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An Interplay of Ridgelet and Linear Canonical Transforms

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Abstract: The present study is the first of its kind, aiming to explore the interface between the ridgelet and linear canonical transforms. To begin with, we formulate a family of linear canonical ridgelet waveforms by suitably chirping a one-dimensional wavelet along a specific direction. The construction of novel ridgelet waveforms is demonstrated via a suitable example supported by vivid graphics. Subsequently, we introduce the notion of linear canonical ridgelet transform, which not only embodies the classical ridgelet transform but also yields another new variant of the ridgelet transform based on the fractional Fourier transform. Besides studying all the fundamental properties, we also present an illustrative example on the implementation of the linear canonical ridgelet transform on a bivariate function.

Keywords: linear canonical transform; radon transform; ridgelet transform; wavelet transform

MSC: 42C40; 42B10; 44A12; 94A12

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1. Introduction

The wavelet transform is a multi-scale integral transform that has emerged as a potent tool for non-stationary signal processing. The intrinsic spatial and scale adaptability of wavelets is of substantial importance in different aspects of time-frequency analysis, wherein the scale and frequency are inverse to each other. Mathematically, the wavelet transform of any univariate function $f \in L^2(\mathbb{R})$ with respect to the mother wavelet $\psi \in L^2(\mathbb{R})$ is defined as

$$\mathscr{W}_\psi\Big[f\Big](a,b) = \int_{\mathbb{R}} f(t)\,\overline{\psi_{a,b}(t)}\,dt, \tag{1}$$

where $\psi_{a,b}(t)$ are the daughter wavelets; that is, the scaled and shifted versions of the mother wavelet $\psi(t)$, and are given by

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right),\ a \in \mathbb{R}^+ \text{ and } b \in \mathbb{R}. \tag{2}$$

The parameter a is called the "scaling parameter," which measures the degree of compression or scale, and b is referred to as the "translation parameter," which determines the time location of the wavelet [1,2]. Each wavelet component $\psi_{a,b}(t)$ acts as a differently scaled band-pass filter in the conventional Fourier domain, and by applying these local decomposition filters, the wavelet transform (1) has proved to be of critical significance in capturing the local characteristics of non-stationary signals. The main feature of such decompositions lies in the fact that they offer a lucid and powerful way of detecting the point-singularities in one-dimensional signals. Owing to the simple and insightful procedure underlying the wavelet decomposition filters, the wavelet transform has paved the way for diverse areas of science and engineering such as signal and image processing, geophysics, astrophysics, quantum mechanics, and medicine [3–9].

Much to our dismay, wavelets do not perform equally well in higher dimensions, mainly because the scalings are isotropic in nature, which are incompatible for capturing the geometric features such as edges and corners in higher-dimensional signals. The quest for the redressal of such limitations stimulated renewed interest into the theory of wavelets and gave birth to notion of steerable wavelets which are not only adapted to location and scale, but can also be oriented along a particular direction in the Euclidean plane $\mathbb{R}^2$. The foremost significant attempt in this direction was reported in the form of ridgelets [10–15]. For any bivariate function $f \in L^2(\mathbb{R}^2)$, the classical ridgelet transform is defined as

$$\mathcal{R}_\psi\Big[f\Big](a,b,\theta) = \Big\langle f,\, \psi_{a,b,\theta}\Big\rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t})\, \overline{\psi_{a,b,\theta}(\mathbf{t})}\, d\mathbf{t}, \tag{3}$$

where $\psi_{a,b,\theta}(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^2$, denote the classical ridgelet waveforms, given by

$$\psi_{a,b,\theta}(\mathbf{t}) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{\Lambda_\theta \cdot \mathbf{t} - b}{a}\right),\; a \in \mathbb{R}^+,\; b \in \mathbb{R},\; \theta \in [0,\, 2\pi) \tag{4}$$

and $\Lambda_\theta = (\cos\theta, \sin\theta)$ is the unit vector in the direction θ. Technically, a ridgelet behaves like a one-dimensional wavelet in a specific direction and is constant along the orthogonal direction. Unlike the classical wavelet transform (1), the ridgelet transform of any bivariate function $f \in L^2(\mathbb{R}^2)$ is determined by a total of three parameters given by scale ($a \in \mathbb{R}^+$), location ($b \in \mathbb{R}$), and orientation ($\theta \in [0,\, 2\pi)$). Consequently, the ridgelet transform is best matched for capturing the line singularities in two-dimensional signals [10,13]. Keeping in view the merits of the ridgelet transform over the classical wavelet transform, the goal of this article is to propose a novel ramification of the notion of ridgelets by blending the ridgelets with the linear canonical transform (LCT). It is pertinent to mention that the LCT is a chirp-based transformation and has dethroned both the classical Fourier and the fractional Fourier transforms in the sense that it corresponds to a twisting operation, which in turn causes a deformation of cells in the time-frequency plane. As such, the prime advantage of the novel ridgelet waveforms lies in the fact that they are chirp-based, wherein the chirp rate is determined by a 2×2 real and unimodular matrix associated with the linear canonical transform. Consequently, the novel ridgelet waveforms enjoy state-of-the-art flexibility while resolving the line singularities in two-dimensional signals. The present study is the first of its kind in the sense that there are no studies reported in the literature that aim at intertwining the merits of the ridgelet and linear canonical transforms. The highlights of this article are as follows:

- The novel ridgelet waveforms coined as “linear canonical ridgelets” are constructed by suitably chirping a typical one-dimensional wavelet along a specific direction in the Euclidean plane.
- The potency of the novel ridgelet waveforms is demonstrated via an example supported by vivid graphics.
- The notion of linear canonical ridgelet transform is introduced, which encapsulates both the classical ridgelet transform and a new variant of the ridgelet transform based on the fractional Fourier transform.
- All the fundamental properties of the linear canonical ridgelet transform are studied in detail abreast of the formulation of orthogonality relation and inversion formula.
- To access the localization characteristics of the linear canonical ridgelet transform, a Heisenberg-type uncertainty inequality is also obtained.
- The article ends with an illustrative example demonstrating the implementation of the linear canonical ridgelet transform on a given bivariate function.

The rest of the article is organized into three sections. Section 2 is an assemblage of the preliminaries including the fundamental notions of the Radon transform and the classical ridgelet transform. In Section 3, we formally introduce the notion of linear canonical ridgelet transform and investigate it using the underlying mathematical theory. A couple of examples on the implications of the proposed work are also presented in the same section. Finally, the conclusion and an impetus for future work is given in Section 4.

2. The Classical Ridgelet Transform

To facilitate the narrative, we subdivide the section into two subsections one each for the prerequisites regarding the Radon and ridgelet transform.

2.1. The Radon Transform

The Radon transform dates back to 1917 with the seminal work of Johann Radon to construct a bivariate function from its integrals over all straight lines in a plane [16]. Furthermore, Radon also generalized this work by reconstructing a bivariate function from its integrals over smooth curves. The Radon transform has found numerous applications across diverse aspects of science and engineering, including medical imaging, astronomy, crystallography, electron microscopy, geophysics, material science, and optics [17,18]. Most importantly, the Radon transform is of substantial importance in computer-assisted tomography, which deals with the problem of finding the internal structure of an object by observations or projections. Mathematically, the Radon transform of a bivariate function $f(\mathbf{x})$ is defined as

$$\mathfrak{R}\Big[f\Big](t,\theta) = \int_{\mathbb{R}^2} f(\mathbf{x})\,\delta(\Lambda_\theta\cdot\mathbf{x} - t)\,d\mathbf{x}, \tag{5}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Furthermore, $\delta(\cdot)$ denotes the well-known Dirac function, and $\Lambda_\theta = (\cos\theta, \sin\theta)$ is the unit vector in the direction θ. The Radon transform (5) maps a bivariate function f to another unique function $\mathfrak{R}[f](t,\theta)$, whose domain is the set of all lines in $\mathbb{R}^2$. In the literature, (5) is also referred to as the X-ray transform due to the fact that in medical imaging, the integral along a line represents a measurement of the intensity of the X-ray beam at the detector after passing through the object being radio-graphed [19].

2.2. The Ridgelet Transform

The ridgelet transform deals with an efficient capturing or resolving of the line singularities in two-dimensional signals. A ridgelet behaves as a one-dimensional wavelet in a specific direction, represented by the unit vector Λ_θ of orientation θ, and remains constant along the orthogonal direction [10,20]. A ridgelet can be translated along its oscillating direction and can also be scaled to obtain a family of ridgelets. Mathematically, let $\psi \in L^2(\mathbb{R})$ be a classical one-dimensional wavelet; then, for $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, the family of ridgelets is denoted by $\mathfrak{F}_\psi(a,b,\theta)$ and is defined as

$$\mathfrak{F}_\psi(a,b,\theta) := \left\{\psi_{a,b,\theta}(\mathbf{t}) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{\Lambda_\theta\cdot\mathbf{t} - b}{a}\right),\ a \in \mathbb{R}^+,\ b \in \mathbb{R},\ \theta \in [0, 2\pi)\right\}, \tag{6}$$

where Λ_θ denotes the unit-vector of orientation θ. From (6), it is quite evident that ridgelets can be roughly visualized as two-dimensional wavelets with the major distinction of being constant along a preferred direction and are thus never admissible with regard to the usual definition of admissibility of two-dimensional wavelets. In fact, any $\psi \in L^2(\mathbb{R})$ is said to be an admissible ridgelet if the following relation holds:

$$C_\psi = (2\pi)^2 \int_{\mathbb{R}} \frac{\left|\mathscr{F}[\psi](\omega)\right|^2}{|\omega|^2}\,d\omega < \infty, \tag{7}$$

where $\mathscr{F}[\psi](\omega)$ denotes the usual Fourier transform of $\psi \in L^2(\mathbb{R})$. With the notion of an admissible ridgelet at hand, we proceed to the formal definition of the ridgelet transform.

Definition 1. *For any $f \in L^2(\mathbb{R}^2)$, the ridgelet transform is denoted by $R_\psi[f]$ and is defined as*

$$R_\psi\Big[f\Big](a,b,\theta) = \Big\langle f,\, \psi_{a,b,\theta} \Big\rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t})\, \overline{\psi_{a,b,\theta}(\mathbf{t})}\, d\mathbf{t}, \tag{8}$$

where $\psi_{a,b,\theta}(\mathbf{t})$ is given by (6).

Remark 1. *The ridgelet transform (8) of any bivariate function $f \in L^2(\mathbb{R}^2)$ is determined by a total of three parameters only; viz, $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, which is justifiable due to the fact that a ridgelet behaves as a one-dimensional wavelet in a specific direction and is constant along the orthogonal direction.*

One of the distinguishing features of the ridgelet transform (8) is that it intertwines with the well-known Radon and wavelet transforms. Next, our objective is to demonstrate the coupling between the Radon and ridgelet transforms by demonstrating that the wavelet transform bridges the gap between the Radon and ridgelet transforms. More precisely, we shall uncover the fundamental fact that the ridgelet transform (8) can be expressed as the wavelet transform of the Radon transform (5) when considered as a function of $t \in \mathbb{R}$. We note that

$$\begin{aligned}
\mathscr{W}_\psi\Big(\mathfrak{R}\Big[f\Big](t,\theta)\Big)(a,b) &= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \mathfrak{R}\Big[f\Big](t,\theta)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, dt \\
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^2} f(\mathbf{x})\, \delta(\Lambda_\theta \cdot \mathbf{x} - t)\, d\mathbf{x} \right\} \overline{\psi\left(\frac{t-b}{a}\right)}\, dt \\
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(\mathbf{x})\, \delta(\Lambda_\theta \cdot \mathbf{x} - t)\, \overline{\psi\left(\frac{t-b}{a}\right)}\, d\mathbf{x}\, dt \\
&= \frac{1}{\sqrt{a}} \int_{\mathbb{R}^2} f(\mathbf{x})\, \overline{\psi\left(\frac{\Lambda_\theta \cdot \mathbf{x} - b}{a}\right)}\, d\mathbf{x} \\
&= R_\psi\Big[f\Big](a,b,\theta),
\end{aligned} \tag{9}$$

which is the desired relationship shared by the wavelet, Radon, and ridgelet transforms. It is interesting to note that, in the case that the given bivariate function has singularity along a single line, then the Radon coefficients at this angle will be constant. As such, because of the vanishing moments of the wavelet $\psi(t)$, the continuous ridgelet coefficients $R_\psi[f](a,b,\theta)$ will also vanish for this direction. Nonetheless, the relationship (9) also serves as the pedestal for extending the ridgelet transform beyond the traditional Fourier domain.

3. Interplay between the Ridgelet and Linear Canonical Transforms

This section constitutes the centerpiece of the article and is solely devoted to extending the classical ridgelet transform to the linear canonical domain. To facilitate the motive, we prepare the linear canonical transform and the associated wavelet transform. For clarity, we subdivide this section into three subsections, one for each recipe for linear canonical ridgelet transform, and provide an example demonstrating the implementation of the linear canonical ridgelet transform on a given bivariate function.

3.1. The Recipes

The origin of the theory of linear canonical transforms dates back to early 1970s with the independent seminal works of Collins Jr. [21] in paraxial optics and Moshinsky and Quesne [22] in quantum mechanics, who studied the conservation of information and uncertainty under linear maps of phase space. It was only in 1990s that both these independent works began to be referred to jointly in the open literature. The linear canonical transform (LCT) is a three-free-parameter class of linear integral transforms, encompassing a number of well-known unitary transformations as well as signal-processing- and optics-related mathematical operations, for example, the Fourier transform, the fractional Fourier transform, the Fresnel transform, and the scaling operations [23,24]. Due to the extra degrees of freedom and simple geometrical manifestation, the LCT is more flexible than other transforms and is as such a suitable and powerful tool for investigating deep problems in science and engineering. The application areas for LCT have been growing within the last two decades at a very rapid rate, and they have been applied in a number of fields, including optics, quantum physics, time-frequency analysis, filter design, phase reconstruction, pattern recognition, radar analysis, holographic three-dimensional television, and many more [21–24]. Prior to the formal definition of the linear canonical transform, we note that herein, any 2×2 matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \mathbb{R}$$

is conveniently written as $M = (A, B; C, D)$. Moreover, we shall remain confined to the realm of real, unimodular matrices, in which case the LCT is a unitaray operator in the Hilbert space $L^2(\mathbb{R})$.

Definition 2. *Given a real, unimodular matrix $M = (A, B; C, D)$, the linear canonical transform of any $f \in L^2(\mathbb{R})$ with respect to the matrix M is defined by*

$$\mathscr{L}_M\Big[f\Big](\omega) = \int_{\mathbb{R}} f(t)\, \mathcal{K}_M(t, \omega)\, dt, \tag{10}$$

where $\mathcal{K}_M(t, \omega)$ is called as kernel of the LCT and is given by

$$\mathcal{K}_M(t, \omega) = \begin{cases} \dfrac{1}{\sqrt{2\pi i B}} \exp\left\{ \dfrac{i(At^2 - 2t\omega + D\omega^2)}{2B} \right\}, & B \neq 0 \\ \sqrt{D} \exp\left\{ \dfrac{iCD\omega^2}{2} \right\} f(D\omega), & B = 0. \end{cases} \tag{11}$$

The orthogonality relation between a pair of functions $f, g \in L^2(\mathbb{R})$ and the the corresponding linear canonical transforms is given by

$$\Big\langle f, g \Big\rangle_2 = \Big\langle \mathscr{L}_M\Big[f\Big], \mathscr{L}_M\Big[g\Big] \Big\rangle_2. \tag{12}$$

For $f = g$, the expression (14) yields the Parseval's formula for the linear canonical transform. Moreover, the inversion formula associated with the integral transformation (10) is given by

$$f(t) = \mathscr{L}_{M^{-1}}\Big(\mathscr{L}_M\Big[f\Big](\omega)\Big)(t) := \int_{\mathbb{R}} \mathscr{L}_M\Big[f\Big](\omega)\, \overline{\mathcal{K}_M(t, \omega)}\, d\omega. \tag{13}$$

For the case $B = 0$, the linear canonical transform (10) corresponds to a scaling transformation coupled with amplitude and quadratic phase modulation. Moreover, multiplication by a Gaussian or chirp function is obtained by switching the matrix to $M = (1, 0; C, 1)$, and the scaling operation can be regarded as a particular case when $M = (1/D, 0; 0, D)$. However, all such cases are of least importance for the subject under consideration; hence, the case $B = 0$ will be omitted throughout the rest of the article.

Based on the notion of linear canonical transform (10), a much more recent ramification of the classical wavelet transform (1) appears in the form of the linear canonical wavelet transform, which relies upon a modified family of daughter wavelets, known as the linear canonical wavelets. Mathematically, for a given unimodular matrix $M = (A, B; C, D)$ and any usual mother wavelet $\psi \in L^2(\mathbb{R})$, the linear canonical wavelets are defined as [25]

$$\psi^M_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)\exp\left\{\frac{iA(b^2-t^2)}{2B}\right\}, \quad a \in \mathbb{R}^+, b \in \mathbb{R}. \tag{14}$$

It is worth noting that the daughter wavelets $\psi^M_{a,b}(t)$ are dictated by the parameters of the given unimodular matrix M. Moreover, for $M = (0,1;-1,0)$ the linear canonical wavelets defined in (14) boil down to the classical wavelets (2), whereas for $M = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, $\alpha \neq n\pi$, $n \in \mathbb{Z}$, the linear canonical wavelets (14) yield a specific class of wavelets, formally referred to as fractional wavelets. Therefore, we infer that the linear canonical wavelets (14) serve as a lucid generalization of both the classical and fractional wavelets.

Definition 3. *Given a unimodular matrix $M = (A, B; C, D)$, the linear canonical wavelet transform of any $f \in L^2(\mathbb{R})$ with respect to the wavelet $\psi \in L^2(\mathbb{R})$ is defined as*

$$\mathscr{W}^M_\psi\Big[f\Big](a,b) = \int_{\mathbb{R}} f(t)\,\overline{\psi^M_{a,b}(t)}\,dt, \tag{15}$$

where $\psi^M_{a,b}(t)$ is given by (14).

The linear canonical wavelet transform $\mathscr{W}^M_\psi[f](a,b)$ is expressible as follows:

$$\mathscr{L}_M\Big(\mathscr{W}^M_\psi\Big[f\Big](a,b)\Big)(\omega) = \sqrt{2\pi a}\,\mathscr{L}_M\Big[f\Big](\omega)\,\overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)}, \tag{16}$$

where the linear canonical transform on the LHS of (16) is computed with respect to b.

3.2. The Linear Canonical Ridgelet Transform

Having prepared the recipes of the linear canonical transform and the associated wavelet transform, we are now in a position to introduce the notion of the linear canonical ridgelet transform. Here, it is imperative to recall that in view of (9), the classical ridgelet transform (8) corresponding to any $f \in L^2(\mathbb{R}^2)$ with respect to a one-dimensional wavelet $\psi \in L^2(\mathbb{R})$ can be expressed as the wavelet transform of the Radon transform (5). Therefore, with the expression (17) in hindsight, the linear canonical ridgelet transform can be formulated by computing the linear canonical wavelet transform (15) of the Radon transform $\mathfrak{R}[f](t,\theta)$, when regarded as a function of $t \in \mathbb{R}$. In fact, we observe that

$$\begin{aligned}
&\mathscr{W}^M_\psi\Big(\mathfrak{R}\Big[f\Big](t,\theta)\Big)(a,b)\\
&= \frac{1}{\sqrt{a}}\int_{\mathbb{R}} \mathfrak{R}\Big[f\Big](t,\theta)\,\overline{\psi\left(\frac{t-b}{a}\right)}\,\exp\left\{\frac{iA(t^2-b^2)}{2B}\right\}dt\\
&= \frac{1}{\sqrt{a}}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^2} f(\mathbf{x})\,\delta(\Lambda_\theta\cdot\mathbf{x}-t)\,d\mathbf{x}\right\}\overline{\psi\left(\frac{t-b}{a}\right)}\,\exp\left\{\frac{iA(t^2-b^2)}{2B}\right\}dt\\
&= \frac{1}{\sqrt{a}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(\mathbf{x})\,\delta(\Lambda_\theta\cdot\mathbf{x}-t)\,\overline{\psi\left(\frac{t-b}{a}\right)}\,\exp\left\{\frac{iA(t^2-b^2)}{2B}\right\}d\mathbf{x}\,dt\\
&= \frac{1}{\sqrt{a}}\int_{\mathbb{R}^2} f(\mathbf{x})\,\overline{\psi\left(\frac{\Lambda_\theta\cdot\mathbf{x}-b}{a}\right)}\,\exp\left\{\frac{iA(\Lambda_\theta\cdot\mathbf{x}-b)(\Lambda_\theta\cdot\mathbf{x}+b)}{2B}\right\}d\mathbf{x},
\end{aligned} \tag{17}$$

Thus, for a given 2×2 real, unimodular matrix $M = (A, B; C, D)$, we define a family of novel ridgelet waveforms (linear canonical ridgelets) governed by the matrix M as

$$\mathfrak{F}^M_\psi(a,b,\theta) := \left\{\psi^M_{a,b,\theta}(\mathbf{t}) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{\Lambda_\theta\cdot\mathbf{t}-b}{a}\right)\exp\left\{-\frac{iA(\Lambda_\theta\cdot\mathbf{t}-b)(\Lambda_\theta\cdot\mathbf{t}+b)}{2B}\right\}\right\}, \tag{18}$$

where $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. From (17), it quite evident that each linear canonical ridgelet behaves like a one-dimensional linear canonical wavelet (14) in a specific direction. Due to the chirping operation in (18), the localizing features of the linear canonical ridgelets are tunable with regard to the parameters of the unimodular matrix $M = (A, B; C, D)$. As such, the linear canonical ridgelets can be of substantial importance in optimizing the resolving power of ridgelet transform for an efficient detection of line singularities appearing in two-dimensional signals, such as images.

Example 1. *Consider the one-dimensional Mexican-hat wavelet $\psi(t) = (1 - t^2)e^{-t^2/2}$. In order to construct the linear canonical ridgelets, we take into consideration the family of analyzing waveforms (18). Consequently, we obtain*

$$\psi^M_{a,b,\theta}(\mathbf{t}) = \frac{1}{\sqrt{a}}\left[1 - \left(\frac{t_1\cos\theta + t_2\sin\theta - b}{a}\right)^2\right]\exp\left\{-\frac{1}{2}\left(\frac{t_1\cos\theta + t_2\sin\theta - b}{a}\right)^2\right\} \times \exp\left\{-\frac{iA\big((t_1\cos\theta + t_2\sin\theta)^2 - b^2\big)}{2B}\right\}. \tag{19}$$

The effect of different choices of the unimodular matrix $M = (A, B; C, D)$ on the "ridges" of the waveforms (19) is shown in Figure 1–6.

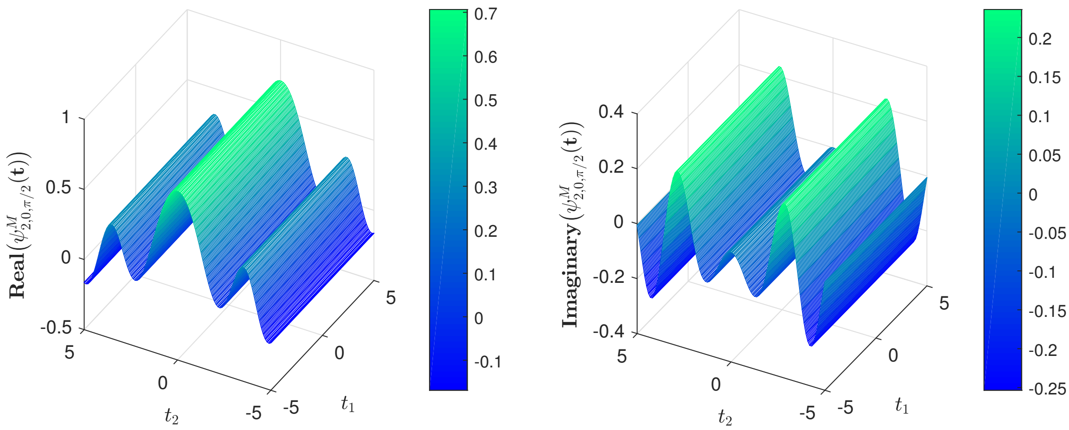

Figure 1. The linear canonical ridgelet waveforms for $M = (1, 2; 1/2, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/2$.

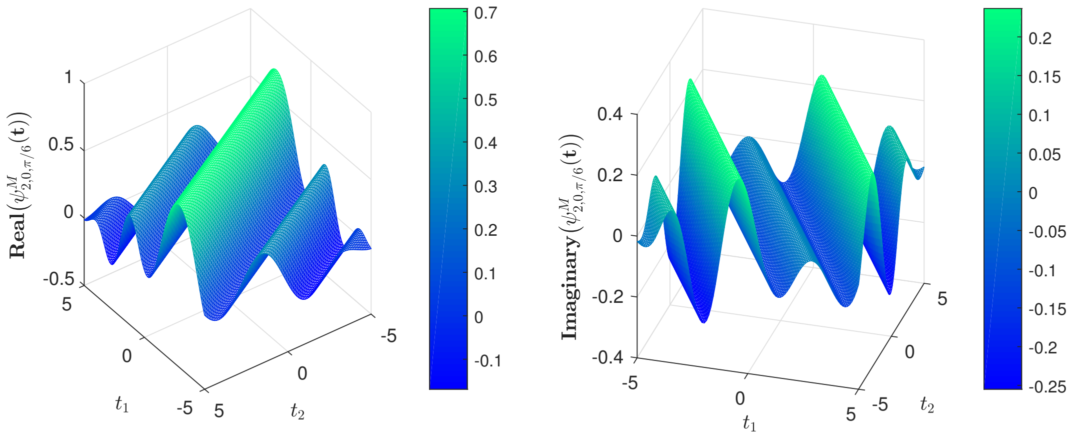

Figure 2. The linear canonical ridgelet waveforms for $M = (1, 2; 1/2, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/6$.

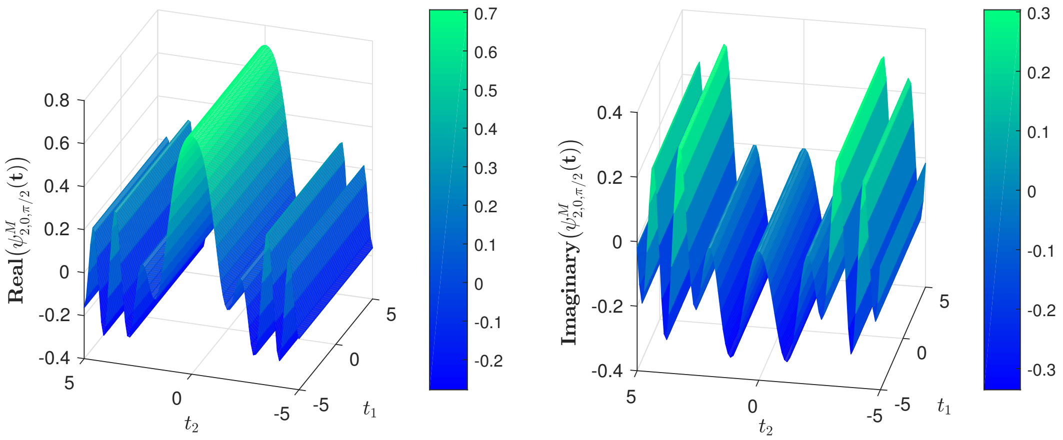

Figure 3. The linear canonical ridgelet waveforms for $M = (1/4, 1/6; -3, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/2$.

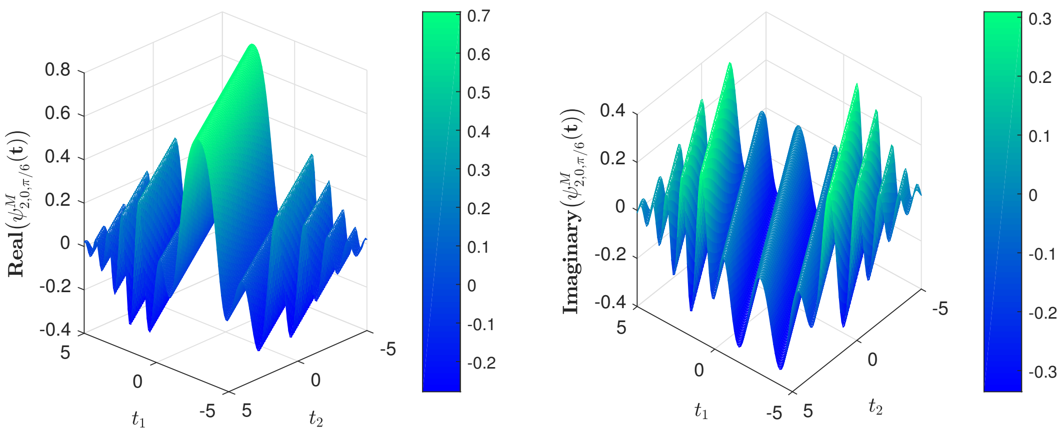

Figure 4. The linear canonical ridgelet waveforms for $M = (1/4, 1/6; -3, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/6$.

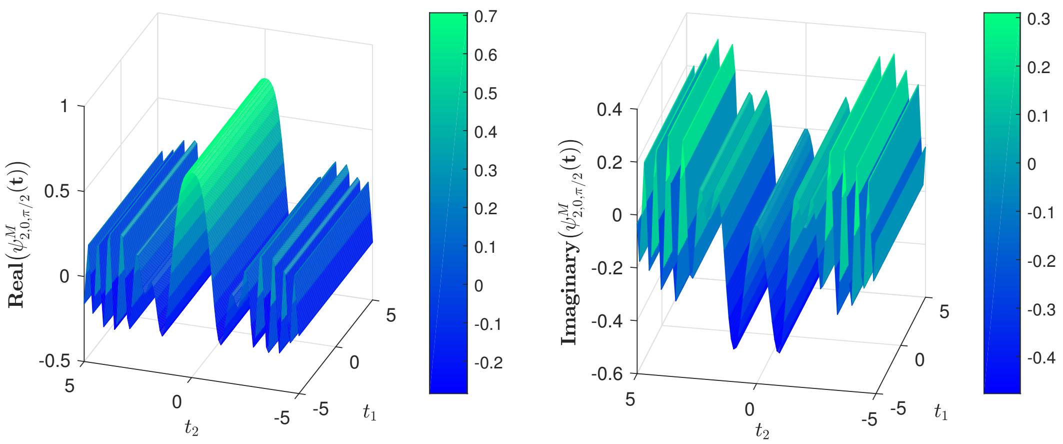

Figure 5. The linear canonical ridgelet waveforms for $M = (3, 1; 5, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/2$.

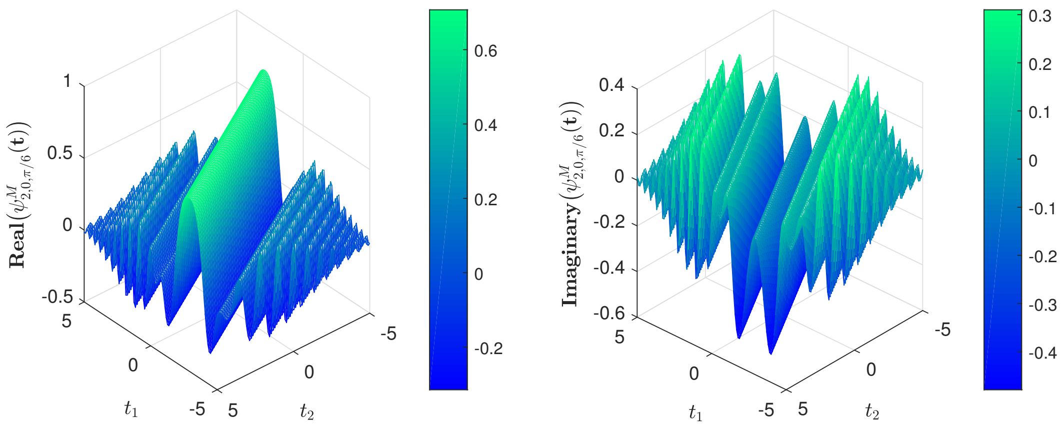

Figure 6. The linear canonical ridgelet waveforms for $M = (3, 1; 5, 2)$ at scale $a = 2$, location $b = 0$, and orientation $\theta = \pi/6$.

The construction of the linear canonical ridgelets as described in Example 1 suggests that the nature of ridges can be effectively altered by varying the underlying unimodular matrix $M = (A, B; C, D)$. With the flexibility of the novel ridgelet waveforms (18) in hindsight, we now present the formal definition of the linear canonical ridgelet transform.

Definition 4. *Given a unimodular matrix $M = (A, B; C, D)$, the linear canonical ridgelet transform of any $f \in L^2(\mathbb{R}^2)$ is denoted by $R^M_\psi\big[f\big]$ and is defined as*

$$R^M_\psi\Big[f\Big](a, b, \theta) = \Big\langle f, \psi^M_{a,b,\theta}\Big\rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t})\, \overline{\psi^M_{a,b,\theta}(\mathbf{t})}\, d\mathbf{t}, \tag{20}$$

where $\psi^M_{a,b,\theta}(\mathbf{t})$ is given by (18).

Some important features of the linear canonical ridgelet transform (20) are given below:

(i) The classical ridgelet transform (8) is just a particular case of linear canonical ridgelet transform (20) and follows by plugging $M = (0, 1; -1, 0)$ into Definition 4.

(ii) For $M = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, $\alpha \neq n\pi$, $n \in \mathbb{Z}$, the linear canonical ridgelet transform given by Definition 4 yields a new variant of the ridgelet transform, namely the fractional ridgelet transform.

(iii) The linear canonical ridgelet transform (20) is endowed with higher degrees of freedom in lieu of the classical ridgelet transform. Nevertheless, it is worth emphasizing that no extra conditions are to be imposed on the given classical one-dimensional wavelet $\psi \in L^2(\mathbb{R})$.

(iv) The linear canonical ridgelet transform (20) can be regarded as an intertwining of the linear canonical wavelet transform (15) and the classical Radon transform (5).

(v) The computational complexity of the linear canonical wavelet transform is completely determined by that of the classical wavelet transform; therefore, from expression (17), we conclude that the computational complexity of the linear canonical ridgelet transform (20) is determined by the computational complexity of classical ridgelet transform.

Below, we shall study some fundamental properties of the linear canonical ridgelet transform defined in (20). To begin with, we fix some important notations. For $\mathbf{k} \in \mathbb{R}^2$, $\lambda \in \mathbb{R}^+$, $\theta \in [0, 2\pi)$, the translation, dilation, and rotation operators acting on any $f \in L^2(\mathbb{R}^2)$ are defined as

$$\mathcal{T}_\mathbf{k} f(\mathbf{t}) = f(\mathbf{t} - \mathbf{k}), \quad \mathcal{D}_\lambda f(\mathbf{t}) = \lambda^{-1/2} f(\lambda^{-1}\mathbf{t}), \quad \text{and} \quad R_\theta f(\mathbf{t}) = f(R_{-\theta}\mathbf{t}),$$

where R_θ denotes the 2×2 rotation matrix effecting the planar rotation by θ radians and is given by

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Moreover, the parity operator acting on any $f \in L^2(\mathbb{R}^2)$ is denoted by $Pf(\mathbf{t})$; that is, $Pf(\mathbf{t}) = f(-\mathbf{t})$.

Theorem 1. *Let $c_1, c_2 \in \mathbb{C}$ and $k \in \mathbb{R}$, then for the functions $f, g \in L^2(\mathbb{R}^2)$ and any pair of one-dimensional wavelets $\psi, \phi \in L^2(\mathbb{R})$, the linear canonical ridgelet transform (20) satisfies the following properties:*

(i) *Linearity:* $R_\psi^M\Big[c_1 f + c_2 g\Big](a,b,\theta) = c_1\, R_\psi^M\Big[f\Big](a,b,\theta) + c_2\, R_\psi^M\Big[g\Big](a,b,\theta)$,

(ii) *Anti-linearity:* $R_{c_1\psi+c_2\phi}^M\Big[f\Big](a,b,\theta) = \overline{c_1}\, R_\psi^M\Big[f\Big](a,b,\theta) + \overline{c_2}\, R_\phi^M\Big[f\Big](a,b,\theta)$,

(iii) *Translation:* $R_\psi^M\Big[\mathcal{T}_\mathbf{k} f\Big](a,b,\theta) = \exp\left\{\dfrac{iA(\Lambda_\theta\cdot\mathbf{k} - b)(\Lambda_\theta\cdot\mathbf{k})}{B}\right\} R_\psi^M\Big[F_M\Big](a, b-\Lambda_\theta\cdot\mathbf{k}, \theta)$,
where $F_M(\mathbf{t}) = \exp\left\{\dfrac{iA(\Lambda_\theta\cdot\mathbf{t})(\Lambda_\theta\cdot\mathbf{k})}{B}\right\} f(\mathbf{t})$,

(iv) *Dilation:* $R_\psi^M\Big[\mathcal{D}_\lambda f\Big](a,b,\theta) = R_\psi^N\Big[f\Big]\left(\dfrac{a}{\lambda}, \dfrac{b}{\lambda}, \theta\right)$, $N = (A, B/\lambda^2; \lambda^2 C, D)$,

(v) *Parity:* $R_{P\psi}^M\Big[Pf\Big](a,b,\theta) = R_\psi^M\Big[f\Big](a,-b,\theta)$,

(vi) *Rotation:* $R_\psi^M\Big[R_{\theta'} f\Big](a,b,\theta) = R_\psi^M\Big[f\Big](a,b,\theta-\theta')$,

(vii) *Translation in Window:* $R_{\mathcal{T}_k\psi}^M\Big[f\Big](a,b,\theta) = \exp\left\{\dfrac{iA\big(2abk + (ak)^2\big)}{2B}\right\} R_\psi^M\Big[f\Big](a, b+ak, \theta)$,

(viii) *Dilation in Window:* $R_{\mathcal{D}_\lambda\psi}^M\Big[f\Big](a,b,\theta) = R_\psi^M\Big[f\Big](a\lambda, b, \theta)$.

Proof. To keep the long story short, the proof is omitted. □

The following proposition offers a lucid and elegant representation of the linear canonical ridgelet transform (20) in the linear canonical domain.

Proposition 1. *For any $f \in L^2(\mathbb{R}^2)$, the linear canonical ridgelet transform (20) can be expressed via the linear canonical transform as*

$$\mathscr{L}_M\left(R_\psi^M\Big[f\Big](a,b,\theta)\right)(\omega) = 2\pi\sqrt{\frac{a}{iB}}\,\exp\left\{\frac{iD\omega^2}{2B}\right\}\mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right)\overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)}, \tag{21}$$

where the linear canonical transform on the L.H.S of (21) is computed with respect to the translation variable b and

$$F_M(\mathbf{t}) = f(\mathbf{t})\exp\left\{\frac{iA(\Lambda_\theta\cdot\mathbf{t})^2}{2B}\right\}, \qquad \frac{\omega\Lambda_\theta}{B} = \left(\frac{\omega\cos\theta}{B}, \frac{\omega\sin\theta}{B}\right). \tag{22}$$

Proof. Note that the linear canonical ridgelet transform defined in (20) can be regarded as the linear canonical wavelet transform of the Radon transform as illustrated in (17). Therefore, by virtue of (16), we express the linear canonical transform of (20) as follows:

$$\mathscr{L}_M\left(R_\psi^M\Big[f\Big](a,b,\theta)\right)(\omega) = \sqrt{2\pi a}\,\mathscr{L}_M\left(\mathfrak{R}\Big[f\Big](t,\theta)\right)(\omega)\,\overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)}. \tag{23}$$

To compute the linear canonical transform of $\mathfrak{R}\big[f\big](t,\theta)$, with respect to variable t, we proceed as

$$\mathscr{L}_M\left(\mathfrak{R}\Big[f\Big](t,\theta)\right)(\omega)$$
$$= \int_\mathbb{R}\mathfrak{R}\Big[f\Big](t,\theta)\,\mathcal{K}_M(t,\omega)\,dt$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi iB}} \int_{\mathbb{R}} \mathfrak{R}\Big[f\Big](t,\theta)\, \exp\left\{\frac{i(At^2 - 2t\omega + D\omega^2)}{2B}\right\} dt \\
&= \frac{1}{\sqrt{2\pi iB}} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^2} f(\mathbf{x})\, \delta(\Lambda_\theta \cdot \mathbf{x} - t)\, d\mathbf{x} \right\} \exp\left\{\frac{i(At^2 - 2t\omega + D\omega^2)}{2B}\right\} dt \\
&= \frac{1}{\sqrt{2\pi iB}} \int_{\mathbb{R}^2} f(\mathbf{x})\, \exp\left\{\frac{i(A(\Lambda_\theta \cdot \mathbf{x})^2 - 2(\Lambda_\theta \cdot \mathbf{x})\omega + D\omega^2)}{2B}\right\} d\mathbf{x} \\
&= \frac{1}{\sqrt{2\pi iB}} \int_{\mathbb{R}^2} f(\mathbf{x})\, \exp\left\{\frac{i(A(\Lambda_\theta \cdot \mathbf{x})^2 - 2(\Lambda_\theta \cdot \mathbf{x})\omega + D\omega^2)}{2B}\right\} d\mathbf{x} \\
&= \frac{1}{\sqrt{2\pi iB}} \int_{\mathbb{R}^2} f(\mathbf{x})\, \exp\left\{\frac{iD\omega^2}{2B}\right\} \exp\left\{\frac{i(A(\Lambda_\theta \cdot \mathbf{x})^2 - 2(\Lambda_\theta \cdot \mathbf{x})\omega)}{2B}\right\} d\mathbf{x} \\
&= \frac{1}{\sqrt{2\pi iB}}\, \exp\left\{\frac{iD\omega^2}{2B}\right\} \int_{\mathbb{R}^2} f(\mathbf{x}) \exp\left\{\frac{iA(\Lambda_\theta \cdot \mathbf{x})^2}{2B}\right\} \exp\left\{-\frac{i(\omega\Lambda_\theta)\cdot \mathbf{x}}{B}\right\} d\mathbf{x} \\
&= \sqrt{\frac{2\pi}{iB}}\, \exp\left\{\frac{iD\omega^2}{2B}\right\} \mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right),
\end{aligned} \tag{24}
$$

where $\dfrac{\omega\Lambda_\theta}{B} = \left(\dfrac{\omega\cos\theta}{B}, \dfrac{\omega\sin\theta}{B}\right)$ and $F_M(\mathbf{t})$ is given by $F_M(\mathbf{t}) = f(\mathbf{t}) \exp\left\{\dfrac{iA(\Lambda_\theta \cdot \mathbf{t})^2}{2B}\right\}$. Finally, using (24) in (23), we obtain the desired expression for the linear canonical transform of the linear canonical ridgelet transform as

$$
\mathscr{L}_M\Big(R^M_\psi\Big[f\Big](a,b,\theta)\Big)(\omega) = 2\pi\sqrt{\frac{a}{iB}}\, \exp\left\{\frac{iD\omega^2}{2B}\right\} \mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right) \overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)}.
$$

This completes the proof of Proposition 1. □

In view of Proposition 1, we shall obtain an orthogonality relation for the linear canonical ridgelet transform defined in (20). As a consequence of such a relation, we can infer that the linear canonical ridgelet transform (20) turns to be an isometry from the space of finite energy signals $L^2(\mathbb{R}^2)$ to the space of three-parameter transformations $L^2(\mathbb{R}^+ \times \mathbb{R} \times [0, 2\pi))$.

Theorem 2. *The linear canonical ridgelet transforms of any pair of functions $f, g \in L^2(\mathbb{R}^2)$ satisfy the following orthogonality relation:*

$$
\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R^M_\psi\Big[f\Big](a,b,\theta)\, \overline{R^M_\phi\Big[g\Big](a,b,\theta)}\, \frac{da\, db\, d\theta}{a^3} = C_{\psi,\phi} \Big\langle f,\, g \Big\rangle_2, \tag{25}
$$

where $C_{\psi,\phi}$ is given by the usual cross-admissibility condition:

$$
C_{\psi,\phi} = (2\pi)^2 \int_{\mathbb{R}} \frac{\overline{\mathscr{F}[\psi](\omega)}\, \mathscr{F}[\phi](\omega)}{|\omega|^2}\, d\omega < \infty. \tag{26}
$$

Proof. Identifying the linear canonical ridgelet transform (20) as a function of the translation variable b and then applying the Parseval's formula for the linear canonical transform together with a change in the order of integration, we have

$$
\begin{aligned}
&\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R^M_\psi\Big[f\Big](a,b,\theta)\, \overline{R^M_\phi\Big[g\Big](a,b,\theta)}\, \frac{da\, db\, d\theta}{a^3} \\
&\quad = \int_0^{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \mathscr{L}_M\Big(R^M_\psi\Big[f\Big](a,b,\theta)\Big)(\omega)\, \overline{\mathscr{L}_M\Big(R^M_\phi\Big[g\Big](a,b,\theta)\Big)(\omega)}\, \frac{d\omega\, da\, d\theta}{a^3} \\
&\quad = \int_0^{2\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left\{ 2\pi\sqrt{\frac{a}{iB}}\, \exp\left\{\frac{iD\omega^2}{2B}\right\} \mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right) \overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)} \right\}
\end{aligned} \tag{27}
$$

$$\times \left\{ 2\pi \sqrt{-\frac{a}{iB}} \exp\left\{ -\frac{iD\omega^2}{2B} \right\} \overline{\mathscr{F}\Big[G_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right)} \mathscr{F}\Big[\phi\Big]\left(\frac{a\omega}{B}\right) \right\} \frac{d\omega\, da\, d\theta}{a^3}$$

$$= \int_0^{2\pi} \int_{\mathbb{R}} \mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right) \overline{\mathscr{F}\Big[G_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right)}$$

$$\times \left\{ \frac{(2\pi)^2}{|B|} \int_{\mathbb{R}^+} \overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)} \mathscr{F}\Big[\phi\Big]\left(\frac{a\omega}{B}\right) \frac{da}{a^2} \right\} d\omega\, d\theta,$$

where the functions $F_M(\mathbf{t})$ and $G_M(\mathbf{t})$ appearing on the RHS of (27) are given by

$$F_M(\mathbf{t}) = f(\mathbf{t}) \exp\left\{ \frac{iA(\Lambda_\theta \cdot \mathbf{t})^2}{2B} \right\} \quad \text{and} \quad G_M(\mathbf{t}) = g(\mathbf{t}) \exp\left\{ \frac{iA(\Lambda_\theta \cdot \mathbf{t})^2}{2B} \right\}.$$

To solve the inner integral on the RHS of (27), we shall make use of the substitution $a\omega/B = \omega'$, so that

$$\frac{(2\pi)^2}{|B|} \int_{\mathbb{R}^+} \overline{\mathscr{F}\Big[\psi\Big]\left(\frac{a\omega}{B}\right)} \mathscr{F}\Big[\phi\Big]\left(\frac{a\omega}{B}\right) \frac{da}{a^2} = \frac{|\omega|}{|B|^2} \left\{ (2\pi)^2 \int_{\mathbb{R}} \frac{\overline{\mathscr{F}[\psi](\omega')}\, \mathscr{F}[\phi](\omega')}{|\omega'|^2} d\omega' \right\}$$

$$= \frac{|\omega|}{|B|^2} C_{\psi,\phi}. \tag{28}$$

Using the estimate (28) in (27) and invoking the cross-admissibility condition (26), we obtain

$$\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R_\psi^M\Big[f\Big](a,b,\theta)\, \overline{R_\phi^M\Big[g\Big](a,b,\theta)} \frac{da\, db\, d\theta}{a^3}$$

$$= C_{\psi,\phi} \int_0^{2\pi} \int_{\mathbb{R}} \mathscr{F}\Big[F_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right) \overline{\mathscr{F}\Big[G_M\Big]\left(\frac{\omega\Lambda_\theta}{B}\right)} \frac{|\omega|}{|B|^2} d\omega\, d\theta \tag{29}$$

$$= C_{\psi,\phi} \int_{\mathbb{R}^2} \mathscr{F}\Big[F_M\Big](\mathbf{w})\, \overline{\mathscr{F}\Big[G_M\Big](\mathbf{w})}\, d(\mathbf{w}),$$

where the last expression is obtained via the substitution $\omega\Lambda_\theta/B = (\omega_1, \omega_2) = \mathbf{w}$ and the fact that the Jacobian of the transformation is ω/B^2. Finally, applying the Parseval's formula for the Fourier transform on the RHS of (29), we obtain the desired orthogonality relation

$$\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R_\psi^M\Big[f\Big](a,b,\theta)\, \overline{R_\phi^M\Big[g\Big](a,b,\theta)} \frac{da\, db\, d\theta}{a^3} = C_{\psi,\phi} \int_{\mathbb{R}^2} \mathscr{F}\Big[F_M\Big](\mathbf{w}) \overline{\mathscr{F}\Big[G_M\Big](\mathbf{w})}\, d\mathbf{w}$$

$$= C_{\psi,\phi} \int_{\mathbb{R}^2} F_M(\mathbf{t})\, \overline{G_M(\mathbf{t})}\, d\mathbf{t}$$

$$= C_{\psi,\phi} \int_{\mathbb{R}^2} f(\mathbf{t})\, \overline{g(\mathbf{t})}\, d\mathbf{t}$$

$$= C_{\psi,\phi} \Big\langle f, g \Big\rangle_2.$$

This completes the proof of Theorem 2. □

Remark 2. *For $f = g$ and $\psi = \phi$, Theorem 2 boils down to the following expression:*

$$\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left| R_\psi^M\Big[f\Big](a,b,\theta) \right|^2 \frac{da\, db\, d\theta}{a^3} = C_\psi \, \|f\|_2^2, \tag{30}$$

where C_ψ is given by the admissibility condition (7). Evidently, if the wavelet ψ satisfies $C_\psi = 1$, then the linear canonical ridgelet transform (20) is indeed an isometry from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^+ \times \mathbb{R} \times [0, 2\pi))$.

Following is the inversion formula associated with the linear canonical ridgelet transform defined in (20).

Theorem 3. *Any function $f \in L^2(\mathbb{R}^2)$ can be reconstructed from the corresponding linear canonical ridgelet transform $R_\psi^M\big[f\big](a,b,\theta)$ via the following formula:*

$$f(\mathbf{t}) = \frac{1}{C_{\psi,\phi}} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R_\psi^M\Big[f\Big](a,b,\theta)\, \phi_{a,b,\theta}^M(\mathbf{t}) \frac{da\, db\, d\theta}{a^3}, \quad a.e., \tag{31}$$

where $C_{\psi,\phi} \neq 0$ is given by the cross-admissibility condition (26).

Proof. We shall accomplish the proof by virtue of Theorem 2. To facilitate the narrative, we choose $g(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{t})$, $\mathbf{t} \in \mathbb{R}^2$ and note that

$$\begin{aligned} R_\phi^M\Big[g\Big](a,b,\theta) &= \frac{1}{\sqrt{a}} \int_{\mathbb{R}^2} \delta(\mathbf{x}-\mathbf{t})\, \overline{\phi\left(\frac{\Lambda_\theta \cdot \mathbf{x} - b}{a}\right) \exp\left\{\frac{iA(\Lambda_\theta \cdot \mathbf{x} - b)(\Lambda_\theta \cdot \mathbf{x} + b)}{2B}\right\}} d\mathbf{x} \\ &= \frac{1}{\sqrt{a}}\, \overline{\phi\left(\frac{\Lambda_\theta \cdot \mathbf{x} - b}{a}\right)} \exp\left\{\frac{iA(\Lambda_\theta \cdot \mathbf{x} - b)(\Lambda_\theta \cdot \mathbf{x} + b)}{2B}\right\} \\ &= \overline{\phi_{a,b,\theta}^M(\mathbf{t})}. \end{aligned} \tag{32}$$

In addition, we note that

$$\Big\langle f, g \Big\rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{x})\, \delta(\mathbf{x}-\mathbf{t})\, d\mathbf{t} = f(\mathbf{t}). \tag{33}$$

Invoking the orthogonality relation (25), then using (32) and (33), we obtain the reconstruction formula for the linear canonical ridgelet transform as

$$f(\mathbf{t}) = \frac{1}{C_{\psi,\phi}} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} R_\psi^M\Big[f\Big](a,b,\theta)\, \phi_{a,b,\theta}^M(\mathbf{t}) \frac{da\, db\, d\theta}{a^3}, \quad a.e.$$

This completes the proof of Theorem 3. □

Towards the end of this section, we shall formulate the Heisenberg-type uncertainty inequality associated with the linear canonical ridgelet transform defined in (20).

Theorem 4. *If $R_\psi^M\big[f\big](a,b,\theta)$ is the linear canonical ridgelet transform of any non-trivial function $f \in L^2(\mathbb{R}^2)$, vanishing at infinity; then the following uncertainty inequality holds:*

$$\begin{aligned} \left\{\int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}^+} b^2 \left| R_\psi^M\Big[f\Big](a,b,\theta)\right|^2 \frac{da\, db\, d\theta}{a^3}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} |\mathbf{w}|^2 \left|\mathscr{F}\Big[F_M\Big](\mathbf{w})\right|^2 d\mathbf{w}\right\}^{1/2} \\ \geq \frac{\sqrt{C_\psi}}{2} \Big\|f\Big\|_2^2, \end{aligned} \tag{34}$$

where C_ψ is given by the admissibility condition (7) and the function $F_M(\mathbf{t})$ is defined as

$$F_M(\mathbf{t}) = f(\mathbf{t}) \exp\left\{\frac{iA(\Lambda_\theta \cdot \mathbf{t})^2}{2B}\right\}.$$

Proof. For any non-trivial function $g \in L^2(\mathbb{R})$, vanishing at $\pm\infty$, Heisenberg's uncertainty principle associated with the linear canonical transform reads [26]:

$$\left\{\int_{\mathbb{R}} t^2 |g(t)|^2 dt\right\}^{1/2} \left\{\int_{\mathbb{R}} \omega^2 \left|\mathscr{L}_M\Big[g\Big](\omega)\right|^2 d\omega\right\}^{1/2} \geq \frac{|B|}{2} \left\{\int_{\mathbb{R}} |g(t)|^2 dt\right\}. \tag{35}$$

We shall identify the linear canonical ridgelet transform $R_\psi^M\big[f\big](a,b,\theta)$ as a function of the translation variable b and then invoke (35) to obtain

$$\left\{\int_{\mathbb{R}} b^2 \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 db\right\}^{1/2} \left\{\int_{\mathbb{R}} \omega^2 \left|\mathscr{L}_M\left(R_\psi^M\left[f\right](a,b,\theta)\right)(\omega)\right|^2 d\omega\right\}^{1/2} \geq \frac{|B|}{2}\left\{\int_{\mathbb{R}} \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 db\right\}. \tag{36}$$

Integrating the inequality (36) with respect to the measure $da\,d\theta/a^3$ yields

$$\int_0^{2\pi}\int_{\mathbb{R}^+}\left\{\int_{\mathbb{R}} b^2 \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 db\right\}^{1/2} \left\{\int_{\mathbb{R}} \omega^2 \left|\mathscr{L}_M\left(R_\psi^M\left[f\right](a,b,\theta)\right)(\omega)\right|^2 d\omega\right\}^{1/2} \frac{da\,d\theta}{a^3} \geq \frac{|B|}{2}\left\{\int_0^{2\pi}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 \frac{db\,da\,d\theta}{a^3}\right\}. \tag{37}$$

Applying the Cauchy–Schwarz inequality, we can redraft inequality (37) as

$$\left\{\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} b^2 \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 \frac{da\,db\,d\theta}{a^3}\right\}^{1/2} \times \left\{\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \omega^2 \left|\mathscr{L}_M\left(R_\psi^M\left[f\right](a,b,\theta)\right)(\omega)\right|^2 \frac{da\,d\omega\,d\theta}{a^3}\right\}^{1/2} \geq C_\psi \frac{|B|}{2}\left\|f\right\|_2^2. \tag{38}$$

By virtue of Proposition 1, we can simplify the second set of integrals on the L.H.S of (38) as

$$\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \omega^2 \left|\mathscr{L}_M\left(R_\psi^M\left[f\right](a,b,\theta)\right)(\omega)\right|^2 \frac{da\,d\omega\,d\theta}{a^3}\\
&= \frac{(2\pi)^2}{|B|}\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \omega^2 \left|\mathscr{F}\left[F_M\right]\left(\frac{\omega\Lambda_\theta}{B}\right)\right|^2 \left|\mathscr{F}\left[\psi\right]\left(\frac{a\omega}{B}\right)\right|^2 \frac{da\,d\omega\,d\theta}{a^2}\\
&= \frac{1}{|B|}\int_0^{2\pi}\int_{\mathbb{R}} \omega^2 \left|\mathscr{F}\left[F_M\right]\left(\frac{\omega\Lambda_\theta}{B}\right)\right|^2 \left\{(2\pi)^2\int_{\mathbb{R}^+}\left|\mathscr{F}\left[\psi\right]\left(\frac{a\omega}{B}\right)\right|^2 \frac{da}{a^2}\right\} d\omega\,d\theta,
\end{aligned} \tag{39}$$

where $F_M(\mathbf{t}) = f(\mathbf{t})\exp\left\{\dfrac{iA(\Lambda_\theta\cdot\mathbf{t})^2}{2B}\right\}$. The inner integral on the R.H.S of (39) can be solved by substituting $a\omega/B = \omega'$ and noting that

$$(2\pi)^2\int_{\mathbb{R}^+}\left|\mathscr{F}\left[\psi\right]\left(\frac{a\omega}{B}\right)\right|^2 \frac{da}{a^2} = \frac{|\omega|}{|B|}\int_{\mathbb{R}} \frac{\left|\mathscr{F}\left[\psi\right](\omega')\right|^2}{|\omega'|^2}\,d\omega' = \frac{|\omega|}{|B|}\,C_\psi, \tag{40}$$

where C_ψ is given by the admissibility condition (7). Using (40) in (39), we obtain

$$\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \omega^2 \left|\mathscr{L}_M\left(R_\psi^M\left[f\right](a,b,\theta)\right)(\omega)\right|^2 \frac{da\,d\omega\,d\theta}{a^3}\\
&= C_\psi\int_0^{2\pi}\int_{\mathbb{R}} \omega^2 \left|\mathscr{F}\left[F_M\right]\left(\frac{\omega\Lambda_\theta}{B}\right)\right|^2 \frac{|\omega|}{|B|^2}\,d\omega\,d\theta\\
&= B^2 C_\psi\int_{\mathbb{R}^2} |\mathbf{w}|^2 \left|\mathscr{F}\left[F_M\right](\mathbf{w})\right|^2 d\mathbf{w},
\end{aligned} \tag{41}$$

where the last expression is obtained via the substitution $\omega\Lambda_\theta/B = (\omega_1,\omega_2) = \mathbf{w}$ and the fact that the Jacobian of the transformation is ω/B^2. Finally, using (41) in (38), we obtain the desired uncertainty inequality for the linear canonical ridgelet transform as

$$\left\{\int_0^{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}^+} b^2 \left|R_\psi^M\left[f\right](a,b,\theta)\right|^2 \frac{da\,db\,d\theta}{a^3}\right\}^{1/2} \left\{\int_{\mathbb{R}^2} |\mathbf{w}|^2 \left|\mathscr{F}\left[F_M\right](\mathbf{w})\right|^2 d\mathbf{w}\right\}^{1/2} \geq \frac{\sqrt{C_\psi}}{2}\left\|f\right\|_2^2.$$

This completes the proof of Theorem 4. □

Remark 3. *For $M = (0,1;-1,0)$ and $M = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, $\alpha \neq n\pi$, $n \in \mathbb{Z}$, the uncertainty inequality (34) boils down to the respective Heisenberg-type inequalities for the classical and fractional variants of the ridgelet transform.*

3.3. An Example

In this subsection, our sole motive is to exemplify the implementation of the proposed linear canonical ridgelet transform on a bivariate function.

Example 2. *Consider the bivariate Gaussian function $f(\mathbf{t}) = K\, e^{-\sigma|\mathbf{t}|^2}$, where $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $K \in \mathbb{C}$, $\sigma > 0$ are scalars. Firstly, we shall find an expression for the Radon transform of the given bivariate function. To keep any confusion of terminology at bay, we temporarily replace the notation $\mathbf{t}$ with $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and then use (5) to obtain*

$$\begin{aligned} \mathfrak{R}\Big[f\Big](t,\theta) &= \int_{\mathbb{R}^2} f(\mathbf{x})\, \delta(\Lambda_\theta \cdot \mathbf{x} - t)\, d\mathbf{x} \\ &= K \int_{\mathbb{R}^2} e^{-\sigma|\mathbf{x}|^2} \delta(\Lambda_\theta \cdot \mathbf{x} - t)\, d\mathbf{x} \\ &= K \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\sigma x_1^2} e^{-\sigma x_2^2}\, \delta(x_1\cos\theta + x_2\sin\theta - t)\, dx_1\, dx_2 \\ &= K \int_{\mathbb{R}} e^{-\sigma x_1^2} \left\{ \int_{\mathbb{R}} e^{-\sigma x_2^2}\, \delta(x_1\cos\theta + x_2\sin\theta - t)\, dx_2 \right\} dx_1. \end{aligned} \tag{42}$$

Making use of the substitution $x_2 \sin\theta = z$, we can simplify the expression (42) as follows

$$\begin{aligned} \mathfrak{R}\Big[f\Big](t,\theta) &= K \int_{\mathbb{R}} e^{-\sigma x_1^2} \left\{ \frac{1}{|\sin\theta|} \int_{\mathbb{R}} e^{-\sigma z^2/\sin^2\theta}\, \delta(x_1\cos\theta - t + z)\, dz \right\} dx_1 \\ &= K \int_{\mathbb{R}} e^{-\sigma x_1^2} \left\{ \frac{1}{|\sin\theta|}\, e^{-\sigma(t - x_1\cos\theta)^2/\sin^2\theta} \right\} dx_1 \\ &= \frac{K}{|\sin\theta|}\, e^{-\gamma t^2} \int_{\mathbb{R}} e^{-(\sigma+\gamma\cos^2\theta)x_1^2 + (-2t\cos\theta)x_1} dx_1 \\ &= \frac{K}{|\sin\theta|}\, e^{-\gamma t^2} \sqrt{\frac{\pi}{\sigma+\gamma\cos^2\theta}}\, \exp\left\{ \frac{(-2t\cos\theta)^2}{4(\sigma+\gamma\cos^2\theta)} \right\}, \end{aligned} \tag{43}$$

where $\gamma = \sigma/\sin^2\theta$ and the last expression follows as an implication of the standard Gaussian integral. Moreover, we can further simplify the expression (43) as

$$\begin{aligned} \mathfrak{R}\Big[f\Big](t,\theta) &= \frac{K}{|\sin\theta|} \sqrt{\frac{\pi\sin^2\theta}{\sigma}}\, \exp\left\{ \left(\frac{\cos^2\theta\,\sin^4\theta - \sigma^2}{\sigma\sin^2\theta} \right) t^2 \right\} \\ &= K\sqrt{\frac{\pi}{\sigma}}\, e^{-\rho t^2}, \quad \rho = \left(\frac{\cos^2\theta\,\sin^4\theta - \sigma^2}{\sigma\sin^2\theta} \right). \end{aligned} \tag{44}$$

Now, in order to compute the ridgelet transform of the given bivariate function, we choose the one-dimensional Morlet wavelet $\psi(t) = e^{i\omega_0 t - t^2/2}$, where $\omega_0 \in \mathbb{R}$. Consequently, in view of (14), the linear canonical daughter wavelets are given by

$$\psi^M_{a,b}(t) = \frac{1}{\sqrt{a}} \exp\left\{ \left(\frac{iA}{2B} - \frac{1}{2a^2} \right) b^2 - \left(\frac{iA}{2B} + \frac{1}{2a^2} \right) t^2 + \left(\frac{i\omega_0}{a} + \frac{b}{a^2} \right) t - \left(\frac{i\omega_0}{a} \right) b \right\}. \tag{45}$$

Finally, using Definition 3 to calculate the linear canonical wavelet transform of the Radon transform obtained in (43), we have

$$\mathscr{W}^M_\psi\left(\mathfrak{R}\Big[f\Big](t,\theta)\right)(a,b) = K\sqrt{\frac{\pi}{a\sigma}}\, \exp\left\{ \left(\frac{1}{2a^2} - \frac{iA}{2B} \right) b^2 + \left(\frac{i\omega_0}{a} \right) b \right\}$$

$$\times \int_{\mathbb{R}} \exp\left\{ -\left(\rho - \frac{iA}{2B} - \frac{1}{2a^2}\right)t^2 - \left(\frac{i\omega_0}{a} + \frac{b}{a^2}\right)t \right\} dt \tag{46}$$

$$= \frac{K\pi}{\sqrt{a\sigma\left(\rho - \frac{iA}{2B} - \frac{1}{2a^2}\right)}} \exp\left\{ \frac{\left(\frac{i\omega_0}{a} + \frac{b}{a^2}\right)^2}{4\left(\rho - \frac{iA}{2B} - \frac{1}{2a^2}\right)} \right\}$$

$$\times \exp\left\{ \left(\frac{1}{2a^2} - \frac{iA}{2B}\right)b^2 + \left(\frac{i\omega_0}{a}\right)b \right\}.$$

Hence, we conclude that the linear canonical ridgelet transform of the given bivariate function with respect to the one-dimensional Morlet wavelet is given by

$$\mathcal{R}_\psi^M\left[f\right](a,b,\theta) = K\pi \frac{\exp\left\{ \left(\frac{1}{2a^2} - \frac{iA}{2B}\right)b^2 + \left(\frac{i\omega_0}{a}\right)b \right\}}{\sqrt{a\sigma\left[\left(\frac{\cos^2\theta\,\sin^4\theta - \sigma^2}{\sigma\sin^2\theta}\right) - \frac{iA}{2B} - \frac{1}{2a^2}\right]}}$$

$$\times \exp\left\{ \frac{\left(\frac{i\omega_0}{a} + \frac{b}{a^2}\right)^2}{4\left[\left(\frac{\cos^2\theta\,\sin^4\theta - \sigma^2}{\sigma\sin^2\theta}\right) - \frac{iA}{2B} - \frac{1}{2a^2}\right]} \right\}. \tag{47}$$

4. Conclusions and Future Work

In the present study, we accomplished the prime objective of entwining the merits of the linear canonical transform and the classical ridgelet transform into a novel integral transform coined as the linear canonical ridgelet transform by making use of the well-known Radon transform and the linear canonical wavelet transform. The efficacy of the linear canonical ridgelet waveforms is demonstrated via an example supported by vibrant graphics, and it is intuitively clear that the underlying unimodular matrix $M = (A, B; C, D)$ significantly affects the nature of ridges. As a supplementation to the fundamental results, we also formulated a Heisenberg-type uncertainty inequality for the newly proposed linear canonical ridgelet transform. Nonetheless, an illustrative example demonstrating the implementation of the linear canonical ridgelet transform on a bivariate function is also presented in support of the undertaken problem. On the flip side, the present article also stimulates interest towards future research in the context of ridgelet analysis; for instance, one of the immediate concerns is to develop the discrete analogue of the proposed linear canonical ridgelet transform. By doing so, the ridgelets can be used in a novel way in many ways where the traditional wavelets fail to perform satisfactorily. In addition, since the linear canonical ridgelet transform is potentially capable of efficiently dealing with line-like phenomena in two-dimensional signals, it is quite effective to extend the proposed transform to arbitrary dimensions by employing the multi-dimensional linear canonical transform. In that case, higher-dimensional singularities, such as plane-like phenomenon in three dimensions, can be be analyzed in an elegant and insightful way.

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