A CLASS OF ANALYTIC FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS[†]

by

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ABSTRACT

The object of the present paper is to derive several interesting properties of the class $\mathcal{T}(\alpha,p_k)$ of analytic functions with fixed finitely many coefficients. These include closure theorems, and theorems involving the radii of starlikeness and convexity for functions belonging to the class $\mathcal{T}(\alpha,p_k)$. A modified convolution product of functions in the class $\mathcal{T}(\alpha,p_k)$ and a certain functional $\mathcal{J}(f)$ of $f(z)$ belonging to $\mathcal{T}(\alpha,p_k)$ are also considered.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Further let $\mathcal{C}(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions satisfying the inequality

$$\text{Re}\{f'(z)\} > \alpha \qquad (z \in \mathcal{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$.

We remark in passing that $\mathcal{C}(\alpha)$ is the sublcass of analytic and univalent functions in the unit disk $\mathcal{U}$, considered by Noshiro [10] and Warschawski [27]; in particular, the class $\mathcal{C}(0)$ was studied by MacGregor [8].

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0). \tag{1.3}$$

We denote by $\mathcal{T}(\alpha)$ the class obtained by taking the intersection of $\mathcal{C}(\alpha)$ with $\mathcal{T}$; that is,

$$\mathcal{T}(\alpha) = \mathcal{C}(\alpha) \cap \mathcal{T}.$$

The class $\mathcal{T}(\alpha)$ was studied by Sarangi and Uralegaddi [21], and by Owa and Uralegaddi [17].

Among several other classes of analytic and univalent functions of the form (1.3) studied in the literature are those introduced by Silverman ([23], [24]), Ahuja and Jain [1], Gupta and Jain ([4], [5]), Silverman and Silvia [25], Pilat [18], Gupta and Ahmad ([2], [3]), Jain and Ahuja [7], and Owa ([11], [12], [13]).

We being by recalling the following lemma due to Sarangi and Uralegaddi [21]:

LEMMA 1. _Let the function_ $f(z)$ _be defined by_ (1.3). _Then_ $f(z)$ _is in the class_ $\mathcal{T}(\alpha)$ _if and only if_

$$\sum_{n=2}^{\infty} n\, a_n \leqq 1 - \alpha. \tag{1.4}$$

In view of Lemma 1, all functions belonging to the class $\mathcal{T}(\alpha)$ satisfy the coefficient inequality

$$a_n \leqq \frac{1-\alpha}{n} \qquad (n \geqq 2). \tag{1.5}$$

Making use of (1.5), we now introduce the following class of functions:

Let $\mathcal{T}(\alpha,p_k)$ denote the subclass of $\mathcal{T}(\alpha)$ consisting of functions of the form

$$(1.6) \qquad f(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} a_n z^n,$$

where

$$a_n \geqq 0, \quad 0 \leqq p_i \leqq 1, \quad \text{and} \quad 0 \leqq \sum_{i=2}^{k} p_i \leqq 1.$$

For $k = 2$, the class $\mathcal{T}(\alpha,p_2)$ was introduced by Owa [14]. Other classes of analytic and univalent functions with fixed second coefficient were studied by Silverman and Silvia [26], and by Owa ([15], [16]).

In the present paper, we prove several interesting (and useful) results for functions belonging to the class $\mathcal{T}(\alpha,p_k)$, and determine the radii of starlikeness and convexity of functions in $\mathcal{T}(\alpha,p_k)$. We also consider a modified convolution product of functions in the class $\mathcal{T}(\alpha,p_k)$, and a certain functional $\mathcal{J}(f)$ of functions $f(z)$ in $\mathcal{T}(\alpha,p_k)$.

2. PROPERTIES OF THE CLASS $\mathcal{T}(\alpha,p_k)$

We state our first result as

THEOREM 1. Let the function $f(z)$ be defined by (1.6). Then $f(z)$ is in the class $\mathcal{T}(\alpha,p_k)$ if and only if

$$(2.1) \qquad \sum_{n=k+1}^{\infty} n\, a_n \leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right),$$

where

$$0 \leqq p_i \leqq 1 \quad \underline{\text{and}} \quad 0 \leqq \sum_{i=2}^{k} p_i \leqq 1.$$

The result (2.1) is sharp.

PROOF. Putting

$$a_i = \frac{(1-\alpha)p_i}{i} \qquad (i = 2,3,\dots,k) \tag{2.2}$$

in Lemma 1, we have

$$\sum_{i=2}^{k} (1-\alpha)p_i + \sum_{n=k+1}^{\infty} n\, a_n \leqq 1 - \alpha, \tag{2.3}$$

which clearly implies (2.1). Further, by taking the function $f(z)$ of the form

$$f(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} z^n \tag{2.4}$$

for $n \geqq k + 1$, we can see that the result (2.1) is sharp.

COROLLARY 1. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{T}(\alpha,p_k)$. Then

$$a_n \leqq \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} \tag{2.5}$$

for $n \geqq k + 1$. The result (2.5) is sharp for the function $f(z)$ given by (2.4).

THEOREM 2. The class $\mathcal{T}(\alpha,p_k)$ is convex.

PROOF. Let the function $f(z)$ be defined by (1.6). Define the function $g(z)$ by

$$g(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} b_n z^n, \tag{2.6}$$

where

$$b_n \geqq 0, \quad 0 \leqq p_i \leqq 1, \quad \text{and} \quad 0 \leqq \sum_{i=2}^{k} p_i \leqq 1.$$

Assuming that $f(z)$ and $g(z)$ are in the class $\mathcal{T}(\alpha,p_k)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \lambda\, f(z) + (1-\lambda)g(z) \qquad (0 \leqq \lambda \leqq 1) \tag{2.7}$$

is also in the class $\mathcal{T}(\alpha,p_k)$.

Since

$$H(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} \{\lambda\, a_n + (1-\lambda)b_n\} z^n, \tag{2.8}$$

we observe that

$$\sum_{n=k+1}^{\infty} n\{\lambda\, a_n + (1-\lambda)b_n\} = \lambda \sum_{n=k+1}^{\infty} n\, a_n + (1-\lambda) \sum_{n=k+1}^{\infty} n\, b_n \tag{2.9}$$

$$\leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)$$

with the aid of Theorem 1. Hence

$$H(z) \in \mathcal{T}(\alpha, p_k).$$

This completes the proof of Theorem 2.

With a view to finding the extreme points of the convex set $\mathcal{T}(\alpha, p_k)$, we next prove

THEOREM 3. *Let*

$$f_k(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i \tag{2.10}$$

and

$$f_n(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} z^n \tag{2.11}$$

for $n \geqq k+1$. *Then* $f(z)$ *is in the class* $\mathcal{T}(\alpha, p_k)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=k}^{\infty} \lambda_n f_n(z), \tag{2.12}$$

where $\lambda_n \geqq 0$ $(n \geqq k)$, *and*

$$\sum_{n=k}^{\infty} \lambda_n = 1. \tag{2.13}$$

PROOF. Assume that $f(z)$ can be expressed in the form (2.12). Then it follows from (2.10), (2.11) and (2.13) that

$$f(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} \frac{\lambda_n(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} z^n. \tag{2.14}$$

Note that

$$(2.15) \qquad \sum_{n=k+1}^{\infty} n\left\{\frac{\lambda_n(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)}{n}\right\} = (1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)\sum_{n=k+1}^{\infty} \lambda_n$$

$$= (1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)(1-\lambda_k)$$

$$\leqq (1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right),$$

which implies that $f(z) \in \mathcal{T}(\alpha,p_k)$.

For the converse, assume that the function $f(z)$ of the form (1.6) belongs to the class $\mathcal{T}(\alpha,p_k)$. Since $f(z)$ satisfies (2.5) for $n \geqq k+1$, we may set

$$(2.16) \qquad \lambda_n = \frac{n\, a_n}{(1-\alpha)\left(1-\sum_{i=2}^{k} p_i\right)} \leqq 1 \qquad (n \geqq k+1)$$

and

$$(2.17) \qquad \lambda_k = 1 - \sum_{n=k+1}^{\infty} \lambda_n.$$

Hence $f(z)$ has the representation (2.12).

This evidently completes the proof of Theorem 3.

COROLLARY 2. The extreme points of $\mathcal{T}(\alpha,p_k)$ are the functions $f_n(z)$ $(n \geqq k)$ given by (2.10) and (2.11).

3. CLOSURE THEOREM

In this section we prove the following closure theorem for functions belonging to the class $\mathcal{T}(\alpha,p_k)$.

THEOREM 4. <u>Let the function</u>

$$f_j(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} a_{n,j} z^n \qquad (a_{n,j} \geqq 0) \tag{3.1}$$

<u>be in the class</u> $\mathcal{T}(\alpha,p_k)$ <u>for every</u> $j = 1, \ldots, m$. <u>Then the function</u> $F(z)$ <u>defined by</u>

$$F(z) = \sum_{j=1}^{m} c_j f_j(z) \qquad (c_j \geqq 0) \tag{3.2}$$

<u>is also in the class</u> $\mathcal{T}(\alpha,p_k)$, <u>where</u>

$$\sum_{j=1}^{m} c_j = 1. \tag{3.3}$$

PROOF. Combining the definitions (3.1) and (3.2), we have

$$F(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} z^i - \sum_{n=k+1}^{\infty} \left(\sum_{j=1}^{m} c_j a_{n,j} \right) z^n, \tag{3.4}$$

where we have also used the relationship (3.3). Since $f_j(z) \in \mathcal{T}(\alpha,p_k)$ for every $j = 1, \ldots, m$, Theorem 1 yields

$$\sum_{n=k+1}^{\infty} n\, a_{n,j} \leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right) \tag{3.5}$$

for $j = 1, \ldots, m$. Thus we obtain

$$\sum_{n=k+1}^{\infty} n\left(\sum_{j=1}^{m} c_j\, a_{n,j}\right) = \sum_{j=1}^{m} c_j\left(\sum_{n=k+1}^{\infty} n\, a_{n,j}\right)$$

$$\leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)$$

which (in view of Theorem 1) implies that

$$F(z) \in \mathcal{T}(\alpha,p_k).$$

4. STARLIKENESS AND CONVEXITY

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be starlike of order β $(0 \leqq \beta < 1)$ if it satisfies the inequality

$$\text{(4.1)} \qquad \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathcal{U})$$

for $0 \leqq \beta < 1$. On the other hand, a function $f(z)$ belonging to the class $\mathcal{A}$ is said to be convex of order β $(0 \leqq \beta < 1)$ if it satisfies the inequality

$$\text{(4.2)} \qquad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \qquad (z \in \mathcal{U})$$

for $0 \leqq \beta < 1$.

These classes were introduced by Robertson [20], and were studied subsequently by Schild [22], MacGregor [9], Pinchuk [19], and Jack [6].

Our result on the radii of starlikeness of order β for functions in the class $\mathcal{T}(\alpha,p_k)$ is contained in

THEOREM 5. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{T}(\alpha,p_k)$. Then $f(z)$ is starlike of order β $(0 \leqq \beta < 1)$ in the disk $|z| < r_1$, where r_1 is the largest value for which

$$\sum_{i=2}^{k} (1-\alpha)\left(1 - \frac{\beta}{i}\right)p_i\, r^{i-1} + \frac{(1-\alpha)(n-\beta)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} r^{n-1} \leqq 1 - \beta \tag{4.3}$$

for $n \geqq k + 1$. The result is sharp.

PROOF. It suffices to prove that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqq 1 - \beta \tag{4.4}$$

for $|z| < r_1$. Note that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \tag{4.5}$$

$$\leqq \frac{\sum_{i=2}^{k} (1-\alpha)\left(1 - \frac{1}{i}\right)p_i\, r^{i-1} - \sum_{n=k+1}^{\infty} (n-1)a_n r^{n-1}}{1 - \sum_{i=2}^{k} \frac{(1-\alpha)p_i}{i} r^{i-1} - \sum_{n=k+1}^{\infty} a_n\, r^{n-1}}$$

$$\leqq 1 - \beta$$

for $|z| \leqq r$ if and only if

$$\sum_{i=2}^{k} (1-\alpha)\left(1 - \frac{\beta}{i}\right)p_i\, r^{i-1} + \sum_{n=k+1}^{\infty} (n-\beta)a_n\, r^{n-1} \leqq 1 - \beta. \tag{4.6}$$

Since $f(z) \in \mathcal{T}(\alpha,p_k)$, by virtue of Theorem 1, we may set

$$a_n = \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{n} \lambda_n \qquad (n \geqq k+1), \tag{4.7}$$

where $\lambda_n \geqq 0$ $(n \geqq k+1)$, and

$$\sum_{n=k+1}^{\infty} \lambda_n \leqq 1. \tag{4.8}$$

For each fixed r, we choose the integer $n_0 = n_0(r)$ for which $(n-\beta)r^{n-1}/n$ is a maximum. Then it follows that

$$\sum_{n=k+1}^{\infty} (n-\beta)a_n r^{n-1} \leqq \frac{(1-\alpha)(n_0-\beta)\left(1 - \sum_{i=2}^{k} p_i\right)}{n_0} r^{n_0-1}. \tag{4.9}$$

Hence $f(z)$ is starlike of order β in $|z| \leqq r_1$ provided that

$$\sum_{i=2}^{k} (1-\alpha)\left(1 - \frac{\beta}{i}\right)p_i r^{i-1} + \frac{(1-\alpha)(n_0-\beta)\left(1 - \sum_{i=2}^{k} p_i\right)}{n_0} r^{n_0-1} \leqq 1 - \beta. \tag{4.10}$$

We find the value r_0 and the corresponding $n_0(r_0)$ so that

$$\sum_{i=2}^{k} (1-\alpha)\left(1 - \frac{\beta}{i}\right)p_i r_0^{i-1} + \frac{(1-\alpha)(n_0-\beta)\left(1 - \sum_{i=2}^{k} p_i\right)}{n_0} r_0^{n_0-1} = 1 - \beta. \tag{4.11}$$

Then this value r_0 is the radius of starlikeness of order β for functions $f(z)$ belonging to the class $\mathcal{T}(\alpha,p_k)$.

Finally, it is easily verified that the assertion of Theorem 5 is sharp for functions $f(z)$ given by (2.4).

In a similar manner, we can prove the following theorem concerning the radius of convexity of order β for functions in the class $\mathcal{T}(\alpha,p_k)$.

THEOREM 6. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{T}(\alpha,p_k)$. Then $f(z)$ is convex of order β $(0 \leqq \beta < 1)$ in the disk $|z| < r_2$, where r_2 is the largest value for which

$$\sum_{i=2}^{k} (1-\alpha)(i-\beta)p_i\, r^{i-1} + (1-\alpha)(n-\beta)\left(1 - \sum_{i=2}^{k} p_i\right) r^{n-1} \leqq 1 - \beta \tag{4.12}$$

for $n \geqq k+1$. The result is sharp for functions $f(z)$ given by (2.4).

5. A MODIFIED CONVOLUTION PRODUCT

Let $f_j(z)$ $(j = 1,2)$ be defined by (3.1). Denote by $f_1 * f_2(z)$ the modified convolution product of two functions $f_1(z)$ and $f_2(z)$; that is,

$$f_1 * f_2(z) = z - \sum_{i=2}^{k} \frac{(1-\alpha)^2 p_i^2}{i^2} z^i - \sum_{n=k+1}^{\infty} a_{n,1}\, a_{n,2} z^n. \tag{5.1}$$

Then we have

THEOREM 7. Let the function $f(z)$ defined by (1.6) be in the class $\mathcal{T}(\alpha,p_k)$. Then $f * f(z)$ belongs to the class $\mathcal{T}(\beta,q_k)$, where

$$\beta = \alpha(2-\alpha) \quad \underline{\text{and}} \quad q_k = p_k^2/k.$$

PROOF. Observing that

$$0 \leqq \beta < 1, \quad 0 \leqq q_i \leqq 1, \quad \text{and} \quad 0 \leqq \sum_{i=2}^{k} q_i \leqq 1,$$

and that the definition (5.1) of the modified convolution product yields

$$f * f(z) = z - \sum_{i=2}^{k} \frac{(1-\beta)q_i}{i} z^i - \sum_{n=k+1}^{\infty} a_n^2 z^n, \tag{5.2}$$

it suffices to prove that

$$\sum_{n=k+1}^{\infty} n\, a_n^2 \leqq (1-\beta)\left(1 - \sum_{i=2}^{k} q_i\right). \tag{5.3}$$

Now, by virtue of Theorem 1, we have

$$\sum_{n=k+1}^{\infty} n\, a_n^2 \leqq (1-\alpha)^2 \frac{\left(1 - \sum_{i=2}^{k} p_i\right)^2}{k+1} \tag{5.4}$$

$$= (1-\beta) \frac{\left(1 - \sum_{i=2}^{k} p_i\right)^2}{k+1}.$$

Hence we need only show that

$$\left(1 - \sum_{i=2}^{k} q_i\right) - \frac{\left(1 - \sum_{i=2}^{k} p_i\right)^2}{k+1} \geqq 0, \tag{5.5}$$

provided that

$$0 \leqq p_i \leqq 1 \quad \text{and} \quad 0 \leqq \sum_{i=2}^{k} p_i \leqq 1.$$

In fact, we have

$$(5.6)\qquad \left(1 - \sum_{i=2}^{k} q_i\right) - \frac{\left(1 - \sum_{i=2}^{k} p_i\right)^2}{k+1}$$

$$= \frac{1}{k+1}\left\{k\left(1 - \sum_{i=2}^{k} \frac{p_i^2}{i}\right) + \left(\sum_{i=2}^{k} p_i - \sum_{i=2}^{k} \frac{p_i^2}{i}\right) + \sum_{i=2}^{k} p_i\left(1 - \sum_{i=2}^{k} p_i\right)\right\}$$

$$\geqq 0$$

which evidently proves the assertion of Theorem 7.

6. THE FUNCTIONAL $\mathcal{J}(f)$

We introduce the functional $\mathcal{J}(f)$ defined by

$$(6.1)\qquad \mathcal{J}(f) = \int_0^z \frac{f(t)}{t}\, dt,$$

which maps the class of starlike functions onto the class of convex functions. Our result involving this functional is contained in

THEOREM 8. <u>Let the function</u> $f(z)$ <u>defined by</u> (1.6) <u>be in the class</u> $\mathcal{T}(\alpha,p_k)$. <u>Then</u> $\mathcal{J}(f)$ <u>belongs to the class</u> $\mathcal{T}(\alpha,q_k)$, <u>where</u> $q_k = p_k/k$.

PROOF. From (1.6) and (6.1), we readily obtain

$$(6.2)\qquad \mathcal{J}(f) = z - \sum_{i=2}^{k} \frac{(1-\alpha)q_i}{i} z^i - \sum_{n=k+1}^{\infty} \frac{a_n}{n} z^n.$$

Since

$$(6.3) \qquad \sum_{n=k+1}^{\infty} a_n \leqq \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{k+1},$$

we find that

$$(6.4) \qquad \sum_{n=k+1}^{\infty} n\left(\frac{a_n}{n}\right) \leqq \frac{(1-\alpha)\left(1 - \sum_{i=2}^{k} p_i\right)}{k+1}$$

$$\leqq (1-\alpha)\left(1 - \sum_{i=2}^{k} \frac{p_i}{i}\right),$$

which, in view of Theorem 1, implies that

$$(6.5) \qquad \mathcal{J}(f) \in \mathcal{T}(\alpha, q_k), \quad q_k = p_k/k,$$

and the proof of Theorem 8 is thus completed.

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