SIGN DETERMINANCY IN LU FACTORIZATION OF P-MATRICES

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DMS-654-IR

January 1994

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Dedicated to John S. Maybee on the occasion of his sixty–fifth birthday.

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[1] The research of this author was supported in part by National Science Foundation grant DMS 92–00899 and by Office of Naval Research contract N00014–90–j–1739.

[2] The research of this author was partially supported by the Natural Sciences and Engineering Research Council of Canada grant A–8214 and the University of Victoria Committee on Faculty Research and Travel.

[3] The research of this author was partially supported by the Natural Sciences and Engineering Research Council of Canada grant A–8965 and the University of Victoria Committee on Faculty Research and Travel.

Abstract

For an n–by–n P–matrix $A = [a_{ij}]$ having LU factorization $A = LU$ with $U = [u_{ij}]$, we determine combinatorial circumstances for which u_{ij} is unambiguously signed for a given pair $i \leq j$, or all such pairs, and give corresponding results for L. A qualitative P–matrix is a sign nonsingular matrix with all diagonal entries positive. For such a matrix A, we give additional sufficient conditions for an entry u_{ij} or the matrix U to be unambiguous. If A is a qualitative P–matrix with A^{-1} unambiguously signed, we prove that the matrices L, U, L^{-1} and U^{-1} are all unambiguous.

1. Introduction and Basic Results

It is well–known (see e.g. [JOV]) that an n–by–n real matrix $A = [a_{ij}]$ has a unique unit LU factorization (one in which $L = [\ell_{ij}]$ is lower triangular with each diagonal entry ℓ_{ii} equal to 1, and $U = [u_{ij}]$ is upper triangular) if and only if every leading principal minor of A of order 1, ..., n–1 is nonzero. All references to LU factorization in this paper are to this (normalized) factorization. In [JOV] we determined combinatorial circumstances for *entry inheritance* in the LU factorization, that is, circumstances under which $u_{ij} = a_{ij}$ for a given pair $i \leq j$ or for all such pairs. Our present interest is in *sign inheritance*, that is the combinatorial circumstances under which $\operatorname{sgn}(u_{ij}) = \operatorname{sgn}(a_{ij})$ for a given pair $i \leq j$ or for all such pairs. As in [BJOV], we use the relation between the LU factorization and the Schur complement, and we utilize results of [JM] concerning qualitative aspects of Schur complements.

An n–by–n matrix A having all principal minors positive is called a *P–matrix*, and we write $A \in P$. Such matrices clearly have a unique LU factorization. An n–by–n array B is a *sign pattern* (*matrix*) if each entry of B is +, – or 0. Matrix A has the sign pattern of B if for all i, j, the value of $\operatorname{sgn}(a_{ij})$ is +1, –1, 0, respectively, when the (i, j) entry of B is +, –, 0, respectively. For a fixed sign pattern B, if A is a P–matrix with the sign pattern of B, we write $A \in P_B$. In general, this places qualitative and quantitative restrictions on the entries of A. If all (main) diagonal entries of B are +, then there exists a matrix $A \in P_B$; that is, the sign pattern B *allows* a P–matrix. If every matrix A with the sign pattern B is in P_B, then we say that B *requires* a P–matrix. A sign pattern B is called *sign nonsingular* if every matrix A having this sign pattern is nonsingular (see e.g. [BMQ]), and such a matrix A is a *sign nonsingular matrix.* A matrix is *combinatorially singular* if it is singular for all choices of the nonzero entries. We note that a matrix is sign nonsingular if and only if it is not combinatorially singular and each nonzero term in its determinant has the same sign.

For a fixed sign pattern B, let $A \in P_B$. In the resulting LU factorization of A, we say that:

u_{ij} is *unambiguous* (-ly +, −, 0, respectively) if for every $A \in P_B$, u_{ij} is uniquely one of +, −, 0, respectively; and

u_{ij} is *ambiguous* if it is not unambiguous.

We use this terminology also for entries of L, U^{-1}, L^{-1} and A^{-1}, and for determinants. If each entry of a matrix is unambiguous, we say that the matrix is unambiguous (-ly signed).

For index sets $\beta, \gamma \subseteq \{1,2,\ldots,n\}$ we denote the submatrix of A lying in rows β and columns γ by $A[\beta|\gamma]$. When $\beta = \gamma$, we denote the principal submatrix by $A[\beta]$. The set $\{1,2,\ldots,n\} - \beta$ is denoted by β^c. For $1 \leq i \leq n$, let $\beta_i = \{1,2,\ldots,i\}$, define $\beta_0 = \phi$, and abbreviate $A[\beta_{i-1} \cup \{j\}]$ by $A[\beta_{i-1} \cup j]$, (and similarly for nonprincipal submatrices).

We record two useful observations about sign nonsingular matrices.

Observation 1.1

Let $\beta, \gamma \subseteq \{1,2,\ldots,n\}$ with $1 \leq |\beta| = |\gamma| \leq n - 1$. Let B be a sign nonsingular pattern and A a matrix with the sign pattern of B. If $\det A[\beta|\gamma]$ is ambiguous, then its complementary submatrix $A[\beta^c \mid \gamma^c]$ must be combinatorially singular.

Proof.

By the Laplace expansion, each term in det A can be written as an appropriately signed product of the form det $A[\beta \mid \gamma]$ det $A[\beta^c \mid \gamma^c]$. If det $A[\beta|\gamma]$ is ambiguous and $A[\beta^c|\gamma^c]$ not combinatorially singular, then two oppositely signed products are possible, contradicting the sign nonsingularity. Thus $A[\beta^c \mid \gamma^c]$ must be combinatorially singular for all A with the sign pattern of B.■

Observation 1.2.

Let B be a sign pattern with all diagonal entries +. Then B is sign nonsingular if and only if B requires a P–matrix.

Proof: If B is a sign nonsingular pattern, and $A \in P_B$ has nonzero diagonal, then the sign of its determinant is the sign of the product of its diagonal entries. This fact together with Observation 1.1 constitute a straightforward proof of the forward implication. The converse is immediate. ■

We call such a matrix $A \in P_B$, for B a sign nonsingular pattern with all diagonal entries positive, a *qualitative P–matrix.* These matrices are the main focus of this paper.

To illustrate the above concepts, consider two examples.

Example 1.3

Let

$$B = \begin{bmatrix} + & - & 0 & 0 \\ + & + & - & 0 \\ + & + & + & - \\ + & + & + & + \end{bmatrix},$$

which is a sign nonsingular pattern with positive diagonal. Thus any matrix A with this sign pattern is a qualitative P–matrix, and it is easily shown that the matrices L and U of its LU factorization have the sign patterns

$$\begin{bmatrix} + & 0 & 0 & 0 \\ + & + & 0 & 0 \\ + & + & + & 0 \\ + & + & + & + \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} + & - & 0 & 0 \\ 0 & + & - & 0 \\ 0 & 0 & + & - \\ 0 & 0 & 0 & + \end{bmatrix},$$

respectively. Both matrices L and U are unambiguous; moreover, $\mathrm{sgn}(u_{ij}) = \mathrm{sgn}(a_{ij})$ for all $i \le j$ and $\mathrm{sgn}(\ell_{ij}) = \mathrm{sgn}(a_{ij})$ for all $i \ge j$. ■

Note that, in this example, the sign patterns of L and U do not depend on the normalization $\ell_{ii} = 1$ of the unit LU factorization, but only on a normalization so that $\ell_{ii} > 0$. This remark applies throughout this paper.

Example 1.4

Consider

$$B = \begin{bmatrix} + & - & - & + & - \\ - & + & - & - & - \\ - & - & + & + & - \\ + & + & + & + & + \\ + & - & 0 & - & + \end{bmatrix},$$

let $A \in P_B$ and $A = LU$. Then (by Theorem 1.5 (i), (iii), below) u_{45} is unambiguously $+$, an example of sign inheritance of one particular entry. However,

u_{24} is ambiguous. Note that this pattern B is *not* sign nonsingular, so there is a quantitative restriction imposed by taking A to be a P–matrix. Note also that the submatrix $A[1,2,3,4 \mid 1,2,3,5]$ is *not* sign nonsingular, but by Theorem 1.5 (ii), it has a positive determinant.■

We next state a well known relationship between the LU factorization of A and the Schur complements of principal submatrices of A. If $\beta \subset \{1,2,\ldots,n\}$ and $A[\beta]$ is nonsingular, then the *Schur complement* of $A[\beta]$ in A is defined as

$$S(\beta) = A[\beta^c] - A[\beta^c \mid \beta]A[\beta]^{-1} A[\beta \mid \beta^c].$$

When $\beta = \beta_i = \{1,2,\ldots,i\}$, the (1, k) entry of this matrix is equal to $u_{i+1,i+k}$, $1 \leq k \leq n-i$, in the LU factorization of A. Recently a qualitative analysis of Schur complements was given in [JM], and we interpret their theorem on Schur complements in terms of the LU factorization. Given a matrix A, a (simple) *path* p *in* A *from* i *to* j via β_{i-1} is a sequence of nonzero entries $a_{t_0t_1}, a_{t_1t_2}, \ldots, a_{t_{\ell-1},t_\ell}$ of A, in which $t_0 = i$ and $t_\ell = j$, with $t_r \in \beta_{i-1}$ and distinct for $r \in [1, \ell-1]$. The *path product of* p, denoted by A[p], is $\prod_{r=0}^{\ell-1} a_{t_rt_{r+1}}$, and the *length of path* p is ℓ. Note that the singleton $a_{ij} \neq 0$ is a path from i to j of length 1 via β_{i-1} (for all $i \geq 1$). The following theorem is essentially stated and proved in [JM] in terms of sign determinancy, but closer analysis reveals the actual signs as stated below.

Theorem 1.5 [JM, Th. 2]

Let B be an n–by–n sign pattern and $A \in P_B$. Then for $A = LU$ and

$i \leq j$, the following are equivalent:

(i) u_{ij} is unambiguously $+$ $\langle$resp. $-$, $0\rangle$;

(ii) $\det A[\beta_{i-1} \cup i \mid \beta_{i-1} \cup j]$ is unambiguously $+$ $\langle$resp. $-$, $0\rangle$;

(iii) every path product $A[p_k]$ from i to j via β_{i-1} is signed $(-1)^{\ell_k-1}$ $\langle$resp. $(-1)^{\ell_k}$, there are no such paths$\rangle$, where ℓ_k is the length of path p_k.

Proof. As stated above, we need only prove that the signs are as indicated. Equation (2.1) of [JOV] gives, for $i \leq j$,

$$u_{ij} = \det A[\beta_{i-1} \cup i \mid \beta_{i-1} \cup j]/\det A[\beta_{i-1}],$$

where $\det A[\phi] = 1$ (the case $i = 1$). As A is a P–matrix, the denominator is positive, thus (i) and (ii) are equivalent. Corollary 8.2 of [MOVW], see also (4) of [JM], gives a path product formula for the numerator, showing that

$$u_{ij} = -\sum_{k=1}^{m} (-1)^{\ell_k} A[p_k]\det A[V(p_k)]/\det A[\beta_{i-1}]. \tag{1.1}$$

Here $\{p_k: 1 \leq k \leq m\}$ denotes the set of all distinct paths in A from i to j via β_{i-1}, ℓ_k is the length of path p_k, and $V(p_k)$ is the set of indices in β_{i-1} *not* on p_k. As A is a P–matrix, $\det A[V(p_k)]$ is positive, thus (ii) and (iii) are equivalent.∎

Observe that if a_{ij} is + or – and u_{ij} is unambiguous, then $\mathrm{sgn}(u_{ij}) = \mathrm{sgn}(a_{ij})$. Also, when $i = j$, then u_{ii} is unambiguously + (as A is a P–matrix). If for matrix A, u_{ij} is unambiguous and then some entries of A are replaced by 0 (retaining the P–matrix property), then the (i, j) entry of U in the resulting LU factorization remains unambiguous.

Implicit in Theorem 1.5 is the result that if $A \in P_B$ and $A = LU$, then $S(\beta_i)$ is unambiguous for all β_i, $1 \leq i \leq n-1$, if and only if U is unambiguous. However, it is possible for U to be unambiguous but $S(\beta)$ ambiguous for some $\beta \subset \{1,2,\ldots,n\}$, $\beta \neq \beta_i$. This is illustrated by the sign pattern B of Example 1.3; if $A \in P_B$, then U is unambiguous, but when $\beta = \{2,3\}$ an entry of $S(\beta)$ is ambiguous.

When $A = LU$, then $A^T = U^T L^T$, so we can formulate a result analogous to Theorem 1.5 for the matrix L (see [JOV, Section 4] for entry inheritance results on L). With the assumptions of Theorem 1.5, the following are equivalent for $i \leq j$:

(i) ℓ_{ji} is unambiguously + ⟨resp. –, 0⟩;

(ii) $\det A[\beta_{i-1} \cup j \mid \beta_{i-1} \cup i]$ is unambiguously + ⟨resp. –, 0⟩;

(iii) every path product $A[p_k]$ from j to i via β_{i-1} is signed $(-1)^{\ell_k - 1}$ ⟨resp. $(-1)^{\ell_k}$, there are no such paths⟩, where ℓ_k is the length of path p_k.

2. LU–Factorization of Sign Nonsingular Matrices

Let B be a sign nonsingular pattern, normalized so that each diagonal entry is *positive.* (Note that this is not the usual normalized form, see e.g. [BMQ], in which each diagonal entry is normalized to be *negative*; however, here we are working with P–matrices.) Let A be a matrix with the same sign pattern as B; then A and every principal submatrix of A are sign nonsingular and are qualitative P–matrices. A sign nonsingular matrix A is said to be *maximal* if every matrix obtained from A by replacing a zero entry with a nonzero is *not* sign nonsingular.

Example 2.1

For fixed n, a maximal sign nonsingular matrix with the minimum number of zero entries is a Hessenberg matrix [Gi], which can be put in the form of the pattern illustrated in Example 1.3 for $n = 4$. In general, let B be the n–by–n pattern with every entry on and below the main diagonal +, every superdiagonal –, and every entry above the superdiagonal 0. Let H be a (Hessenberg) matrix with the sign pattern of B, and $H = LU$. Then U is bidiagonal, L is full, and every entry in each factor is unambiguous and has its sign (+, – or 0) inherited from H. See Example 1.3 for $n = 4$, with L and U explicitly given.■

Not every sign nonsingular matrix does give U unambiguously, as the following example shows.

Example 2.2

Let A be a matrix with the sign nonsingular pattern

$$B = \begin{bmatrix} + & - & + & 0 \\ + & + & + & - \\ - & 0 & + & 0 \\ + & 0 & + & + \end{bmatrix},$$

and suppose $A = LU$. Then both a_{23} and $a_{21}a_{13}$ are positive. Thus, by

Theorem 1.5 (i), (iii), the entry u_{23} is ambiguous. Note that the minor $\det A[1, 2| 1, 3]$ is ambiguous, but does *not* enter into $\det A$ because its complementary submatrix $A[3, 4| 2, 4]$ is combinatorially singular; see Observation 1.1.■

We remark that the conclusion of Example 2.1 can be deduced from Theorem 1.5, or by using *symbolic Gaussian elimination* on the sign pattern B. That is, the sign patterns of L, U can be determined by imitating the Gaussian elimination procedure on B and assuming that the product of like ⟨unlike⟩ signs is + ⟨–⟩. But even for a sign nonsingular pattern, symbolic Gaussian elimination is not, in general, sharp enough to determine whether an entry u_{ij} is unambiguous. For example, consider B in Example 2.2. Symbolic Gaussian elimination on this pattern gives an unknown sign for u_{33}. But, by an observation from Theorem 1.5, we know that u_{33} is unambiguously +. We also remark that the "symbolic product" of the patterns for L and U may not yield the initial pattern for A. For example, let H have the sign pattern B in Example 2.1 with $n \geq 2$. If $H = LU$, then L and U are unambiguous. However, in the symbolic product of the sign pattern of L and the sign pattern of U, there are some ambiguously signed entries (e.g., all diagonal entries except the (1, 1) entry).

Applying Theorem 1.5 to qualitative P–matrices, we have the following results.

Theorem 2.3

Let A be a qualitative P–matrix with $A = LU$, and let $i \leq j$. If $A[\beta^c_{i-1} - i \mid \beta^c_{i-1} - j]$ is not combinatorially singular, then u_{ij} is unambiguous.

Proof.

If u_{ij} is ambiguous, then det $A[\beta_{i-1} \cup i \mid \beta_{i-1} \cup j]$ is ambiguous (by Theorem 1.5). Thus, by Observation 1.1, its complementary submatrix must be combinatorially singular, giving the contrapositive of the statement.■

Note that for a matrix A with the pattern of Example 2.2, the entry u_{24} is unambiguously –, although the complementary submatrix $A[3, 4 \mid 2, 3]$ is combinatorially singular. Thus Theorem 2.3 gives only a sufficient condition, which we use in the following.

Theorem 2.4

Let A be a qualitative P–matrix with $A = LU$.

(i) If $a_{ji} \neq 0$, $j > i$, then u_{ij} is unambiguous, and $\mathrm{sgn}(u_{ij}) \neq \mathrm{sgn}(a_{ji})$.

(ii) If $a_{j,j-1}, a_{j-1,j-2}, \ldots, a_{i+1,i} \neq 0$, $j > i$, then u_{ij} is unambiguous.

(iii) If $a_{k+1,k} \neq 0$ for all $k = 2, 3, \ldots, n-1$, then the entire matrix U is unambiguous.

(iv) If A has a simple path of length $\geq n - 2$, then there is a permutation matrix Q such that QAQ^T has an unambiguous U in its LU factorization.

Proof.

Each of the conditions (i) and (ii) is sufficient for the minor in Theorem 2.3 not to be combinatorially zero. For (i), the complementary submatrix contains a transversal using a_{ji} and main diagonal entries of A. By the observation immediately after Theorem 1.5 and as $a_{ij}a_{ji}$ cannot be positive in a qualitative

P–matrix, result (i) follows. For (ii), the complementary submatrix contains a transversal using the given nonzero terms and $a_{j+1,j+1}, \ldots, a_{nn}$.

Condition (iii) simply implies that (ii) holds for all $j > i$ $(2 \leq i \leq n-1$ and $3 \leq j \leq n)$, and thus ensures that the appropriate complementary submatrices are not combinatorially singular for all such pairs i, j. Since all entries u_{ij} are necessarily unambiguous, result (iii) follows. Condition (iv) introduces a (possible) reordering of the rows and columns of A, so that condition (iii) holds in the resulting matrix. A path of length $n-2$ is sufficient, since inheritance of all entries in the first row is automatic.■

Note that the Hessenberg matrices in Example 2.1 satisfy condition (iii). A matrix A with the pattern given by Example 2.2 has the path $a_{43}a_{31}$ of length $n - 2 = 2$. Let Q be the permutation that interchanges 1 and 2; then QAQ^T has an unambiguous U in its LU factorization, illustrating condition (iv).

Conjecture 2.5. Given any sign nonsingular matrix A, there exist permutation matrices Q_1, Q_2 and a signature matrix S such that SQ_1AQ_2 has a positive diagonal and the matrix U of its LU factorization is unambiguous.

Theorem 2.4 (iv) gives a positive answer to this conjecture for a special class of sign nonsingular matrices. It is known [J. Maybee, private communication] that for $n \leq 9$, every sign nonsingular matrix has an equivalent form Q_1AQ_2 with a Hamilton cycle and a nonzero diagonal; thus conjecture 2.5 is true for at least $n \leq 9$. However, it is also known [J. Maybee, private communication] that not every sign nonsingular matrix has such an equivalent form, the smallest known example being 12–by–12.

In this section we have stated results only for the matrix U in the LU factorization of A. However, as explicitly given in section 1, analogous results hold for L.

3. Sign Determined Inverses

Let B be a sign nonsingular pattern with positive diagonal, A a matrix with the sign pattern of B, and $A^{-1} = [\alpha_{ij}]$. We are interested in combinatorial conditions for which an entry α_{ij} is unambiguous, or the entire matrix A^{-1} is unambiguous; for A irreducible, see [T] for a forbidden digraph characterization, and [LM], [S]. We work with path products, and begin by giving in our notation a well known result.

Theorem 3.1

Let A be a qualitative P–matrix with $A^{-1} = [\alpha_{ij}]$. Then α_{ii} is positive, and for $i \neq j$, α_{ij} is unambiguously + ⟨–, 0⟩ if and only if every path product $A[p_k]$ from i to j is signed $(-1)^{\ell_k}$ $\langle(-1)^{\ell_k - 1}$, there are no such paths⟩, where ℓ_k is the length of path p_k.

Proof. By Jacobi's theorem, $A \in P$ implies $A^{-1} \in P$, thus $\alpha_{ii} > 0$ for all i. For $i \neq j$, the result follows from the cofactor expansion

$$(3.1) \qquad \alpha_{ij} = \sum_{k=1}^{r} (-1)^{\ell_k} A[p_k] \det A[V(p_k)]/\det A,$$

with the notation as in (1.1) except that $\left\{p_k: 1 \le k \le r\right\}$ is the set of *all* distinct paths in A from i to j.■

Note that if the path product condition in this theorem is satisfied for all pairs i, j (i≠j), then A^{-1} is unambiguous, and A is called *inverse sign determined.* If A is an inverse sign determined, qualitative P–matrix, then $(A[\beta])^{-1}$ is unambiguous for all $\beta \subset N$. If A is an *irreducible*, inverse sign determined, qualitative P–matrix, then every entry in A^{-1} is unambiguously + or –.

Using Theorem 1.5, we have the following relation between A^{-1} and the matrices in the LU factorization of A.

Corollary 3.2

Let A be a qualitative P–matrix with $A^{-1} = [\alpha_{ij}]$, and assume $A = LU$. If α_{ij} is unambiguously $+ \langle -, 0\rangle$, then

(i) for $j > i$, u_{ij} is unambiguously – or 0 $\langle$+ or 0, 0$\rangle$, and

(ii) for $i > j$, ℓ_{ij} is unambiguously – or 0 $\langle$+ or 0, 0$\rangle$.

Proof. If $j > i$ and α_{ij} is unambiguously +, then by (3.1), $(-1)^{\ell_k} A[p_k] > 0$ for all paths p_k from i to j in A. Thus, from (1.1), either $u_{ij} < 0$ or, if there are no such paths via β_{i-1}, $u_{ij} = 0$. The other cases follow similarly.■

This corollary shows that, for a qualitative P–matrix, if A^{-1} is unambiguous, then L and U are unambiguous (since all three matrices have positive diagonal entries). The converse is not in general true, as can be seen from Example 1.3 in which the (3, 1), (4, 1) and (4, 2) entries of the inverse of a matrix with the sign pattern of B are ambiguous. However, for $A \in P_B$, A^{-1} is unambiguous if and only if the Schur complement $S(\beta)$ is unambiguous for all $\beta \subset \{1,2,\ldots,n\}$. If A is an inverse sign determined qualitative P–matrix, then Conjecture 2.5 is true with Q_1 arbitrary, $Q_2 = Q_1^T$ and $S = I$.

We are now able to prove a result about the inverses of matrices L and U in the LU factorization of A. Such a result cannot be proved solely on information about path product signs in A^{-1}; there are subtle quantitative interrelationships (see Example 3.4).

Theorem 3.3

Let A be a qualitative P–matrix that is inverse sign determined, $A = LU$, $L^{-1} = [\lambda_{ij}]$ and $U^{-1} = [\mu_{ij}]$. Then U^{-1} and L^{-1} are unambiguous.

Proof. All diagonal entries of U^{-1} and L^{-1} are positive. By the argument in [Ga, II §1] (cf. the result for u_{ij} in Theorem 1.5), as $A^{-1} = U^{-1} L^{-1}$,

$$\lambda_{ij} = \frac{\det A^{-1}[i,\ i+1,\ \ldots,\ n \mid j,\ i+1,\ \ldots,\ n]}{\det A^{-1}[i+1,\ \ldots,\ n]},\ i > j,$$

$$= \frac{(-1)^{i+j} \det A[1,\ \ldots,\ j-1,\ j+1,\ \ldots,\ i \mid 1,\ \ldots,\ i-1]}{\det A^{-1}[i+1,\ \ldots,\ n] \det A}$$

by Jacobi's theorem. As $A \in P$, the sign of λ_{ij} is given by the sign of the numerator. But the determinant in the numerator is an almost principal minor, thus the numerator has the sign of

$$(-1)^{i+j} (-1)^{i+j} \sum_{k=1}^{m} (-1)^{\ell_k} A[p_k]$$

where $\{p_k\colon 1 \le k \le m\}$, as in (1.1) denotes the set of all distinct paths in A from i to j via β_{i-1} (but now $i > j$). Thus λ_{ij} is unambiguous, and similarly μ_{ij}, $i < j$, is unambiguous.■

Note that if A^{-1} has an ambiguous entry, then this result may fail, as illustrated again by Example 1.3.

Example 3.4

Any matrix A with the sign pattern

$$B = \begin{bmatrix} + & - & 0 \\ + & + & - \\ + & 0 & + \end{bmatrix}$$

is a qualitative P–matrix. For $A = LU$, matrices L and U are unambiguous with sign patterns $\begin{bmatrix} + & 0 & 0 \\ + & + & 0 \\ + & + & + \end{bmatrix}$ and $\begin{bmatrix} + & - & 0 \\ 0 & + & - \\ 0 & 0 & + \end{bmatrix}$, respectively. Also A is inverse sign determined, and A^{-1}, L^{-1}, U^{-1} have sign patterns $\begin{bmatrix} + & + & + \\ - & + & + \\ - & - & + \end{bmatrix}$, $\begin{bmatrix} + & 0 & 0 \\ - & + & 0 \\ - & - & + \end{bmatrix}$ and $\begin{bmatrix} + & + & + \\ 0 & + & + \\ 0 & 0 & + \end{bmatrix}$, respectively. Clearly, not all matrices with the sign pattern of A^{-1} have inverses with the sign pattern B. Much quantitative information is contained in the entries of A^{-1}; e.g., all of its minors are unambiguous. Similarly, L, U and their inverses contain quantitative information; e.g., $\det L[23 \mid 12] < 0$.■

We conclude with two special cases. Let $A = LU$ be a P–matrix with $a_{ij} \le 0$ for all $i \ne j$ (i.e., A is a nonsingular M–matrix). If A is irreducible, then (3.1) verifies the well known fact that A^{-1} is entrywise positive. Corollary 3.2

then gives $u_{ij} \leq 0$ for $j > i$, and $\ell_{ij} \leq 0$ for $i > j$. Thus our path conditions give a proof of the known fact that L and U are again M–matrices; see [FP]. Consider now an n–by–n *unipathic pattern* [M], i.e. a pattern that has exactly one path from i to j for all pairs $i, j \in \{1,\ldots,n\}$, $i \neq j$. If A is a P–matrix with a unipathic pattern, then $A = LU$ is inverse sign determined and both L and U are unambiguous. This class includes combinatorially symmetric P–matrices with tree graphs.

Acknowledgement. We thank T.J. Lundy for discussions on Conjecture 2.5 and sign determined inverses.

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