

SOME CHARACTERIZATIONS OF UNIVALENT, STARLIKE
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E.P. Merkes and W.T. Scott [Proc. Amer. Math. Soc. 12(1961), 885-888] proved an interesting result characterizing starlike hypergeometric functions. More recently, B.C. Carlson and D.B. Shaffer [SIAM J. Math. Anal. 15(1984), 737-745] introduced a linear operator and demonstrated how this operator can be applied systematically to the study of various interesting classes of starlike, convex, and prestarlike hypergeometric functions. The object of the present paper is to establish several characterization theorems involving univalent, starlike, and convex hypergeometric functions in the unit disk. Various inequalities involving hypergeometric series are also proved.

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1. INTRODUCTION

A single-valued function $f(z)$ is said to be univalent in a domain \mathcal{D} if it never takes on the same value twice, that is, if $f(z_1) = f(z_2)$ for $z_1, z_2 \in \mathcal{D}$ implies that $z_1 = z_2$. A set \mathcal{E} is said to be convex if the line segment joining any two points of \mathcal{E} lies entirely within \mathcal{E} . If a function $f(z)$ maps \mathcal{D} onto a convex domain, then we say that $f(z)$ is a convex function in \mathcal{D} .

Let \mathcal{A} denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Then a function $f(z)$ belonging to the class \mathcal{A} is said to be close-to-convex if there is a convex function $g(z)$ such that

$$(1.2) \quad \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > 0$$

for all $z \in \mathcal{U}$. We note that $f(z)$ is not required to be univalent, and $g(z)$ need not be a function belonging to the class \mathcal{A} . It is readily observed that every close-to-convex function is univalent (cf. Duren [2], p. 47, Theorem 2.17).

Merkes and Scott [4] proved an interesting result characterizing starlike hypergeometric functions. More recently, Carlson and Shaffer [1] studied various interesting classes of starlike, convex, and prestarlike hypergeometric functions. In the present paper we establish several interesting (and useful) characterization theorems involving univalent, starlike, and convex hypergeometric functions in the unit disk \mathcal{U} . We also prove various inequalities involving hypergeometric series.

2. UNIVALENT HYPERGEOMETRIC FUNCTIONS

Let a, b, c be complex numbers with $c \neq 0, -1, -2, \dots$. Then the hypergeometric function $F(a, b; c; z)$ is defined by

$$(2.1) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(2.2) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1) \dots (\lambda+n-1), & \text{if } n \in \mathcal{N} = \{1, 2, 3, \dots\}. \end{cases}$$

We begin by stating the following result characterizing univalent hypergeometric functions in the unit disk \mathcal{U} .

THEOREM 1. Let the hypergeometric function $F(a, b; c; z)$ defined by (2.1) satisfy the condition

$$(2.3) \quad \left| F'(a, b; c; z) - \frac{ab}{c} \right|^{1-\alpha} \left| \frac{zF''(a, b; c; z)}{F'(a, b; c; z)} \right|^\alpha < \left(\frac{ab}{c} \right)^{1-\alpha} \left(\frac{1}{2} \right)^\alpha$$

for some $\alpha \geq 0$, where $abc > 0$. Then $F(a, b; c; z)$ is univalent in the unit disk \mathcal{U} .

PROOF. Our proof of Theorem 1 depends upon a result due to Jack [3], which we recall here as

LEMMA 1. Let $w(z)$ be regular in the unit disk \mathcal{U} , with $w(0) = 0$.

Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point z_1 , we can write

$$(2.4) \quad z_1 w'(z_1) = k w(z_1),$$

where k is real and $k \geq 1$.

Define a function $G(z)$ by

$$(2.5) \quad G(z) = \frac{c}{ab} \{F(a, b; c; z) - 1\} \quad (z \in \mathcal{U}).$$

Then the condition (2.3) becomes

$$(2.6) \quad |G'(z) - 1|^{1-\alpha} \left| \frac{zG''(z)}{G'(z)} \right|^\alpha < \left(\frac{1}{2} \right)^\alpha,$$

and $G(z)$ is in \mathcal{A} . If we let

$$(2.7) \quad w(z) = G'(z) - 1 \quad (z \in \mathcal{U}),$$

then it is easily seen that $w(0) = 0$ and $w(z) \in \mathcal{A}$. Replacing $G(z)$ in (2.6) by $w(z)$, we have

$$(2.8) \quad |w(z)|^{1-\alpha} \left| \frac{zw'(z)}{1+w(z)} \right|^\alpha < \left(\frac{1}{2} \right)^\alpha,$$

whence

$$(2.9) \quad |w(z)| \left| \frac{zw'(z)}{w(z)} \cdot \frac{1}{1+w(z)} \right|^\alpha < \left(\frac{1}{2} \right)^\alpha.$$

Assume that there exists a point $z_1 \in \mathcal{U}$ such that

$$(2.10) \quad \max_{|z| \leq |z_1|} \{|w(z)|\} = |w(z_1)| = 1.$$

Then we can put

$$(2.11) \quad \frac{z_1 w'(z_1)}{w(z_1)} = k \geq 1,$$

by virtue of Lemma 1, and we have

$$(2.12) \quad |w(z_1)| \cdot \left| \frac{z_1 w'(z_1)}{w(z_1)} \cdot \frac{1}{1 + w(z_1)} \right|^\alpha \cong \left(\frac{k}{2}\right)^\alpha \cong \left(\frac{1}{2}\right)^\alpha.$$

This evidently contradicts (2.9). Thus we can show that

$$(2.13) \quad |w(z)| = |G'(z) - 1| < 1,$$

which implies that $\operatorname{Re}\{G'(z)\} > 0$ for $z \in \mathcal{U}$. This shows that $G(z)$ is a close-to-convex function in the unit disk \mathcal{U} , since

$$(2.14) \quad \operatorname{Re} \left(\frac{G'(z)}{g'(z)} \right) > 0 \quad (z \in \mathcal{U})$$

for $g(z) = z$. Consequently, for $g(z) = z$,

$$(2.15) \quad \operatorname{Re} \left(\frac{F'(a, b; c; z)}{g'(z)} \right) > 0 \quad (z \in \mathcal{U})$$

provided that $abc > 0$. Thus we can conclude that the hypergeometric function $F(a, b; c; z)$ is close-to-convex in the unit disk \mathcal{U} , and hence further that

$F(a,b;c;z)$ is univalent in the unit disk \mathcal{U} (cf. Duren [2], p. 47, Theorem 2.17). This completes the proof of Theorem 1.

Setting $\alpha = 0$ in Theorem 1 we have the following result:

COROLLARY 1. Let the hypergeometric function $F(a,b;c;z)$ defined by
(2.1) satisfy the condition

$$(2.16) \quad \left| F'(a,b;c;z) - \frac{ab}{c} \right| < \frac{ab}{c},$$

where $abc > 0$. Then $F(a,b;c;z)$ is univalent in the unit disk \mathcal{U} .

Furthermore, taking $\alpha = 1$ in Theorem 1 we obtain

COROLLARY 2. Let the hypergeometric function $F(a,b;c;z)$ defined by
(2.1) satisfy the condition

$$(2.17) \quad \left| \frac{zF''(a,b;c;z)}{F'(a,b;c;z)} \right| < \frac{1}{2},$$

where $abc > 0$. Then $F(a,b;c;z)$ is univalent in the unit disk \mathcal{U} .

3. STARLIKE HYPERGEOMETRIC FUNCTIONS

A set \mathcal{E} is said to be starlike with respect to $w_0 \in \mathcal{E}$ if the line segment joining w_0 to every other point $w \in \mathcal{E}$ lies entirely within \mathcal{E} . If a function $f(z)$ maps \mathcal{D} onto a domain that is starlike with respect to w_0 , then $f(z)$ is said to be starlike with respect to w_0 . In particular,

if w_0 is the origin, we say that $f(z)$ is a starlike function.

Let \mathcal{S} denote the subclass of \mathcal{A} consisting of analytic and univalent functions in the unit disk \mathcal{U} . Then a function $f(z)$ belonging to the class \mathcal{S} is said to be starlike if and only if

$$(3.1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0 \quad (z \in \mathcal{U}).$$

We denote by \mathcal{S}^* the subclass of \mathcal{S} consisting of all starlike functions in the unit disk \mathcal{U} . The following result can be proven with the aid of Lemma 1.

LEMMA 2. Let the function $f(z)$ defined by (1.1) satisfy the condition

$$(3.2) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-\alpha} \left| \frac{zf''(z)}{f'(z)} + 1 \right|^\alpha < \left(\frac{3}{2} \right)^\alpha \quad (z \in \mathcal{U})$$

for some $\alpha \geq 0$. Then $f(z)$ is in the class \mathcal{S}^* .

PROOF. Following the technique employed by Singh and Singh [6], we define the function $w(z)$ by

$$(3.3) \quad \frac{zf'(z)}{f(z)} = \frac{1+w(z)}{1-w(z)}$$

for $z \in \mathcal{U}$. Clearly, $w(z)$ has a zero at the origin. Differentiating both sides of (3.3) logarithmically, we get

$$(3.4) \quad \frac{zf''(z)}{f'(z)} = \frac{2zw'(z)}{[1+w(z)][1-w(z)]} + \frac{1+w(z)}{1-w(z)} - 1,$$

which readily yields

$$(3.5) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right|^{1-\alpha} \left| \frac{zf''(z)}{f'(z)} + 1 \right|^\alpha \\ = \left| \frac{2w(z)}{1-w(z)} \right| \left| \frac{zw'(z)}{w(z)} \cdot \frac{1}{1+w(z)} + \frac{1}{2} + \frac{1}{2w(z)} \right|^\alpha.$$

If

$$(3.6) \quad \operatorname{Re} \left(\frac{z_1 f'(z_1)}{f(z_1)} \right) = 0$$

for $z_1 \in \mathcal{U}$, and if (3.1) holds true for $|z| < |z_1|$, then we have

$$|w(z)| < |w(z_1)| = 1 \quad \text{for } |z| < |z_1| \quad \text{and } w(z_1) \neq 1.$$

Applying Lemma 1 to $w(z)$ at $z_1 \in \mathcal{U}$, and putting

$$(3.7) \quad z_1 w'(z_1) = kw(z_1) \quad (k \geq 1),$$

we obtain

$$(3.8) \quad \left| \frac{z_1 f'(z_1)}{f(z_1)} - 1 \right|^{1-\alpha} \left| \frac{z_1 f''(z_1)}{f'(z_1)} + 1 \right|^\alpha \geq \left(1 + \frac{k}{2} \right)^\alpha \geq \left(\frac{3}{2} \right)^\alpha$$

from (3.5). This obviously contradicts the condition (3.2). Hence (3.1) holds true for all $z \in \mathcal{U}$, that is, $f(z) \in \mathcal{S}^*$.

Next we establish

THEOREM 2. Let the hypergeometric function $F(a,b;c;z)$ defined by (2.1) satisfy the condition

$$(3.9) \quad \left| \frac{zF'(a,b;c;z)}{F(a,b;c;z)} \right| < 1 \quad (z \in \mathcal{U}).$$

Then the function $zF(a,b;c;z)$ is in the class \mathcal{S}^* .

PROOF. Define a function $H(z)$ by

$$(3.10) \quad H(z) = zF(a,b;c;z) \quad (z \in \mathcal{U}).$$

Then (3.9) implies that

$$(3.11) \quad \left| \frac{zH'(z)}{H(z)} - 1 \right| < 1$$

for $z \in \mathcal{U}$. Taking $\alpha = 0$ in Lemma 2, we immediately have $H(z) \in \mathcal{S}^*$, which is the assertion of Theorem 2.

THEOREM 3. Let the hypergeometric function $F(a,b;c;z)$ defined by

(2.1) satisfy the condition

$$(3.12) \quad \left| \frac{zF''(a,b;c;z)}{F'(a,b;c;z)} - 1 \right| < \frac{3}{2} \quad (z \in \mathcal{U}),$$

where $abc \neq 0$. Then $F(a,b;c;z)$ is starlike with respect to $c/(ab)$ in the
unit disk \mathcal{U} .

PROOF. Let the function $G(z)$ be defined by (2.5). Then $G(z) \in \mathcal{A}$
and

$$(3.13) \quad \left| \frac{zG''(z)}{G'(z)} - 1 \right| = \left| \frac{zF''(a,b;c;z)}{F'(a,b;c;z)} - 1 \right| < \frac{3}{2}$$

for $z \in \mathcal{U}$. Hence, by setting $\alpha = 1$ in Lemma 2, we find that $G(z) \in \mathcal{S}^*$.
Thus $G(z)$ is starlike with respect to the origin, and it follows from the

definition (2.5) that $F(a,b;c;z)$ is starlike with respect to $c/(ab)$ in the unit disk \mathcal{U} .

4. CONVEX HYPERGEOMETRIC FUNCTIONS

A function $f(z)$ belonging to the class \mathcal{S} is said to be convex if and only if

$$(4.1) \quad \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0 \quad (z \in \mathcal{U}).$$

We denote by \mathcal{K} the subclass of \mathcal{S} consisting of all convex functions in the unit disk \mathcal{U} . Then it is known that $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

Our next result (Theorem 4 below) involves a generalized hypergeometric function ${}_pF_q$ (with p numerator and q denominator parameters) defined by

$$(4.2) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q + 1),$$

where $(\lambda)_n$ is the Pochhammer symbol introduced in (2.2). Thus the hypergeometric function $F(a,b;c;z)$ defined by (2.1) can be rewritten, more precisely, as

$${}_2F_1(a,b;c;z).$$

Now we state and prove

THEOREM 4. Let the hypergeometric function $F(a,b;c;z)$ defined by
 (2.1) satisfy the condition (3.9). Then the function

$$z {}_3F_2(a,b,1;c,2;z)$$

is in the class \mathcal{H} .

PROOF. Since $z F(a,b;c;z)$ is in the class \mathcal{S}^* in view of the condition (3.9), we readily observe that

$$(4.3) \quad \int_0^z F(a,b;c;z) dz \in \mathcal{H},$$

and since

$$(4.4) \quad \int_0^z F(a,b;c;z) dz = z {}_3F_2(a,b,1;c,2;z),$$

Theorem 4 follows immediately.

5. UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

Let \mathcal{F} denote the class of functions of the form

$$(5.1) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0),$$

which are analytic and univalent in the unit disk \mathcal{U} . For this class of functions, Silverman [5] proved the following result:

LEMMA 3. Let the function $f(z)$ be defined by (5.1). Then $f(z)$ is in the class \mathcal{T} if and only if

$$(5.2) \quad \sum_{n=2}^{\infty} n a_n \leq 1.$$

On the other hand, the following useful result is due to Merkes and Scott [3]:

LEMMA 4. Let $0 \leq b \leq 2$ and $b \leq a < c$. Also let the hypergeometric function $F(a,b;c;z)$ be defined by (2.1). Then $z F(a,b;c;z)$ is univalent and starlike with respect to the origin in the unit disk \mathcal{U} .

Making use of Lemmas 3 and 4, we shall prove various inequalities involving hypergeometric series, which are given by Theorems 5 and 6 below.

THEOREM 5. Let the hypergeometric function $F(a,b;c;z)$ defined by (2.1) satisfy the condition (2.3) for some $\alpha \geq 0$, where $abc > 0$. Then

$$(5.3) \quad \sum_{n=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n (n-1)!} \leq \frac{ab}{c},$$

that is,

$$(5.4) \quad F(a+1, b+1; c+1; 1) \leq 1,$$

provided further that $c > a + b$.

PROOF. First of all we note that, in view of Theorem 1, $F(a,b;c;z)$ is univalent in the unit disk \mathcal{U} . Consequently, the function $\phi(z)$ defined by

$$(5.5) \quad \Phi(z) = 2z + \frac{c}{ab} \{1 - F(a,b;c;z)\}$$

is also univalent in the unit disk \mathcal{U} , and it has the following series expansion

$$(5.6) \quad \Phi(z) = z - \frac{c}{ab} \sum_{n=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

Thus $\Phi(z)$ belongs to the class \mathcal{F} , and upon applying Lemma 3 to $\Phi(z)$, we arrive at the assertion (5.3) of Theorem 5.

REMARK. By Gauss's theorem:

$$(5.7) \quad F(\lambda, \mu; \nu; 1) = \frac{\Gamma(\nu)\Gamma(\nu-\lambda-\mu)}{\Gamma(\nu-\lambda)\Gamma(\nu-\mu)}, \quad \operatorname{Re}(\nu) > \operatorname{Re}(\lambda+\mu),$$

the assertion (5.4) of Theorem 5 immediately leads to

COROLLARY 3. Under the hypotheses of Theorem 5,

$$(5.8) \quad \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq 1,$$

provided further that $c > a + b$.

THEOREM 6. Let $0 \leq b \leq 2$ and $b \leq a < c$. Also let the hypergeometric function $F(a,b;c;z)$ be defined by (2.1). Then

$$(5.9) \quad \sum_{n=1}^{\infty} \frac{(n+1) (a)_n (b)_n}{(c)_n n!} \leq 1,$$

that is,

$$(5.10) \quad {}_3F_2(a, b, 2; c, 1; 1) \leq 2,$$

provided further that $c > a + b + 1$.

PROOF. Since $z F(a, b; c; z)$ is univalent in the unit disk \mathcal{U} , by virtue of Lemma 4, the function $\Psi(z)$ defined by

$$(5.11) \quad \begin{aligned} \Psi(z) &= 2z - z F(a, b; c; z) \\ &= z - \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^{n+1}}{n!} \end{aligned}$$

is also univalent in the unit disk \mathcal{U} . Thus the proof of Theorem 6 is completed by merely appealing to Lemma 3.

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