C*–ALGEBRA EXTENSIONS OF C(X)

by

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Contents

Abstract

We show that the Weyl-von Neumann theorem for unitaries holds for σ-unital AF-algebras and their multiplier algebras. By studying $E(X, A)$, the quotient of $\mathbf{Ext}(C(X), A)$ by a special class of trivial extensions which we call totally trivial extensions, we give a BDF-type classification for extensions of $C(X)$ by a σ-unital purely infinite simple C^*-algebra with trivial K_1-group (e.g., the C^*-algebra $O_n \otimes \mathcal{K}$). We also show that every extension of $C(X)$, where X is a compact subset of the plane, by a finite matroid C^*-algebra is totally trivial. Classification of these extensions for nice spaces is given. Some other versions of the Weyl-von Neumann-Berg theorem are also given.

Key words: Extensions, C^*-algebra with real rank zero, BDF-theory, the Weyl-von Neumann-Berg theorem

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§ 0. Introduction

In 1970's, motivated by the problem of classifying essential normal operators on the separable, infinite dimensional Hilbert space, Brown, Douglas and Fillmore developed their celebrated BDF-theory which classifies the C^*-algebra extensions of the following form:

$$0 \to \mathcal{K} \to E \to C(X) \to 0,$$

where $\mathcal{K}$ is the C^*-algebra of compact opereators on the separable Hilbert space and X is a compact metric space. Since then the theory of C^*-algebra extensions have been developed rapidly. (It is difficult to give a complete list of references. We refer the readers to [Bl] for reference.) The theory provides many effective ways to compute the Ext-group $Ext(B, A)$ which gives nice pictures of the following extension of B by A :

$$0 \to A \to E \to B \to 0.$$

However, unlike the original BDF-theory which gives the complete classification of extensions of $C(X)$ by $\mathcal{K}$, $Ext(B, A)$ does not give a complete information for the extensions in general due to the following facts:
(1) in general there are many different equivalence classes of trivial extensions;
(2) if τ is such an extension and $[\tau] = 0$ in $Ext(B, A)$, it is not clear at all that τ is in fact trivial;
(We know only that there is a trivial extension τ' such that $\tau \oplus \tau'$ is trivial.)
(3) if A is not stable, $Ext(B, A)$ gives very little useful information for the above extension, if it gives any.
That is the main reason for us to study $E(X, A)$.

Let A be a (σ-unital) C^*-algebra. We denote by $M(A)$ the multiplier algebra (the set $\{x \in A^{**} : xa, ax \in A, \forall a \in A\}$) of A. When $A = \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite dimensional

Hilbert space H, it is well known that $M(A) = B(H)$, the C^*-algebra of the bounded operators on H.
Let there be a short exact sequence

$$0 \to A \to E \to B \to 0.$$

The C^*-algebra A sits in E as an ideal of E. Hence there is a $*$-homomorphism σ from E into $M(A)$ (see the proof of [Pd1, 3.12.8]). The map σ is injective if and only if A is essential in E. If we compose σ with the quotient map $\pi : M(A) \to M(A)/A$, we obtain a *-homomorphism τ from $E/A \cong B$ into $M(A)/A$. The map τ is injective if and only if A is essential in E. We call τ essential if τ is injective; if B is unital, and τ is a unital map, we call τ unital. We call the map τ an extension corresponding to the exact sequence. All extensions in this paper are unital and essential extensions. There are a number of equivalence relations concerning extensions. In this paper we consider only the following equivalence: if τ_1 and τ_2 are two extensions then τ_1 is (unitarily) equivalent to τ_2 if and only if there exists a unitary $u \in M(A)$ such that

$$\tau_2(x) = \pi(u)^*\tau_1(x)\pi(u)$$

for all $x \in B$.

Let $\mathbf{Ext}(B, A)$ denote the equivalence classes of (unital essential) extensions of B by A, and $Ext(B, A)$ be the quotient of $\mathbf{Ext}(B, A)$ by the set of trivial extensions (an extension $\tau : B \to M(A)/A$ is trivial if there is a monomorphism $\sigma : B \to M(A)$ such that $\tau = \pi \circ \sigma$). Applying the theory of Ext-group (and KK-theory), one can calculate, in many cases, the group $Ext(B, A)$ when A is some separable, stable and nuclear C^*-algebra. In the case that $A = \mathcal{K}$ and $B = C(X)$, where X is a compact metric space, from BDF-theory ([BDF1] and [BDF2]) we know that all trivial extensions are equivalent. Therefore those extensions are completely determined by $Ext(C(X), A)$ (up to unitary equivalence). An important fact related to this case is the Weyl-von Neumann-Berg theorem holds:

for any normal element $N \in M(\mathcal{K})$ there exist $D \in M(\mathcal{K})$ and $a \in \mathcal{K}$ such that

$$N = D + a,$$

where D is a diagonal normal element with spectrum $sp(D) = sp(\pi(N))$, (π is the quotient map from $B(H)$ onto $B(H)/\mathcal{K}$). The Weyl-von Neumann theorem for self-adjoint elements for other C^*-algebras has been studied by a number of authors (see [BP],[Zh1,2,3,4],[HR] and [Lin2,3]). It is known (see [Zh1]) that this problem is equivalent to ask if the multiplier algebras have real rank zero. (Recall that a C^*-algebra is said to have real rank zero if the set of self-adjoint elements with finite spectra is dense in $A_{s.a.}$. If A has real rank zero, we will write $RR(A) = 0$.(see [BP])) The problem whether the Weyl-von Neumann theorem holds for AF-algebras had been open since George A. Elliott first raised this quesion in 1974 at Tôhoku (The question was also raised formally by L.G.Brown and G.K. Pedersen in 1988 [BP]). It was recently shown by the author that the multiplier algebras of AF-algebras (and many others) have real rank zero (see [Lin2]). So if $x \in M(A)_{s.a.}$, where A is a σ-unital AF-algebra, then there exist an approximate identity $\{e_n\}$ for A consisting of projections and $a \in A$ such that

$$x = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}) + a,$$

where $\{\lambda_n\}$ is a bounded sequence of real numbers (and $\|a\|$ is smaller than any given number). A unital C^*-algebra A is said to have (FU) if the set of unitaries with finite spectra is dense in the unitary group of A. If $M(A)$ has (FU), then the Weyl-von Neumann theorem holds for unitaries. (See [Lin3].) The problem when $M(A)$ has (FU) has been studied in [Lin3]. The Berg theorem, i.e. the Weyl-von Neumann theorem for normal elements, turns out much more complicated. We show in the section 2 that the Weyl-von Neumann theorem for unitaries holds for all σ-unital AF-algebras and other C^*-algebras with (FU) (see § 2). One of our main results is that the Berg theorm holds for (nonunital) separable purely infinite simple C^*-algebras with $K_1(A) = 0$ (for example, $A \cong O_n \times \mathcal{K}$) and for (nonunital) separable simple AF-algebras with unique normalized trace (up to scalar multiples) (for example, finite matroid algebras). By studying $E(X, A)$ the quotinet of $\mathbf{Ext}(C(X), A)$ by a special class of trivial extensions E which we call totally trivial extensions (see the definition in Section 1), we show that BDF-theory holds for σ-unital purely infinite simple C^*-algebras A with $K_1(A) = 0$ (see Section 7), in particular, for $A = O_n \otimes \mathcal{K}$. We show that $E(X, A)$ becomes a group and $E(\cdot, A)$ is a homotopy invariant covariant functor from metrizable spaces to abelian groups, provided that A is a separable, stable C^*-algebra with $RR(M(A)) = 0$.

While our methods are undoubtedly based on original BDF-theory, other techniques involving corona constructions, strict topology, the generalized Weyl-von Neumann theory, etc., are used.

The article is organized as follows. Chapter one gives definition of totally trivial extensions and some elementary properties of totally trivial extensions. The semi-group $E(X, A)$ is introduced in § 1. In the same chapter, we show that the multiplier algebras $M(A)$ of σ-unital AF-algebras and other C^*-algebras with (FU) has the property (FU). So the Weyl-von Neumann theorem for unitaries holds for these algebras. In chapter two, we study the semi-group $E(X, A)$. After some preliminaries in § 3, we show the exactness of the sequence

$$E(F, A) \to E(X, A) \to E(X/F, A)$$

in § 4 for σ-unital stable C^*-algebra A with $RR(M(A)) = 0$. Under the same assumptions, we show that (in § 5) that $E(X, A)$ is a group and (in § 6) that $E(\cdot, A)$ is a homotopy invariant covariant functor from metrizable spaces to abelian groups. In chapter three, we give some BDF type classification for extensions of $C(X)$ by σ-unital purely inifinite simple C^*-algebras with $K_1(A) = 0$ and other Weyl-von Neumann theorem for some one dimensional spaces. The last chaper, the chapter four, is contributed solely to finite matroid algebras. § 10 contains some absorption lemmas which are needed in § 11. In § 11, We show that every extension of $C(X)$, where X is a compact subset of the plane, by a finite matroid algebra is totally trivial. (We notice that, as in [Lin4,3.12] is certainly not the case for stable matroid algebras.) Consequently, the Weyl-von Neumann-Berg theorem holds for finite matroid C^*algebras. Finally, in § 12, we give some classification of extensions of C(X) for some nice spaces by finite matroid algebras. For example, we show that all these extensions are equivalent for disks, annuli and "Hawaiian ear rings". We also determine the equivalence classes of extensions for the cases that X has finitely many such components.

Before we end this the section, we would like to give some examples of C^*-algebras which are considered in this paper.

(1) C^*-algebras with $RR(M(A)) = 0$:

i) All σ-unital AF-algebras (see [Lin2]).

ii) All σ-unital purely simple C^*-algebras with $K_1(A) = 0$ (see [Zh2] and see [LZ] for equivalent definitions for these algebras).

(2) C^*-algebras with the property that $M(A)$ has (FU) :

i) All matroid algebras. Those are the C^*-algebras defined by the inductive limits

$$M_{n(1)} \to M_{n(2)} \to \dots \to M_{n(k)} \to \dots .$$

ii) All σ-unital purely simple C^*-algebras with $K_1(A) = 0$.

iii) We will show in the section 2 that the multiplier algebra of any σ-unital AF-algebra has the property (FU).

(3) A typical example of separable stable purely infinite simple C^*-algebra with $K_1(A) = 0$ is $O_n \otimes \mathcal{K}$ (or a nonunital hereditary C^*-subalgebra of O_n), where O_n is the C^*-algebra generated by n isometries $s_1, s_2, ..., s_n$ such that $s_i^* s_i = 1, s_i s_i^* = p_i$ and $\sum_{i=1}^n p_i = 1$. (See [Cu].)

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1 Totally Trivial Extensions

§ 1. Totally trivial extensions

Before we begin our discussion, we would like to review a few terminologies.

Definition 1.1. Let A be a (non-unital) C^*-algebra, B a unital C^*-algebra. An *extension of B by A* is a unital $*$-monomorphism τ from B into the corona algebra $M(A)/A$, where $M(A)$ is the multiplier algebra of A. We would like to remind the readers that there are more general definition for extensions, but for the simplicity, we consider only these (unital and essential) extensions. An extension $\tau : B \to M(A)/A$ gives the following short exact sequence

$$0 \to A \to E \to B \to 0$$

which is uniquely determined by the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & E & \to & B & \to & 0 \\ & & \| & & \downarrow & & \downarrow \tau & & \\ 0 & \to & A & \to & M(A) & \to & M(A)/A & \to & 0 \end{array}$$

Since we will work in the multiplier algebras very often, we would remind the readers of the strict topology. The *strict topology* on the multiplier $M(A)$ of a (non-unital) C^*-algebra A is the topology generated by the semi-norms:

$$p_a(x) = \|a \cdot x\| + \|x \cdot a\|$$

for $a \in A$ and $x \in M(A)$(see [Bu]).

Definition 1.2. Recall that an extension $\tau : B \to M(A)/A$ is *trivial* if there is a unital $*$-monomorphism $\sigma : B \to M(A)$ such that $\tau = \pi \circ \sigma$.

Definition 1.3. Two extensions τ_1 and τ_2 are said to be *(unitarily) equivalent* if there is a unitary $u \in M(A)$ such that

$$u^* \tau_1(f) u = \tau_2(f)$$

for all $f \in B$.
Suppose that $B_1 = pAp$ and $B_2 = qAq$, where $p, q \in M(A)$ are projections and

$$\tau_1 : B \to M(B_1)/B_1 \quad , \tau_2 : B \to M(B_2)/B_2.$$

Notice that $M(B_1) = pM(A)p$,$M(B_1)/B_1 \cong \pi(p)(M(A)/A)\pi(p)$ and $M(B_2) = qM(A)q$, $M(B_2)/B_2 \cong \pi(q)M(A)/A\pi(q)$. We say τ_1 is (unitarily) equivalent to τ_2 if there is a partial isometry $v \in M(A)$ such that $v^* \tau_1 v = \tau_2$.
As shown in [Lin4], trivial extensions are in general not necessarily equivalent. So we need to study trivial extensions further. This, with many other reasons, leads us to the following definition.

Definition 1.4. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$ and let $\tau : C(X) \to M(A)/A$ be an extension of $C(X)$ by A. The extension τ is said to be totally trivial if $\tau(C(X))$ is contained in an abelian (separable) AF-C^*-subalgebra of $M(A)/A$.*

It should be noted that if B is an abelian (separable) AF-C^*-algebra then either $B \cong C(X)$ (or $B \cong C_o(X)$ if B is not unital), where X is a totally disconnected compact (or locally compact) Hausdorff space. Since we consider only those unital extensions , we may assume that the abelian AF-algebra in the definition is unital.

Proposition 1.5. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$. If $\tau : C(X) \to M(A)/A$ is a totally trivial extension, then there exist an approximate identity $\{e_n\}$ for A consisting of projections and a dense* *sequence $\{\lambda_n\}$ contanined in X with isolated points repeated infinitely often, and a monomorphism $\sigma : C(X) \to M(A)$ such that*

$$\sigma(f) = \sum_{i=1}^{\infty} f(\lambda_i)(e_i - e_{i-1})$$

and $\sigma = \pi \circ \tau$, where the sum converges in the strict topology.

Proof: By the definition, there is an abelian AF-C^*-subalgebra $B \subset M(A)/A$ such that $\tau(C(X)) \subset B$. We may assume that $1 \in B$. So B is generated by projections and therefore B is generated by a single non-negative function g ([R, p 293]). Then B is isomorphic to $C(\tilde{X})$, where a totally disconnected compact subset of $\mathbf{R}$. We may also assume that $sp(g) = \tilde{X}$. Since $RR(M(A)) = 0$, by [Zh1,3.1], there is $h \in M(A)_{s.a.}$ such that $\phi(h) = g$. It follows from [Lin4,2.4] that we may assume that

$$h = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}),$$

where $\lambda_n \in \tilde{X}$, $\{e_n\}$ is an approximate identity for A consisting of projections and the sum converges in the strict topology. If we identify B with $C(\tilde{X})$, then the functional calculus $\sigma'(f) = f(h)$ defines a monomorphism $\sigma' : B \to M(A)$. The inclusion $im\tau(C(X)) \subset B \cong C(\tilde{X})$ gives a surjection $\varphi : \tilde{X} \to X$. So for any $f \in C(X)$

$$\sigma'(\tau(f)) = \sigma'(\tau(f) \circ \varphi) = f \circ \varphi(h).$$

Now let $\sigma = \sigma'|_{C(X)}$, then

$$\sigma(f) = f \circ \varphi(h) = \sum_{i=1}^{\infty} f(\varphi(\lambda_i))(e_n - e_{n-1}).$$

Notice that $\{\varphi(\lambda_n)\}$ is dense in X and $\tau = \pi \circ \sigma$. The requirement that $\lambda = \varphi(\lambda_n)$ for infinitely many n when λ is isolated is automatic, for otherwise we would have $\tau(f) = 0$ for f being the characteristic function of $\{\lambda\}$.
Q.E.D.

The following is the converse of 1.5.

Proposition 1.6. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$. Suppose that $\tau : C(X) \to M(A)/A$ is an extension. If there is an monomorphism $\sigma : C(X) \to M(A)$ such that*

$$\sigma(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1}),$$

where $\{e_n\}$ is an approximate identity for A consisting of projections and $\{\lambda_n\}$ is a dense sequence contained in X (with isolated points repeated infinitely often), and $\pi \circ \tau = \sigma$. Then τ is totally trivial.

Proof: Select a countable basis $\{O_k\}$ for the topology of X. Set

$$p_k = \sum_{\lambda_{k(n)} \in O_k} (e_{k(n)} - e_{k(n)-1}).$$

For any $a \in A$ and $\varepsilon > 0$, there is an integer N such that

$$\|e_n a - a\| < \varepsilon \,, \|a e_n - a\| < \varepsilon,$$

if $n \geq N$. So

$$\| \sum_{\lambda_{k(n)} \in O_k \,, k(n) > N} (e_{k(n)} - e_{k(n)-1}) a \| < \varepsilon$$

and

$$\|a \sum_{\lambda_{k(n)} \in O_k \,, k(n) > N} (e_{k(n)} - e_{k(n)-1}) \| < \varepsilon.$$

Therefore the sum converges in the strict topology and p_k is a projection in $M(A)$. So $\{p_k\}$ generates an abelian AF-algebra. Therefore its image B under π is an abalian AF-C^*-subalgebra of $M(A)/A$. Moreover, it is easy to see that $im\tau \subset B$.

Q.E.D.

Corollary 1.7. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$. If X is a compact subset of the real line then any extension $\tau : C(X) \to M(A)/A$ is totally trivail.*

Proof: This follows immdiately from [Lin4,2.4].
Q.E.D.

Corollary 1.8. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$. If X is totally disconnected, then every extension $\tau : C(X) \to M(A)/A$ is totally trivial.*

Proof: If X is totally disconnected, then $C(X)$ is generated by projections and therefore is generated by a single non-negative function g. [R,p.293]. Then X is homeomorphic to $im(g)$ and 1.4 applies.
Q.D.E.

For the rest of this section A is always a stable,σ-unital C^*-algebra with $RR(M(A)) = 0$.

1.9. Fix an isomorphism of A with $M_2(A)$; this isomorphism induces an isomorphism $M(A) \cong M_2(M(A))$, $M(A)/A \cong M_2(M(A)/A)$. These isomorphisms are called *standard* isomorphisms and are uniquely determined up to unitary equivalence. If τ_1,τ_2 are extensions of B by A, then the sum $\tau_1 + \tau_2 : B \to M(A)/A \oplus M(A)/A \subset M_2(M(A)/A) \cong M(A)/A$ is also an extension of B by A, where the last isomorphism is a standard isomorphism.

Proposition 1.10. *The sum of two totally trivial extensions is a totally trivial extension.*

Proof: The direct sum of two (orthogonal) abelian AF C^*-subalgebras in $M(A)/A$ is an abelian AF-C^*-algebra.
Q.E.D.

Definition 1.11. *Let* $\mathbf{Ext}(C(X), A)$ *denote the (unitary) equivalence classes of (unital essential) extensions by A of $C(X)$. It is known that* $\mathbf{Ext}(C(X), A)$ *is an abelian semigroup. By Proposition 1.10, the set of equivalence classes of totally trivial extensions forms a subsemigroup. We denote by $E(X, A)$ the quotient of* $\mathbf{Ext}(C(X), A)$ *by the subsemigroup of totally trivial extensions. If we denote by E the subsemigroup of the equivalent classes of totally trivial extensions, then $[a] = [b]$ if and only if there are $e, e' \in E$ such that $a + e = b + e'$. We notice now that $E(X, A)$ form an abelian semigroup by the obvious addition.*

Proposition 1.12. *$E(\cdot, A)$ is a covariant functor on compact metrizable spaces to abelian semigroups.*

Proof: Given a continuous function $f : X \to Y$ and an extension $\tau : C(X) \to M(A)/A$. Let $\tau_Y : C(Y) \to M(A)/A$ be a totally trivial extension and define

$$f_*(\tau) = \tau \circ f^* + \tau_Y .$$

If τ and τ' are equivalent, then it is clear that $f_*(\tau)$ is equivalent to $f_*(\tau')$. If $\tau'_Y : C(Y) \to M(A)/A$ is another totally trivial extension, then

$$\tau \circ f^* + \tau_Y + \tau'_Y = \tau \circ f^* + \tau'_Y + \tau_Y .$$

So $[\tau \circ f^* + \tau_Y] = [\tau \circ f^* + \tau'_Y]$ in $E(Y, A)$. It is clear that f^* preserves the semigroup structure, and that $(id_*)_* = id_{E(X,A)}$ and $(gf)_* = g_* f_*$, where $g : Y \to Z$ is continuous.

Proposition 1.13. *If $f : X \to Y$ be a continuous map and $\tau : C(X) \to M(A)/A$ is a totally trivial, then $f_*(\tau)$ is totally trivial.*

Proof: This is obvious.
Q.E.D.

§ 2. The Weyl-von Neumann Theorem for Unitaries

As indicated in the introduction, the main result in this section is that if u is a unitary in the multiplier algebra $M(A)$ of a σ-unital AF-algebra A, then, for any $\varepsilon > 0$, there exist an approximate identity $\{e_n\}$ for A consisting of projections and $a \in A$ such that

$$u = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1}) + a \, , (e_o = 0)$$

where $\|\lambda_n\| = 1$, the sum converges in the strict topology and $\|a\| < \varepsilon$.

We start with a few technical terminologies which may be found in [Ph1].

Definition 2.1.([Ph1,1.1.2]) Let A be a unital C^*-algebra. The *exponetial rank* of A, written $cer(A)$, is the largest element of the set of symbols

$$\{1, 1+\varepsilon, 2, 2+\varepsilon, ..., \infty\}$$

(with the obvious order) consistent with the following restrictions:
(1)$cer(A) \leq n$ if every $u \in Uo(A)$ is the product

$$exp(ih_1) \cdot exp(ih_2) \cdot ... \cdot exp(ih_n)$$

for some $h_1, ..., h_n \in A_{s.a.}$.
(2) $cer(A) \leq n+\varepsilon$ if every $u \in U_o(A)$ is norm limit of the products as in (1). For non-unital C*-algebra A, set $cer(A) = cer(\tilde{A})$, where $\tilde{A}$ is the C^*-algebra obtained by joining an identity to A.

Definition 2.2. (cf [Ph1,1.1]) A C^*-algebra A is said to have weak (FU) if A has real rank zero and $cer(A) \leq 1+\varepsilon$. A C^*-algebra A is said to have (FU) if A has weak (FU) and the unitary group of $\tilde{A}$ is connected; Or equivalently, the set of unitaries in $\tilde{A}$ with finite spectra is (norm) dense in the unitary group of $\tilde{A}$. A C^*-algebra is said to have stable (weak) (FU) if for every n, $M_n(A)$, the $n \times n$ matrices over A, has (weak) (FU). We notice that if A has stable (weak) (FU), then by [Ph1,1.7], $A \otimes \mathcal{K}$ has (weak) (FU).

Lemma 2.3. *Let A be a unital C^*-algebra with $RR(A) = 0$ and let u be a unitary in A. Suppose that there is a projection p in A such that $pu = up$ and the spectrum of pu, as an element in pAp, is not the whole unit circle. If u can be approximated (in the norm topology) by unitaries in A with finite spectra, then $(1-p)u$ can also be approximated (in the norm topology) by unitaries in $(1-p)A(1-p)$ with finite spectra.*

Proof: Suppose that the spectrum of pu, as an element in pAp, is F. Suppose that $S^1 \setminus F$ contains an arc with length $d > 0$. Fix a closed subset

Ω of $S^1 \setminus F$ such that Ω contains an arc with length at least $d/2$. Let f and g be two continuous functions defined on S^1 such that $0 \le f \le 1$, $f(z) = 0$ if $z \in F$, $f(z) = 1$ if $z \in \Omega$, and $0 \le g \le 1, g(z) = 1$ if $z \in F$ and $g(z) = 0$ if $z \in \Omega$. For any $\delta > 0$, by [Ch,Lamma 2], there is $0 < \eta \le \delta$ such that for any two unitaries u,w in A if $\|w - u\| < \eta$, then

$$\|f(w) - f(u)\| < \delta$$

and

$$\|g(w) - g(u)\| < \delta.$$

There are mutually orthogonal projections $p_1, p_2, ..., p_n$ in A and complex numbers $\lambda_1, \lambda_2, ..., \lambda_n$ with $|\lambda_i| = 1$, $i = 1, 2, ..., n$ such that

$$\|u - \sum_{i=1}^{n} \lambda_i p_i\| < \eta.$$

Set $w = \sum_{i=1}^{n} \lambda_i p_i$, $w_1 = \sum_{\lambda_i \in \Omega} \lambda_i p_i$, $w_2 = \sum_{\lambda_i \notin \Omega} \lambda_i p_i$ and $q = \sum_{\lambda_i \in \Omega} p_i$. Since q commutes with w and p commutes with u, we have $f(w)q = qf(w) = q$ and $g(u)p = pg(u) = p$. Therefore

$$\|qp\| = \|qf(w)g(u)p\|$$

$$\le \|qf(u)g(u)p\| + \|q[f(w) - f(u)]p\| < \delta$$

and

$$\|q - (1-p)q(1-p)\| < 2\delta.$$

If $\delta < 1/4$, by [Eff,A8], there is a projection $q' \le 1 - p$ such that

$$\|q' - q\| < 2\delta$$

and there is a unitary $v_1 \in A$ such that

$$\|v_1 - 1\| < 4\delta \, , v_1^* q v_1 = q'.$$

Thus

$$\|v_1^* w_1 v_1 - w_1\| < 8\delta.$$

Then

$$\begin{aligned}
&\|(1-p)w_2 - w_2(1-p)\| \\
&\le \|(1-p)qw_1 - w_1(1-p)\| + \|(1-p)w - w(1-p)\| \\
&\le \|(1-p)qw_1 - w_1q(1-p)\| + 2\eta \\
&< 2(\delta+\eta).
\end{aligned}$$

Since

$$\|v_1^* w_2 v_1 - w_2\| < 8\delta,$$

we obtain

$$\begin{aligned}
&\|(1-p)(v_1^* w_2 v_1) - (v_1^* w_2 v_1)(1-p)\| \\
&\le 2(\delta+\eta) + 16\delta = 32\delta + 2\eta.
\end{aligned}$$

Put $p' = (1-p) - q'$. Since

$$q' v_i^* w_2 v_1 = v_1^* w_2 v_1 q' = 0,$$

then

$$\|p'(v_1^* w_2 v_1) - (v_1^* w_2 v_1)p'\| < 32\delta + 2\eta.$$

Since Ω contains an arc with length $> d/2$, for any $\varepsilon > 0$, by Lin2,1.2], if δ is small enough (δ depends only on d), there is $h \in A_{s.a.}$ such that

$$\|p' + \sum_{n=1}^{\infty} (ip'hp')^n/n! - p' v_1^* w_2 v_1 p'\| < \varepsilon/4$$

and $\delta < \varepsilon/2$. Then

$$\|(1-p)u - v_1^* w_1 v_1 - (p' + \sum_{n=1}^{\infty} (ip'hp')^n/n!)\| < \varepsilon/2.$$

It follows from [BP,2.8] that $p'Ap'$ has real rank zero. Hence the selfadjoint element $p'hp'$ can be approximated by selfadjoint elements in $p'Ap'$ with finite spectra.Therefore there are mutually orthogonal projections $q_1, q_2, ..., q_m$ in

$p'Ap'$ with $\sum_{k=1}^m q_k = p'$ and complex numbers $\alpha_1, \alpha_2, ..., \alpha_m$ with $|\alpha_i| = 1$ such that

$$\|p' + \sum_{n=1}^{\infty}(ip'hp')^n/n! - \sum_{k=1}^{m} \alpha_k q_k\| < \varepsilon/2.$$

Finally, we conclude that

$$\|(1-p)u - \sum_{\lambda_i \in \Omega} \lambda_i(v_i^* p_i v_i) - \sum_{k=1}^{m} \alpha_k q_k\| < \varepsilon.$$

Q.E.D.

Lemma 2.4. *Let A be a σ-unital C^*-algebra with*

$$RR(M(A \otimes \mathcal{K})/A \otimes \mathcal{K}) = 0$$

and $u \in U_o(M(A)/A)$. Suppose that $\{e_{ij}\}$ is a matrix unit for $\mathcal{K}$, and π is the quotient map: $M(A \otimes \mathcal{K}) \to M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$. If we identify A with $(1 \otimes e_{11})A(1 \otimes e_{11})$, then for any dense sequence $\{\lambda_n\}_{n=2}^{\infty}$ in the unit circle and $\varepsilon > 0$, there is unitary $v \in M(A \otimes \mathcal{K})/A \otimes \mathcal{K})$ with finite spectrum such that

$$\|u + \sum_{n=2}^{\infty} \lambda_n 1 \otimes e_{nn} - v\| < \varepsilon.$$

Proof: Suppose that $\{a_n\}$ is an approximate identity for A consisting of projections. Set

$$e_n = \sum_{i=2}^{n} a_n \otimes e_{ii}\, , n = 2, 3, ...$$

It is easy to see that $\{e_n\}$ forms an approximate identity for $p(A \otimes \mathcal{K})p$, where $p = \sum_{i=2}^{\infty} 1 \otimes e_{ii}$ is a projection in $M(A)$. Moreover, one can check that $\sum_{i=2}^{m} 1 \otimes e_{ii}$ coverges to p in the strict topology.

Now define

$$u_o = \sum_{n=2}^{\infty} \lambda_n 1 \otimes e_{nn}.$$

Since $\sum_{i=2} 1 \otimes e_{ii}$ converges to p strictly, one sees that u_o is a unitary in $p(M(A \otimes \mathcal{K})p$. Suppose that $\lambda_n = e^{i\delta_n}$, $0 \le \delta_n \le 2\pi$. Then $u_o = p(exp(ih))$, where $h = \sum_{n=2} \delta_n 1 \otimes e_{nn} \in p(M(A \otimes \mathcal{K})p$. Let $\bar{h} = \pi(h)$, then $\pi(u_o) = \pi(p)exp(i\bar{h})$. Clearly, $u + \pi(u_o) \in U_o(M(A \otimes \mathcal{K})/A \otimes \mathcal{K}))$. For any $\varepsilon > 0$, there are $u_1, u_2, ..., u_k \in U_o(M(A \otimes \mathcal{K})/A \otimes \mathcal{K})$ along a path connecting $u + \pi(u_o)$ to the identity such that

$$\|u_{i+1} - u_i\| < \epsilon/3 \, , \|u_k - 1\| < \epsilon/3$$

and

$$\|u + \pi(u_o) - u_1\| < \epsilon/3.$$

By [Ph2,5], and the fact that

$$RR(M_k(M(A \otimes \mathcal{K})/A \otimes \mathcal{K})) = 0,$$

there are mutually orthogonal projections

$$q_2, q_3, ..., q_m \in M_{2k+1}(M(A \otimes \mathcal{K})/A \otimes \mathcal{K}))$$

and complex numbers $\alpha_2, \alpha_3, ..., \alpha_m$, $|\alpha_i| = 1, i = 2, 3, ..., m$ such that

$$\|u' - \sum_{i=2}^{m} \alpha_i q_i\| < \epsilon/3,$$

where $u' = diag(u_1^*, u_1, u_2^*, u_2, ..., u_k^*, u_k, 1)$. Without loss of generality, we may assume that $\alpha_2 = \lambda_2$, $\alpha_3 = \lambda_3, ..., \alpha_m = \lambda_m$. Since A is stable, for each n,($n = 2, ..., m$) there are partial isometries w_n and w_n' in $M_{2k+1}(M(A \otimes \mathcal{K})/A \otimes \mathcal{K})$ such that

$$w_n^* w_n = 1 \otimes e_{nn} \, , w_n w_n^* = p_n,$$

$$w_n'^* w_n = q_n' \, , w_n' w_n^* = q_n$$

and $1 \otimes e_{nn} = p_n + q_n'$.

Notice that for any j,

$$M_j(M(A \otimes \mathcal{K})/A \otimes \mathcal{K}) \cong M(A \otimes \mathcal{K})/A \otimes \mathcal{K}$$

via an isometry. Therefore there is a partial isometry v such that

$$v^*(u \oplus \pi(u_o))v = [u \oplus u\pi(u_o)] \oplus \sum_{i=2}^{m} \alpha_i q_i.$$

Put

$$u'' = diag(u \oplus \pi(u_o), [u \oplus \pi(u_o)]^*, u_1, u_1^*, ..., u_k, u_k^*).$$

Then

$$\|u \oplus \pi(u_o) \oplus u' - u''\| < \epsilon/3$$

and

$$\|u \oplus \pi(u_o) \oplus \sum_{i=2}^{m} \alpha_i q_i - u''\| < \epsilon/3 + \epsilon/3.$$

By [Ph2,5] again, there are mutually orthogonal projections

$$d_1, d_2, ..., d_l \in M(A \otimes K)/A \otimes K$$

and complex numbers $\beta_1, \beta_2, ..., \beta_l$ with $|\beta_i| = 1$ such that

$$\|u'' - \sum_{i=1}^{l} \beta_i d_i\| < \epsilon/3.$$

So

$$\|u \oplus \pi(u_o) - \sum_{i=1}^{l} \beta_i v d_i v^*\| < \epsilon.$$

Notice that vd_iv^* are mutually orthogonal projections.
Q.E.D.

Theorem 2.5. *Let A be a σ-unital stable C^*-algebra with $RR(M(A)/A) = 0$. Then*

$$cer(M(A)/A) \leq 1 + \epsilon.$$

Consequently, $M(A)/A$ has weak (FU).

Proof: Let $u \in U_o(M(A)/A)$. We will keep the notations used in the proof of Lemma 2.4. Put

$$g = \sum_{\lambda_i \in F} 1 \otimes e_{ii},$$

where F is the part of the unit circle in the closed right plane. It is easily checked that the sum converges to g in the strict topology. Hence $g \in M(A\otimes\mathcal{K})$. Moreover, both $\sum_{\lambda_i\in F}\lambda_i 1\otimes e_{ii}$ and $\sum_{\lambda_i\notin F}\lambda_i 1\otimes e_{ii}$ are in $M(A)$. Set $\bar{g}=\pi(g)$. Clearly the spectrum of $\pi(\sum_{\lambda_i\in F}\lambda_i 1\otimes e_{ii})$, as an element in $\bar{g}M(A)/A\bar{g}$, lies in the closed right plane. By lemma 2.3, we conclude that

$$u\oplus\pi(\sum_{\lambda_i\notin F}\lambda_i 1\otimes e_{ii})$$

can be approximated by unitaries in $(1-\bar{g})M(A\otimes\mathcal{K})/A\otimes\mathcal{K}(1-\bar{g})$ with finite spectra. Since

$$sp[\pi(\sum_{\lambda_i\notin F}\lambda_i 1\otimes e_{ii})]\subseteq S^1\cap\{z: re(z)\le 0\},$$

by applying 2.3 again, we see that u can be approximated by unitaries in $M(A)$ with finite spectra. Since unitaries with finite spectra are of exponentials, we conclude that

$$cer(M(A)/A)\le 1+\epsilon.$$

Q.E.D.

Theorem 2.6. *Let A be a σ-unital C^*-algebra with $RR(M(A\otimes\mathcal{K})=0$. Then both $M(A)$ and $M(A\otimes\mathcal{K})$ has (FU).*

Proof: The proof is similar to that of Theorem 2.5. Let $u\in U(M(A\otimes\mathcal{K}))$. It follows from [Mi] that

$$U(M(A\otimes\mathcal{K}))=U_o(M(A\otimes\mathcal{K})).$$

As in the proof of Lemma 2.4, we can show that $u+\sum_{n=2}\lambda_i 1\otimes e_{ii}$ can be approximted in norm by unitaries in $M(A\otimes\mathcal{K})$ with finite spectra. Then, as in the proof of Theorem 2.5, by applying Lemma 2.3 twice, we see that u can be approximated in norm by unitaries in $M(A\otimes\mathcal{K})$ with finite spectra. This implies that $M(A\otimes\mathcal{K})$ has (FU). Since

$$M(A)\cong(1\otimes e_{11})M(A\otimes\mathcal{K})(1\otimes e_{11}),$$

by [Lin2,1.2], (or by Lemma 2.3 again) $M(A)$ has (FU).
Q.E.D.

Theorem 2.7. *Let A be a σ-unital C^*-algebra with $RR(M(A\otimes\mathcal{K}))=0$. Then*

$$cer(M(A)/A)=1.$$

Proof: For any unitary $u\in U_o(M(A)/A)$, there are $h_1,h_2,...,h_n$ in $(M(A)/A)_{s.a.}$ such that

$$u=exp(ih_1)exp(ih_2)\cdot ... \cdot exp(ih_n).$$

There are $a_1,a_2,...,a_n$ in $M(A)_{s.a.}$ such that

$$\pi(a_k)=h_k\,,k+1,2,...,n$$

where $\pi: M(A)\to M(A)/A$ is the quotient map. Set

$$v=exp(ia_1)exp(ia_2)\cdot ... \cdot exp(ia_n).$$

Then $v\in U(M(A)), \pi(v)=u$. It follows from Theorem 2.6 that $M(A)$ has (FU). By [Lin3,4.2], there are an approximte identity $\{e_n\}$ for A consisting of projections and $a\in A$ such that

$$v=\sum_{n=1}^{\infty}\lambda_n(e_n-e_{n-1})+a,$$

where $|\lambda_n|=1$. Therefore

$$\pi(\sum_{n=1}^{\infty}\lambda_n(e_n-e_{n-1}))=u.$$

Suppose that $\lambda_n=e^{i\alpha_n}$, $0\le\alpha_n\le 2\pi$. Then it is easy to see that

$$h=\sum_{n=1}^{\infty}\alpha_n(e_n-e_{n-1})\in M(A)_{s.a.}$$

and $exp(ih)=\sum_{n=1}^{\infty}\lambda_n(e_n-e_{n-1})$. Moreover

$$exp(i\pi(h))=\pi(exp(ih))=u.$$

Q.E.D.

Theorem 2.8. *Let A be a σ-unital C^*-algebra with stable rank one. Then the multiplier algebra $M(A)$ has stable (FU) if and only if A has stable (FU).*

Proof: Suppose that A has stable rank one and stable (FU). Then $A \otimes \mathcal{K}$ has stable rank one too (see [Bl,6]). Moreover, $A \otimes \mathcal{K}$ has (FU). It follows from [Lin2,3.5] that $M(A \otimes \mathcal{K})$ has real rank zero. By Theorem 2.6, $M(A)$ has (FU). Since $M_k(M(A)) \cong M(M_k(A))$ and $M_k(A)$ has the same properties that A has, we conclude that $M_k(M(A))$ has (FU).

Now suppose that $M(A)$ has stable (FU). Then, by [Lin2,1.4], A has stable (FU).

Q.E.D.

Corollary 2.9. *Let A be a σ-unital AF C^*-algebra. Suppose that u is a unitary element in $M(A)$, then, for any $\varepsilon > 0$, there are an approximate identity $\{e_n\}$ for A consisting of projections and an element $a \in A$ such that*

$$u = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1}) + a \quad (e_o = 0),$$

where $|\lambda_n| = 1$ and $\|a\| < \varepsilon$.

Proof: This is an immediate consequence of Theorem 2.8 and [Zh1,3.1].

Q.E.D.

Corollary 2.10. *If A is a σ-unital AF C^*-algebra then the corona algebra $M(A)/A$ has real rank zero and*

$$cer(M(A)/A) = 1.$$

Proof: This follows from Corollary 2.9 and Theorem 2.7.

Corollary 2.11. *If A is a finite matroid algebra, then for any unitary $u \in M(A)/A$ there is a selfadjoint element $h \in M(A)/A$ such that $0 \leq h \leq 2\pi$ and*

$$u = exp(ih).$$

Proof: It follows from [Ell] that the unitary group of the corona algebra $M(A)/A$ is connected. So this corollary follows from Corollary 2.10.
Q.E.D.

Corollary 2.12. *Let A be a σ-unital C^*-algebra with $RR(M(A\otimes\mathcal{K})) = 0$. Then any trivial extension $\tau : C(S^1) \to M(A)/A$ is totally trivial.*

Proof: Since $RR(M(A\otimes\mathcal{K})) = 0$, $RR(M(A)) = 0$. Thus $RR(M(A)/A) = 0$. On the other hand, by 2.6, the unitary group is connected. It follows from 2.7 that $cer(M(A)/A) = 1$. Therefore $\pi(u)$ is contained in an abelain AF C^*-subalgebra for any unitary $u \in M(A)$. This shows that every trivial extension $\tau : C(S^1) \to M(A)/A$ is totally trivial.
Q.E.D.

Corollary 2.13. *Let A be a σ-unital C^*-algebra with stable (FU) and stable rank one. Then any trivial extension $\tau : C(S^1) \to M(A)/A$ is totally trivial.*

Proof: This follows from [Lin2,3.5] and 2.12.
Q.E.D.

A question is now naturally raised: Does the Weyl-von Neumann-Berg theorem hold for AF-algebras (or more generally, C^*-algebras with stable rank one and stable (FU)) and their multiplier algebras? How about the normal elements with spectra which is homeomorphic to, say, the figure eight curve? or the θ curve? or even the so called " Hawaiian ear ring? We will give (partial) answers to these questions in § 9 and § 11.

2 The Functor $E(\cdot, A)$

§ 3. Some Operations on $E(X, A)$

In this section, A is a σ-unital C^*-algebra with $RR(M(A)) = 0$ unless otherwise stated.

One should compare our 3.2, 3.3, 3.4, 3.6 with 4.8, 7.3, 7.4, 8.1 in [BDF1], respectively.

Definition 3.1. *The disjoint sum of extensions $\tau_1 : C(X) \to M(A)/A$ and $\tau_2 : C(X) \to M(A)/A$ is an extension $\tau_1 \vee \tau_2$ of $C(X \vee Y)$ by A, where $X \vee Y$ is the disjoint union of X and Y, defined by*

$$(\tau_1 \vee \tau_2)(f) = \tau_1(f|_X) + \tau_2(f|_Y).$$

(Notice that $M_2(M(A)/A) \cong M(A)/A$.)

Proposition 3.2. *The operation $\bigvee$ induces an isomorphism*

$$\lambda : E(X, A) \oplus E(Y, A) \to E(X \vee Y, A).$$

Proof: It is easily shown that the class of $\tau_1 \vee \tau_2$ depends only on those of τ_1 and τ_2, and that the induced map is a homomorphism. Moreover, from the definition, the map λ is injective.

To show it is surjective, let $\tau : C(X \vee Y) \to M(A)/A$ be an extension, let p be the projection $\tau(\chi_1)$, where χ_1 is the characteristic function for X, and define $\tau_1 : C(X) \to p(M(A)/A)p$ by $\tau_1 = \tau(\cdot|_{C(X)})$. Here we regard $C(X)$ as a C^*-subalgebra of $C(X \vee Y)$ by making the functions in $C(X)$ zero in Y. The extension $\tau_2 : C(X) \to (1-p)(M(A)/A)(1-p)$ is obtained similarly. Let $t_1 : C(X) \to (1-p)(M(A)/A)(1-p)$ and $t_2 : C(Y) \to p(M(A)/A)p$ be two trivial extensions. Define $\tau_1' = \tau_1 + t_1$ and $\tau_2' = \tau_2 + t_2$. Then

$$\lambda((\tau_1', \tau_2')) = [\tau_1' + \tau_2'] = [\tau_1 + \tau_2 + t_1 + t_2] = [\tau].$$

Threfore λ is bijective. Q.E.D.

The following is a generalized form of [BDF1,7.3] and an easy consequence of [OP,3.4]. As indicated in [OP], it is a version of Kasparov's Technical Theorem [K,§3, Theorem 2].

Lemma 3.3. *Let A be a σ-unital C^*-algebra. For any self-adjoint elements $h_0, h_1, h_2, \ldots$ of $M(A)/A$ such that h_0 commutes with all h_n there exists $a \in M(A)/A$ such that $0 \leq a \leq 1$, a commutes with all h_n and $af(h_0) = f(h_0)$ for all continuous functions f vanishing on $[1/2, \infty)$ and $af(h_0) = 0$ for all continuous functions vanishing on $(-\infty, 1/2]$.*

Proof: Let $f_n \nearrow \chi_{(-\infty,1/2)}$, $g_n \searrow \chi_{(-\infty,1/2]}$, $0 \leq f_n \leq g_n \leq 1$. Then $f_n(h_0)$ are increasing, and $g_n(h_0)$ are decreasing $0 \leq f_n(h_0) \leq g_n(h_0)$, $f_n(h_0)$ commutes with $D = \{h_n\}$. It follows from [OP,3.4] that there is an $a \in (M(A)/A)_{s.a}$ such that

$$f_n(h_0) \leq a \leq g_n(h_0)$$

and a commutes with each element in D. It is then clear that a is a required element.
Q.E.D.

We denote by I^ω the Hilbert cube $\prod_{n=1}^{\infty}[0, 1]$.

Lemma 3.4. *If $a = [\tau] \in E(I^\omega, A)$, there exists $a' \in E(I^\omega, A)$ and $b \in E(B \vee C, A)$ with $f_*(b) = a + a'$, where $f : B \vee C \to I^\omega$ is the obvious map and $B = [0, 1/2] \times \prod_{n=2}^{\infty} I, C = [1/2, 1] \times \prod_{n=2}^{\infty} I$.*

Proof: Let $[\tau] \in E(I^\omega, A)$ and $\{g_n\} \subset C(I^\omega)$ be the coordinate maps, let $h_{n-1} = \tau(g_n)$ and let a be as provided by Lemma 3.3.

Define $\varphi : I^\omega \to \{1/2\} \times \prod_{n=2}^{\infty} I$ such that $g_1(\varphi) = 1/2, g_n(\varphi) = g_n$. Put $\tau_o = \tau \circ \varphi$, τ_o is a homomorphism from $C(I^\omega)$ into $M(A)/A$. Let $\tau' = \tau + \tau_o$.

Now

$$p = \begin{pmatrix} a & (a(1-a))^{1/2} \\ (a(1-a))^{1/2} & 1-a \end{pmatrix}$$

is a projection that commutes with $im\tau'$. So $\tau_1 = p\tau'$ defines a homomorphism from $C(X)$ into $p(M(A)/A)p$ and $\tau_2 = (1-p)\tau'$ defines a homomorphism from $C(X)$ into $(1-p)(M(A)/A)(1-p)$. Suppose that B_1, C_1 are two closed subsets of I^ω such that

$$ker\tau_1 = \{f \in C(X) : f(x) = 0, x \in B_1\},$$

$$ker\tau_2 = \{f \in C(X) : f(x) = 0, x \in C_1\}.$$

We now show that $B_1 \subset B$ and $C_1 \subset C$. For if $f \in C[0,1]$ vanishes precisely on $[0, 1/2]$ and $g \in C(I^\omega)$ defined by $g(x) = f(g_1(x))$, then

$$\tau_1(g) = p([\tau(g)] \oplus \tau[g(\varphi)])$$

$$= \begin{pmatrix} a & (a(1-a))^{1/2} \\ (a(1-a))^{1/2} & 1-a \end{pmatrix} \begin{pmatrix} \tau(f(g_1)) & 0 \\ 0 & \tau(f(g_1) \circ \varphi) \end{pmatrix}$$

$$= 0.$$

Therefore $g \in ker\tau_1$. So

$$B_1 \subset \{x : g(x) = 0\} = B.$$

In the same way $C_1 \subset C$. Now let $i_1 : B_1 \to B, i : B \to I^\omega, j_1 : C_1 \to C$, and $j : C \to I^\omega$ be the inclusion maps. Then $i \circ i_1$ and $j \circ j_1$ are the inculsions of B_1 and C_1 into X. Then $\tau_1 \circ i_1^* + \tau_2 \circ j_1^*$ is a unital homomorphism from $C(B \vee C)$ into $M(A)/A$. Take a totallyt trivial extension $t : C(B \vee C) \to M(A)/A$, then

$$b = [\tau_1 \circ i_1^* + \tau_2 \circ j_1^* + t]$$

determines an element in $E(B \vee C, A)$. Let $a' = [\tau_o' + t]$, then we conclud that

$$f_*(b) = a + a'.$$

Q.E.D.

Remark 3.5. It is easy to see that $B_1 \cup C_1 = X$, and the interior of B_1 is the same as that of B and the interior of C_1 is the same as that of C.

Lemma 3.6. *Let A be a (non-unital) C^*-algebra, $\{p_n\}$ be an orthogonal sequence of projections in $M(A)$. If $\sum_{k=1}^n p_k$ converges weakly to 1 in A^{**}, then $\sum_{k=1}^n$ converges to 1 in strict topology.*

Proof: Let $x \in A$. In A^{**}, $\sum_{k=1}^n p_k \nearrow 1$ weakly. Then $\{x - \sum_{k=1}^n xp_k\}$ converges to zero in σ-weak topology in A^{**}. But since A in σ-weak topology and norm topology has the same continuous functionals we conclude that from Hahn-Banach's theorem that the closure of the convext hull of $\{c_n\}$ contains zero as a limit point in norm, where $c_n = x - \sum_{k=1}^n xp_k$. Consequently, for any $\epsilon > 0$, there are $\alpha_1, \alpha_2, ..., \alpha_m$ such that

$$0 \le \alpha_i < 1, \sum_{i=1}^m \alpha_i = 1$$

and

$$\|x - \sum_{i=1}^m \alpha_i (\sum_{k=1}^{n_i} xp_k)\| < \epsilon/2.$$

(We may assume that $n_{i+1} > n_i$.)
So

$$\|\sum_{j=1}^N \beta_j xp_j - \sum_{k=N+1}^\infty xp_k\| < \epsilon/2$$

for some $0 < \beta_j < 1$.
Since $\{p_k\}$ are mutually orthogonal, the above inequality implies that

$$\|\sum_{k=N+1}^\infty xp_k\| < \epsilon.$$

Thus

$$\|x(1 - \sum_{k=1}^N p_k)\| \to 0,$$

as $N \to 0$. Similarly,

$$\|(1 - \sum_{k=1}^{N} p_k)x\| \to 0,$$

as $N \to 0$. Hence $\sum_{k=1}^{n} p_k \to 1$ strictly. Q.E.D.

Definition 3.7. *If $\{X_n\}$ is a sequence of closed subsets of X such that $diam(X_n) \to 0$ and $\tau_n : C(X_n) \to \bar{p}_n(M(A)/A)\bar{p}_n$, where $\bar{p}_n$ are the images of projections $p_n \in M(A)/A$ such that*

$\sum_{n=1}^{\infty} p_n = 1$(weak convergence in A^{**}).

Then $\sum \tau : C((\bigcup X_n)^-) \to M(A)/A$ is defined as follows. Fix $x_n \in X_n$ for each n. Given $f \in C((\bigcup X_n)^-)$ choose $T_n \in p_n M(A) p_n$ such that

$$\pi(T_n) = \tau_n(f|_{X_n})$$

and

$$\|T_n - f(x_n)\| \to 0.$$

Then put $(\sum \tau_n)(f) = \pi(\sum T_n)$.

The definition needs the following remarks:

(1) Since $diam(X_n) \to 0$,

$$\|\tau(f|_{X_n} - f(x_n)1)\| = \|f(|_{X_n}) - f(x_n)1\|_\infty \to 0.$$

It follows that required T_n's exist.

(2) For each $f \in C((\bigcup X_n)^-)$, $\sum \tau_n$ converges strictly. If $\{e_k^{(i)}\}$ is an approximate identity for $p_k A p_k$, put

$$e_n = \sum_{i=1}^{n} e_1^{(i)} + \sum_{i=1}^{n-1} e_2^{(i)} + ... + e_n^{(1)}.$$

Since, by lemma 3.6, $\sum_{k=1}^{n} p_k \to 1$ strictly, $\{e_n\}$ forms an approximate identity for A. For each m,

$$\sum_{n=1}^{\infty} T_n e_m = \sum_{n=1}^{m} T_n e_m$$

and

$$e_m \sum_{n=1}^{\infty} T_n = e_m \sum_{n=1}^{m} T_n,$$

$\sum_{n=1}^{k} T_n e_m \to \sum_{n=1}^{\infty} T_n e_m$, $e_m \sum_{n=1}^{k} T_n \to e_m \sum_{n=1}^{\infty} T_n$ *as* $k \to \infty$ *in norm, so* $\sum T_n$ *converges strictly.*

(3) The definition is independent of the choices of x_n'*s and* T_n'*s, for if* $\{x_n'\}$ *and* $\{T_n'\}$ *is another such choice, then* $T_n - T_n' \in A$ *and*

$$\|T_n - T_n'\| \leq \|T_n - f(x_n)\| + \|T_n' - f(x_n')\| + |f(x_n) - f(x_n')| \to 0.$$

So $\sum(T_n - T_n') \in A$.

(4) It is clear that $\sum \tau_n$ *is a *-isomorphism.*

(5) If $\varphi : X \to Y$ *is continuous,* $Y_n = \varphi(X_n)$*, and* $\varphi_n = \varphi|_{X_n}$*, then*

$$\varphi'_*(\sum \tau_n) = \sum (\varphi_n)_*(\tau),$$

where $\varphi' = \varphi|_{(\bigcup X_n)^-}$.

(6) Notice that $A \otimes \mathcal{K} \cong A$. *So we may allow the case that* $p_n = 1$. *In fact* $\sum_{i=1}^{n} 1 \otimes e_{ii}$ *is weakly converges to* $\sum_{i=1}^{\infty} 1 \otimes e_{ii}$ *in* $(A \otimes K)^{**} (\cong A^{**})$*, where* $\{e_{ij}\}$ *is a martix unit for* $\mathcal{K}$.

§ 4. Exactness of the Sequence

$$E(F, A) \to E(X, A) \to E(X/F, A)$$

Let F be a closed subset of a compact metrizable space X. We have the short exact sequence

$$0 \to F \to X \to X/F \to 0.$$

In [BDF1], Brown, Douglas and Fillmore showed that when $A = \mathcal{K}$, the sequence

$$Ext(C(F), A) \to Ext(C(X), A) \to Ext(X/F, A)$$

is exact. This fact plays an important role in the BDF-theory. We will show in this section that the sequence

$$E(F,A) \to E(X,A) \to E(X/F,A)$$

is exact whenever A is a σ-unital stable C^*-algebra with $RR(M(A)) = 0$.

One should compare this section with Section 6 in [BDF1]. Notice that $M(A)$ is no longer a W^*-algebra. So the spectral theroy does not hold in $M(A)$. Since $M(A)$ is not weakly closed in general, one has to use the strict topology carefully.

Lemma 4.1. *Let A be a σ-unital C^*-algebra with $RR(M(A)) = 0$. Suppose that X/F is totally disconnected and X is the disjoint union of F and clopen sets $X_1, X_2, \ldots$ with $diam(X_n) \to 0$, and $\tau : C(X) \to M(A)/A$ is an extension. Then there exist mutually orthogonal projections $p_n \in M(A)$ such that $\tau(\chi_n) = \pi(p_n)$, where χ_n is the characteristic function of X_n, and $\sum_{n=1}^{\infty} p_n$ converges strictly. Moreover, with $p_o = 1 - \sum p_n$,*
(1) $\tau(g \circ r) = \tau(g \circ r)\pi(p_o) + \pi(\sum g(x_n)p_n)$;
(2)$\tau' : g \to \tau(g \circ r)\pi(p_o)$ is an isomorphism
for $g \in C(F)$, where r is a retraction from X to F defined by sending each X_n to a nearest point $x_n \in F$.

Proof: Let

$$I = \{f \in C(X) : f(x) = 0, x \in F\}.$$

Since $X \setminus F$ is totally disconnected, $\tau(I)$ is an abelian AF- algebra. Since $\tilde{I}$, the C^*-algebra obtained by adjoining the identity to I, is an abelian AF-algebra, there is $S \subset \mathbf{R}$, such that S is homeomorphic to $X \setminus F$. One can extend τ to a unital and injective homomorphism from $\tilde{I}$ into $M(A)/A$. We will still use τ for the extension of the original τ. Let $\varphi : S \to X \setminus F$ is the homeomorphism. Suppose that $g \in \tilde{I}$ such that $g \circ \varphi^{-1}(x) = \varphi^{-1}(x)$ for all $x \in X \setminus F$. Since we assume that $RR(M(A) = 0$, by [Lin4,2.4], there is an

$h \in M(A)_{s.a}$ such that

$$h = \sum_{i=1}^{\infty} \lambda_i p_i,$$

and $p_i \le e_i - e_{i-1}$, where $\{e_i\}$ is an approximate identity for A consisting projections and $\{\lambda_i\}$ is a dense subset of S, and $\pi(h) = \tau(g)$. Let $p = \sum_{i=1}^{\infty} p_i$, since $p_i \le e_i - e_{i-1}$, one can check that the sum converges in the strict topology. Therefore p is a projection in $M(A)$. There is a continuous function $f_n \in C(R)$ such that $\chi_n = f_n(g)$. Let

$$q_n = f_n(h) = \sum_{i=1}^{\infty} f_n(\lambda_i) p_i.$$

Then $\{q_n\}$ is a set of mutually orthogonal projections in $pM(A)p$ such that $\pi(q_n) = \tau(\chi_n)$. As in the proof of 1.6, one sees that $\sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} f_n(h)$ converges to p in the strict topology.

Let $\{g_m\}$ be a countable dense subset of $C(F)_{s.a.}$. Choose $h_m \in M(A)_{s.a.}$ such that $\pi(h_m) = \tau(g_m \circ r)$. Then

$$\pi(h_m q_n - g_m(x_n) q_n) = \tau[(g_m \circ r)\chi_n - g_m(x_n)\chi_n] = 0.$$

So

$$h_m q_n - g_m(x_n) q_n, h_m q_n - q_n h_m \in A$$

for all n, m.
Let $\{e_n^{(k)}\}$ be an approximate identity for $q_n A q_n$ consisting of projections. We have

$$(q_n h_m q_n - g_m(x_n) q_n)(q_n - e_n^{(k)}) \to 0,$$

$$(q_n - e_n^{(k)})(q_n h_m q_n - g_m(x_n) q_n) \to 0,$$

$$(q_n h_m - h_m q_n)(q_n - e_n^{(k)}) = (q_n h_m q_n - h_m q_n)(q_n - e_n^{(k)}) \to 0$$

and

$$(q_n - e_n^{(k)})(q_n h_m - h_m q_n) = (q_n - e_n^{(k)})(q_n h_m - q_n h_m q_n) \to 0$$

as $k \to \infty$.

Therefore,

$$q_n h_m (q_n - e_n^{(k)}) - h_m (q_n - e_n^{(k)}) \to 0,$$

$$(q_n - e_n^{(k)}) h_m - (q_n - e_n^{(k)}) h_m q_n \to 0,$$

$$(q_n - e_n^{(k)}) h_m q_n - h_m (q_n - e_n^{(k)}) \to 0$$

and

$$q_n h_m (q_n - e_n^{(k)}) - (q_n - e_n^{(k)}) h_m \to 0$$

as $k \to \infty$.

Hence

$$(q_n - e_n^{(k)}) h_m - h_m (q_n - e_n^{(k)}) \to 0.$$

Moreover

$$(q_n - e_n^{(k)}) h_m - h_m (q_n - e_n^{(k)}) \in A.$$

For each n, let $\epsilon_n = q_n - e_n^{(k)}$ for sufficient large k, we may assume that

$$\|\epsilon_n h_m - h_m \epsilon_n\| < (1/n)^2$$

for all $n \geq m$.

Fix $a \in A$, since $a(\sum_{n=1}^{N} q_n) \to ap$ in norm, $\|a(\sum_{n=N+1}^{\infty} q_n)\| \to 0$. Thus

$$\|a(\sum_{n=N+1}^{\infty} e_n^{(k)})\|^2 = \|a(\sum_{n=N+1}^{\infty} e_n^{(k)})(\sum_{n=N+1}^{\infty} e_n^{(k)})^* a^*\|$$

$$\leq \|a(\sum_{n=N+1}^{\infty} q_n)(\sum_{n=N+1}^{\infty} q_n)^* a^*\|$$

$$= \|a(\sum_{n=N+1}^{\infty} q_n)\|^2 \to 0.$$

So $\sum_{n=1}^{\infty} \epsilon_n = \sum_{n=1}^{\infty} q_n - \sum_{n=1}^{\infty} e_n^{(k)}$ converges strictly. Consequently, $\sum_{n=1}^{\infty} \epsilon_n$ is a projection in $M(A)$. Set $\epsilon_o = 1 - \sum_{n=1}^{\infty} \epsilon_n$. Since

$$\epsilon_k h_m \epsilon_k = \epsilon_k (h_m \epsilon_n - \epsilon_n h_m) \epsilon_n$$

for $n > k \geq 0$. So $\epsilon_k h_m \epsilon_n \in A$ for $n > k \geq 0$. Consequently, $\sum_{k=1}^{m} \epsilon_k h_m \epsilon_n \in A$. If $k \geq m$, then

$$\|\sum_{n>k} \epsilon_k h_m \epsilon_n\| < (1/k)^2.$$

Hence

$$\sum_{k=1}^{\infty} \sum_{n>k} \epsilon_k h_m \epsilon_n \in A.$$

Similarly

$$\sum_{n=1}^{\infty} \sum_{n<k} \epsilon_k h_m \epsilon_n \in A.$$

(Since $\epsilon_k h_m \epsilon_n = \epsilon_k(h_m \epsilon_k - \epsilon_k h_m)\epsilon_n$ if $k > n \geq 0$)
So

$$h_m = \sum_{n=0}^{\infty} \epsilon_n h_m \epsilon_n + a_1$$

for some $a_1 \in A$. Moreover

$$\epsilon_n h_m \epsilon_n = g(x_n)\epsilon_n + b_n$$

for some $b_n \in A$ and $n \geq 1$. Since

$$(q_n h_m q_n - g_m(x_n) q_n)(q_n - e_n^{(k)}) \to 0,$$

we have $\|b_n\| \to 0$.
Therefore

$$h_m = \epsilon_o h_m \epsilon_o + \sum_{n=1}^{\infty} g_m(x_n)\epsilon_n + a_2,$$

where $a_2 \in A$. Finally, let $\varepsilon_n = \epsilon_n - d_n$ where d_n is a nonzero projection in $\epsilon_n A \epsilon_n$ for all $n \geq 1$, and let $\varepsilon_o = 1 - \sum_{n=1}^{\infty} \varepsilon_n$, (The same argument used to show that $\sum_{n=1}^{\infty} \epsilon_n$ converges strictly shows that $\sum_{n=1}^{\infty} \varepsilon_n$ converges strictly. Hence it is a projection in $M(A)$.) then $\pi(\varepsilon_n) = \tau(\chi_n)$ for $n \geq 1$. So

$$\tau(g_m \circ r) = \tau(g_m \circ r)\pi(\varepsilon_o) + \pi(\sum_{n=1}^{\infty} g_m(x_n)\varepsilon_n)$$

for all m. Hence
(1) $\tau(g \circ r) = \tau(g \circ r)\pi(\varepsilon_o) + \pi(\sum_{n=1}^{\infty} g_m(x_n)\varepsilon_n)$
for all $g \in C(F)$.

Next we observe that $\tau(g \circ r)$ commutes with $\pi(\varepsilon_o)$ for $g = g_m$ by the construction, and hence for all $g \in C(F)$. Therefore

$$\tau' = r_*(\tau)\pi(\varepsilon_o)$$

is a *-homomorphism. From our final construction that $\varepsilon_n = \epsilon_n - d_n$, we notice that

$$|g_m(x_n)| \leq \|\tau(g_m \circ r)\pi(\varepsilon_o)\|.$$

So it holds for all real $g's$. Suppose that $k \in ker(\tau')$, without loss of generality, we may assume that k is real valued. Then $k(x_n) = 0$. Thus by (1),

$$\tau(k \circ r) = 0.$$

This implies that $r_*\tau(k) = 0$, since r is surjective, $k = 0$. We then conclude that τ' is a monomorphism.
Q.E.D.

Lemma 4.2. *Let $\varphi : X \to Y$ be a continuous surjection, $\tau : C(X) \to M(A)/A$ be an extension such that $\varphi_*(\tau)$ is totally trivial. Suppose that $\varphi_*(\tau)[C(Y)] \subset B$, where B is isomorphic to $C(\tilde{Y})$, $\tilde{Y}$ is totally disconnected, and $u : \tilde{Y} \to Y$ is the mapping induced by inclusion $\varphi_*(\tau)[C(Y)] \subset B$. For each clopen set $C \subset \tilde{Y}$, if $\partial[u(C)]$, the boundary of $u(C)$, contains no point with multiple preimage in X, then χ_C commutes with $im\tau$.*

Proof: We may identify B with $C(\tilde{Y})$. Let $F = \varphi^{-1}(\partial u(C)), U = \varphi^{-1}(int(u(C))$ and $V = \varphi^{-1}(Y \setminus u(C))$. We first notice that it is enough to prove that χ_C commutes with $\tau(f)$ when f vanishes on F. In fact, since φ is injective on F, for any $f \in C(X)$ there is $g \in C(Y)$ such that $g \circ \varphi$ agrees with f on F. Since χ_C commutes with $\tau(g \circ \varphi)$, we may replace f by $f - g \circ \varphi$.

Now since any f vanishing F can be approximated uniformly by functions vanishing on a neighborhood of F, we may further assume that f vanishes on a neighborhood of F. So we may assume that $f = f_U + f_V$ with f_U supported in U and f_V in V. For f_U, choose $g \in C(Y)$ supported in $u(C)$ such that $g \circ \varphi = 1$ on the support of f_U. Then, since

$$f_U(g \circ \varphi) = f_U,$$

$$\chi_C \tau(f_U) = [\chi_C \cdot \tau(g \circ \varphi)]\tau(f_U) = \tau(g \circ \varphi)\tau(f_U)$$
$$= \tau(f_U)$$

and

$$\tau(f_U)\chi_C = \tau(f_U)[\tau(g \circ \varphi)\chi_C] = \tau(f_U)\tau(g \circ \varphi) = \tau(f_U).$$

Similarly χ_C commutes with $\tau(f_V)$.
Q.E.D.

Lemma 4.3. *Let A be a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. If X/F is totally disconnected then i_* is an isomorphism.*

Proof: Write X as the disjoint union of F and clopen sets X_n with $diam(X_n) \to 0$. Define a retraction $r : X \to F$ by sending each X_n to a nearest point $x_n \in F$. Then $r_* i_* = id_{E(F,A)}$ by proposition 1.12.
Next we show that $i_* \circ f_* = id_{E(X,A)}$. Fix an extension $\tau : C(X) \to M(A)/A$. We will keep the notations used in the proof of 4.3. The decomposition $f = (f - f \circ r) - (f \circ r)$ shows that $C(X)$ is the linear direct sum of the ideal I and the subalgebra $r^*[C(F)]$. Since each X_n is totally disconnected, there is a *-monomorphism $\sigma_1 : I \to (1 - \varepsilon_o)M(A)(1 - \varepsilon_o)$ such that

$$\pi(0 \oplus \sigma_1) = \tau|_I$$

and $\sigma(\chi_n) = \varepsilon_n\,, n \geq 1$, where 0 is the zero map into $\varepsilon_o M(A)\varepsilon_o$. Moreover there exsist an approximate identity $\{d_k\}$ for $(1 - \varepsilon_o)A(1 - \varepsilon_o)$ consisting of projections and a dense sequence $\{\lambda_k\}$ in $\cup_{k=1}^{\infty} X_k$ such that

$$\sigma_1(f) = \sum_{k=1}^{\infty} f(\lambda_k)(d_k - d_{k-1})$$

for all $f \in I$. We also define
$\sigma_2 : r^*[C(F)] \to (1-\varepsilon_o)M(A)(1-\varepsilon_o)$ by

$$\sigma_2(g \circ r) = \sum_{n=1}^{\infty} g(x_n)\varepsilon_n$$

and then let $\sigma(f) = \sigma_1(f - f \circ r) + \sigma_2(f \circ r)$. This map is clearly *-linear. To show that σ is a homomorphism , it is enough to show that

$$\sigma_1((g \circ r)f) = \sigma_2(g \circ r)\sigma_1(f)$$

for all $f \in I, g \in C(F)$.
For $f \in I$, the expansion $f = \sum f\chi_n$ converges in norm, so by the linearity and continuity it is enough to establish the above relation for $f's$ satisfying $f\chi_n = f$. Then $(g \circ r)f = g(x_n)f$. So

$$\sigma_1((g \circ r)f) = g(x_n)\sigma_1(f).$$

On the otherhand,

$$\sigma_1(f) = \sigma_1(f)\sigma_1(\chi_n) = \sigma_1(f)\varepsilon_n.$$

Therefore

$$\sigma_2(g \circ r)\sigma_1(f) = \sigma_2(g \circ r)\varepsilon_n\sigma_1(f)$$
$$= g(x_n)\varepsilon_n\sigma_1(f) = g(x_n)\sigma_1(f).$$

We also notice that $\pi \circ \sigma$ is totally trivial. We set $\sigma_\infty = \sum \sigma$, countably many copies of σ. Then $\pi \circ \sigma_\infty : C(X) \to M(A)/A$ (A is stable) is a totally trivial extension. We need to show that

$$[\tau] = [\tau \circ r^* \circ i^* + \pi \circ \sigma_\infty].$$

We have
(1) $(\tau + \pi \circ \sigma_\infty)(g \circ r) = \tau(g \circ r)\pi(\varepsilon_o) + \pi \circ \sigma_2(g \circ r) + \pi \circ \sum \sigma_2(g \circ r)$
for $g \in C(F)$.

$(2)(\tau + \pi \circ \sigma_\infty)(f) = \pi \circ \sum \sigma_1(f)$
for $f \in I$.
There is a patial isometry $u \in M(A)$ such that

$$u^*u = \sum_{i=1}^{\infty}(1 - \varepsilon_o) \otimes e_{ii}$$

and

$$uu^* = \sum_{i=2}^{\infty}(1 - \varepsilon_o) \otimes e_{ii},$$

where $\{e_{ij}\}$ is a matrix unit for $\mathcal{K}$. Therefore, since A is stable, from (1) and (2), we conclude that $\tau + \pi \circ \sigma_\infty$ is unitarily equivalent to $\tau \circ r^* \circ i^* + \pi \circ \sigma_\infty$
So

$$[\tau] = [\tau \circ r^* \circ i^* + \pi \circ \sigma_\infty].$$

Q.E.D.

Lemma 4.4. *Suppose that A is a σ-unital stable C^*-algebra with $RR(M(A)) = 0$ then*

$$ker\varphi_* \subset i_*(ker\varphi'_*),$$

where $\varphi' = \varphi|_F$, $F = \varphi^{-1}(G)$ and G is a closed subset of Y which contains all points with multiple preimage in X.

Proof: Let $\tau : C(X) \to M(A)/A$ be such that $\varphi_*(\tau)$ is totally trivial. So $\varphi(\tau)[C(Y))]$ is contained in a C^*-subalgebra of $M(A)/A$ which is isomorphic to $C(\tilde{Y})$, where $\tilde{Y}$ is a totally disconnected space. We may identify this C^*-subalgebra with $C(\tilde{Y})$. Let $u : \tilde{Y} \to Y$ be the surjective map induced by the inclusion $\varphi_*(\tau)(C(Y)) \subset C(\tilde{Y})$. So $u^{-1}(Y \setminus G) = \tilde{Y} \setminus u^{-1}(G)$ is an open subset of $\tilde{Y}$ and it is totally disconnected. The C^*-subalgebra

$$I = \{f \in C(\tilde{Y}) : f(x) = 0, x \in u^{-1}(G)\}$$

is an abelian AF-algebra. Let $\{p_n\}$ be the projections of I which generates I, by lemma 4.2, p_n commutes with $im\tau$. Denote by B the C^*-subalgebra generated by $\{p_n\}$ and $im\tau$. Assume that $B \cong C(\tilde{X})$. Since $\tau(C(X)) \subset B$,

we get a surjection $v : \tilde{X} \to X$ and a *-isomorphism $\tilde{\tau} : C(\tilde{X}) \to B$ such that $\tau = v_*(\tilde{\tau})$. We claim that v is a homeomorphism on $v^{-1}(F)$.
It is clear that $G \cap u(C_n') = \emptyset$, where $p_n = \chi_{C_n'}$ and C_n' is viewed as a clopen subset of $\tilde{Y}$. Since both X and Y are compact metrizable spaces and the map φ maps $X \setminus F$ one-to-one and onto $Y \setminus G$, φ^* maps I_G isomorphically onto I_F, where

$$I_G = \{f \in C(Y) : f(y) = 0, y \in G\}$$

and

$$I_F = \{f \in C(X) : f(x) = 0, x \in F\}.$$

Moreover $\phi_* \circ \tau(I_G) = \tau(I_F)$. Therefore it is easy to see that the ideal of $C(\tilde{X})$ generated by $(v^*)^{-1}(I_F)$ is I. Now, if $x \in \tilde{X}$ such that $v(x) \in F$ then $v^*(I_F) \subset ker\hat{v}(x)$ ($\hat{v}(x)$ is the complex homomorphism induced by $v(x)$.). Since $\hat{x}|_{C(X)} = \hat{v}(x)$ and I is the ideal generated by $(v^*)^{-1}(I_F)$, $I \subset ker\hat{x}$. This shows that $F \cap v(C_n) = \emptyset$, where $p_n = \chi_{C_n}$ and C_n are clopen subsets of $\tilde{X}$. It is easy to see that if $v(x) \notin v(C_n)$ then $\chi_{C_n}(x) = 0$. For $v(x) \notin v(C_n)$ implies that $\varphi(v(x)) \notin \varphi(C_n)$, so there exists $g \in C(Y)$ with $g(\varphi(v(x)) = 1$ and $g = 0$ on $\varphi(v(C_n))$, and therefore

$$0 = \tau(g \circ \varphi)(x)\chi_{C_n}(x)$$

$$= g(\varphi(v(x))\chi_{C_n}(x) = \chi_{C_n}(x).$$

In particular, if $v(x) \in F$, then x annihilates all p_n. Now suppose $x_1, x_2 \in v^{-1}(F)$ and $v(x_1) = v(x_2)$. So $\hat{x}_i(p_n) = 0$. If $x_1 \neq x_2$, there is $f \in C(\tilde{X})$ such that

$$\hat{x}_1(f) \neq \hat{x}_2(f).$$

Since $\hat{x}_i(f) = 0$ for $f \in I$ and $i = 1, 2$, there must be $f \in im\tau$ such that $\hat{x}_1(f) \neq \hat{x}_2(f)$. But if $f \in im\tau$, $\hat{x}_i(f) = f(v(x_i))$. Hence $x_1 = x_2$. This implies that v is one-to-one on $v^{-1}(F)$. Since both F and $v^{-1}(F)$ are compact metrizable spaces, v is a homeomorphism on $v^{-1}(F)$. This end the proof of the claim. So we may identify $v^{-1}(F)$ with F. Since φ^* maps I_G isomorphically onto I_F, one sees easily that $\tilde{X} \setminus F$ is totally disconnected.

Let $j = v|F$. Since $\tilde{X} \setminus F$ is totally disconnected, the map $j_* : E(F, A) \to E(\tilde{X}, A)$, by 4.3 and 1.12, is bijectve. There is $\tau' \in E(F, A)$ such that $j_*(\tau') = \tilde{\tau}$. Since $\tau = v_* j_*(\tau')$ and $v_* j_* = i_*$, it remains to show that $\varphi'_*(\tau')$ is totally trivial. Notice that φ_* is a map from $E(F, A)$ into $E(Y, A)$. We also observe that $C(\tilde{Y}) \subset C(\tilde{X})$, the inclusion gives a surjective map $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ and we have

$$\tilde{\varphi} \circ j = l \circ \varphi',$$

where $l : G \to \tilde{Y}$ is the inclusion (u is a homeomorphism on $u^{-1}(G)$). Now $\tilde{\varphi}_* j_*(\tau')$ is totally trivial ($\tilde{Y}$ is totally disconnected). So $l_* \varphi'_*(\tau')$ is totally trivial, and therefore $[\varphi'_*(\tau')] = 0$ (see 1.12).
Q.E.D.

Theroem 4.5. *Suppose that A is a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. If X is a compact and metrizable space, F is a closed subset of X, then*

$$E(F, A) \to E(X, A) \to E(X/F, A)$$

is exact.

Proof: We have $ker\varphi_* \subset imi_*$ by 4.4 and other inclusion is obvious.
Q.E.D.

§ 5. $E(X, A)$ as a Group

We now show that $E(X, A)$ is a group when A is a σ-unital stable C^*-algebra with $RR(M(A)) = 0$, as in Section 9 of [BDF1].

Recall that the inverse limit of a sequence $f_n : X_{n+1} \to X_n (n \geq 1)$ of spaces and continuous maps may be identified as

$$lim_{\leftarrow} X_n = \{x \in \prod_{n=1}^{\infty} X_n : f_n(x_{n+1}) = x_n, n \geq 1\},$$

and that the corresponding statement holds for inverse system of groups and homomorphisms. In particular one always has a homomorphism

$$P : E(lim_{\leftarrow} X_n) \longrightarrow lim_{\leftarrow} E(X_n, A),$$

defined by $P(a) = \{(p_n)_*(a)\}$, where the $p_k : lim_{\leftarrow} X_n \to X_k$ are the coordinate projections. As in [BDF2,2.5] we have the following:

Theorem 5.1 *If $\{X_n, f_n\}$ is an increasing system of compact metrizable spaces and continuous surjections, then the induced map*

$$P : E(lim_{\leftarrow} X_n, A) \longleftarrow lim_{\leftarrow} E(X_n, A)$$

is surjective.

Proof: See the proof of 2.5 of [BDF2].
Q.E.D.

Lemma 5.2 *Let A be a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. Then $E(I^{\omega}, A)$ is a group.*

Proof: Let $a = [\tau] \in E(I^{\omega}, A), G = [0, 1/2] \times \prod_{n=2}^{\infty}, G = [1/2, 1] \times \prod_{n=2}^{\infty}$, by lemma 3.4, there exist $a' \in E(I^{\omega}, A)$ and $b \in E(F \vee G, A)$ with $f_*(b) = a + a'$, where $f : F \vee G \to I^{\omega}$ is the obvious map. Since F and G are homeomorphic to I^{ω}, we can repeat this argument for each of the "pieces" of b (relative to the isomorphism between $E(F, A) \oplus E(G, A)$ and $E(F \vee G, A)$). Iteration then produces the following situation:
i) a sequence $f_n : X_{n+1} \to X_n (n \geq 1)$ of spaces and surjective maps, where $X_1 = I^{\omega}$, X_n has 2^{n-1} components each homeomorphic to I^{ω}, and the diameter of the components tends to zero.
ii) $a_n, a_n' \in E(X_n, A)$ such that $(f_n)_*(a_{n+1}) = a_n + a_n'$, where $a_1 = a$ is given. Since the diameter of the components tends to zero, we conclude that the inverse limit X of $\{X_n, f_n\}$ is totally disconnected, and there exist well-defined

elements

$$b_n = a_n + a'_n + (f_n)_*(a'_{n+1} + (f_n)_*(f_{n+1}))_*(a'_{n+2}) + ...$$

of $E(X_n, A)$ by the definition 3.7 (because each a'_k can be decomposed into 2^{k-1} pieces corresponding to the components of X_n). Moreover,

$$\begin{aligned}(f_n)_*(b_{n+1}) &= (f_n)_*(a_{n+1}) + (f_n)_*(a'_{n+1}) + (f_n)_*(f_{n+1}))_*(a'_{n+1}) + ... \\ &= a_n + a'_n + (f_n)_*(a'_{n+1}) + ... \\ &= b_n.\end{aligned}$$

Therefore $\{b_n\}$ is an element of $lim_{\leftarrow} E(X_n, A)$, so by Theorem 5.1, there exists $c \in E(X, A)$ such that $(\varphi_n)_*(c) = b_n, n \geq 1$. By 1.8, $E(X, A) = 0$, because X is totally disconnected. So $b_1 = 0$ and

$$a'_1 + (f_1)_*(a'_2) + (f_1)_*(f_2)_*(a'_2) + ...$$

is the required inverse of a.
Q.E.D.

Theorem 5.3. *Let A be σ-unital stable C^*-algebra with $RR(M(A)) = 0$. Let F and G be closed subsets of X with $X = F \cup G$, and let i_1, i_2, j_1, j_2 be the inclusion maps of $F \cap G$ in $F, F \cap G$ in G, F in X, and G in X. Then*

$$E^{-1}(F \cap G, A) \xrightarrow{\alpha} E(F, A) \oplus E(G, A) \xrightarrow{\beta} E(X, A)$$

is exact, where $E^{-1}(F, A)$ denotes the group of invertible elements in $E(F \cap G, A)$, and α and β are defined by

$$\alpha(a) = ((i_1)_*(a), (i_2)_*(-a)), a \in E^{-1}(F \cap G, A);$$

$$\beta((b, c)) = (j_1)_*(b) + (j_2)_*(c), b \in E(F, A), c \in E(G, A).$$

Proof: We have $\beta \cdot \alpha = 0$, since $j_1 \circ i_1 = j_2 \circ i_2$. Suppose that $\beta((b, c)) = 0$ for $(b, c) \in E(F \vee G, A)$. Let $\varphi : F \vee G \to X$ be the obvious map. By

Proposition 3.2 , it follows that $\beta = \varphi_* \circ \lambda$. So $\varphi(b \vee c) = 0$. By Lemma 4.4, there is $d \in E(S \vee S, A)$ such that $i_*(d) = b \vee c$ and $\varphi'_*(d) = 0$, where $S = F \cap G$, $i : S \vee S \to F \vee G$ is the inclusion, and $\varphi' : S \vee S \to S$ is the restriction of φ. By Proposition 3.2, we get $d = a_1 \vee a_2$ for some a_1 and $a_2 \in E(S, A)$, and $a_1 + a_2 = \varphi'_*(d) = 0$. Thus $a_1 \in E^{-1}(S, A)$ with inverse a_2. But

$$(i_1)_*(a_1) \vee (i_2)_*(a_2) = i_*(a_1 \vee a_2) = i_*(d) = b \vee c,$$

and therefore

$$\alpha(a_1) = ((i_1)_*(a_1), (i_2)_*(a_2)) = (b, c),$$

by the Proposition 3.2 again.
Q.E.D.

Corollary 5.4. *If $E(X, A)$ is a group, then $E(F, A)$ is a group for any closed subset of F of X.*

Proof: From Theorem 5.3, we have the exact sequence

$$E^{-1}(F, A) \xrightarrow{\alpha} E(F, A) \oplus E(X, A) \xrightarrow{\beta} E(X, A).$$

If $a \in E(F, A)$ and $i : F \to X$ is inclusion, then $\beta(a, -i_*(a)) = 0$, and consequently there exists $a' \in E^{-1}(F, A)$ such that

$$\alpha(a') = (a, -i_*(a)).$$

That is

$$(a', i_*(-a')) = (a, -i_*(a)).$$

In particular, $a' = a$ and a is invertible.
Q.E.D.

Theorem 5.5. *Let A be a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. For any compact metrizable space X, $E(X, A)$ is a group.*

Proof: The space X can be regarded as a closed subset of I^ω. So the result follows 5.2 and 5.4.
Q.E.D.

§ 6. $E(\cdot, A)$ as a Homotopy Funtcor

Throughout this section, A is always a σ-unital stable C^*-algebra with $RR(M(A)) = 0$.

In section 2 of [BDF2], Brown, Douglas and Fillmore show the following theorem:

Theorem 6.1.(section 2 of [BDF2]) *Let h be a covariant functor from compact metrizable spaces to abelian groups such that*
1) $h(F) \to h(X) \to h(X/F)$ is exact for F a closed subset of X;
2) If $X = lim_{\to} X_n$, where each of the maps $X_{n+1} \to X_n$ is onto, then the natural map $P : h(X) \to lim_{\to} h(X_n)$ is onto; and
3) If X and $\{X_n\}$ are as in 2) and each X_n has only finitely many points, then P is an isomorphism.
Then h is homotopy invariant.

From section 4, 5 and Theorem 6.1, we have the following:

Theorem 6.2. *The functor $E(., A)$ is homotopy invariant.*

Proof: It follows from Proposition 1.12 and Theorem 5.5 that $E(\cdot, A)$ is a covariant functor from compact metrizable spaces to abelian groups. By Theorem 4.5 and Theorem 5.1, this functor satisfies condition 1) and 2) in 6.1. Furthermore, if X and X_n are as in 3) of 6.1, then X is totally disconnected. Therefore in this case, $E(X, A) = E(X_n, A) = 0$ for all n. Hence P is an isomorphism. By Theorem 5.1, $E(\cdot, A)$ is homotopy invariant.
Q.E.D.

While we omit the detailed proof of Theorem 6.1, which we simply quote from [BDF2], we would like to include proofs of the next two lemma for the completeness.

Lemma 6.3. *If X is the union of closed subsets F and G, and $F \cap G$ is a retract of F, then*

$$E(F, A) \oplus E(G, A) \longrightarrow E(X, A)$$

is a surjection.

Proof: Let $r_o : F \to F \cap G$ be a retraction, let $r : X \to G$ be defined by $r(x) = x$ for $x \in G$ and $r(x) = r_o(x)$ for $x \in F$. Suppose that $i : G \to X$ and $j : F \to X$ are the inclusions and $q : X \to X/F$ is the quotient map. Then we see that $q \circ i \circ r = q$, so that if $a \in E(X, A)$ is given,

$$q_*(a - i_* r_*(a)) = 0.$$

By Theorem 4.5, there is $f \in E(F, A)$ such that

$$j_*(f) = a - i_* r_*(a).$$

Then $r_*(a) \oplus f$ maps to a as required. Q.E.D.

Corollary 6.4. *Supposed X is the union of closed subsets F, G and $F \cap G$ is a retract of G. If $E(F \cap G, A) = 0$, then the map*

$$E(F, A) \oplus E(G, A) \longrightarrow E(X, A)$$

is an isomorphism.

Proof: It follows from 5.3 that the map is surjective. We also have the following exact sequence:

$$E(F \cap G, A) \to E(F, A) \oplus E(G, A) \to E(X, A).$$

Since $E(F \cap G) = 0$, we conclude that the map is injective, and therefore it is an isomorphism.
Q.E.D.

Lemma 6.5 *Suppose that X is the union of closed subsets $\{X_n\}$ such that $diam(X_n) \to 0$ and $E(F_n, A) = 0$, where $F_n = \cup_{m=n+1}^{\infty}(X_n \cap X_m)$. If there is retraction $r_n : X \to X_n$ for each n, then the map*

$$S : \prod_{n=1}^{\infty} E(X_n, A) \to E(X, A)$$

defined by $(\tau_1, \tau_2, ...) = \sum_{n=1}^{\infty} \tau_n$ *is an isomorphism.*

Proof: Clearly, $(r_m)_*(\sum \tau_n) = \tau_m$. So if $\sum_{n=1}^{\infty} \tau_n$ is totally trivial, every τ_n is totally trivial. This implies that the map S is injective.

Now define closed sets Y_n by $Y_n = \cup_{m>n} X_m$, and let $\tau : C(X) \to M(A)/A$ be an extension. Since $X = X_1 \cup X_2 \cup ... \cup X_n \cup Y_n$, by repeated application of Corollary 6.4 we obtain mutually orthogonal projections $p_1, p_2, ... \in M(A)/A$ and extension $\tau_n = p_n \tau$ and $\tau_n' = (1 - \sum_{k=1}^{n} p_k)\tau$ so that $\tau = (q_n)_*(\tau_1 \vee \tau_2 \vee ... \vee \tau_n \vee \tau_n')$. (Here $q_n : X_1 \vee X_2 \vee ... \vee X_n \vee Y_n \to X$ is the natural map.) Since $RR(M(A)) = 0$, by [BP,2.5] , $RR(A) = 0$. It follows from [BP,3.14] that every projection $p \in M(A)/A$ lifts to a projection $P \in M(A)$. By repeated application of this fact, we obtain orthogonal projections $P_n \in M(A)$ such that $\pi(P_n) = p_n$ and $\sum_{n=1}^{\infty} P_n = 1$ (converges in strict topology). We may view τ_n as an extension from $C(X_n) \to p_n(M(A)/Ap_n$. By 3.7, $\sum_{n=1}^{\infty} \tau_n$ does define an extension from $C(X) \to M(A)/A$. We claim that $\tau = \sum_{n=1}^{\infty} \tau_n$. In fact it is easy to see that $\tau(f) = (\sum_{n=1}^{\infty} \tau_n)(f)$ if f is constant on some Y_n, and such functions are dense in $C(X)$. Hence S is surjective and thus is an isomorphism.

Q.E.D.

Corollary 6.6. *Suppose that X is a contractive compact metric space. Then*

$$E(X, A) = \{0\}.$$

Proof: It follows from 1.7 that

$$Ext([0,1], A) = \{0\}.$$

By 6.2,

$$Ext(X, A) = E([0,1], A) = \{0\}.$$

Q.E.D.

3 BDF Theory for C^*-algebras with Real Rank Zero

§ 7. The Index Map

7.1 By Brown's universal coefficient theorem ([Bn2]), we know that

$$0 \to Ext_Z^1(K_o(C(X), Z) \to Ext(C(X), \mathcal{K}) \to Hom(K^1(X), Z) \to 0,$$

and when X is a compact subset of the plane, as in [BDF1, Section 10], the index map

$$\gamma : Ext(C(X), \mathcal{K}) \to Hom(K^1(X), Z) \to 0$$

is an isomorphism.
We will show in this section that a similar result holds for $E(X, A)$ at least for some A. Now we assume that A is a σ-unital stable C^*-algdebra. We will use the Mingo's definition for index. If $f \in \mathcal{F}_A$, where

$$\mathcal{F}_A = \{f \in M(A) : \pi(f) \text{ invertible}\},$$

the (generalized Fredholm) index for f is defined by

$$ind(f) = [1 - v^*v] - [1 - vv^*] \in K_o(A),$$

where $f = g + a$ for some $a \in A$ and g has closed range and finitely generated kernel and co-kernel such that $g = v|g|$ for a partial isometry $v \in M(A)$ with $kerv = kerg$(see [M]). Mingo shows that this index is well defined. Moreover $ind : \mathcal{F}_A \to K_o(A)$ is locally constant. If, in particular, f and g are connected by a norm continuious path in $\mathcal{F}_A$, then $ind(f) = ind(g)$. If $[\mathcal{F}_A]$ denotes the set of path component in $\mathcal{F}_A$, then

$$ind : [\mathcal{F}_A] \longrightarrow K_o(A)$$

is an isomorphism. An extension τ of A by $C(X)$ extends uniquely to an extension $\tilde{\tau} : M_2(C(X)) \to M_2(M(A)/A)(\cong M(A)/A)$. We define our map γ as follows:

$$\gamma([\tau])(f) = ind(\tilde{\tau}(f)),$$

where $\tau : C(X) \to M(A)/A$ is any extension and $f \in C(X, M_n)$ any unitary. It is clear that $ind(\tilde{\tau}(f))$ depends only on the equivalence class $[\tau]$ of τ and the homotopy class $[f]$ of f.

It is clear that if τ is totally trivial, then $ind(\tilde{\tau}(f)) = 0$ for unitary $f \in C(X, M_n)$. We also have

$$ind(\tau(fg)) = ind(\tau(f)(\tau(g)) = ind(\tau(f)) + ind(\tau(g)),$$

so $\gamma([\tau]) : K^1(X) \to K_o(A)$ is a homomorphism; and since

$$ind(\tau_1 + \tau_2)(f) = ind(\tau_1(f) \oplus \tau_2(f)) = ind(\tau_1(f)) + ind(\tau_2(f)),$$

γ is a homomorphism.

Lemma 7.2 *Let A be a σ-unital C^*-algebra with $RR(M(A)/A) = 0$. Suppose that X is the union of closed subsets F, G and $F \cap G = \{x_o\}$. Then for any extension $\tau : C(X) \to M(A)/A$, there is a projection $p \in M(A)/A$ such that p commutes with $im\tau$. Moreover,*

$$\tau_1(f) = \tau(f \cdot r_1)p$$

(for $f \in C(F)$) and

$$\tau_2(f) = \tau(f \cdot r_2)(1 - p)$$

(for $f \in C(G)$) are monomorphisms from $C(F)$ and $C(G)$ into $p(M(A)/A)p$ and $(1-p)(M(A)/A)(1-p)$, respectively, where r_1 and r_2 are retractions from X onto F and G, respectively.

Proof: Let $O_1 = F \setminus \{x_o\}$, $O_2 = G \setminus \{x_o\}$ and let $f, g \in C(X)$ such that $f(z) > 0$, if $z \in O_1$ and $f(z) = 0$ elsewhere; and $g(z) > 0$, if $z \in O_2$ and

$g(z) = 0$ elsewhere. Then, by [Bn3], there is a projection $p \in M(A)/A$ such that $p\tau(f) = \pi(f)$ and $p\tau(g) = 0$. For any $f \in C(X)$, let $\lambda = f(x_o)$, then

$$f - \lambda \in I = \{f \in C(X) : f(x_o) = 0\}.$$

Clearly p commutes with each element in $\tau(I)$. So

$$p(f - \lambda) = (f - \lambda)p.$$

Hence p commutes with $im\tau$. The rest of the proof is clear (see the proof of 3.4 and Remark 3.5).
Q.E.D.

In the rest of this section, A is always a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. So, by 2.6, $M(A)$ has (FU).

We notice that if X is a metric space with $dim(X) \leq 2$, then $K_1(C(X)) \cong U_1(C(X))/U_o(C(X))$.

Lemma 7.3 *If F is a closed subset of $[0,1]$ and $X = [0,1]/F$, then γ_X is injective.*

Proof: By considering the components of the complement of F, we see that X is the union of a sequence of closed subsets X_n, each homeomorphic to an interval or a circle and $diam(X_n) \to 0$. Moreover, there is $x_o \in X$ such that $X_n \cap X_m = \{x_o\}$ for all $n \neq m$, and X can be regarded as a subset of $\mathbf{C}$.

The lemma is true when X is an interval, since then $E(X, A) = 0$. If X is a circle, by 2.12,

$$E(X, A) = Ext(X, A).$$

Let z be the identity function on S^1 and $\tau_1, \tau_2 : C(S^1) \to M(A)/A$ be two extensions. It is enough to show that if

$$inf(\tau_1(z)) = ind(\tau_2(z))$$

then $[\tau_1] = [\tau_2]$ in $E(S^1, A)$. For this let u_1, u_2 be two partial isometies in $M(A)$ such that $\pi(u_1) = \tau_1(z)$ and $\pi(u_2) = \tau_2(z)$, respectviely. So $1 - u_1^*u_1, 1 - u_1u_1^*, 1 - u_2^*u_2, 1 - u_2u_2^* \in A$, and

$$[1 - u_1^*u_1] - [1 - u_1u_1^*] = [1 - u_2^*u_2] - [1 - u_2u_2^*].$$

Set $v = u_1 \oplus u_2$. Then $[1 - v^*v] = [1 - vv^*]$. Let w be a partial isometry in $M_2(A) \cong A$ such that $w^*w = 1 - v^*v$ and $ww^* = 1 - vv^*$. Then $w + v$ is a unitary. Similarly there is $w' \in M_2(A)$ such that $w' + v'$ is a unitary , where $v' = u_2 \oplus u_2^*$. Let $\sigma : C(S^1) \to M(A)/A$ be an extension determined by u_2^*, then $[\sigma] = [\tau_2]^{-1}$ and $[\sigma] = [\tau_1]^{-1}$ in $E(S^1, A)$, since all trivail extensions are totally trivial in this case. So $[\tau_1]^{-1} = [\tau_2]^{-1}$, and hence $[\tau_1] = [\tau_2]$.
So lemma is true when X is a circle. We will reduce the general case to these two cases. Let $\tau : C(X) \to M(A)/A$ be an extension in $ker\gamma$, and let $Y_n = \bigcup_{i>n} X_i$ (a closed subset). By induction and repeated application of Lemma 7.2, we obtain mutually orthogonal projections $\{p_k\}$ in $M(A)/A$ such that $\tau_n = p_n\tau$ give extensions $\tau_n : C(X_n) \to p_n(M(A)/A)p_n$ and $\tau_n' : C(Y_n) \to (1 - \sum_{k=1}^n p_k)(M(A)/A)(1 - \sum_{k=1}^n p_k)$. Moreover,

$$\tau = q_n(\tau_1 \vee \tau_2 \vee ... \vee \tau_n \vee \tau_n'),$$

where $q_n : X_1 \vee X_2 \vee ... \vee X_n \vee Y_n \to X$ is the natural map. If $f \in C(X)$ is constant on Y_n, we have

$$\tau(f) = \tau_1(f|_{X_1}) \oplus \tau_2(f|_{X_2}) \oplus ... \oplus \tau_n(f|_{X_n}) \oplus f(x_o).$$

By taking f properly, one sees easily that τ_n is in $ker\gamma$ for each n. Thus each τ_n is totally trivail. Suppose that $B_n \subset p_n(M(A)/A)p_n$ is the abelian AF-C^*-subalgebra containing $\tau_n(C(X_n)$. Since A is stable, the closure of

$$\bigcup_{m=1}^{\infty} ((\oplus_{n=1}^m B_n) \oplus \mathbf{C}(1 - \sum_{n=1}^m p_n))$$

is an abelian AF-C^*-subalgebra of $M(A)/A$. We denote it by B. For functions f that are constant on some Y_n, $\tau(f) \in B$. Since such functions are dense in

$C(X)$, $\tau(C(X)) \subset B$. Therefore τ is totally trivial. So γ_X is injective.
Q.E.D.

Lemma 7.4. *If X is a subset of the plane, F and G are the intersection of X with the closed half-plane determined by a straight line L, and*

$$\beta : E(F,A) \oplus E(G,A) \to E(X,A)$$

is the usual map, then

$$ker(\gamma_X \beta) \subset ker\gamma_F \oplus ker\gamma_G.$$

Proof: Suppose that τ_1 and τ_2 are in $E(F,A)$ and $E(G,A)$, respectively, such that

$$ind(\tau(f|_F) \oplus \tau_2(f|_G)) = 0$$

for all $f : X \to S^1$. It is sufficient to show that $ind\tau_1(g) = 0$ for all $g : F \to S^1$. Extend g to $g' : F \cup L \to S^1$ by linearity on the component of $L \setminus F \cap L$, steering around $\{0\}$ is necessary; and let $f = g' \circ p$, where $p : G \to L$ is the projection and $p|_F = id$. Then $f|_G$ is null-homotopic so that $ind\tau_2(f|_G) = 0$. Therefore

$$0 = ind(\tau_1(f|_F) \oplus \tau_2(f|_G))$$
$$ind\tau_1(f|_F) = ind\tau_1(g).$$

Q.E.D.

Lemma 7.5. *If X is a compact subset of the plane, the γ_X is injective.*

Proof: Let $a \in E(X,A)$ with $\gamma(a) = 0$, let F, G, L be as in Lemma 7.4, and let J be a compact interval on L that contains $X \cap L$. We have the following commutative diagram:

$$\begin{array}{ccccc} F \cup J & \xrightarrow{i_1'} & X \cup J & \xleftarrow{i_2'} & G \cup J \\ \uparrow j_1 & & j \uparrow \uparrow j_2 & & \\ F & \xrightarrow{i_1} & X & \xleftarrow{i_2} & G \end{array}$$

By lemma 6.3, the map

$$E(F \cup J, A) \oplus E(G \cup J, A) \to E(X \cup J, A)$$

is surjective. So there are $b' \in E(F \cup J, A)$ and $c' \in E(G \cup J, A)$ with $j_*(a) = (i_1')_*(b') + (i_2')_*(c')$. Then $j_*(a) \in ker\gamma$, Hence $b', c' \in ker\gamma$,by lemma 7.4. Therefore the images of b', c' in $E(F \cup J/F, A)$ and $E(G \cup J/G, A)$ are in $ker\gamma$. Since both $F \cup J/F$ and $G \cup J/G$ are homeomorphic to $J/F \cap G$, by lemma 7.3. the images of b', c' are totally trivial. However

$$E(F, A) \to E(F \cup J, A) \to E(F \cup J/F, A)$$

$$E(G, A) \to E(G \cup J, A) \to E(G \cup J/G, A)$$

are exact by theorem 4.5. So there are $b \in E(F, A), c \in E(G, A)$ such that $b' = (j_1)_*(b)$ and $c' = (j_2)_*(c)$. We have

$$j_*(((i_1)_*(b) + (i_2)_*(c))) = j_*(a).$$

Since $E(J, A) = E(X \cap J, A) = 0$, it follows from the exact sequence of

$$E(X \cap J, A) \to E(X, A) \oplus E(J) \to E(X \cup J, A)$$

that j_* is injective. Thus

$$(i_1)_*(b) + (i_2)_*(c) = a.$$

By lemma 7.3, $b, c \in ker\gamma$.

Now we use the iterated spliting argument of 5.2. We can produce the following:

1) a sequence $f_n : X_{n+1} \to X_n$ $(n \geq 1)$ of spaces and surjective continuous maps, where $X_1 = X$, X_n has 2^{n-1} components each homeomorphic to a compact subset of the plain and diameter of the component tends to zero.

2) $a_n \in E(X_n, A)$ such that $(f_n)_*(a_{n+1}) = a_n$, where $a_1 = a$ is given.

3) $a_n \in ker\gamma$.

Since the diameter of the components tends to zero, we conclude that $lim_{\leftarrow} X_n$ of $\{X_n, f_n\}$ is totally disconnected. Therefore $E(lim_{\leftarrow} X_n, A) = 0$. By Theorem 5.1, $\{a_n\} = 0$ as an element in $lim_{\leftarrow} E(X_n, A)$. Hence $a = a_1 = 0$. Q.E.D.

Theorem 7.6. *If X is a compact subset of the plane, then the index map*

$$E(X, A) \xrightarrow{\gamma} Hom(K^1(X), K_o(A))$$

is an isomorphism.

Proof: It follows from [Zh1,2.3] that $K_1(A) = 0$. From universal cofficient theorem ([RS]), since $K_1(A) = 0$, we have the following diagram:

$$\begin{array}{ccc} Ext(C(X), A) & \xrightarrow{\gamma} Hom(K^1(X), K_o(A)) \to 0 & \\ \uparrow q & \nearrow & , \\ E(X, A) & & \end{array}$$

where q is the quotient map. Since, by lemma 7.5, the map $\gamma : E(X, A) \to Hom(K^1(X), K_o(A))$ is injective, the map is in fact an isomorphism.
Q.E.D.

Remark 7.7. There is an alternative proof without using the universal coefficient theorem. The proof is based on the following fact: If X is a finite simplicial complex in $\mathbf{C}$, then γ_X is surjective.

Suppose that $f_1, f_2, ..., f_n : X \to S^1$ is a system of free generators for $K_1(X)$. It suffices to show that for any projection $p \in A$, there are extensions $\tau_1, ..., \tau_n$ such that $ind\tau_i(f_i) = [p]$ and $ind\tau_i(f_j) = 0$ for $i \neq j$. Since A is a σ-unital stable C^*-algebra, there is a partial isometry $v \in M(A)$ such that $v^*v = 1, vv^* = 1 - p$. In fact, we may assume that $A = B \otimes \mathcal{K}$, where B is unital. By [Eff,A8], we may assume that $p \in M_n(B)$ for some n. Since $M_n(B) \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, we may further assume that $p \in B$. Since there is partial isometry $v_1 \in M(B \otimes \mathcal{K}$ such that

$$v_1^* v_1 = \sum_{i=1}^{\infty} (1 - p) \otimes e_{ii} \ , v_1 v_1^* = \sum_{i=2}^{\infty} (1 - p) \otimes e_{ii},$$

there is a partial isometry $v_2 \in M(B \otimes \mathcal{K})$ such that

$$v_2^* v_2 = (1-p) \otimes e_{11} \oplus \sum_{i=2}^{\infty} 1 \otimes e_{ii}\,, \; v_2 v_2^* = \sum_{i=2}^{\infty} 1 \otimes e_{ii}.$$

Therefore there is a partial isometry $v \in M(A)$ such that

$$v^* v = 1 \;\; , vv^* = 1 - p.$$

Choose $g_1, ..., g_n : S^1 \to X$ such that $f_i \circ g_j$ is homotopic to the identity map for $i = j$ and to the constant map for $i \neq j$, choose a diagonal normal element $x \in M(A)$ with $sp(\pi(x)) = X$. Define $\tau_i(f) = f(g_i(\pi(v)) \oplus \pi(x))$. Then

$$\begin{aligned} ind\tau_i(f_j) &= ind[f_j(g_i(\pi(v)) + f_j(\pi(x))] \\ &= indf_j \circ g_j(\pi(v)). \end{aligned}$$

So if $i = j$, $ind\tau_i(f_j) = ind\pi(v) = [p]$ and if $i \neq j$, $ind\tau_i(f_j) = 0$.

Once we have this, we can use the argument of 10.5 in [BDF1] to show that γ_X is bijective for any compact subset of $\mathbf{C}$.

Corollary 7.8. *If X is a compact subset of the plane,*

$$E(X, A) = Ext(C(X), A).$$

Proof: The quotient map q in the diagram of 7.5 must be injective. Q.E.D.

§ 8. Applications to Purely Simple C^*-Algebras

First we notice that every (non-unital) σ-unital purely infinite simple C^*-algebra is stable (see [Zh3,part I,1.2]).

Theorem 8.1.(cf [Lin 4,3.5] *Let A be a σ-unital purely infinite simple C^*-algebra with $K_1(A) = 0$, X be a compact metrizable space. Then any two totally trivial extensions are unitarily equivalent.*

Proof: Let $\{\lambda_n\}$ be a countable dense subset of X, where each point occurs infinitely often. Suppose that $\{e_n\}$ is an approximate identity for A consisting of projections. Let $\sigma : C(X) \to M(A)$, by

$$\sigma(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1}).$$

Then $\tau = \pi \circ \sigma$ defines a totally trivial extension of $C(X)$ by A. Suppose that $\{\gamma_n\}$ is another dense subset of X and $\{\epsilon_n\}$ is another approximate identity for A consisting of projections and $\sigma' : C(X) \to M(A)$ is define by

$$\sigma'(f) = \sum_{n=1}^{\infty} f(\gamma_n)(\epsilon_n - \epsilon_{n-1}).$$

We will show that $\tau' = \pi \circ \sigma'$ is unitarily equivalent to τ. It is easy to see that there is a permutation ρ on the set of natural integers $\mathbf{N}$ satistying $d(\lambda_n, \gamma_{\rho(n)}) \to 0$. Suppose that $\{\lambda_n\} = \bigcup\{\lambda_{(k,i)}\}$, where $\lambda_{(k,i)} = \lambda_{(k,1)}$ and $\lambda_{(k,i)} \neq \lambda_{(j,i)}$, if $k \neq j$. Let $p_k = \sum_i (e_{(k,i)} - e_{(k,i)-1})$ and $q_k = \sum(\epsilon_{(k,i)} - \epsilon_{(k,i)-1})$. By [Zh 4, 3.3], there is a partial isomety $v_k \in M(A)$ such that

$$v_k^* v_k = p_k, v_k v_k^* = q_k.$$

Set $u_n = \sum_{k=1}^{n} v_k$. For each m,

$$u_n e_m = u_{n+k} e_m,$$

for $n > n(m)$ some $n(m)$, since $p_n e_m = 0$ for large k. Consequently, u_n converges left strictly to a unitary in $LM(A)$. Clearly $\{u_{n(m)} e_m u_{n(m)}^*\}$ forms an approximate identity for A. Notice for large n,

$$u_{n(m)}e_m u^*_{n(m)}u_{n+k} = u_{n(m)}e_m = u_{n(m)}e_m u^*_{n(m)}u_n.$$

Thus u_n converges strictly to a unitary in $M(A)$. For any $f \in C(X)$,

$$\sigma(f) - u^*\sigma'(f)u = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}[f(\lambda_{(k,i)}) - f(\gamma_{\rho[(k,i)]}))](e_{(k,i)} - e_{(k,i)-1}).$$

Since $\lambda_{(k,i)} = \lambda_{(k,1)}$,

$$lim_{i\to\infty}[f(\lambda_{(k,i)}) - f(\lambda_{\gamma[\rho[(k,i)]}] = 0.$$

Therefore

$$\sum_{i=1}^{\infty}[f(\lambda_{(k,i)}) - f(\gamma_{\rho[(k,i)]})](e_{(k,i)} - e_{(k,i)-1}) \in A.$$

Since $d(\lambda_n, \gamma_{\rho(n)}) \to 0$, we conclude that

$$\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}[f(\lambda_{(k,i)}) - f(\gamma_{\rho[(k,i)]}))](e_{(k,i)} - e_{(k,i)-1}) \in A.$$

Thus

$$\pi[\sigma(f) - u^*\sigma'(f)u] = 0.$$

By Proposition 2.2, every totally trivial extension of A by $C(X)$ is equal to one of those (modulo A).

Q.E.D.

Corollary 8.2. *If A is a σ-unital purely infinite simple C^*-algebra with $K_1(A) = 0$ and X is a compact metric space, then*

$$E(C(X), A) = \mathbf{Ext}(C(X), A).$$

Proof: As in [Lin4,2.7], applying 8.1, one sees that $\tau + \tau_o$ is unitarily equivalent to τ for any extension $\tau : C(X) \to M(A)/A$ and totally trivial

extension $\tau_o C(X) \to M(A)/A$.
Q.E.D.

Corollary 8.3. *Let A be a σ-unital purely infinite simple C^*-algebra with $K_1(A) = 0$ and X be a contractive compact metric space. Then every extension $\tau : C(X) \to M(A)/A$ is totally trivail.*

Proof: By 6.7, $E(X, A) = 0$. Then, by 8.2, there is only one unitarily equivalence class of extensions of $C(X)$ by A. Therefore every extension of $C(X)$ by A is totally trivial.
Q.E.D.

Corollary 8.4. *If A is a σ-unital purely infinite simple C^*-algebra with $K_1(A) = 0$ and X is a compact subset of the plane, then*

$$E(X, A) = Ext(C(X), A) = \mathbf{Ext}(C(X), A).$$

Proof: It is immediate consequence of 7.8 ,8.2.
Q.E.D.

Corollary 8.5. (The Berg Theorem)*Suppose that A is a σ- unital purely infinite simple C^*-algebra with $K_1(A) = 0$. Let $T \in M(A)$ be a normal element. Then there exist an approximate identity $\{e_n\}$ for A consisting of projections and $a \in A$ such that*

$$T = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}) + a,$$

where $\{\lambda_n\}$ is a dense subset of the spectrum of $sp(\pi(T))$.

Proof: If is immediate consequence of 8.4 and 1.5.
Q.D.E.

Theorem 8.6.(The BDF Theorem) *Let A be a σ-unital purely simple C^*-algebra with $K_1(A) = 0$. If T_1 and T_2 are two elements in $M(A)$ such that both $\pi(T_1)$ and $\pi(T_2)$ are normal in $M(A)/A$, then a necessary and* *sufficient condition that T_1 is unitarily equivalent to T_2 molulo A is that* $sp(\pi(T_1) = sp(\pi(T_2) = \Omega$ *and*

$$ind(T_1 - \lambda I) = ind(T_2 - \lambda I)$$

for all $\lambda \notin \Omega$.

Proof: The necessary is obvious now. For sufficiency, let τ_1 and τ_2 be the extensions by A determined by T_1 and T_2. By Theorem 7.5 and Theorem 8.1, it is enough to show that $\gamma_\Omega[\tau_1] = \gamma_\Omega[\tau_2]$. But this follows from a standard algebraic topology argument (See the proof of [BDF1, 11.1]).
Q.E.D.

Corollary 8.7. *Let A be a σ-unital purely infinite simple C^*-algebra with* $K_1(A) = 0$.. *If $T \in M(A)$ such that $\pi(T)$ is normal and $ind(T - \lambda I) = 0$ for all λ not in $sp(\pi(T))$, then T is in*

$$\mathcal{N} + A = \{n + a : n \text{ normal}, a \in A\}.$$

Moreover there exist an approximate identity $\{e_n\}$ for A consisting projections and $a \in A$ such that

$$T = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}) + a,$$

where $\{\lambda_n\}$ is a dense subset of $sp(\pi(T))$.

Corollary 8.8. *Let A be a σ-unital purely infinite simple C^*-algebra with* $K_1(A) = 0$. . *Then the the subset $\mathcal{N} + A$ is morm-closed in $M(A)$.*

Proof: The set $\pi(\mathcal{N} + A)$ is clearly norm-closed. Then this corollary follows from the continuity of the index and 8.7.
Q.E.D.

Now let us give some computation for $\mathbf{Ext}(C(X), A)$ (not $Ext(C(X), A)$!).

Example 8.8. Let $A = O_n \otimes \mathcal{K}$.

(1) X is homotopic to S^1.

$$\mathbf{Ext}(C(X), A) = E(X, A) = E(S^1, A)$$

$$\cong Hom(K^1(S^1), \mathbf{Z}/(n-1))$$

$$= Hom(\mathbf{Z}, \mathbf{Z}/(n-1)) \cong \mathbf{Z}/(n-1).$$

(2) X is homotopic to the figure eight curve.

$$\mathbf{Ext}(C(X), A) = E(X, A)$$

$$\cong Hom(K^1(X), \mathbf{Z}/(n-1))$$

$$= Hom(\mathbf{Z} \oplus \mathbf{Z}, \mathbf{Z}/(n-1)) = \mathbf{Z}/(n-1) \oplus \mathbf{Z}/(n-1).$$

(3) X is homotopic to the Hawaiian ear ring. Applying 6.5, (and the results in this section),we have

$$\mathbf{Ext}(C(X), A) = E(X, A)$$

$$\cong \prod_{k=1}^{\infty} Ext(S^1, A)$$

$$\cong \prod_{k=1}^{\infty} Hom(K^1(S^1), \mathbf{Z}/(n-1)) = \prod_{k=1}^{\infty} Hom(\mathbf{Z}, \mathbf{Z}/(n-1))$$

$$= \prod_{k=1}^{\infty} \mathbf{Z}/(n-1).$$

(4) For $n = 2$, $Hom(K^1(X), \mathbf{Z}/(n-1)) = 0$. So if $A = O_2 \otimes \mathcal{K}$ and X is a compact subset of the plane, then all extensions of $C(X)$ by A are totally trivial and unitarily equivalent.

§ 9. **Weyl-von Neumann Theorem for Normal Elements with One Dimensional Spectra**

Let A be a σ-unital stable C^*-algebra with $RR(M(A) = 0$. By section 2, $M(A)$ has (stable) (FU). Moreover, $cer(M(A)/A) = 1$. This implies that if $[\tau] = 0$ in $Ext(C(X), A)$ then τ is totally trivial. Suppose that X is a compact subset of the plane and $\tau : C(X) \longrightarrow M(A)/A$ is an extension. If $[\tau] = 0$ in $Ext(C(X), A)$, then by Collaray 7.8, $[\tau] = 0$ in $E(X, A)$. Is τ totally trivial? From our definition, there is a totally trivial extension σ such that $\tau + \sigma$ is totally trivial. It is not clear at all that this implies that τ is totally trivial in general. We will show in this section that at least for some one dimensional compact subsets X of the plane that $[\tau] = 0$ in $Ext(C(X), A)$ implies that τ is totally trivial. This also generalizes the Weyl-von Neumann theorem in section 2.

Theorem 9.1 *Let A be a σ-unital stable C^*-algebra with $RR(M(A)) = 0$, let F_i be a closed subset of the unit interval $[0,1]$, $i = 1, 2, ..., n$, and let X be a space which is homeomorphic to the disjoint union of $[0,1]/F_i$. Suppose that $\tau : C(X) \longrightarrow M(A)/A$ is an extension and $[\tau] = 0$ as an element in $Ext(C(X), A)$, then τ is totally trivial.*

Proof: If X is homeomorphic to $[0,1]/F_1$, then the theorem follows immediately from the proof of 7.3. One can then trivially extend this situation to the disjoint union of finitely many such sets.
Q.E.D.

Theorem 9.2 *Let A be a σ-unital C^*-algebra with $RR(M(A \otimes K)) = 0$, let F_i be a closed subset of the unit interval $[0,1]$, $i = 1, 2, ..., k$ and let X be a compact space which is homeomorphic to the disjoint union of $[0,1]/F_i$. Suppose that x is a normal element in $M(A)$ with $sp(\pi(x)) \subset X$. Then there exist an approximate identity $\{e_n\}$ for A consisting of projections and $a \in A$ such that*

$$x = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1}) + a \, , (e_0 = 0)$$

where $\{\lambda_n\}$ is dense in $sp(\pi(x))$.

Proof: Let $\{e_{ij}\}$ denote a matrix unit for $\mathcal{K}$. We identify A with $(1 \otimes e_{11})A \otimes \mathcal{K}(1 \otimes e_{11})$. Set $y = x + \sum_{n=2}^{\infty} \lambda 1 \otimes e_{nn}$, where λ is a nonzero point outside of the spectrum. It follows from 9.1 that $\pi(y)$ is contained in an abelian AF C^*-subalgebra B of $M(A\otimes\mathcal{K})/A\otimes\mathcal{K}$ such that $\pi(y)\in B$. Therefore $\pi(\sum_{n=2}^{\infty}1\otimes e_{nn})$ is a projection in B. So $\pi(x)\in(1\otimes e_{11})B(1\otimes e_{11})$ and $(1\otimes e_{11})B(1\otimes e_{11})$ is an abelian AF C^*-subalgebra of $M(A)/A$. Then the conclusion follows from 1.5.

Q.E.D.

Theorem 9.3 *Let A be a σ-unital stable C^*-algebra with $RR(M(A)) = 0$. Suppose that X is homeomorphic to an one dimensional finite simplicial complex on the plane. Suppose that $\tau : C(X) \to M(A)/A$ is an extension such that $[\tau] = 0$ as an element in $Ext(C(X), A)$. Then τ is totally trivial.*

Proof: It follows from 7.8 that $[\tau] = 0$ as an element in $E(X, A)$. By Theorem 7.6, $\gamma(\tau) = 0$. Notice that if X is homeomorphic to the figure eight curve, then the result follows from 9.2. Now we assume that X is homeomorphic to figure θ curve. We may further assume that $X = S^1 \cup L$, where $L = \{z = ai : -1 \le a \le 1\}$. Let ϕ be the continuous surjection which maps the right half of the circle symetrically to the left half. Since $\gamma(\phi_*(\tau)) = 0$, it follows from 7.3 that $\phi_*(\tau)$ is totally trivial. By 1.5, there exist an approximate idnetity $\{e_n\}$ for A consisting of projections and a dense sequence $\{\lambda_n\}$ contained in X, and a monomorphism $\sigma : C(X) \to M(A)$ such that

$$\sigma(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1})\,(e_0 = 0)$$

and $\sigma = \pi \circ \phi_*(\tau)$. Set

$$p = \sum_{\lambda_n \in L} (e_n - e_{n-1}).$$

Then p is a projection in $M(A)$. Moreover, as in 1.6, $\pi(p)$ is in an abelian AF C^*-subalgebra B containing $im\phi_*(\tau)$. It follows from Lemma 4.2 that $\pi(p)$ commutes with $im\tau$. It is easily verified that $\tau_1 = \pi(p)\tau$ gives a monomorphism from $C(L)$ into $\pi(p)M(A)/A\pi(p)$ and $\tau_2 = (1-\pi(p))\tau$ gives a monomorphism from $C(S^1)$ into $(1-\pi(p))M(A)/A(1-\pi(p))$. Moreover, if $f \in C(X)$, then

$$\tau(f) = \tau_1(f) \oplus \tau_2(f).$$

As in 7.3, $\gamma(\tau) = 0$ implies that both τ_1 and τ_2 are totally trivial. Or equivalently, there are abelian AF-C^*-subalgebras $B_1 \subseteq \pi(p)M(A)/A\pi(p)$ and $B_2 \subseteq (1-\pi(p))M(A)/A(1-\pi(p))$ such that $im\tau_1 \subseteq B_1$ and $im\tau_2 \subseteq B_2$. This implies that $im\tau \subseteq B_1 \oplus B_2$. Hence τ is totally trivial. Now for general finite simplicial complex we can use induction to deduce the general case to the case of the figure θ curve or the case of figure eight curve.
Q.E.D.

Corollary 9.4 *Let A be a σ-unital C^*-algebra with $RR(M(A \otimes \mathcal{K}) = 0$, and let X be finite simplicial complex. Suppose that x is an normal element in $M(A)$ with $sp(\pi(x))$ contained in X. Then there exist an approximate identity $\{e_n\}$ for A consisting of projections and $a \in A$ such that*

$$x = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}) + a,$$

where $\{\lambda_n\}$ is a dense sequence in $sp(\pi(x))$.

Proof: If A is stable, then the corollary follows immediately from 9.3. Since 4.2 works for non-stable case, we see that the proof of 9.3 implies the corollary.
Q.E.D.

4 Extensions by Finite Matroid Algebras

In this Chapter, we will study the extensions determined by the following short extact sequence

$$0 \to A \to E \to C(X) \to 0,$$

where A is a finite matroid algebra.

§ 10. Absorption Lammas and $\tilde{E}(X, A)$

10.1. Let X be a compact metric space. We may assume that the diameter of X is 1. There are n_1 disjoint open subsets $O_1, O_2, ..., O_{n_1}$ such that the closure of the union $\cup_{i=1}^{n_1} O_i$ is X. For each $\bar{O}_i$, the closure of O_i, there are $2(i)$ many disjoint open subsets $O_{i,j}$, $j = 1, 2, ..., 2(i)$ such that the closure of the union of $O_{i,j}$ is $\bar{O}_i$. So we have a sequence of open subsets $\{O_{i_1,i_2,...,i_k}\}$ satisfying the following:
(1) $O_{i_1,i_2,...,i_k,j} \subset O_{i_1,i_2,...,i_k}$;
(2) $O_{i_1,i_2,...,i_k} \cap O_{j_1,j_2,...,j_k} = \emptyset$, if $i_s \neq j_s$ for some $1 \leq s \leq k$;
(3) the closure of $\cup_{j=1}^{k(i_1,i_2,...,i_{k-1})} O_{i_1,i_2,...,i_{k-1},j}$ is $\bar{O}_{i_1,i_2,...,i_{k-1}}$.
Now we fix this sequence. Let $\tau : C(X) \to M(A)/A$ be a totally trivial extension. By 1.5, there exist an approximate identity $\{e_n\}$ for A consisting of projections and a dense sequence $\{\lambda_n\}$ contained in X with isolated points repeated infinitely often, and a monomorphism $\sigma : C(X) \to M(A)$ such that

$$\sigma(f) = \sum_{n=1} f(\lambda_n)(e_n - e_{n-1})$$

and $\sigma = \pi \circ \tau$. We may also assume that $\lambda_n \subset O_{i_1,i_2,...,i_k}$ for some $O_{i_1,i_2,...,i_k}$.

Set

$$p_{i_1,i_2,\ldots,i_k} = \sum_{\lambda_n \in O_{i_1,i_2,\ldots,i_k}} (e_n - e_{n-1}).$$

Clearly the sum converges in the strict topology, whence $p_{i_1,i_2,\ldots,i_k}$ is a projection in $M(A)$. Let B be the abelian C^*-subalgebra of $M(A)/A$ generated by projections $\{p_{i_1,i_2,\ldots,i_k}\}$. Then B is an abelian AF-algebra and there is a totally disconnected space $\tilde{X}$ such that B is isomorphic to $C(\tilde{X})$.

We will use this special construction later.

10.2. Recall that a matroid C^*-algebra is the C^*-inductive limit of the following (cf [Dix]):

$$M_{n_1} \longrightarrow M_{n_2} \longrightarrow \ldots \longrightarrow M_{n_k} \longrightarrow \ldots.$$

It is known (see [Dix])that if A is a matroid C^*-algebra then A has unique semifinite trace up to scalar multiples. If A has a finite trace, we say A is a finite matroid C^*-algebra (see [Dix]). Suppose that A is a finite matroid C^*-algebra. Let T be the unique normalized trace. Let $D(A)$ denote the range of T on the set of projections, and

$$D = \mathbf{Z}D(A) = \{n \cdot d : n \in \mathbf{Z}, d \in D(A)\}.$$

From [Dix], we know that D is a dense subgroup of $\mathbf{R}$. It follows from [Zh,2.5] that $D_o(M(A)/A)_+ \cong [0,1]/D$, where $D_o(M(A)/A)_+$ denotes the unitary equivalence classes of nonzero projections. We would also like to mention the following facts about finite matroid algebras which we may use later:

(1) The unitary group of $M(A)/A$ is connected (see [Ell]).

(2) Let $p, q \in M(A) \setminus A$ be two projections. Then p is equivalent to q if and only if $T(p) = T(q)$ (see [EH]).

(3) If $p, q \in M(A) \setminus A$ and

$$\rho \circ T(p) = \rho \circ T(q),$$

where ρ is the quotient map from $\mathbf{R}$ onto $\mathbf{R}/D$, then there is a unitary $u \in M(A)$ such that

$$\pi(u^*pu) = \pi(q).$$

Suppose that $1 \equiv 0(D)$. We can construct projections $p_{i_1,i_2,\ldots,i_k}$ such that

$$\sum_{j=1}^{k(i_1,i_2,\ldots,i_{k-1})} p_{i_1,i_2,\ldots,j} = p_{i_1,i_2,\ldots,i_{k-1}}$$

and $[p_{i_1,i_2,\ldots,j}] \equiv 0(D)$. Let $\lambda_{i_1,i_2,\ldots,j}$ be a point in $X_{i_1,i_2,\ldots,j}$. For each $f \in C(X)$, define $\tau(f)$ to be the limit of

$$\sum f(\lambda_{i_1,i_2,\ldots,i_k})p_{i_1,i_2,\ldots,i_k}.$$

This τ defines a totally trivial extension. Moreover, every projection p in the C^*-subalgebra generated by projections $\{p_{i_1,i_2,\ldots,i_k}\}$ has the property that $[p] \equiv 0(D)$. Such totally trivial extensions are called *null.*

Corollary 10.3. *Let A be a finite matroid C^*-algebra and X be a compact metric space. Then two null extensions of $C(X)$ by A are unitarily equivalent. Moreover if $\sigma : C(X) \to M(A)/A$ is any totally trivial extensiton and $\tau : C(X) \to M(A)/A$ is null, totally trivial extension then $\sigma \oplus \tau$ is unitarily equivalent to σ.*

Proof: Let τ_1 and τ_2 be two null extensions and $p_{i_1,i_2,\ldots,i_k}$ and $q_{i_1,i_2,\ldots,i_k}$ be the projections in 10.1 associated with τ_1 and τ_2 respectively. It follows from [Lin4,3.1] that there is a unitary $u \in M(A)/A$ such that

$$u^*p_{i_1,i_2,\ldots,i_k}u = q_{i_1,i_2,\ldots,i_k}.$$

Since $\tau_1(f)$ is the norm limit of

$$\sum f(\lambda_{i_1,i_2,\ldots,i_k})p_{i_1,i_2,\ldots,i_k}$$

and $\tau_2(f)$ is the limit of

$$\sum f(\lambda_{i_1,i_2,\ldots,i_k}) q_{i_1,i_2,\ldots,i_k}$$

for all $f \in C(X)$, where $\lambda_{i_1,i_2,\ldots,i_k}$ is the same as in 10.2, we see that $u^*\tau_1(f)u = \tau_2(f)$ for all $f \in C(X)$.
For the second part of the corollary, let $p_{i_1,i_2,\ldots,i_k}$ and $q_{i_1,i_2,\ldots,i_k}$ be the projections constructed in 10.2 associated with τ and τ_o respectively. Then

$$[p_{i_1,i_2,\ldots,i_k} \oplus q_{i_1,i_2,\ldots,i_k}] \equiv [p_{i_1,i_2,\ldots,i_k}](D).$$

So, by [Lin4,3.1] , $\tau \oplus \tau_o$ is unitarily equivalent to τ.
Q.E.D.

Lemma 10.4. *Let A be a finite matroid C^*-algebra and X be a compact metric space. Suppose that $\tau : C(X) \to M(A)/A$ is an extensions and $\tau_o : C(F) \to M(pAp)/pAp$ is another extension, where p is a projection in $M(A) \setminus A$ and F is a compact subset of X which is homeomorphic to the unit interval such that τ_o is totally trivial and $[\pi(p)] \equiv 0(D)$, then $\tau \oplus \tau_o \circ j$ is unitarily equivalent to τ, where j is the inclusion.*

Proof: It follows from [Lin4.2.6] that there is a nonzero projection $q \in M(A)/A$ such that q commutes with $im\tau$ and $q\tau$ is totally trivial extension of $C(F)$ by $q'Aq'$, where $q' \in M(A)$ is a projection such that $\pi(q') = q$. It follows from 1.5 that there exist an approximate identity $\{e_n\}$ for $q'Aq'$ consisting of projections and a dense sequence of $\{\lambda_n\}$ in F with isolated points repeated infinitely often, and a monomorphism $\sigma : C(F) \to q'M(A)q'$ such that

$$\sigma(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1})$$

and $\pi(\tau) = \sigma$.
There is a nonzero projection $\epsilon_n \leq (e_n - e_{n-1})$ for each n such that

$$q'' = \sum_{n=1}^{\infty} \epsilon_n$$

and $T(q'') \in D$, where T is the unique normalized trace on A. So $[\pi(q'')] \equiv 0(D)$. Moreover $\pi(q'')$ commutes with $im\tau$ and $\pi(q'')\tau$ is a totally trivial extension of $C(F)$ by $q''Aq''$. There is a unitary $u \in M_2(M(A)/A)$ such that $u^*(q'' \oplus p)u = q''$. It follows from [Lin4,3.10] that $u^*(q''\tau \oplus \tau_o)u$ is unitarily equivalent to $q''\tau$. Consequently $\tau \oplus \tau_o$ is unitarily equivalent to τ.
Q.E.D.

Lemma 10.5. *Let A be a finite matroid C^*-algebra and X be a compact metric space. Suppose that p is a projection in $M(A)/A$, $\tau_1 : C(X) \to pM(A)/Ap$, $\tau_2 : C(X) \to M(A)/A$ are two totally trivial extensions. Define $\tau = \tau_1 \oplus \tau_2 : C(X) \to qM_2(M(A)/A)q$, where $q = p \oplus 1$. Then there is a partial isometry $v \in M_2(M(A)/A)$ such that*

$$v^*v = q\,, \quad vv^* = p_o \leq p$$

(we identify p with $p \oplus 0$.), p_o commutes with $im\tau_1$ and $v^*\tau v = p_o\tau_1$.

Proof: Let $\tilde{X}$ be as in 10.1. We will keep the notations in 10.1. There is $0 \leq \delta \leq 1$ such that

$$[p] + 1 + \delta \equiv [p](D).$$

There is a projection $d \leq 1 - p$ in $M(A)/A$ such that there exists a projection $d' \in M(A)$ with $\pi(d') = d$ and $T(d') = \delta$. Let $p_{i_1,i_2,...,i_k}$ and $q_{i_1,i_2,...,i_k}$ be the projections in 10.1 associated with τ_1 and τ, respectively. There are projections $d_1, d_2, ..., d_{n_1} \in M(A)/A$ such that

$$d_1 + d_2 + ... + d_{n_1} = d$$

and

$$[d_i] + [q_i] \equiv [p_i],$$

$i = 1, 2, ..., n_1$. Continuing this way, we obtain projections $\{d_{i_1,i_2,...,i_k}\}$ in $M(A)/A$ such that

$$\sum_{j=1}^{k(i_1,i_2,...,i_{k-1})} d_{i_1,i_2,...,j} = d_{i_1,i_2,...,i_{k-1}}$$

and

$$[d_{i_1,i_2,\ldots,j}] + [q_{i_1,i_2,\ldots,j}] \equiv [p_{i_1,i_2,\ldots,i_j}].$$

Since $[p] + 1 + \delta \equiv [p]$, there is a partial isometry $v_1 \in M_2(M(A)/A)$ such that

$$v_1^* v_1 = p \oplus 1 \oplus d\,, \quad v_1 v_1^* = p.$$

Denote by B the abelian AF C^*-subalgebra generated by projections

$$\{q_{i_1,i_2,\ldots,i_k} \oplus d_{i_1,i_2,\ldots,i_k}\}$$

and $B_1 = v_1 B v_1^*$. Since

$$[v_1 q_{i_1,i_2,\ldots,i_k} v_1^*] + [v d_{i_1,i_2,\ldots,i_k} v_1^*] = [p_{i_1,i_2,\ldots,i_k}]$$

in $D_o(pM(A)/Ap)$, by [Lin4,3.1], there is a unitary $u \in pM(A)/Ap$ such that

$$u^*(v_1(q_{i_1,i_2,\ldots,i_k} \oplus d_{i_1,i_2,\ldots,i_k})v^*)u = p_{i_1,i_2,\ldots,i_k}.$$

Now set $p_o = u^* v_1 q v_1^* u$ and $v = v^* u$. Then $p_o \leq p$, p_o commutes with $im\tau_1$ and $v^* \tau v = p_o \tau_1$.
Q.E.D.

Suppose that A is a finite matroid C^*-algebra with $T(1) \in D$. Then $M_2(M(A)/A) \cong M(A)/A$ via a partial isometry in $M_2(M(A)/A)$. So addition of two extensions can be defined in a obvious way. (Addition of extensions by non-stable C^*-algebras has been studied by [EH].) We use the notation $\tilde{E}(X, A)$ for the semigroup of the quotient of $\mathbf{Ext}(C(X), A)$ by the totally trivial extensions. Similar to the stable case, we have the following properties for $\tilde{E}(X, A)$.

10.6. *Propostion 3.2 holds for* $\tilde{E}(X, A)$.

10.7. *Lemma 4.3 holds for* $\tilde{E}(X, A)$.

Proof: We will keep the notations in the proof of 4.1 and 4.3. For each n, let ε_n^o be a projection in $M(A)$ such that $\varepsilon_n^o \leq \varepsilon_n$ and $[\pi(\varepsilon_n^o)] \equiv 0(D)$. Define

$$\tau_n : C(X_n) \to M(\varepsilon_n^o A \varepsilon_n^o)/\varepsilon_n^o A \varepsilon_n^o.$$

Since $\sum_{n=1}^{\infty} \varepsilon_n$ converges to $(1-\varepsilon_o)$ in the strict topology, $\sum_{n=1}^{\infty} \varepsilon_n^o$ converges in the strict topology too. Moreover, $\tau' = \sum \tau_n$ defines a monomorphisim from I into $(1-\varepsilon_o)M(A)/A(1-\varepsilon_o)$. Now define

$$\tau''(f) = \pi \circ \sigma(f \circ r) + \tau'(f - f \circ r)$$

for $f \in C(X)$. As in 4.3, to show this is a homomorphism, it is enough to show that

$$\tau'((g \circ r)f) = \pi \circ \sigma_2(g \circ r)\tau'(f)$$

for all $f \in I$, $g \in C(F)$. As in 4.3, it is enough to establish the above relation for f's satisfying $f\chi_n = f$. Then $(g \circ r)f = g(x_n)f$. So

$$\tau'((g \circ r)f) = g(x_n)\tau'(f).$$

On the other hand,

$$\tau'(f) = \tau'(f)\tau'(\chi_n) = \tau'(f)\pi(\varepsilon_n^o).$$

Therefore

$$\begin{aligned}\pi \circ \sigma_2(g \circ r)\tau'(f) &= \pi \circ \sigma_2(g \circ r)\pi(\varepsilon_n^o)\tau'(f)\\ &= g(x_n)\pi(\varepsilon_n^o)\tau'(f) = g(x_n)\tau'(f).\end{aligned}$$

We now show that

$$[\tau \oplus \tau''] = [\tau \circ r^* \circ i^* \oplus \pi \circ \sigma].$$

For $f \in I$,

$$(\tau \oplus \tau'')(f) = \tau(f) + \tau'(f) = \sum_{n=1}^{\infty} \tau(f)\pi(\varepsilon_n) + \sum_{n=1}^{\infty} \tau_n(f\chi_n)$$

and

$$(\tau \circ r^* \circ i^* \oplus \pi \circ \sigma)(f) = \tau(f).$$

By 10.5, for each n, there is a partial isometry

$$v_n \in (\pi(\varepsilon_n^o) \oplus \pi(\varepsilon_n) M_2(M(A)/A))(\pi(\varepsilon_n^o) \oplus \pi(\varepsilon_n))$$

such that

$$v_n^*[\tau(f)\varepsilon_n + \tau_n(f\chi_n)]v = \tau(f)$$

for all $f \in I$. There is a partial isometry $u_n \in (\pi(\varepsilon_n^o) \oplus \varepsilon_n) M_2(M(A))\pi(\varepsilon_n^o) \oplus \varepsilon_n)$ such that $\pi(u_n) = v_n$. Since $\sum_{n=1}^{\infty} \varepsilon_n$ converges in the strict topology, we conclude that $\sum_{n=1}^{\infty} u_n$ converges to a partial isometry

$$u \in (\sum_{n=1}^{\infty} \varepsilon_n^o \oplus (1-\varepsilon_o)) M_2(M(A))(\sum_{n=1}^{\infty} \varepsilon_n^o \oplus (1-\varepsilon_o).$$

Therefore

$$\pi(u)^*(\tau(f) + \tau''(f))\pi(u) = \tau(f)$$

for all $f \in I$.

Similarly, there is a partial isometry

$$w \in (1-\varepsilon_o)M(A)(1-\varepsilon_o)$$

such that $\pi(w^*)\pi(\varepsilon_n - \varepsilon_n^o)\pi(w) = \pi(\varepsilon_n)$. Notice that

$$\pi(u)^*\pi(\sum_{n=1}^{\infty} g(x_n)(\varepsilon_n - \varepsilon_n^o))\pi(u) = \pi(\sum_{n=1}^{\infty} g(x_n)\varepsilon_n)$$

and

$$\pi(w)^*\pi(\sum_{n=1}^{\infty} g(x_n)(\varepsilon_n - \varepsilon_n^o))\pi(w) = \pi(\sum_{n=1}^{\infty} \varepsilon_n).$$

Since, for any $f \in C(X)$,

$$(\tau o r^* o i^* \oplus \pi o \sigma)(f) = \tau(f o r)\pi(\varepsilon_o) \oplus [\tau(f - f o r) = \pi(\sum f(x_n)\varepsilon_n)] \oplus \pi(\sum f(x_n)\varepsilon_n)$$

$$(\tau \oplus \tau'')(f) = \tau(f o r) \oplus [\tau(f - f o) + \tau''(f - f o r) + \pi(\sum f(x_n)(\varepsilon_n \oplus \varepsilon_n^o))] \oplus \pi(\sum f(x_n)(\varepsilon_n - \varepsilon_n^o)),$$

we obtain

$$\pi(w_o)^*(\tau(f) + \tau''(f))\pi(w_o) = (\tau \circ r^* \circ i^* \oplus \pi \circ \sigma)(f)$$

for all $f \in C(X)$. This implies that

$$[\tau \oplus \tau''] = [\tau \circ r^* \circ i^* \oplus \pi \circ \sigma].$$

Since both τ'' and $\pi \circ \sigma$ are totally trivial. We finally conclude that $[\tau] = [i_* \circ \tau \circ r^*]$.
¿From this and the proof of 4.3, we see that Lemma 4.3 holds for $\tilde{E}(X, A)$.

10.8. *Lemma 4.4 holds for* $\tilde{E}(X, A)$.

By applying 10.7, we can prove 10.8 in the same way as that of 4.4.

10.9. *Thoerem 4.5 holds for* $\tilde{E}(X, A)$.

This follows from 10.7 and 10.8.

10.10. *Theorem 5.3 holds for* $\tilde{E}(E, A)$.

Since now we have 10.6 and 10.8, the proof of 10.10 follows that of 5.3.

10.11. *Corollary 5.4 holds for* $\tilde{E}(X, A)$.

§ 11. Extensions by Finite Matroid C^*-Algebras and the Weyl-von Neumann-Berg Theorem

We will show in this section that if A is a finite matroid C^*-algebra, then every extension $\tau : C(X) \to M(A)/A$, where X is a compact subset of the plane, is totally trivial. Consequentely, the Weyl-von Neumann-Berg theorem holds for finite matroid C^*-algebras.

Lemma 11.1. *Let A be a finite matroid C^*-algebra with $T(1) \in D$, let X be a closed subset of the unit square I^2 such that X contains $\{1/2\} \times I$, let $F = X \cap ([0,1/2] \times I)$, and let $G = X \cap ([1/2,1] \times I)$. Suppose that $\tau : C(X) \to M(A)/A$ is an extension and z is the identity function. Then there is a projection $q \in M(A)/A$ such that q commutes with $\tau(z)$ and the spectrum of $\tau(z)p$ is F_1 and the spectrum of $\tau(z)(1-p)$ is G_1, where $F_1 \subset F$, $G_1 \subset G$ and $F_1 \cup G_1 = X$.*

Proof: Let $\tau(z) = h_o + ih_1$ with h_o, h_1 selfadjoint, and let a be as provided by Lemma 3.3 (with $h_n = h_1, n \geq 1$). Then $\tau_o(f) = f(1/2 + ih_1)$ is well defined, since $\{1/2\} \times I \subset X$. As in 4.4,

$$p = \begin{pmatrix} a & (a(1-a))^{1/2} \\ (a(1-a))^{1/2} & 1-a \end{pmatrix}$$

is a projection that commutes with $im(\tau + \tau_o)$. In particular p commutes with $\tau(z) \oplus (1/2 + ih_1)$.

The maps $\tau_1 = p(\tau + \tau_o)$ and $\tau_2(1-p)(\tau + \tau_o)$ are homomorphisms from $C(X)$ to $pM_2(M(A)/A)p$ and to $(1-p)M_2(M(A)/A)(1-p)$ respectively. As in 4.4, $ker\tau_1 = F_1$ and $ker\tau_2 = G_2$, where $F_1 \subset F$, $G_1 \subset G$ and $F_1 \cup G_1 = X$. By Lemma 10.4, there is a partial isometry $v \in M_2(M(A)/A)$ such that

$$vv^* = 1 \oplus p_o \, , v^*v = 1$$

and

$$v^*[\tau(z) \oplus (1/2 + ih_1)]v = \tau(z).$$

Take $q = v^*pv$.

Q.E.D.

Lemma 11.2. *Let A be a finite matroid C^*-algebra with $T(1) \in D$ and X be the unit squre I^2. Then every extension $\tau : C(X) \to M(A)/A$ is totally trivial.*

Proof: Notice that, in the proof of 11.1, since now $X = I^2$, $F_1 = F$ and $G_1 = G$. Moreover, both F and G are homeomorphic to X. By repeated application of 13.1, one can produce, for each n, 2^n mutually orthogonal projections $\{p_{k,n} \in M(A)/A, k = 1, 2, ..., 2^n\}$ and 2^n closed subsets $X_{k,n}$ of X such that

(1) projections in $\{p_{k,n}\}, k = 1, 2, ..., 2^n$ and $n = 1, 2, ...,$ are mutually commutative;

(2) the union of these 2^n subsets $X_{k,n}$ is X and intersection of interiors of any two different subsets is empty;

(3) each $X_{k,n}$ is a retract of X and is homeomorphic to I^2 and the diameter tends to zero;

(4) $p_{k,n}\tau \circ j_{k,n}$ is a monomorphism from $C(X_{k,n})$ into $p_{k,n}(M(A)/A)p_{k,n}$, where $j_{k,n}$ is the inclusion from $X_{k,n}$ into X.

Suppose that B is the C^*-subalgebra of $M(A)/A$ generated by projections $\{p_{k,n}\}$. Then B is an abelian AF-algebra and $im\tau \subset B$. By our definition, τ is totally trivial.

Q.E.D.

Corollary 11.3. *Let A be a finite matroid C^*-algebra with $T(1) \in D$. Then $\tilde{E}(I^2, A)$ is a group and $\tilde{E}(I^2, A) = 0$.*

Proof: This follows immediately from 11.2.

Q.E.D.

Corollary 11.4. *Let A be a finite matroid C^*-algebra with $T(1) \in D$. Then $\tilde{E}(X, A)$ is a group for any compact subset of the plane.*

Proof: This follows from 10.11 and 11.3.

Q.E.D.

Lemma 11.5. *Let A be a finite matroid C^*-algebra and X is homeomorphic to $[0,1]/F$, where F is a compact subset of the unit interval $[0,1]$. Then every extension $\tau : C(X) \to M(A)/A$ is totally trivial.*

Proof: Let $\{e_{ij}\}$ be a matrix unit for $\mathcal{K}$. Define

$$\tau'(f) = \tau(f \circ j) \oplus f(x_o) \sum_{n=2}^{\infty} e_{ii}$$

for $f \in C(X \vee \{x_o\})$, where $j : X \to X \vee \{x_o\}$ is the inclusion. Since the unitary group of $M(A)/A$ is connected, $\gamma[\tau'] = 0$. Thus, by 7.3, $[\tau'] = 0$ in $Ext(C(X), A \otimes \mathcal{K})$. By the proof of 9.1, we see that τ is totally trivial.
Q.E.D.

Theorem 11.6. *Let A be a finite matroid C^*-algebra and X be a compact subset of the plane. Then every extension $\tau : C(X) \to M(A)/A$ is totally trivial.*

Proof: Let L be a straight line which divides X into two closed subsets F and G. We will show that there is a projection $p \in M(A)/A$ such that p commutes with $im\tau$ and $p\tau$ gives a monomorphism from $C(F)$ into $pM(A)/Ap$ and $(1-p)\tau$ gives a monomorphism from $C(G)$ into $(1-p)M(A)/A(1-p)$. Once this is done, we can apply this repeatedly as in 11.2 to show that that τ is totally trivial.

Let $\lambda \in \mathbf{C} \setminus X$. Suppose that $d \in M(A)/A$ is a projection such that $[d] + 1 \equiv 0(D)$. Define

$$\bar{\tau}(f) = f(\lambda)d \oplus \tau(f \circ j)$$

where $f \in C(X \cup \{\lambda\})$ and j is the inclusion from X into $X \cup \{\lambda\}$. If a projection $p \in (d \oplus 1)M_2(M(A)/A)(d \oplus 1)$ commutes with $im\bar{\tau}$, by taking functions vanishing at λ, we know that p commutes with $0 \oplus (im\tau)$, Then, by a direct computation, one sees that $p = p_1 \oplus p_2$, where $p_1 \leq d$ and $p_2 \leq 1$ are projections. So p_2 commutes with $im\tau$. Thus we may now assume that

$T(1) \in D$.

Let J be a compact interval on L that contains $X \cap L$. Let $j : X \to X \cup J$ be the inclusion. From 11.5, we know that $\tilde{E}(J/F \cap G, A) = 0$. Now we proceed the proof of 7.5. So there is $\tau_1 : C(F) \to M(A)/A$ and $\tau_2 : C(G) \to M(A)/A$ such that

$$[(i_1)_*(\tau_1)] + [(i_2)_*(\tau_2)] = [\tau].$$

Therefore we may assume that there are totally trivial extensions $\tau_o : C(X) \to M(A)/A$ and $\tau_o' : C(X) \to M(A)/A$ such that

$$\tau_1 \circ i_1 \oplus \tau_2 \circ i_2 \oplus \tau_o = \tau \oplus \tau_o'.$$

Since τ_o is totally trivial, by 1.5, there exist an approximate ideneity $\{e_n\}$ for A consisting of projections and a dense sequence $\{\lambda_n\}$ in X such that

$$\sigma(f) = \sum_{n=1}^{\infty} f(\lambda_n)(e_n - e_{n-1})$$

and $\pi \circ \sigma = \tau_o$. Let

$$p_1 = \pi[\sum_{\lambda_n \in F} (e_n - e_{n-1})].$$

Then p_1 commutes with $im\tau_o$ and $p_1\tau_o$ gives a monomorphism from $C(F)$ into $p_1 M(A)/A p_1$ and $(1 - p_1)\tau_o$ gives a monomorphism from $C(G)$ into $(1 - p_1)M(A)/A(1 - p_1)$. Thus there is a projection $q \in M_2(M(A)/A)$ such that q commutes with $im\tau \oplus \tau_o'$ such that $q(\tau \oplus \tau_o')$ gives a monomorphism from $C(F)$ into $qM_2(M(A)/A)q$ and $(1 - q)(\tau \oplus \tau_o')$ gives a monomorphism from $C(G)$ into $(1 - q)M_2(M(A)/A)(1 - q)$. It follws from [Lin4,2.6] that we can write $\tau = \tau' \oplus \tau_{oo}$, where $\tau_{oo} : C(X) \to gM(A)/Ag$ is totally trivial and g is a nonzero projection in $M(A)/A$. By 10.5, we may write $\tau_{oo} = \tau_{oo}' \oplus \tau_{oo}''$ such that τ_{oo}' is unitarily equivalent to $\tau_{oo} \oplus \tau_o'$ So there is a partial isometry $v_1 \in M_2(M(A)/A)$ such that

$$v_1^*(\tau' \oplus \tau_{oo}')v = \tau \oplus \tau_o'.$$

Hence the projection $q' = v_1 q v_1^*$ commutes with $im\tau' \oplus \tau'_{oo}$ such that $q'(\tau' \oplus \tau'_{oo})$ gives a monomorphism from $C(F)$ into $M(A)/A$ and $(1-q')(\tau' \oplus \tau'_{oo})$ gives a monomorphism from $C(G)$ into $M(A)/A$. Since τ''_{oo} is totally trivial, there is a projection q'' such that q'' commutes with $im\tau''_{oo}$, $q''\tau''_{oo}$ gives a monomorphism from $C(F)$ into $M(A)/A$ and $(1-q'')\tau''_{oo}$ gives a monomorphism from $C(G)$ into $M(A)/A$. Set

$$p = q' \oplus q''.$$

Then p commutes with $im\tau$ and $p\tau$ gives a monomorphism from $C(F)$ into $pM(A)/Ap$ and $(1-p)\tau$ gives a monomorphism from $C(G)$ into $(1-p)M(A)/A(1-p)$. This completes the proof.
Q.E.D.

The following is a version of the Weyl-von Neumann-Berg theorem.

Corollary 11.7. *Let A be a finite matroid C^*-algebra and $x \in M(A)/A$ be a normal element. Then there is a normal element $y \in M(A)$ such that there exist an approximate identity $\{e_n\}$ for A consisting of projections and a dense squence $\{\lambda_n\}$ in $sp(x)$ such that*

$$y = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1}),$$

$\pi(y) = x$ and $sp(y) = sp(x)$.

Remark 11.8. The results of 11.6 and 11.7 hold for separable *AF*-algebra with unique normalized trace.

Theorem 11.9. *Let A be a finite matroid C^*-algebra and $B \cong C(X) \otimes M_n$, where X is a compact subset of the plane. Then every extension $\tau : B \to M(A)/A$ is trivial.*

Proof: We identify B with the C^*-algebra $C(X, M_n)$, the continuous functions from X to M_n. Let D denote the constant functions. So $D \cong M_n$.

It follows from [Eff,9.8] that there is a C^*-subalgebra B_o of $M(A)$ such that $B_o \cong M_n$ and $\pi(B_o) = D$. Let $\{e_{ij}\}$ be a mtrix unit for M_n. Set $f(z) = z \cdot \tau(e_{11})$ in $C(X, M_n)$. Let $\bar{a} = \tau(f)$. Then the C^*-subalgebra generated by D and $\bar{a}$ is $\tau(B)$. If we also use e_{11} for the obvious constant function in D, then there is $d_{11} \in B_o$ such that $\pi(d_{11}) = \tau(e_{11})$. It follows from 11.6 that there is a normal element $a \in d_{11}M(A)d_{11}$ such that $sp(a) = sp(\bar{a})$ and $\pi(a) = \bar{a}$. Let B_1 be the C^*-subalgebra of $M(A)$ generated by B_1 and a. Then $B_1 \cong B$ and $\pi(B_1) = \tau(B)$. This implies that τ is trivial.
Q.E.D.

Remark 11.10. Let A be a finite matroid C^*-algebra and $X = S^1$. Then from KK-theory, and Universal Coefficient Theorem, one has that $Ext(C(X), A \otimes \mathcal{K}) = K_o(A)$. This gives an example that extensions by A is quite different from extensions by $A \otimes \mathcal{K}$.

Finally, we show that a type of spectral theorem holds for corona alegebras of finite matroid algebras.

Theorem 11.11 *Let A be finite matroid algebra. Suppose. that $x \in M(A)/A$ is a normal element. Then there is an abelian AF-algebra $B \subset M(A)/A$ such that $x \in B$. Consequentely, x can be approximated by normal elements in $M(A)/A$ with finite spectra.*

Proof : This is immediate consequence of 11.6.
Q.E.D.

§ 12. Classification of Extensions by Finite Matroid Algebras

In section 11, we show that every extension of $C(X)$, where X is a compact metric space, by finite matroid algebras is totally trivial. Are they equivalnet? We will show in this section that the answer is yes for many connected X and no for disconnected X. We also determined the different equivalence classes of extensions of $C(X)$, where X has finitely many nice components.

Lemma 12.1 *Let A be a C^*-algebra with approximate identity $\{e_n\}$ consisting of projectins. Suppose that*

$$x = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1})$$

such that $\{\lambda_n\}$ is a dense sequence in $sp(\pi(x))$. If $\{\beta_n\}$ is another dense sequence in $sp(\pi(x))$, then there is a permutation α on the natural numbers and $a \in A$ such that

$$x = \sum_{n=1}^{\infty} \beta_{\alpha(n)}(e_n - e_{n-1}) + a.$$

Proof : There is a permutation α on the natural numbers such that

$$dis(\lambda_n, \beta_{\alpha(n)}) \to 0.$$

Set

$$y = \sum_{n=1}^{\infty} \beta_{\alpha(n)}(e_n - e_{n-1})$$

and $a = x - y$. Since

$$dis(\lambda_n, \beta_{\alpha(n)}) \to 0,$$

$$a = \sum_{n=1}^{\infty} (\lambda_n - \beta_{\alpha(n)})(e_n - e_{n-1})$$

is in A. Q.E.D.

Lemma 12.2 *Let A be a finite (non-unital) matroid C^*-algebra and X be a compact subset of the plane which is homeomorphic to the unit squre I^2. Suppose that x and y be two (essentailly) normal elements in $M(A)$ with $sp(\pi(x)) = sp(\pi(y)) = X$. Then there is a unitary $u \in M(A)$ such that*

$$u^*xu = y + a,$$

where $a \in A$.

Proof : We may assume that $sp(\pi(x)) = S$, where S is the unit squre $I^2 = \{\lambda : 0 \le Re\lambda \le 1, 0 \le Im\lambda \le 1\}$. By 1.5, we may further assume that

$$x = \sum_{n=1}^{\infty} \lambda_n(e_n - e_{n-1}),$$

where $\{\lambda_n\}$ is a dense squence in $X = sp(\pi(x))$. Let

$$S_1^{(1)} = \{\lambda : \lambda \in S, 0 \le Re\lambda \le 1/2\},$$

$$S_1^{(1)'} = \{\lambda : \lambda \in S, 0 \le Rm\lambda < 1/2\}$$

and

$$S_2^{(1)} = \{\lambda : \lambda \in S, 1/2 \le Re\lambda \le 1\}.$$

Denote

$$p_1^{(1)} = \sum_{\lambda_k \in S_1^{(1)'}} (e_k - e_{k-1})$$

and $p_2^{(1)} = 1 - p_1^{(1)}$. Let

$$x_1 = p_1^{(1)} x = \sum_{\lambda_k \in S_1^{(1)'}} \lambda_k(e_k - e - k - 1).$$

and $x_2 = p_2^{(1)} x$. Then $sp(\pi(x_1)) = S_1^{(1)}$ and $sp(\pi(x_2)) = S_1^{(2)}$ (we view x_i as an element in $p_i^{(1)} A p_i^{(1)}$).

We now cut these two rectangular in the same way as we have just done but along the y-axis. So we cut S into four equal size squre: $S_1^{(2)}, S_2^{(2)}, S_3^{(2)}$ and $S_4^{(2)}$. Corresponding to these squres, we have four projections $p_i^{(2)} \in M(A)$, $i = 1, 2, 3, 4$ such that $p_i^(2)$ commutes with x and $sp(\pi(xp_i^(2))) = S_i^(2)$, $i = 1, 2, 3, 4$. If we continue to cut these squres, we will get a sequence of squres $\{S_i^(k)\}$ and a sequence of projections $\{p_i^{(k)}\} \in M(A)$ such that $\{p_i^{(k)}\}$ generates an abelian AF-subalgebra of $M(A)$ and

$$x = lim_{k \to \infty} \sum_{i=1}^{2^k} \lambda_{k,i} p_i^{(k)},$$

where $\lambda_{k,i}$ is the center point in the squre $S_i^{(k)}$.

Now we assume that

$$y = \sum_{n=1}^{\infty} \beta_n(\epsilon_n - \epsilon_{n-1}),$$

where $\{\beta_n\}$ is a dense sequence in X and $\{\epsilon_n\}$ is an approximate identity for A consisting of projections. For each straight line segment L_m which cuts the squres as above, we may assume, by 12.1, that $\{\beta\}$ is dense in L_m. Set

$$q_1^{(1)'} = \sum_{\beta_k \in S_1^{(1)'}} (\epsilon_k - \epsilon_{k-1}),$$

$$q_3^{(1)} = \sum_{Re\beta_k = 1/2} (\epsilon_k - \epsilon_{k-1})$$

and $q_2^{(1)'} = 1 - q_1(1)' - q_3^{(1)}$. By our assumption, $q_3^{(1)} \neq 0$. Suppose that $T(q_1^{(1)'}) = a$, $T(q_3^{(1)}) = \delta(> 0)$ and $T(p_1^{(1)}) = \alpha$. Let $\rho : [0,1] \to [0,1]/D$ be the quotient map. Then

$$\rho([a, a+\delta]) = [0,1]/D.$$

So there is $0 \leq b \leq \delta$ such that

$$\rho(\alpha) = \rho(a+b).$$

Since $q_3^{(1)} \notin A$, one can find $0 \leq d_k \leq \epsilon_k - \epsilon_{k-1}$, where $Re\beta_k = 1/2$ such that

$$T(\sum_{Re\beta_k = 1/2} d_k) = b.$$

Now set $q_1^{(1)} = q_1^{(1)'} + \sum_{Re\beta_k = 1/2} d_k$. Then

$$q_1^{(1)} x = x q_1^{(1)}$$

and $sp(q_1^{(1)} x) = S_1^{(1)}$. Moreover, if we let $q_2^{(1)} = 1 - q_1^{(1)}$ then $sp(q_2^{(1)} x) = S_2^{(1)}$. Furthermore,

$$[\pi(q_1^{(1)})] = [\pi(p_1^{(1)}], \; [\pi(q_2^{(1)})] = [\pi(p_2^{(1)})].$$

We continue this procesure. We will get a sequence of projections $\{q_i^{(k)}\}$ corresponding to the squres $S_i^{(k)}$ such that $\{q_i^{(k)}\}$ generates an abelian AF-C^*-subalgebra of $M(A)$,

$$[\pi(q_i^{(k)})] = [\pi(p_i^{(k)})]$$

and

$$y = lim_{k\to\infty} \sum_{i=1}^{2^k} \lambda_{k,i} q_i^{(k)}.$$

It follows from the proof of [Lin] that there is a unitary $u \in M(A)$ such that

$$\pi(u^* y u) = \pi(x).$$

Q.E.D.

Corollary 12.3 *Let A be a finite matroid algebra and X be a connected compact subset of the plane. Then*

$$\mathbf{Ext}(C(X), A) = \{0\}.$$

In other words, all extensions $\tau : C(X) \to M(A)/A$ are totally trivial and equivalent.

Proof: This follows immediately from 11.6 and 12.2.
Q.E.D.

Theorem 12.4 *Let A be a finite matroid algebra and let X be a connected compact subset of the plane. Suppose that there are (at most) countably many mutually disjoint open subsets $\{O_n\}$ of X such that $X = \cup_{n=1}^{\infty} \bar{O}_n$, each $\bar{O}_n$ is either homeomorphic to the unit interval or the unit disk, $X_k = \cup_{i=k}^{\infty} \bar{O}_i$ is connected and $lim_{k\to\infty} diam(\bar{O}_k) = 0$. Then*

$$\mathbf{Ext}(C(X), A) = \{0\}.$$

In other words, all extensions $\tau : C(X) \to M(A)/A$ are totally trivial and equivalent.

Proof : Let

$$x = \sum_{n=1}^{\infty} \lambda_n (e_n - e_{n-1})$$

and

$$y = \sum_{n=1}^{\infty} \beta_n (\epsilon_n - \epsilon_{n-1},$$

where $\{e_n\}$ and $\{\epsilon_n\}$ are two approximate identities consisting of projections, and $\{\lambda_n\}$ and $\{\beta_n\}$ are two dense sequences in X. It is enough to show that there is a unitary $u \in M(A)$ such that

$$\pi(u^* x u) = \pi(y).$$

For each i, set

$$L_i = (\bar{O}_i \setminus \cup_{j=1}^{i-1} \bar{O}_j) \setminus O_i,$$

$$F_1 = O_1, \;\; F_i = \bar{O}_i \setminus \cup_{j=i+1}^{\infty} \bar{O}_j, \text{if } i > 1$$

and

$$G_i = O_i \cup F_i.$$

Since X_i is connected, $L_i \neq \emptyset$. Moreover, G_i is an open subset of X_i. We may assume, by 12.1, that $\{\beta_n\} \cap L_i$ is dense in L_i. Let

$$p_i = \sum_{\lambda_k \in G_i} (e_k - e_{k-1}),$$

$i = 1, 2, \ldots.$ Then the sum $\sum_{i=1}^{n} p_i$ converges to the identity in the strict topology. We also have $p_i x = x p_i$,

$$x = \sum_{i=1} p_i x$$

and $sp(\pi(p_i x)) = \bar{O}_i$. Let

$$q_1' = \sum_{\beta_k \in G_1} (\epsilon_k - \epsilon_{k-1}).$$

If $\rho \circ T(p_1) = \rho \circ T(q_1')$, we set $q_1 = q_1'$. Otherwise,let

$$q_1'' = \sum_{\beta_k \in L_1} (\epsilon_k - \epsilon_{k-1}).$$

Since $L_1 \neq \emptyset$, $q_1'' \in M(A) \setminus A$. As in the proof of 12.2, there are projections $d_k \in A$ such that

$$d_k \leq \epsilon_k - \epsilon_{k-1},$$

where $\lambda_k \in L_1$, such that

$$[q_1' + \sum_{\lambda_k \in L_1} d_k)] = [p_1].$$

Set $q_1 = q_1 + \sum_{\lambda_k \in L_1} d_k$. Then q_1 commutes with y, $sp(q_1 y) = \bar{O}_1$ and $sp((1-q_1)y) = X_2$. By induction, we have a sequence of mutually orthogonal projections $\{q_k\}$ in $M(A)$ such that

(1) $q_k y = y q_k$ for $k = 1, 2, ...$;

(2) $sp(q_k y) = \bar{O}_k$;

(3) $\sum_{k=1}^{n} q_k$ converges to the identity in the strict topology;

(4) $[q_k] = [p_k]$.

It follows from [Lin,] and 12.2 that there are partial isometries $v_k \in M(A)$ such that

$$a_k = v_k^* p_k x v - q_k y \in A,$$

$k = 1, 2, ...$ It is clear that $\sum_{k=1}^{n} v_k$ converges to a unitary $u \in M(A)$. Since $lim_{k \to \infty} diam(\bar{O}_k) = 0$,

$$lim_{k \to \infty} a_k = 0.$$

Therefore $a = \sum_{k=1}^{\infty} a_k \in A$. Now it is ready to check that

$$u^* x u = y + a.$$

Q.E.D.

Remark 12.5 Many compact connected subsets in the plane satisfy the conditions in 12.3. For examples, every annulus and every disk with finitely many holes satisfy the conditions. The so called "Hawaiian ear ring" also satisfy the conditions.

Theorem 12.6 *Let A be finite matroid algebra. Suppose that X is a compact subset of the plane and X has k connected components X_i which satisfies the conditions in 12.2, $i = 1, 2, ..., k$. Then*

$$\mathbf{Ext}(C(X), A) \cong \{([p_1], [p_2], ..., [p_k]) : [p_i] \in D_0(M(A)/A), \sum_{i=1}^{k}[p_i] = 1\}.$$

Therefore, $\mathbf{Ext}(C(X), A)$ *has uncountably many equivalence classes.*

Proof : We may write

$$X = \vee_{i=1}^{k} X_i.$$

Let $f_i \in C(X)$ such that

$$f(\lambda) = 1, \text{ for } \lambda \in X_i,$$

and

$$f(\lambda) = 0, \text{ for } \lambda \in X_j, j \neq i.$$

Let $\tau : C(X) \to M(A)/A$ and $\sigma : C(X) \to M(A)/A$ be two extensions. Set $p_i = \tau(f_i)$ and $q_i = \sigma(f_i)$, $i = 1, 2, ..., k$. Clearly

$$\sum_{i=1}^{k}[p_i] = 1, \text{ and } \sum_{i=1}^{k}[q_i] = 1.$$

Moreover, it τ is (unitarily) equivalent to σ, there is a unitary $u \in M(A)/A$ such that

$$\pi(u^*)p_i\pi(u) = (q_i).$$

So $[p_i] = [q_i]$, $i = 1, 2, ..., k$.

Conversely, if $[p_i] = [q_i]$, $i = 1, 2, ..., k$ then there is a unitary $v \in M(A)$ such that

$$\pi(v^*)p_i\pi(v) = q_i, \ i = 1, 2, ..., k.$$

This implies that, since the unitary group of $M(A)/A$ is connected, there are mutually orthogonal projections p_i' and q_i in $M(A)$ such that $v^*p_i'v = q_i'$,

$\pi(p_i') = p_i$ and $\pi(q_i') = q_i$. For each i, it follows from 12.3 that the extensions $\pi(v^*)\tau \circ p_i\pi(v) : C(X_i) \to M(q_i'Aq_i')/q_i'Aq_i'$ are equivalent to the extensions $\sigma \circ q_i : C(X_i) \to M(q_i'Aq_i')/q_i'Aq_i')$. Therefore τ is equivalent to σ.

Finally, suppose that $a_1, a_2, ..., a_k \in D_o(M(A)/A)$ such that $\sum_{i=1}^k = 1$. There are projections $p_1, p_2, ..., p_k \in M(A)$ such that $\sum_{i=1}^k = 1$. Let $\tau_i : C(X_i) \to M(p_iAp_i)/p_iAp_i$ be extensions. Then $\tau = \sum_{i=1}^k \tau_i$ is an extension of $C(X)$ by A.

Q.E.D.

Theorem 12.7 *Let A be a finite matroid C^*-algebra and $B \cong C(X) \otimes M_n$, where X is a connected compact subset of the plane satisfies the conditions in 12.4. Then*

$$\mathbf{Ext}(B, A) = \{0\}.$$

In other words, all extensions of B by A are equivalent (and trivial).

Proof : Suppose $\tau : B \to M(A)/A$ is an extension. Then it is trivial as shown in 11.9. We will keep the notations in 11.9. Now suppose that $\sigma : B \to M(A)/A$ is another extension. There are mutually orthogonal projections $\{d_{ii}\} \in M(A)$ such that $\pi(d_{ii}') = \sigma(e_{ii})$, $i = 1, 2, ..., n$. Clearly,

$$\rho \circ T(d_{ii}') = \rho \circ T(d_{ii}), \ i = 1, 2, ..., n.$$

Since the unitary group of $M(A)/A$ is connected, there is a unitary $w \in M(A)$ such that $\pi(w^*)\sigma(e_{ii})\pi(w) = \tau(e_{ii})$. Then we have $\pi(w^*)\sigma\pi(u)$ maps $e_{11}Be_{11}$ to $M(d_{11}Ad_{11})/d_{11}Ad_{11}$. It follows from 12.4 that there is a unitary $v_1 \in M(d_{11}Ad_{11})$ such that

$$\pi(v_1^*w^*)\sigma(f)\pi(wv_1) = \tau(f),$$

where $f(z) = z \cdot e_{11}$ as in the proof of 11.9. Set

$$v = diag(v_1, v_1, ..., v_1)$$

and $u = wv$. Then one can check easily that

$$\pi(u^*)\tau\pi(u) = \sigma.$$

Q.E.D.

Corollary 12.8 *Let A be a finite matroid algebra and $B \cong C(X) \otimes M_n$, where X has k connected components which satisfy the conditions in 12.4. Then*

$$\mathbf{Ext}(B, A) \cong \{([p_1], [p_2], ..., [p_k]) : [p_i] \in D_0(M(A)/A), \ \sum_{i=1}^{k}[p_i] = 1\}.$$

Proof: The proof is the same as that of 12.7 but instead of applying 12.4 we will apply 12.6.
Q.E.D.

Further classification of thses extensions will have to appear elsewhere.

References

[Bl] B. Blackadar, K-Theory for Operator Algebra, Springer Verlag, New York/Berlin/Heidelberg/London/Paris/Tokyo,1986.

[Bn1] L. G. Brown, Stable isomorphism of hereditary subalgebras, Pacific J. Math. 71(1977),335-348.

[Bn2] L. G. Brown, The universal coefficient theorem for Ext and quasidiagonality, Proc. International Conf. on Operator Algebras and Group Representations, Neptun, Romania 1980.

[Bn3] L. G. Brown, Interpolation by projections in C^*-algebras of real rank zero, preprint.

[Bn4] L. G. Brown, Extensions of AF algebras; the projection lifting problem, Operator Algebras and Applications, Proc. Symp. Pure Math.38 (A.M.S., Providence, 1981), Part I, 175-176.

[BDF1] L.Brown, R. Douglas and P.Fillmore, Unitary equivalence modulo the compact operators and extensions of C^*-algebras, Proc. Conf. on Operator Theory, Springer Lecture Notes in Math. 345(1973),58-128..

[BDF2] L. Brown, R.Douglas and P.Fillmore, Extensions of C^*-algebras and K-homology, Ann. of Math. 105(1977), 265-324.

[BP] L.Brown and G.K. Pedersen, C^*-algebras of real rank zero, J.Funct. Anal. , to appear.

[Bu] R. C. Busby, Double centralizers and extensions of C^*-algebras, Trans. Amer. Math. Soc., 132(1968),79-99.

[Ch] M.-D. Choi, Lifing projections from quotient C^*-algebras, J.Operator Theory 10(1983),21-30.

[**Cu**] J. Cuntz, Simple C^*-algebras generated by isometries, Comm. Math.Phys., 57(1977),173-185.

[**D**] K. Davidson, Almost commuting Hermitian matrices, Math. Scand., 56(1985),222-240.

[**Dix**] J.Dixmier, On some C^*-algebras considered by Glimm, J. Funct. Anal. 1(1967) 183-203.

[**Eff**] E. Effros, Dimensions and C^*-algebras, CBMS reginal Conf. Ser. in Math., no.46, Amer. Math. Soc., Provindence, R.I., 1981, pp.1-74.

[**Ell**] G. A. Elliott, Derivations of Matroid C^*-algebras, II, Ann. of Math. (2) 100(1974) 407-422.

[**EH**] G. A. Elliott and D. E. Handelman, Addition of C^*-algebra extensions, Pacific J. Math.,137(1989) 87-121.

[**HR**] N.Higson and M.Rϕrdam, The Weyl-von Neumann theorem for multipliers of some AF-algebras, preprint.

[**K**] G. G. Kasparov, The operator K-functor and extensions of C^*-algebras, Math. USSR Izv. 16(1981),513-572.

[**M**] J. Mingo, K-theory and multipliers of stable C^*-algebras, Trans. Amer.Math.Soc., 299 (1987),397-411.

[**Lin1**] Ideals of multiplier algebras of simple AF-C^*-algebras, Proc. Amer.Math.Soc., 104 (1988),239-244.

[**Lin2**] Generalized Weyl-von Neumann theorems , Inter. J. Math., to appear.

[**Lin3**] Generalized Weyl-von Neumann theorems II, Preprint.

[**Lin4**] Extensions by C^*-algebras with real rank zero, preprint.

[OP] C.L.Olsen and G.K. Pedersen, Corona C^*-algebras and their applications to lifting problems Math. Scand., to appear.

[R] C.Rickart, General Theory of Banach Algebras, Van Nostrand, Princeton, 1960.

[RS] J.Rosenberg and C.Schochet, The Kunneth Theorem and universal coefficient theorem for Kasparov's generalized K-functor, MSRI Preprint.

[Pd1] G. K. Pedersen, C^*-Algebras and their Automorphism Groups, Academic Press, London/New York, 1979.

[Pd2] G.K. Pedersen, SAW^*-algebras and corona C^*-algebras, contributions to non-commutative topology, J. Operator Theory, 15 (1986),15-32.

[Ph1] N. C. Phillips, Simple C^*-algebras with property weak (FU), to appear.

[Ph2] N. C. Phillips, How many exponentials?, to appear.

[Ph3] Approximation by unitaries with finite spectrum in purely infinite C^*-algebras, preprint.

[Zh1] S, Zhang, K_1-groups, quasidiagonality and interpolation by multiplier projections, Trans.Amer. Math. Soc., to appear.

[Zh2] S. Zhang, C^*-algebras with real rank zero and their corona and multiplier algebras, Part III, Cand.J. Math., to appear.

[Zh3] S. Zhang, C^*-algebras with real rank zero and their multiplier algebras, Part I,II and IV, preprint.

[Zh4] S.Zhang, A riesz decomposition property and ideal structure of multiplier algebras, J.Operator Theory, to appear.

[Zh5] S. Zhang, Diagonalizing projections in multiplier algebras over C^*-algebra, Pacific J. Math. 145(1990),181-200.

[Zh6] S. Zhang, On the structure of projections and ideals of corona algebras, Can. J. Math. 41(1989)721-742.

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