Let $w(z) \in \mathcal{H}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0 , then

$$z_0 w'(z_0) = m w(z_0) \quad (m \geq 1).$$

3. Two Sufficient Conditions for p -Valently Starlikeness

Nunokawa [7] proved that each of the following two conditions:

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \frac{1}{p} \left| \frac{zf'(z)}{f(z)} \right| \log(4e^{p-1}) \quad (f \in \mathcal{A}(p); z \in \mathcal{U}) \quad (2)$$

and

$$\left| 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \right| < \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right| \log 4 \quad (f \in \mathcal{A}(p); z \in \mathcal{U}) \quad (3)$$

implies that $f(z)$ is p -valently starlike in \mathcal{U} , that is, $f(z) \in \mathcal{S}_p^*(0) = \mathcal{S}_p^*$.

Nunokawa [7] also proposed the problem of finding the best constants in the above two starlikeness conditions. Subsequently, Nunokawa [10, pp. 206-211] showed that the same conclusion holds true if the constants in (2) and (3) are replaced by $(p + \frac{1}{2})/p$ and $\frac{3}{2}$, respectively. In this section, we shall improve the above results as follows.

Theorem 1. Let $f(z) \in \mathcal{A}(p)$ with $f(z) \neq 0$ for $0 < |z| < 1$. If $f(z)$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} \right| < \left| \frac{zf'(z)}{f(z)} \right| k(p) \quad (z \in \mathcal{U}), \quad (4)$$

where $k(p) \in (1, 2)$ is the unique root of the equation:

$$1 + p(1 - k^2) \ln \left(1 + \frac{1}{k} \right) = 0, \quad (5)$$

then $f(z) \in S_p^*$.

Proof. Let $g(z) = pf(z)/(zf'(z))$. Then, from the assumption, $g(z) \in \mathcal{H}$ with $g(0) = 1$ and $g(z) \neq 0$ in \mathcal{U} , and

$$|p - zg'(z)| < p k(p) \quad (z \in \mathcal{U}). \quad (6)$$

Let $w(z)$ be defined by

$$zg'(z) = (1 - k^2) \frac{pw(z)}{k + w(z)} \quad (z \in \mathcal{U}), \quad (7)$$

where $k = k(p) > 1$.

It follows from (6) that

$$\left| \frac{1 + kw(z)}{k + w(z)} \right| < 1 \quad (z \in \mathcal{U}),$$

which is equivalent to $|w(z)| < 1$ in \mathcal{U} .

From (7), we have

$$zg'(z) < (1 - k^2) \frac{pz}{k + z}, \quad (8)$$

and by Lemma 1, we easily get

$$g(z) < 1 + p(1 - k^2) \ln \left(1 + \frac{z}{k} \right). \quad (9)$$

Since

$$\Re \left\{ 1 + p(1 - k^2) \ln \left(1 + \frac{z}{k} \right) \right\} > 1 + p(1 - k^2) \ln \left(1 + \frac{1}{k} \right) = 0 \quad (z \in \mathcal{U}),$$

we deduce from (9) that

$$\Re\{g(z)\} > 0 \quad (z \in \mathcal{U}),$$

which shows that $f(z) \in \mathcal{S}_p^*$. This evidently completes the proof of Theorem 1.

It may be of interest to note that, if we let

$$\phi(k) = 1 + p(1 - k^2) \ln \left(1 + \frac{1}{k} \right),$$

then

$$\phi'(k) < 0 \quad (k \in (1, \infty)), \quad \phi(2) < 0, \quad \text{and} \quad \phi \left(1 + \frac{7}{10p} \right) > 0 \quad (p \in \mathbb{N}).$$

Thus the constant in (4) satisfies the inequality:

$$k(p) > \frac{1}{p} \left(p + \frac{7}{10} \right).$$

In the cases when $p = 1$ and $p = 2$, we have

$$k(1) = 1.809\dots \quad \text{and} \quad k(2) = 1.3857\dots$$

Theorem 2. Let $f(z) \in \mathcal{A}(p)$ with $f^{(p-1)}(z) \neq 0$ in $0 < |z| < 1$. If f satisfies

$$\left| 1 + \frac{z f^{(p+1)}(z)}{f^{(p)}(z)} \right| < \left| \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} \right| k(1) \quad (z \in \mathcal{U}), \quad (10)$$

then $f(z) \in \mathcal{S}_p^*(p-1)$.

Proof. Let $g(z) = f^{(p-1)}(z)/p!$. Then $g(z) \in \mathcal{A}(1) = \mathcal{A}$, $g(z) \neq 0$ in $0 < |z| < 1$, and

$$\left| 1 + \frac{z g''(z)}{g'(z)} \right| < \left| \frac{z g'(z)}{g(z)} \right| k(1) \quad (z \in \mathcal{U}).$$

Applying Theorem 1, we have

$$\Re \left\{ \frac{z f^{(p)}(z)}{f^{(p-1)}(z)} \right\} > 0 \quad (z \in \mathcal{U}). \quad (11)$$

Let

$$q_j(z) = \frac{z f^{(p-j)}(z)}{(j+1) f^{(p-j-1)}(z)} \quad (j = 0, 1, \dots, p-1).$$

Then $q_j(0) = 1$ and

$$q_j(z) + \frac{z q_j'(z)}{(j+1) q_j(z)} = \frac{1 + j q_{j-1}(z)}{j+1} \quad (j = 1, 2, \dots, p-1). \quad (12)$$

It follows from (11) and a result of Nunokawa [10, pp. 206-211] that

$$\Re\{q_j(z)\} > 0 \quad (z \in \mathcal{U}; j = 0, 1, \dots, p-1)$$

i.e.,

$$f^{(p-j-1)}(z) \in \mathcal{S}_{j+1}^* \quad (j = 0, 1, \dots, p-1).$$

However, if we apply Lemma 2, we find that (11) or $q_0(z) \prec (1+z)/(1-z)$ implies that

$$q_j(z) + \frac{zq'_j(z)}{(j+1)q_j(z)} \prec \frac{1}{j+1} \left(j + \frac{1+z}{1-z} \right) \quad (j = 1, \dots, p-1), \quad (13)$$

which yields

$$\Re\{(j+1)q_j(z)\} > j \quad (j = 1, \dots, p-1; z \in \mathcal{U})$$

or, equivalently,

$$f^{(p-j-1)}(z) \in \mathcal{S}_{j+1}^*(j) \quad (j = 1, \dots, p-1; z \in \mathcal{U}).$$

Hence the required result follows. This completes the proof of Theorem 2.

4. Sufficient Conditions for Starlikeness

Let $f(z) \in \mathcal{A}$, $\beta + \gamma \geq 0$, and $\beta \geq 0$. Obradović and Owa [10, pp. 220-233] considered a problem of Ruscheweyh and some related topics. They showed that, if

$$\left| \frac{zf''(z)}{f'(z)} \right|^\beta \left| \frac{zf'(z)}{f(z)} - 1 \right|^\gamma < \frac{1}{2^\beta 3^\gamma} \quad (z \in \mathcal{U}), \quad (14)$$

then $f(z)$ is starlike in \mathcal{U} , that is, $f(z) \in \mathcal{S}^*(0) = \mathcal{S}^*$. In this section, we shall extend this result and some related results.

Theorem 3. *Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ in $0 < |z| < 1$, $\frac{1}{2} \leq \alpha < 1$, $\beta + \gamma \geq 0$, and $\beta \geq 0$. If $f(z)$ satisfies the inequality:*

$$\left| \frac{zf''(z)}{f'(z)} \right|^\beta \left| \frac{zf'(z)}{f(z)} - 1 \right|^\gamma < 2^\beta (1-\alpha)^{\beta+\gamma} \quad (z \in \mathcal{U}), \quad (15)$$

then

$$f(z) \in S^*(\alpha) \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1 - \alpha)z}.$$

Proof. Let us put

$$\frac{zf'(z)}{f(z)} = \frac{\alpha}{\alpha - (1 - \alpha)w(z)}. \quad (16)$$

Then $w(z) \in \mathcal{H}$ with $w(0) = 0$ and, by an easy calculation, we have

$$\frac{|zw'(z) + w(z)|^\beta |w(z)|^\gamma}{|\alpha - (1 - \alpha)w(z)|^{\beta+\gamma}} < 2^\beta \quad (z \in \mathcal{U}). \quad (17)$$

To prove Theorem 3, it suffices to show that $|w(z)| < 1$ in \mathcal{U} . If this is not the case, then there is a point z_0 with $|z_0| = \rho < 1$, such that

$$\max_{|z| \leq \rho} |w(z)| = |w(z_0)| = 1.$$

By Lemma 3, there exists a real number $m \geq 1$ such that

$$z_0 w'(z_0) = m w(z_0).$$

Thus we get

$$\frac{|z_0 w'(z_0) + w(z_0)|^\beta |w(z_0)|^\gamma}{|\alpha - (1 - \alpha)w(z_0)|^{\beta+\gamma}} = \frac{(m+1)^\beta}{|\alpha - (1 - \alpha)w(z_0)|^{\beta+\gamma}} \geq 2^\beta, \quad (18)$$

which contradicts (17). Therefore, we have $|w(z)| < 1$ in \mathcal{U} , and it follows from (16) that

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1 - \alpha)z} \quad \text{and} \quad \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathcal{U})$$

This completes the proof of Theorem 3.

The following theorem can be proved along similar lines and so we omit its proof.

Theorem 4. Let $f(z) \in \mathcal{A}$ with $f(z) \neq 0$ in $0 < |z| < 1$, $\frac{1}{2} \leq \alpha < 1$, $\beta > 0$, and $\beta + \gamma \geq 0$.

If $f(z)$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \right|^\beta \left| \frac{zf'(z)}{f(z)} \right|^\gamma < 2^\beta \alpha^\gamma (1 - \alpha)^\beta \quad (z \in \mathcal{U}), \quad (19)$$

then $f(z) \in S^*(\alpha)$ and

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1 - \alpha)z} \quad (z \in \mathcal{U}).$$

Several cases of Theorem 3 and Theorem 4, for special values of the parameters α , β , and γ , will improve some other interesting known results.

In the rest of this section we restrict our discussion to the case when $\beta = 1$ and $\gamma = 0$. It follows from Theorem 3 or Theorem 4 that, if $\frac{1}{2} \leq \alpha < 1$, then

$$\left| \frac{zf''(z)}{f'(z)} \right| < 2(1 - \alpha) \Rightarrow f(z) \in \mathcal{S}^*(\alpha) \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \frac{\alpha}{\alpha - (1 - \alpha)z}. \quad (20)$$

Owa [8] showed that

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{(1 - \alpha)(3 - \alpha)}{2 - \alpha} \Rightarrow f(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1).$$

In the case when $0 < \alpha < \frac{2}{3}$, Li Jian Lin [3] improved Owa's result and showed that

$$\left| \frac{zf''(z)}{f'(z)} \right| < M(\alpha) \Rightarrow f(z) \in \mathcal{S}^*(\alpha) \quad \left(0 < \alpha < \frac{2}{3} \right),$$

where

$$M(\alpha) := \begin{cases} \frac{3}{2} - \alpha & (0 < \alpha < \frac{1}{2}) \\ (1 - \alpha)(1 + \frac{1}{2\alpha}) & (\frac{1}{2} \leq \alpha < \frac{2}{3}). \end{cases}$$

Obviously, in the case when $\frac{1}{2} \leq \alpha < 1$, (20) is much better than Owa's assertions. The following theorem will improve this result further.

Theorem 5. *Let $f(z) \in \mathcal{H}$ with $f(z) \neq 0$ in $0 < |z| < 1$.*

(i) *If*

$$0 < \mu \leq 1 \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| \leq \mu \quad (z \in \mathcal{U}),$$

then

$$f(z) \in \mathcal{S}^*(\rho) \quad \left(\rho = \frac{\mu}{e^{\mu} - 1} \right).$$

The value of ρ is the best possible.

(ii) *If*

$$\frac{1}{e - 1} \leq \alpha < 1 \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| \leq \mu(\alpha),$$

where $\mu(\alpha) \in (0, 1]$ is the unique root of the equation

$$\mu + \alpha(1 - e^{\mu}) = 0,$$

then $f(z) \in \mathcal{S}^*(\alpha)$. This sufficient condition for starlikeness of order α is sharp.

Proof. (i) If we put

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathcal{U}),$$

then

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \mu$$

reduces to

$$p(z) + \frac{zp'(z)}{p(z)} \prec 1 + \mu z. \quad (21)$$

By using Lemma 2, we have the best dominant

$$p(z) \prec \frac{\mu z e^{\mu z}}{e^{\mu z} - 1}. \quad (22)$$

Ruscheweyh and Singh [9] showed that

$$g(z) := \frac{e^{\mu z} - 1}{\mu} \in \mathcal{S}^*(\rho),$$

that is, that

$$\Re \left\{ \frac{\mu z e^{\mu z}}{e^{\mu z} - 1} \right\} > \frac{\mu}{e^\mu - 1} \quad (z \in \mathcal{U}),$$

and this result is sharp. Combining this result with (22), we have the desired assertion of Theorem 5(i).

(ii) Let

$$\phi(\mu) = \frac{\mu}{e^\mu - 1}.$$

Then

$$\phi'(\mu) < 0 \quad (0 < \mu < 1), \quad \phi(0) = 1, \quad \text{and} \quad \phi(1) = \frac{1}{e-1}.$$

If

$$\frac{1}{e-1} \leq \alpha < 1,$$

then (ii) follows from (i) by letting

$$\frac{\mu}{e^\mu - 1} = \alpha.$$

This completes the proof of Theorem 5.

Miller and Mocanu [10, pp. 171-178] showed that, if $\mu \leq 4.046\dots$, then (21) implies (22).

Hence, if

$$\mu(\alpha) = \sup \left\{ \mu : \frac{e^{\mu z} - 1}{\mu} \in S^*(\alpha) \right\},$$

then

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \mu(\alpha) \implies f(z) \in S^*(\alpha) \quad (0 \leq \alpha < 1),$$

and this sufficient condition is sharp. Anisiu and Mocanu [1] proved that $\mu(0) = 2.83\dots$.

5. A Result Involving the Alexander Integral Operator

The integral operator introduced by J.W. Alexander is defined by

$$f(z) = \int_0^z \frac{g(t)}{t} dt \quad (z \in \mathcal{U}; g(z) \in \mathcal{A}). \quad (23)$$

Recently, Mocanu [6] proved that, if $g(z)$ satisfies the inequality:

$$|g'(z) - 1| < \frac{8}{2 + \sqrt{15}} \quad (z \in \mathcal{U}),$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}).$$

In this section, we shall improve this result to the following form.

Theorem 6. For $f(z)$ and $g(z)$ given by (23), let

$$|g'(z) - 1| < \frac{3}{2} \quad (z \in \mathcal{U}). \quad (24)$$

Then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}).$$

Proof. It is easily seen from (24) that

$$g'(z) \prec 1 + \frac{3}{2}z.$$

By Lemma 1, we also have

$$f'(z) \prec 1 + \frac{3}{4}z. \quad (25)$$

Applying Lemma 1 once again, we obtain

$$\frac{f(z)}{z} \prec 1 + \frac{3}{8}z$$

or, equivalently,

$$\left| \frac{f(z)}{z} - 1 \right| < \frac{3}{8} \quad (z \in \mathcal{U}). \quad (26)$$

Let

$$f'(z) - \frac{f(z)}{z} = \frac{5}{8}w(z). \quad (27)$$

Then $w(z) \in \mathcal{H}$ with $w(0) = 0$. If we suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1,$$

then, from Lemma 3, we have

$$z_0 w'(z_0) = m w(z_0) \quad (m \geq 1).$$

Since

$$g'(z) = \frac{f(z)}{z} + \frac{5}{8} \{z w'(z) + 2w(z)\},$$

we have

$$|g'(z_0) - 1| = \left| \frac{f(z_0)}{z_0} - 1 + \frac{5}{8}(m+2)w(z_0) \right| < \frac{3}{2}. \quad (28)$$

The inequality in (28) yields

$$\left| \frac{f(z_0)}{z_0} - 1 \right| > \frac{5}{8}(m+2) - \frac{3}{2} \geq \frac{3}{8},$$

which contradicts (26). Therefore, we have $|w(z)| < 1$ ($z \in \mathcal{U}$) or, equivalently,

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{5}{8} \quad (z \in \mathcal{U}). \quad (29)$$

Finally, we observe from (26) that

$$\left| \frac{f(z)}{z} \right| > \frac{5}{8} \quad (z \in \mathcal{U}). \quad (30)$$

The inequality (30), in conjunction with (29), yields

$$\left| f'(z) - \frac{f(z)}{z} \right| < \left| \frac{f(z)}{z} \right| \quad (z \in \mathcal{U}),$$

which shows that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}).$$

This evidently completes the proof of Theorem 6.

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