A CERTAIN FRACTIONAL DERIVATIVE OPERATOR AND ITS APPLICATIONS TO A NEW CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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A CERTAIN FRACTIONAL DERIVATIVE OPERATOR AND ITS APPLICATIONS TO A NEW CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS.II

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Recently, the authors made use of a certain operator of fractional derivatives in order to introduce (and initiate a systematic study of) a novel subclass $T_p(\alpha, \beta, \lambda)$ of analytic and p-valent functions with negative coefficients. In this sequel to the aforementioned work, they prove a number of closure and inclusion theorems and determine the radii of p-valently close-to-convexivity, starlikeness, and convexity for the class $T_p(\alpha, \beta, \lambda)$. They also obtain a class-preserving integral operator of the form:

$$F(z) = \left(J_{\gamma,p} f\right)(z) := \frac{\gamma + p}{z^{\gamma}} \int_0^z t^{\gamma - 1} f(t)\,dt \qquad (\gamma > -p)$$

for the class studied here.

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1. Introduction, Definitions, and Preliminaries

Let S_p denote the class of functions $f(z)$ of the form (*cf.*, *e.g.*, Goodman [1]; see also Srivastava and Owa [6]):

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} := \{1,2,3,...\}), \tag{1.1}$$

which are analytic and p-valent in the *open* unit disk

$$U := \{z: \ z \in \mathbb{C} \ \text{and} \ |z| < 1\}.$$

Also let T_p denote the subclass of S_p consisting of analytic and p-valent functions $f(z)$ which can be expressed in the form:

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad \left(a_{p+n} \geq 0; \ p \in \mathbb{N}\right). \tag{1.2}$$

Following Srivastava and Aouf [4], we say that a function $f \in T_p$ is in the class $T_p(\alpha, \beta, \lambda)$ if and only if

$$\left| \frac{\Omega_z^{(\lambda,p)} f(z) - 1}{\Omega_z^{(\lambda,p)} f(z) - 2\alpha + 1} \right| < \beta \qquad (z \in U; \ 0 \leq \alpha < 1; \ 0 < \beta \leq 1; \ 0 \leq \lambda \leq 1), \tag{1.3}$$

where, for convenience,

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p - \lambda + 1)}{\Gamma(p + 1)} z^{\lambda - p} D_z^{\lambda} f(z) \tag{1.4}$$

in terms of the *fractional derivative operator* D_z^{λ} of order λ, studied by Owa [2] and others (*cf.*, *e.g.*, Srivastava and Owa [5]), with

$$D_z^0 f(z) = f(z) \quad \text{and} \quad D_z^1 f(z) = f'(z). \tag{1.5}$$

By suitably specializing the parameters λ, α, β, and p, the class $T_p(\alpha, \beta, \lambda)$ can be reduced to several interesting subclasses of analytic functions with negative coefficients, which were studied by various other authors (see, for details, Srivastava and Aouf [4, p. 2]). The object of the present sequel to the aforementioned work of Srivastava and Aouf [4] is to obtain a number of new results for the general class $T_p(\alpha, \beta, \lambda)$ which involve, for example, closure properties, radii of p-valently close-to-convexity, starlikeness, and convexity, and modified Hadamard products. We also obtain a class-preserving integral operator of the form:

$$F(z) = \left(J_{\gamma,p} f\right)(z) := \frac{\gamma + p}{z^{\gamma}} \int_0^z t^{\gamma - 1} f(t)\,dt \tag{1.6}$$

for the class $T_p(\alpha, \beta, \lambda)$. Conversely, when $F \in T_p(\alpha, \beta, \lambda)$, we determine the radius of p-valence of $f(z)$ defined by (1.6).

The following coefficient theorem for the class $T_p(\alpha, \beta\ \lambda)$ will be required in our investigation.

Lemma 1 (Srivastava and Aouf [4, p. 3, Theorem 1]). *A function* $f(z)$ *defined by* (1.2) *is in the class* $T_p(\alpha, \beta, \lambda)$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1) a_{p+n} \leq 2\beta(1-\alpha). \tag{1.7}$$

This result is sharp for the function

$$f(z) = z^p - \frac{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} z^{p+n} \quad (n \in \mathbb{N}). \tag{1.8}$$

2. Closure Theorems

We begin by defining the functions $f_j(z)$ $(j = 1, \ldots, m)$ by

$$f_j(z) := z^p - \sum_{n=1}^{\infty} a_{p+n,j}\, z^{p+n} \quad \left(a_{p+n,j} \geq 0;\ p \in \mathbb{N};\ z \in \mathrm{U}\right). \tag{2.1}$$

Theorem 1. *Let the functions* $f_j(z)$ $(j = 1, \ldots, m)$ *defined by* (2.1) *be in the classes* $\mathrm{T}_p(\alpha_j, \beta_j, \lambda)$ $(j = 1, \ldots, m)$, *respectively.*

Then the function $h(z)$ *defined by*

$$h(z) := z^p - \frac{1}{m} \sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} a_{p+n,j} \right) z^{p+n} \tag{2.2}$$

is in the class $\mathrm{T}_p(\alpha, \beta, \lambda)$, *where*

$$\alpha = \min_{1 \leq j \leq m} \{\alpha_j\} \quad \text{and} \quad \beta = \max_{1 \leq j \leq m} \{\beta_j\} \tag{2.3}$$

$$\left(0 \leq \alpha_j < 1;\quad 0 < \beta_j \leq 1;\quad j = 1, \ldots, m\right).$$

Proof. Since $f_j \in \mathrm{T}_p(\alpha_j, \beta_j, \lambda)$ $(j = 1, \ldots, m)$, by applying Lemma 1 to the definition (2.1), we observe that

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} \left(\frac{1}{m} \sum_{j=1}^{m} a_{p+n,j} \right) \\ &= \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)}\, a_{p+n,j} \right) \\ &\leq \frac{1}{m} \sum_{j=1}^{m} \left(\frac{2\beta_j(1-\alpha_j)}{\beta_j + 1} \right) \\ &\leq \frac{2\beta(1-\alpha)}{\beta+1}, \end{aligned} \tag{2.4}$$

where α and β are given by (2.3).

Thus we have

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1)\left(\frac{1}{m}\sum_{j=1}^{m} a_{p+n,j}\right) \leq 2\beta(1-\alpha), \tag{2.5}$$

which, in view of Lemma 1 again, implies that [*cf.* Definition (2.2)]

$$h \in T_p(\alpha, \beta, \lambda),$$

and the proof of Theorem 1 is completed.

Next we state and prove

Theorem 2. *The class* $T_p(\alpha, \beta, \lambda)$ *is convex.*

Proof. Suppose that the functions $f_j(z)$ $(j = 1,2)$ defined by (2.1) are in the class $T_p(\alpha, \beta, \lambda)$. Then it is sufficient to show that the function

$$h(z) = \mu f_1(z) + (1-\mu)f_2(z) \qquad (0 \leq \mu \leq 1) \tag{2.6}$$

or, equivalently,

$$h(z) = z^p - \sum_{n=1}^{\infty} \{\mu a_{p+n,1} + (1-\mu)a_{p+n,2}\} z^{p+n} \qquad (0 \leq \mu \leq 1) \tag{2.7}$$

is also in the class $T_p(\alpha, \beta, \lambda)$.

Now, from our hypothesis and Lemma 1, it follows readily that

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1)\{\mu a_{p+n,1} + (1-\mu)a_{p+n,2}\}$$

$$\leq 2\beta(1-\alpha),$$

which evidently proves Theorem 2.

Theorem 3. *Let*

$$f_p(z) = z^p \qquad (p \in \mathbb{N}) \tag{2.8}$$

and

$$f_{p+n}(z) = z^p - \frac{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} z^{p+n} \tag{2.9}$$

$$(0 \le \alpha < 1; \;\; 0 < \beta \le 1; \;\; 0 \le \lambda \le 1).$$

Then a function $f(z)$ *is in the class* $\mathrm{T}_p(\alpha, \beta, \lambda)$ *if and only if it can be expressed in the form*:

$$f(z) = \sum_{n=0}^{\infty} c_{p+n} f_{p+n}(z) \tag{2.10}$$

$$\left(c_{p+n} \ge 0; \quad \sum_{n=0}^{\infty} c_{p+n} = 1 \right).$$

Proof. First of all, let us suppose that $f(z)$ is given by (2.10), that is, by

$$f(z) = z^p - \sum_{n=1}^{\infty} \frac{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} c_{p+n} z^{p+n} \tag{2.11}$$

$$\left(\sum_{n=0}^{\infty} c_{p+n} = 1; \quad c_{p+n} \ge 0 \right).$$

Then, since

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1) \cdot \left\{\frac{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} c_{p+n}\right\}$$

$$= 2\beta(1-\alpha) \sum_{n=1}^{\infty} c_{p+n}$$

$$= 2\beta(1-\alpha)(1-c_p)$$

$$\leq 2\beta(1-\alpha), \tag{2.12}$$

Lemma 1 implies that $f \in T_p(\alpha, \beta, \lambda)$.

Conversely, assume that the function $f(z)$ defined by (1.2) is in the class $T_p(\alpha, \beta, \lambda)$. Then, writing the assertion (1.7) of Lemma 1 in the form:

$$a_{p+n} \leq \frac{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} \qquad (n \in \mathbb{N}), \tag{2.13}$$

and setting

$$c_{p+n} = \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)} a_{p+n} \qquad (n \in \mathbb{N}) \tag{2.14}$$

and

$$c_p = 1 - \sum_{n=1}^{\infty} c_{p+n}, \tag{2.15}$$

we can readily see that $f(z)$ can be expressed precisely as in (2.10).

This evidently completes the proof of Theorem 3.

Corollary 1. *The extreme points of the class* $T_p(\alpha, \beta, \lambda)$ *are the functions*

$$f_{p+n}(z) \qquad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right)$$

given by Theorem 3.

3. Class-Preserving Integral Operators

In this section we first prove that the integral operator $J_{\gamma,p}$ defined by (1.6) is indeed a class-preserving operator for the class $T_p(\alpha, \beta, p)$.

Theorem 4. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$. *Also let* $\gamma > -p$.

Then the function $F(z)$ *defined by* (1.6) *is also in the class* $T_p(\alpha, \beta, \lambda)$.

Proof. From the definitions (1.6) and (1.2), it is easily seen that

$$F(z) = z^p - \sum_{n=1}^{\infty} A_{p+n} z^{p+n}, \tag{3.1}$$

where

$$A_{p+n} := \left(\frac{\gamma + p}{\gamma + p + n}\right) a_{p+n} \qquad (n \in \mathbb{N}). \tag{3.2}$$

Since $\gamma > -p$, we have

$$0 \le A_{p+n} < a_{p+n} \qquad (n \in \mathbb{N}), \tag{3.3}$$

which is fairly obvious from (3.2), and Theorem 4 follows immediately from Lemma 1.

Theorem 4 simplifies considerably when we set $\gamma = 1 - p$, and we thus obtain

Corollary 2. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$.

Then

$$G(z) := z^{p-1} \int_0^z \frac{f(t)}{t^p}\, dt \in \mathrm{T}_p(\alpha, \beta, \lambda). \tag{3.4}$$

Next we prove

Theorem 5. *Let* $\gamma > -p$. *Also let* $F(z)$ *be in the class* $\mathrm{T}_p(\alpha, \beta, \lambda)$. *Then the function* $f(z)$ *given by* (1.6) *is p-valent in the disk* $|z| < R$, *where*

$$R := \inf_{n \in \mathbb{N}} \left\{ \frac{(\beta+1)(\gamma+p)\Gamma(n+p)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)(\gamma+p+n)\Gamma(p)\Gamma(n+p-\lambda+1)} \right\}^{1/n}. \tag{3.5}$$

The result is sharp.

Proof. Assuming that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \qquad (b_{p+n} \geq 0;\ p \in \mathbb{N}), \tag{3.6}$$

and applying the definition (1.6), we get

$$\begin{aligned} f(z) &= \frac{z^{1-\gamma}}{\gamma+p} \frac{d}{dz}\{z^{\gamma} F(z)\} \\ &= z^p - \sum_{n=1}^{\infty} \left(\frac{\gamma+p+n}{\gamma+p}\right) b_{p+n} z^{p+n} \qquad (\gamma > -p). \end{aligned} \tag{3.7}$$

In order to prove the main assertion of Theorem 5, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p \qquad (|z| < R), \tag{3.8}$$

where R is given by (3.5). Indeed we have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{n=1}^{\infty} (p+n)\left(\frac{\gamma+p+n}{\gamma+p}\right) b_{p+n}\, z^n\right|$$

$$\leq \sum_{n=1}^{\infty} (p+n)\left(\frac{\gamma+p+n}{\gamma+p}\right) b_{p+n}\, |z|^n,$$

which yields the desired inequality in (3.8), provided that

$$\sum_{n=1}^{\infty} \frac{(p+n)(\gamma+p+n)}{p(\gamma+p)}\, b_{p+n}\, |z|^n \leq 1. \tag{3.9}$$

But, since the function $F(z)$ defined by (3.6) is in the class $T_p(\alpha, \beta, \lambda)$, Lemma 1 gives us

$$\sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}\, b_{p+n} \leq 1. \tag{3.10}$$

Thus the inequality (3.9), and hence also the inequality (3.8), will hold true if

$$\frac{(p+n)(\gamma+p+n)}{p(\gamma+p)}\, |z|^n \leq \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)} \qquad (n \in \mathbb{N}), \tag{3.11}$$

that is, if

$$|z| \leq \left\{\frac{(\beta+1)(\gamma+p)\Gamma(n+p)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)(\gamma+p+n)\Gamma(p)\Gamma(n+p-\lambda+1)}\right\}^{1/n} \qquad (n \in \mathbb{N}), \tag{3.12}$$

which leads us precisely to the main assertion of Theorem 5.

The assertion of Theorem 5 is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{2\beta(1-\alpha)(\gamma+p+n)\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)(\gamma+p)\Gamma(n+p+1)\Gamma(p-\lambda+1)}\, z^{p+n}. \tag{3.13}$$

4. Radii of Close-to-Convexity, Starlikeness, and Convexity

Our results in this section (Theorem 6, Theorem 7, and Corollary 3 below) would provide the radii of p-valently close-to-convexity, starlikeness, and convexity for the class $T_p(\alpha, \beta, \lambda)$.

Theorem 6. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$. *Then* $f(z)$ *is p-valently close-to-convex of order* δ $(0 \leq \delta < p)$ *in the disk* $|z| < r_1$, *where*

$$r_1 = r_1(p, \alpha, \beta, \lambda, \delta) := \inf_{n \in \mathbb{N}} \left\{ \frac{(\beta + 1)(p - \delta)\Gamma(n + p)\Gamma(p - \lambda + 1)}{2\beta(1 - \alpha)\Gamma(p + 1)(n + p - \lambda + 1)} \right\}^{1/n}, \tag{4.1}$$

The result is sharp with the extremal function $f(z)$ *given by* (1.8).

Proof. From the definition (1.2), we easily get

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{n=1}^{\infty} (p + n) a_{p+n} |z|^n. \tag{4.2}$$

Thus we have the desired inequality:

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, \tag{4.3}$$

if

$$\sum_{n=1}^{\infty} \left(\frac{p + n}{p - \delta} \right) a_{p+n} |z|^n \leq 1, \tag{4.4}$$

that is, if

$$\left(\frac{p + n}{p - \delta} \right) |z|^n \leq \frac{(\beta + 1)\Gamma(n + p + 1)\Gamma(p - \lambda + 1)}{2\beta(1 - \alpha)\Gamma(p + 1)\Gamma(n + p - \lambda + 1)} \qquad (n \in \mathbb{N}), \tag{4.5}$$

where we have made use of the assertion (1.7) of Lemma 1.

The last inequality (4.5) leads us immediately to the disk $|z| < r_1$, where r_1 is given by (4.1), and the proof of Theorem 6 is completed.

Theorem 7. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$. *Then* $f(z)$ *is p-valently starlike of order* δ $(0 \le \delta < p)$ *in the disk* $|z| < r_2$, *where*

$$r_2 = r_2(p, \alpha, \beta, \lambda, \delta) := \inf_{n \in \mathbb{N}} \left\{ \frac{(\beta + 1)(p - \delta)\Gamma(n + p + 1)\Gamma(p - \lambda + 1)}{2\beta(1 - \alpha)(n + p + \delta)\Gamma(p + 1)\Gamma(n + p - \lambda + 1)} \right\}^{1/n}. \tag{4.6}$$

The result is sharp with the extremal function $f(z)$ *given by* (1.8).

Proof. Making use of the definition (1.2), we readily observe that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \le \frac{\sum_{n=1}^{\infty} n\, a_{p+n} |z|^n}{1 - \sum_{n=1}^{\infty} a_{p+n} |z|^n} \tag{4.7}$$

$$\le p - \delta,$$

if

$$\sum_{n=1}^{\infty} \left(\frac{n + p - \delta}{p - \delta} \right) a_{p+n} |z|^n \le 1,$$

that is, if

$$\left(\frac{n + p - \delta}{p - \delta} \right) |z|^n \le \frac{(\beta + 1)\Gamma(n + p + 1)\Gamma(p - \lambda + 1)}{2\beta(1 - \alpha)\Gamma(p + 1)\Gamma(n + p - \lambda + 1)} \qquad (n \in \mathbb{N}), \tag{4.8}$$

where we have also applied the assertion (1.7) of Lemma 1.

The last inequality (4.8) leads us precisely to the disk $|z| < r_2$, where r_2 is given by (4.6), and the proof of Theorem 7 is evidently completed.

Corollary 3. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$. *Then* $f(z)$ *is p-valently convex of order* δ $(0 \le \delta < p)$ *in the disk* $|z| < r_3$, *where*

$$r_3 = r_3(p, \alpha, \beta, \lambda, \delta) := \inf_{n \in \mathbb{N}} \left\{ \frac{(\beta + 1)(p - \delta)\Gamma(n + p)\Gamma(p - \lambda + 1)}{2\beta(1 - \alpha)(n + p - \delta)\Gamma(p + 1)\Gamma(n + p - \lambda + 1)} \right\}^{1/n}. \quad (4.9)$$

The result is sharp with the extremal function $f(z)$ *given by* (1.8).

5. Inclusion Theorems Involving Modified Hadamard Products

Throughout this section we let

$$(f_1 * f_2)(z) := z^p - \sum_{n=1}^{\infty} a_{p+n,1}\, a_{p+n,2}\, z^{p+n} \quad (5.1)$$

denote the *modified Hadamard product* of the functions $f_1(z)$ and $f_2(z)$ defined by (2.1). The proof of one of our results involving the modified Hadamard products (Corollary 4 below) is based upon

Lemma 2 (Srivastava and Aouf [4, p. 10, Theorem 8]). *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (2.1) *be in the class* $\mathrm{T}_p(\alpha, \beta, \lambda)$.

Then $(f_1 * f_2)(z)$ *is in the class* $\mathrm{T}_p(\gamma, \beta, \lambda)$, *where*

$$\gamma = \gamma(p, \alpha, \beta, \lambda) := 1 - \frac{2\beta(1 - \alpha)^2(p - \lambda + 1)}{(\beta + 1)(p + 1)}. \quad (5.2)$$

The result is sharp for the functions

$$f_j(z) = z^p - \frac{2\beta(1 - \alpha)(p - \lambda + 1)}{(\beta + 1)(p + 1)} z^{p+1} \qquad (j = 1,2). \quad (5.3)$$

Employing a technique used earlier by Schild and Silverman [3], we shall first prove a *mild* generalization of Lemma 2, which is contained in

Theorem 8. *For the functions* $f_j(z)$ $(j = 1,2)$ *defined by* (2.1), *let*

$$f_1(z) \in T_p(\alpha, \beta, \lambda) \quad \textit{and} \quad f_2(z) \in T_p(\xi, \beta, \lambda). \tag{5.4}$$

Then

$$(f_1 * f_2)(z) \in T_p(\kappa, \beta, \lambda), \tag{5.5}$$

where

$$\kappa = \kappa(p, \alpha, \beta, \xi, \lambda) := 1 - \frac{2\beta(1-\alpha)(1-\xi)(p-\lambda+1)}{(\beta+1)(p+1)}. \tag{5.6}$$

The result is the best possible for

$$f_1(z) = z^p - \frac{2\beta(1-\alpha)(p-\lambda+1)}{(\beta+1)(p+1)} z^{p+1} \tag{5.7}$$

and

$$f_2(z) = z^p - \frac{2\beta(1-\xi)(p-\lambda+1)}{(\beta+1)(p+1)} z^{p+1}. \tag{5.8}$$

Proof. In view of Lemma 1, it suffices to prove that

$$\sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\kappa_0)\Gamma(p+1)\Gamma(n+p-\lambda+1)} a_{p+n,1}\, a_{p+n,2} \leq 1 \qquad (\kappa_0 \leq \kappa), \tag{5.9}$$

where κ is defined by (5.6). On the other hand, under the hypotheses (5.4), it follows from (1.7) *and* the Cauchy-Schwarz inequality that

$$\sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta\sqrt{\{(1-\alpha)(1-\xi)\}}\ \Gamma(p+1)\Gamma(n+p-\lambda+1)} \sqrt{a_{p+n,1}\, a_{p+n,2}} \leq 1. \tag{5.10}$$

Thus we need to find the largest κ_0 such that

$$\sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\kappa_0)\Gamma(p+1)\Gamma(n+p-\lambda+1)} a_{p+n,1}\, a_{p+n,2}$$

$$\leq \sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta\sqrt{\{(1-\alpha)(1-\xi)\}}\,\Gamma(p+1)\Gamma(n+p-\lambda+1)} \sqrt{a_{p+n,1}\, a_{p+n,2}}\,,$$

or, equivalently, that

$$\sqrt{a_{p+n,1}\, a_{p+n,2}} \leq \frac{1-\kappa_0}{\sqrt{\{(1-\alpha)(1-\xi)\}}} \qquad (n \in \mathbb{N}). \tag{5.11}$$

By virtue of (5.10), it is sufficient to find the largest κ_0 such that

$$\frac{2\beta\sqrt{\{(1-\alpha)(1-\xi)\}}\,\Gamma(p+1)\Gamma(n+p-\lambda+1)}{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)} \leq \frac{1-\kappa_0}{\sqrt{\{(1-\alpha)(1-\xi)\}}} \qquad (n \in \mathbb{N}). \tag{5.12}$$

The inequality (5.12) yields

$$\kappa_0 \leq 1 - \frac{2\beta(1-\alpha)(1-\xi)}{\beta+1}\,\Phi(n) \qquad (n \in \mathbb{N}), \tag{5.13}$$

where, for convenience,

$$\Phi(n) := \frac{\Gamma(p+1)\Gamma(n+p-\lambda+1)}{\Gamma(n+p+1)\Gamma(p-\lambda+1)} \qquad (n \in \mathbb{N}). \tag{5.14}$$

Since $\Phi(n)$ defined by (5.14) is a *decreasing* function of n $(n \in \mathbb{N})$ for fixed λ, we have

$$\kappa_0 \leq \kappa = \kappa(p, \alpha, \beta, \xi, \lambda) = 1 - \frac{2\beta(1-\alpha)(1-\xi)}{\beta+1}\,\Phi(1)$$

$$= 1 - \frac{2\beta(1-\alpha)(1-\xi)(p-\lambda+1)}{(\beta+1)(p+1)},$$

which completes the proof of Theorem 8.

In its special case when $\xi = \alpha$, Theorem 8 would immediately reduce to Lemma 2. In fact, by applying Lemma 2 in conjunction with Theorem 8, it is not difficult to prove

Corollary 4. *Let the functions* $f_j(z)$ $(j = 1,2,3)$ *defined by* (2.1) *be in the class* $T_p(\alpha, \beta, \lambda)$.

Then

$$(f_1 * f_2 * f_3)(z) \in T_p(\eta, \beta, \lambda),$$

where

$$\eta = \eta(p, \alpha, \beta, \lambda) := 1 - \frac{4\beta^2(1-\alpha)^3(p-\lambda+1)^2}{(\beta+1)^2(p+1)^2}. \tag{5.15}$$

The result is the best possible for

$$f_j(z) = z^p - \frac{2\beta(1-\alpha)(p-\lambda+1)}{(\beta+1)(p+1)} z^{p+1} \qquad (j = 1,2,3). \tag{5.16}$$

Theorem 9. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p(\alpha, \beta, \lambda)$. *Also let*

$$g(z) := z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n} \qquad \left(|b_{p+n}| \leq 1;\ p \in \mathbb{N}\right). \tag{5.17}$$

Then $(f * g)(z) \in T_p(\alpha, \beta, \lambda)$.

Proof. Since

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1) \left|a_{p+n}\, b_{p+n}\right|$$

$$= \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1)\, a_{p+n} \left|b_{p+n}\right|$$

$$\leq \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)\Gamma(p-\lambda+1)}{\Gamma(p+1)\Gamma(n+p-\lambda+1)} (\beta+1)\, a_{p+n}$$

$$\leq 2\beta(1-\alpha),$$

by Lemma 1, it follows that

$$(f * g)(z) \in \mathrm{T}_p(\alpha, \beta, \lambda),$$

again by virtue of Lemma 1.

Finally, an obvious variant of Theorem 9 may be stated as

Corollary 5. *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $\mathrm{T}_p(\alpha, \beta, \lambda)$. *Also let*

$$g(z) := z^p - \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n} \qquad \left(0 \leq b_{p+n} \leq 1;\ p \in \mathbb{N}\right). \tag{5.18}$$

Then $(f * g)(z) \in \mathrm{T}_p(\alpha, \beta, \lambda)$.

6. A Further Inclusion Property of the Class $\mathrm{T}_p(\alpha, \beta, \lambda)$

For functions in the class $\mathrm{T}_p(\alpha, \beta, \lambda)$, we prove yet another inclusion property contained in

Theorem 10. *Under the hypotheses of Lemma* 2, *the function* $h(z)$ *defined by*

$$h(z) := z^p - \sum_{n=1}^{\infty} \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) z^{p+n} \qquad (p \in \mathbb{N}) \tag{6.1}$$

is in the class $T_p(\zeta, \beta, \lambda)$, *where*

$$\zeta = \zeta(p, \alpha, \beta, \lambda) := 1 - \frac{4\beta(1-\alpha)^2(p-\lambda+1)}{(\beta+1)(p+1)}. \tag{6.2}$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ *defined by* (5.3).

Proof. In view of Lemma 1, it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\zeta_0)\Gamma(p+1)\Gamma(n+p-\lambda+1)} \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \leq 1 \qquad \left(\zeta_0 \leq \zeta\right), \tag{6.3}$$

where ζ is defined by (6.2).

Since $f_j(z) \in T_p(\alpha, \beta, \lambda)$ $(j = 1,2)$, we find from the definition (2.1) and Lemma 1 that

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{\frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}\right\}^2 a_{p+n,j}^2 \\ &\leq \left\{\sum_{j=1}^{\infty} \frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)} a_{p+n,j}\right\}^2 \\ &\leq 1 \qquad (j = 1,2), \end{aligned} \tag{6.4}$$

which would readily yield

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{\frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}\right\}^2 \left(a_{p+n,1}^2 + a_{p+n,2}^2\right) \leq 1. \tag{6.5}$$

By comparing (6.3) and (6.5), it is easily seen that the inequality (6.3) will be satisfied if

$$\frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\zeta_0)\Gamma(p+1)\Gamma(n+p-\lambda+1)}$$

$$\leq \frac{1}{2}\left\{\frac{(\beta+1)\Gamma(n+p+1)\Gamma(p-\lambda+1)}{2\beta(1-\alpha)\Gamma(p+1)\Gamma(n+p-\lambda+1)}\right\}^2 \qquad (n \in \mathbb{N}),$$

that is, if

$$\zeta_0 \leq 1 - \frac{4\beta(1-\alpha)^2}{\beta+1}\,\Phi(n) \qquad (n \in \mathbb{N}), \tag{6.6}$$

where $\Phi(n)$ is given by (5.14).

Just as in the proof of Theorem 8, we conclude from (6.6) that

$$\zeta_0 \leq \zeta = \zeta(p, \alpha, \beta, \lambda) = 1 - \frac{4\beta(1-\alpha)^2}{\beta+1}\,\Phi(1)$$

$$= 1 - \frac{4\beta(1-\alpha)^2(p-\lambda+1)}{(\beta+1)(p+1)},$$

which completes the proof of Theorem 10.

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