A UNIFIED PRESENTATION OF CERTAIN CLASSES OF STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

The authors introduce and study a unified class $\mathcal{A}(\alpha,\beta,n)$ of starlike and convex functions of order α in the open unit disk $\mathcal{U}$. They prove a number of theorems involving, for example, sharp distortion inequalities for, and modified Hadamard product (or convolution) of, functions belonging to the class $\mathcal{A}(\alpha,\beta,n)$. It is also shown how these theorems would apply to yield various results given in the literature.

1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

Let $\mathcal{A}(n)$ denote the class of functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \qquad (a_k \geqq 0;\ n \in \mathbb{N} = \{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the <u>open</u> unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}(n)$ is said to be <u>starlike of order</u> α if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{S}_\alpha(n)$ the subclass of $\mathcal{A}(n)$ consisting of functions which are starlike of order α in $\mathcal{U}$.

A function $f(z)$ belonging to the class $\mathcal{A}(n)$ is said to be <u>convex of order</u> α if and only if

$$\text{(1.3)} \qquad \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{C}_\alpha(n)$ the subclass of $\mathcal{A}(n)$ consisting of all convex functions of order α in $\mathcal{U}$. We also note that

$$\text{(1.4)} \qquad \mathcal{T}_\alpha(n) \subseteq \mathcal{T}_\beta(n) \quad (0 \leqq \beta \leqq \alpha < 1),$$

$$\text{(1.5)} \qquad \mathcal{C}_\alpha(n) \subseteq \mathcal{C}_\beta(n) \quad (0 \leqq \beta \leqq \alpha < 1),$$

and

$$\text{(1.6)} \qquad f(z) \in \mathcal{C}_\alpha(n) \Leftrightarrow zf'(z) \in \mathcal{T}_\alpha(n) \quad (0 \leqq \alpha < 1).$$

The classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ were introduced by Chatterjea [1], and were studied subsequently by Srivastava, Owa, and Chatterjea [5], and by Srivastava, Saigo, and Owa [6]. In particular, $\mathcal{T}_\alpha(1)$ and $\mathcal{C}_\alpha(1)$ are the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively, introduced by Silverman [4]. (For a systematic presentation of various other interesting subclasses of analytic functions, see (for example) Duren [2].)

For the general classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$, Chatterjea [1] gave the following lemmas which (as already observed by Srivastava, Owa, and Chatterjea [5, p. 117], and also by Srivastava, Saigo, and Owa [6, p. 415, Remark 1]) are immediate consequences of certain results of Silverman [4].

LEMMA 1. A function $f(z)$ defined by (1.1) is in the class $\mathcal{T}_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left[\frac{k-\alpha}{1-\alpha}\right] a_k \leqq 1 \qquad (n \in \mathbb{N}). \tag{1.7}$$

The result (1.7) is sharp.

LEMMA 2. A function $f(z)$ defined by (1.1) is in the class $\mathcal{C}_\alpha(n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left[\frac{k(k-\alpha)}{1-\alpha}\right] a_k \leqq 1 \qquad (n \in \mathbb{N}). \tag{1.8}$$

The result (1.8) is sharp.

In view of Lemma 1 and Lemma 2, it would seem to be natural to introduce and study an interesting unification of the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$. Thus we say that a function $f(z)$ defined by (1.1) belongs to the class $\mathcal{A}(\alpha,\beta,n)$ if and only if

$$\sum_{k=n+1}^{\infty} \left[\frac{(k-\alpha)(1-\beta+\beta k)}{1-\alpha}\right] a_k \leqq 1 \qquad (n \in \mathbb{N}) \tag{1.9}$$

for some α $(0 \leqq \alpha < 1)$ and some β $(0 \leqq \beta \leqq 1)$. Then, since

$$1-\beta+\beta k \geqq 1 \qquad (0 \leqq \beta \leqq 1;\ k = n+1, n+2, n+3, \cdots;\ n \in \mathbb{N}), \tag{1.10}$$

$$\mathcal{A}(\alpha,0,n) = \mathcal{T}_\alpha(n) \quad \text{and} \quad \mathcal{A}(\alpha,1,n) = \mathcal{C}_\alpha(n) \qquad (n \in \mathbb{N}). \tag{1.11}$$

The object of this paper is to present a unified study of the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ by proving various interesting properties and characteristics of the general class $\mathcal{A}(\alpha,\beta,n)$.

2. DISTORTION INEQUALITIES

Our main distortion inequalities for functions belonging to the class $\mathcal{A}(\alpha,\beta,n)$ are contained in

THEOREM 1. If a function f(z) defined by (1.1) is in the class $\mathcal{A}(\alpha,\beta,n)$, then

$$(2.1) \qquad |z| - \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)}|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)}|z|^{n+1}$$

and

$$(2.2) \qquad 1 - \frac{(n+1)(1-\alpha)}{(n+1-\alpha)(1+\beta n)}|z|^{n} \leqq |f'(z)| \leqq 1 + \frac{(n+1)(1-\alpha)}{(n+1-\alpha)(1+\beta n)}|z|^{n}$$

for $z \in \mathcal{U}$. The results (2.1) and (2.2) are sharp.

PROOF. Observe from (1.9) that

$$(2.3) \qquad \sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)} \qquad (n \in \mathbb{N}).$$

Making use of the inequality (2.3), we readily have

$$(2.4)\qquad |f(z)| \geqq |z| - |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \geqq |z| - \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)} |z|^{n+1}$$

and

$$(2.5)\qquad |f(z)| \leqq |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k \leqq |z| + \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)} |z|^{n+1},$$

which evidently prove the assertion (2.1) of Theorem 1.

Further, since

$$(2.6)\qquad \sum_{k=n+1}^{\infty} k\, a_k \leqq (n+1) \sum_{k=n+1}^{\infty} a_k \leqq \frac{(n+1)(1-\alpha)}{(n+1-\alpha)(1+\beta n)} \qquad (n \in \mathbb{N}),$$

we have

$$(2.7)\qquad |f'(z) \geqq 1 - |z|^{n} \sum_{k=n+1}^{\infty} k\, a_k$$

$$\geqq 1 - \frac{(n+1)(1-\alpha)}{(n+1-\alpha)(1+\beta n)} |z|^{n}$$

and

$$(2.8) \qquad |f'(z)| \leqq 1 + |z|^n \sum_{k=n+1}^{\infty} k\, a_k$$

$$\leqq 1 + \frac{(n+1)(1-\alpha)}{(n+1-\alpha)(1+\beta n)} |z|^n ,$$

which prove the assertion (2.2) of Theorem 1.

Finally, taking the function

$$(2.9) \qquad f(z) = z - \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)} z^{n+1} ,$$

we see that the results (2.1) and (2.2) are sharp.

Remark 1. Setting $\beta = 0$ and $\beta = 1$ in Theorem 1, we obtain the corresponding results for the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$, given earlier by Srivastava, Owa, and Chatterjea [5, p. 117, Theorem 1; p. 119, Theorem 2].

3. MODIFIED HADAMARD PRODUCTS OR CONVOLUTIONS

For functions $f_j(z)$ $(j = 1,2)$ belonging to the class $\mathcal{A}(n)$ and given by

$$(3.1) \qquad f_j(z) = z - \sum_{k=n+1}^{\infty} a_{j,k} z^k \qquad (a_{j,k} \geqq 0;\ j = 1,2;\ n \in \mathbb{N}),$$

we denote by $f_1 \star f_2(z)$ the modified Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ defined by

(3.2) $$f_1 \star f_2(z) = z - \sum_{k=n+1}^{\infty} a_{1,k}\, a_{2,k}\, z^k \qquad (n \in \mathbb{N}),$$

and prove

THEOREM 2. Let the functions $f_j(z)$ $(j = 1,2)$ be in the class $\mathcal{A}(\alpha,\beta,n)$. Then $f_1 \star f_2(z)$ belongs to the class $\mathcal{A}(\gamma,\beta,n)$, where

(3.3) $$\gamma = \frac{(n+1-\alpha)^2(1+\beta n) - (n+1)(1-\alpha)^2}{(n+1-\alpha)^2(1+\beta n) - (1-\alpha)^2}.$$

PROOF. Employing the technique used earlier by Schild and Silverman [2], we need to find the largest γ such that

(3.4) $$\sum_{k=n+1}^{\infty} \left[\frac{(k-\gamma)(1-\beta+\beta k)}{1-\gamma}\right] a_{1,k}\, a_{2,k} \leqq 1 \qquad (n \in \mathbb{N}).$$

Since the functions $f_j(z)$ $(j = 1,2)$ belong to $\mathcal{A}(\alpha,\beta,n)$, we have

(3.5) $$\sum_{k=n+1}^{\infty} \left[\frac{(k-\alpha)(1-\beta+\beta k)}{1-\alpha}\right] a_{j,k} \leqq 1 \qquad (j = 1,2;\ n \in \mathbb{N}).$$

Therefore, by the Cauchy–Schwarz inequality, we find that

(3.6) $$\sum_{k=n+1}^{\infty} \left[\frac{(k-\alpha)(1-\beta+\beta k)}{1-\alpha}\right] \sqrt{a_{1,k}\, a_{2,k}} \leqq 1.$$

Thus it is sufficient to show that

$$\text{(3.7)} \qquad \left[\frac{(k-\gamma)(1-\beta+\beta k)}{1-\gamma}\right] a_{1,k}\, a_{2,k} \leqq \left[\frac{(k-\alpha)(1-\beta+\beta k)}{1-\alpha}\right] \sqrt{a_{1,k}\, a_{2,k}} \quad (k \geqq n+1),$$

that is, that

$$\text{(3.8)} \qquad \sqrt{a_{1,k}\, a_{2,k}} \leqq \frac{(k-\alpha)(1-\gamma)}{(1-\alpha)(k-\gamma)} \qquad (k \geqq n+1).$$

Since (3.6) implies that

$$\text{(3.9)} \qquad \sqrt{a_{1,k}\, a_{2,k}} \leqq \frac{1-\alpha}{(k-\alpha)(1-\beta+\beta k)} \qquad (k \geqq n+1),$$

we need only to prove that

$$\text{(3.10)} \qquad \frac{1-\alpha}{(k-\alpha)(1-\beta+\beta k)} \leqq \frac{(k-\alpha)(1-\gamma)}{(1-\alpha)(k-\gamma)} \qquad (k \geqq n+1),$$

that is, that

$$\text{(3.11)} \qquad \gamma \leqq \frac{(k-\alpha)^2(1-\beta+\beta k) - k(1-\alpha)^2}{(k-\alpha)^2(1-\beta+\beta k) - (1-\alpha)^2} \qquad (k \geqq n+1).$$

If we set

$$\text{(3.12)} \qquad \Phi(k) = \frac{(k-\alpha)^2(1-\beta+\beta k) - k(1-\alpha)^2}{(k-\alpha)^2(1-\beta+\beta k) - (1-\alpha)^2} \qquad (k \geqq n+1),$$

we easily see that $\Phi(k)$ is an increasing function of k. Thus, letting $k = n+1$ in (3.11), we obtain

(3.13) $$\gamma \leqq \Phi(n+1) = \frac{(n+1-\alpha)^2(1+\beta n) - (n+1)(1-\alpha)^2}{(n+1-\alpha)^2(1+\beta n) - (1-\alpha)^2},$$

which evidently proves the assertion of Theorem 2.

Finally, by taking the functions given by

(3.14) $$f_j(z) = z - \frac{1-\alpha}{(n+1-\alpha)(1+\beta n)} z^{n+1} \qquad (j = 1,2;\ n \in \mathbb{N}),$$

we can see that the result of Theorem 2 is sharp.

REMARK 2. Putting $\beta = 0$ and $\beta = 1$ in Theorem 2, we obtain the corresponding results for the classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$, given earlier by Srivastava, Owa, and Chatterjea [5, p. 120, Theorem 3; p. 122, Theorem 4].

Finally, we prove

THEOREM 3. Let the functions $f_j(z)$ $(j = 1,2)$ defined by (3.1) be in the class $\mathcal{A}(\alpha,\beta,n)$. Then the function $h(z)$ defined by

(3.15) $$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2) z^k$$

belongs to the class $\mathcal{A}(\gamma,\beta,n)$, where

(3.16) $$\gamma = \frac{(n+1-\alpha)^2(1+\beta n) - 2(n+1)(1-\alpha)^2}{(n+1-\alpha)^2(1+\beta n) - 2(1-\alpha)^2}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1,2)$ given by (3.14).

Proof. Since

$$\sum_{k=n+1}^{\infty} \frac{(k-\alpha)^2(1-\beta+\beta k)^2}{(1-\alpha)^2} a_{j,k}^2 \leqq \left[\sum_{k=n+1}^{\infty} \frac{(k-\alpha)(1-\beta+\beta k)}{1-\alpha} a_{j,k}\right]^2 \leqq 1, \tag{3.17}$$

we have

$$\sum_{k=n+1}^{\infty} \frac{(k-\alpha)^2(1-\beta+\beta k)^2}{(1-\alpha)^2} (a_{1,k}^2+a_{2,k}^2) \leqq 2. \tag{3.18}$$

Thus it is sufficient to find the largest γ such that

$$\frac{(k-\gamma)(1-\beta+\beta k)}{1-\gamma} \leqq \frac{(k-\alpha)^2(1-\beta+\beta k)^2}{2(1-\alpha)^2}, \tag{3.19}$$

that is, that

$$\gamma \leqq \frac{(k-\alpha)^2(1-\beta+\beta k) - 2k(1-\alpha)^2}{(k-\alpha)^2(1-\beta+\beta k) - 2(1-\alpha)^2} \qquad (k \geqq n+1). \tag{3.20}$$

Note that the function

$$\Psi(k) = \frac{(k-\alpha)^2(1-\beta+\beta k) - 2k(1-\alpha)^2}{(k-\alpha)^2(1-\beta+\beta k) - 2(1-\alpha)^2} \qquad (k \geqq n+1) \tag{3.21}$$

is an increasing function of k. This implies that

(3.22) $$\gamma \leqq \Psi(n+1) = \frac{(n+1-\alpha)^2(1+\beta n) - 2(n+1)(1-\alpha)^2}{(n+1-\alpha)^2(1+\beta n) - 2(1-\alpha)^2},$$

which completes the proof of Theorem 3.

Remark 3. Taking $\beta = 0$ on Theorem 3, we obtain the corresponding result for the class $\mathscr{C}_\alpha(n)$, given earlier by Srivastava, Owa, and Chatterjea [5, p. 122, Theorem 5].

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