Arnold Diffusion in the Elliptic Restricted Planar Three-Body Problem

by

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ABSTRACT

In 1964, V. Arnold conjectured that a chaotic phenomenon, now known as Arnold Diffusion, exists in the three-body problem. In 1993, Z. Xia gave a partial confirmation of the conjecture, showing that Arnold Diffusion exists in the elliptic restricted three-body problem. Xia later generalized his proof to the planar three-body problem. In this thesis, we work towards an understanding of Xia's proof of the existence of Arnold Diffusion in the elliptic restricted three-body problem. The equations of motion of the restricted planar three-body problem are formulated in position-momentum coordinates so that the circular problem is a perturbation of the unperturbed problem, and the elliptic problem is a perturbation of the circular problem. These equations of motion are transformed into a form more suited to an analysis of its parabolic solutions. The transformed unperturbed problem is explicitly solved for its parabolic solutions. Under a small enough perturbation from the transformed unperturbed problem to the transformed circular problem, the parabolic solutions of the transformed unperturbed problem are used to give sufficient conditions under which a twist map exists in the discretized transformed circular problem. Under a small enough perturbation from the transformed circular problem to the transformed elliptic problem, KAM Theory is applied to the twist map, near which is then shown the existence of Arnold diffusion.

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Introduction

The three-body problem from celestial mechanics has been an active area of research since its beginning. Its difficulty has attracted the attention of some of the best mathematicians and astronomers. In their efforts to understand the dynamics of the three-body problem, they have dealt with simplified versions of the problem. One of those simplified versions is the restricted planar three-body problem. It was Poincaré who first noticed that complicated dynamics exist in the restricted three-body problem (see Poincaré [1899]).

In 1964, V. Arnold introduced a new concept of dynamical behavior to Dynamical Systems Theory. Working with a periodically forced Hamiltonian system with two degrees of freedom, Arnold [1964] showed the existence of a chaotic phenomenon which he called *diffusion*. He held the belief that such a diffusion exists in the three-body problem. Xia [1993] gave a partial confirmation of Arnold's belief, showing that that diffusive phenomenon, now known as *Arnold Diffusion*, exists in the elliptic restricted planar three-body problem. Xia's proof uses "... advanced mathematical techniques ..." which "... are hard to understand in detail (at the present time) even for mathematicians trained in the field." (Bakker and Diacu [1993]) To overcome this deficiency is the purpose of this thesis.

We approach this task by dividing Xia's proof into seven sections, each one containing an important stepping stone to the final result. As much as has been permitted by the restraint of time, we have filled in many of the gaps, discovered and corrected some minor errors, and have written in great detail many of calculations not found in Xia's proof.

In Section 1 and the appendix, we derive the equations of motion, in position-momentum coordinates, for the restricted planar three-body problem under the condition that the two primaries move in elliptic orbits. These equations depend on two parameters, in which the unperturbed restricted planar three-body problem (which is really a two body problem), the circular restricted planar three-body problem, and the elliptic restricted planar three-body problem correspond to certain values of the two parameters. For the sake of brevity, we refer to the these three problems as the unperturbed, the circular, and the elliptic problem. By the derivation, the circular problem is a perturbation of the unperturbed problem, and the elliptic problem is a perturbation of the circular problem. The method of analysis is to proceed from the unperturbed problem to the cicular problem to the elliptic problem.

In Section 2, we make use of a coordinate transformation to put the equations of motion into a form more suited to the analysis of its solutions which are parabolic. The transformation of the three problems are thus referred to as the transformed unperturbed, transformed circular, and the transformed elliptic problems. It is the parabolic solutions of the transformed unperturbed problem that play a vital role in Xia's proof.

In Section 3, the transformed unperturbed problem is explicitly solved for its parabolic solutions. It is the explicit form of these solutions that is needed in some of the calculations done in the following sections.

In Section 4, we show that a certain functional is a first integral of the transformed circular problem. This first integral is the well-known Jacobi integral but is in a different form because the equations of motion have been transformed. By the Jacobi integral and a new time variable, the transformed circular problem is reduced to a two-dimensional system of nonautonomous equations. This is the starting point for an analysis of the qualitative aspects of the transformed circular problem.

In Section 5, we further reduce the reduced transformed circular problem to a Poincaré map. Working with this map the existence of transverse symmetric homoclinic orbits in the flow of the reduced transformed circular problem is shown. These orbits are homoclinic to an artificial periodic orbit at infinity which does not exist in the physical plane. We will draw upon results of McGehee [1973] and Xia [1992] to find these transverse symmetric homoclinic orbits, which, in some sense, are closely approximated by the parabolic solutions of the transformed unperturbed problem. Some of the consequences of the existence of these transverse symmetric homoclinic orbits are then exploited through the use of symbolic dynamics and a result of Moser [1973] to yield the existence of periodic solutions, of arbitrarily large period, of the reduced transformed circular problem.

In Section 6, the equations of motion of the reduced transformed circular problem are put into a special form for which the perturbation from the circular to the elliptic problem serves our purposes. Working with the Poincaré map associated with the flow of the transformed circular problem in this special form, the results of the previous section are used to show the existence of quasiperiodic solutions in the transformed circular problem. At the same time the existence of a twist map is shown. It is this twist map that plays the vital role in the perturbation from the transformed circular problem to the transformed elliptic problem.

In Section 7, KAM Theory (see Guckenheimer and Holmes [1983]) is applied to the twist map of the transformed circular problem. It is here that, for a small enough perturbation, the essential ingredients that compose the mechanism of Arnold Diffusion are shown to exist in the transformed elliptic problem.

We finish with a conclusion giving some of the consequences of the existence of Arnold Diffusion in the elliptic problem, and indicate some possible directions for further studies.

1. Formulation of the Equations

We begin by formulating the equations of motion of the restricted planar three-body problem as a two-parameter family of ordinary differential equations.

Let P_1, P_2 and P_3 be three point masses in the plane. Let P_1 have mass $1-\mu$, and P_2 have mass μ, where $0 \leq \mu < 1$. Let P_3 be a test particle. So, P_3 does not affect the motion of the *primaries* P_1 and P_2. We shall call P_3 the *zero-mass.* Suppose the primaries move according to Newton's law of gravity, and that the center of mass is fixed at the origin. Under these conditions, the primaries move in elliptic (which includes circular), parabolic or hyperbolic orbits. Suppose the primaries move in elliptic orbits with eccentricity $0 \leq e < 1$. Note that if $e = 0$ then the primaries move in circular orbits. Let $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2$ be the position vector of the zero mass P_3, and let $\mathbf{p} = (p_1, p_2) \in \mathbf{R}^2$ be its momentum vector. The distance between the primaries is the usual Euclidean norm of the vector $(r_1, r_2) \in \mathbf{R}^2$, where $r_i : \mathbf{R} \times [0,1) \to \mathbf{R}$, for $i = 1,2$ are given by (see appendix)

$$r_1 = r_1(T(t;e);e) = (1 - e\cos T)\cos T + O(e^2), \tag{1}$$

$$r_2 = r_2(T(t;e);e) = (1 - e\cos T)\sin T + O(e^2), \tag{2}$$

and where $T : \mathbf{R} \times [0,1) \to \mathbf{R}$ is given by (see appendix)

$$T(t;e) = t + e\sin t + O(e^2). \tag{3}$$

Since the mass of P_3 is zero, the *potential function* $U : \mathbf{R}^3 \times [0,1)^2 \to \mathbf{R}$ is given by

$$\begin{aligned} U(q_1,q_2,t;\mu,e) = &\frac{1-\mu}{\sqrt{\left(q_1-\mu r_1\right)^2+\left(q_2-\mu r_2\right)^2}} \\ &+\frac{\mu}{\sqrt{\left(q_1+(1-\mu)r_1\right)^2+\left(q_2+(1-\mu)r_2\right)^2}}, \end{aligned} \tag{4}$$

where μ and e are the parameters, and r_1 and r_2 are given by (1) and (2) respectively. The potential function is ill-defined on the set $\mathcal{C}$ of points in the domain of U where one of the denominators in (4) equals zero. Points in $\mathcal{C}$ correspond to *binary collisions* of the zero mass with either of the primaries. However, $\mathcal{C}$ is of measure zero, and so, properly speaking, the potential function is defined by (4) modulo this set $\mathcal{C}$ of measure zero.

With the prime denoting differentiation with respect to the independent time variable t, the equations of motion for P_3 are (suppressing the variables of U)

$$\begin{cases} q_1' = p_1 \\ q_2' = p_2 \\ p_1' = \partial U/\partial q_1 \\ p_2' = \partial U/\partial q_2. \end{cases} \tag{5}$$

When $\mu = 0$ the equations (5) describe the planar two-body problem with P_1 at the origin having mass 1 and the zero mass P_3 moving under its attraction. Here the potential function (4) does not depend on t and so the phase space of this two-body problem is $\mathbf{R}^4$. When $\mu \in (0,1)$ and $e = 0$ the equations (5) describe a perturbation of the two-body problem called the circular restricted planar three-body problem with the primaries moving in circular orbits and P_3 moving under the mutual attraction of the primaries. Here, the potential function (4) depends on t and so its generalized phase space is $\mathbf{R}^5$. When $\mu \in (0,1)$ and $e \in (0,1)$, the equations (5) describe a perturbation of the circular problem called the elliptic restricted planar three-body problem. Here the potential function (4) also depends on t and so its generalized phase space is $\mathbf{R}^5$. For convenience in writing, we will refer to these as the *unperturbed*, *circular*, and *elliptic* problems respectively.

For each of these three problems, there is the *collision set* $\tilde{C}$ such that for any solution of (5) with an initial condition in $\tilde{C}$ there is a time $\tilde{t} \in \mathbf{R}$ such that the solution will experience a binary collision at time $\tilde{t}$. We refer to such solutions as the *collision solutions.* All the points given by a collision solution (on whatever interval of time it is defined for) are also in the collision set $\tilde{C}$. However, $\tilde{C}$ is of measure zero, and so we will, in general, exclude collision solutions from our study. Under this assumption, the solutions we will consider exist for all $t \in \mathbf{R}$, are unique, and are real analytic in t. These solutions we refer to as *non-collision solutions.* When we do include solutions which experience a binary collision, we will analytically regularize such solutions.

2. A Transformation of the Equations of Motion

In this section we transform equations of motion (5) into a form more suitable for analysis of parabolic solutions by defining a set of new dependent variables. These new dependent variables will be given by a real analytic diffeomorphism Λ which is defined on "most" of the space of the dependent variables (q_1, q_2, p_1, p_2). We will use the first order approximation of the potential function (4) in terms of μ in computing the transformation of the equations of motion into the new variables.

Proposition 1: *Let $(\mu, e) \in [0,1)^2$. The first order approximation of the potential function* (4) *in terms of μ is*

$$\begin{aligned} U(q_1, q_2, t; \mu, e) = &\frac{1}{\sqrt{q_1^2 + q_2^2}} \\ &+ \mu\left(\frac{-1}{\sqrt{q_1^2+q_2^2}} + \frac{q_1 r_1 + q_2 r_2}{(q_1^2+q_2^2)^{3/2}} + \frac{1}{\sqrt{(q_1+r_1)^2 + (q_2+r_2)^2}}\right) \\ &+ O(\mu^2), \end{aligned} \tag{6}$$

where r_1 and r_2 are given by (1) *and* (2) *respectively.*

Proof: With $\mu = 0$, the potential function (4) is

$$U(q_1, q_2, t; 0, e) = \frac{1}{\sqrt{q_1^2 + q_2^2}}. \tag{7}$$

Write $U = U_1 + U_2$, where U_1 and U_2 are the first and second terms on the right-hand side of (4). Now

$$\begin{aligned}\frac{\partial U_1}{\partial \mu}(q_1, q_2, t; \mu, e) = &\frac{-1}{\sqrt{(q_1 - \mu r_1)^2 + (q_2 - \mu r_2)^2}} \\ &+ (1-\mu)\left(\frac{(q_1 - \mu r_1) r_1 + (q_2 - \mu r_2) r_2)}{((q_1 - \mu r_1)^2 + (q_2 - \mu r_2)^2)^{3/2}}\right),\end{aligned} \tag{8}$$

and

$$\begin{aligned}\frac{\partial U_2}{\partial \mu}(q_1, q_2, t; \mu, e) = &\frac{1}{\sqrt{(q_1 + (1-\mu) r_1)^2 + (q_2 + (1-\mu) r_2)^2}} \\ &+ \mu\left(\frac{(q_1 + (1-\mu) r_1) r_1 + (q_2 + (1-\mu) r_2) r_2}{((q_1 + (1-\mu) r_1)^2 + (q_2 + (1-\mu) r_2)^2)^{3/2}}\right).\end{aligned} \tag{9}$$

Setting $\mu = 0$ in (8) and (9) we have

$$\left.\frac{\partial U_1}{\partial \mu}\right|_{\mu=0} = \frac{-1}{\sqrt{q_1^2 + q_2^2}} + \frac{q_1 r_1 + q_2 r_2}{(q_1^2 + q_2^2)^{3/2}}, \tag{10}$$

and

$$\left.\frac{\partial U_2}{\partial \mu}\right|_{\mu=0} = \frac{1}{\sqrt{(q_1 + r_1)^2 + (q_2 + r_2)^2}}. \tag{11}$$

Combining (10) and (11) with (7) completes the proof.

Since the equations of motion (5) depend on the partial derivatives of the potential function (4), we will need the following.

Proposition 2: *Let $(\mu, e) \in [0,1)^2$. To the first order approximation in μ,*

$$\begin{aligned}\frac{\partial U}{\partial q_1}(q_1, q_2, t; \mu, e) = &\frac{-q_1}{(q_1^2 + q_2^2)^{3/2}} \\ &+ \mu\Bigg(\frac{q_1}{(q_1^2 + q_2^2)^{3/2}} + \frac{(q_1^2 + q_2^2)^{3/2} r_1 - 3(q_1 r_1 + q_2 r_2)(q_1^2 + q_2^2) q_1}{(q_1^2 + q_2^2)^3} \\ &- \frac{q_1 + r_1}{((q_1 + r_1)^2 + (q_2 + r_2)^2)^{3/2}}\Bigg) + O(\mu^2),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\frac{\partial U}{\partial q_2}(q_1, q_2, t; \mu, e) = &\frac{-q_2}{(q_1^2 + q_2^2)^{3/2}} \\ &+ \mu\Bigg(\frac{q_2}{(q_1^2 + q_2^2)^{3/2}} + \frac{(q_1^2 + q_2^2)^{3/2} r_2 - 3(q_1 r_1 + q_2 r_2)(q_1^2 + q_2^2) q_2}{(q_1^2 + q_2^2)^3} \\ &- \frac{q_2 + r_2}{((q_1 + r_1)^2 + (q_2 + r_2)^2)^{3/2}}\Bigg) + O(\mu^2),\end{aligned} \tag{13}$$

where r_1 and r_2 are given by (1) and (2) respectively.

Proof: By differentiating the expression (6) with respect to q_1 one obtains (12), and differentiating the expression (6) with respect to q_2 one obtains (13). This completes the proof.

Now we introduce the real analytic diffeomorphism Λ. Let $A = (0, \infty) \times S^1 \times \mathbf{R}^2$ and $B = (\mathbf{R}^2 \setminus (0,0)) \times \mathbf{R}^2$. Define $\Lambda : A \to B$ by $\Lambda(x, \theta, y, \rho) = (q_1, q_2, p_1, p_2)$, where

$$\begin{cases} q_1 = q_1(x,\theta) = x^{-2}\cos\theta \\ q_2 = q_2(x,\theta) = x^{-2}\sin\theta \\ p_1 = p_1(x,\theta,y,\rho) = y\cos\theta - x^2\rho\sin\theta \\ p_2 = p_2(x,\theta,y,\rho) = y\sin\theta + x^2\rho\cos\theta. \end{cases} \tag{14}$$

The inverse $\Lambda^{-1} : B \to A$ is given by $\Lambda^{-1}(q_1, q_2, p_1, p_2) = (x, \theta, y, \rho)$, where

$$\begin{cases} x = x(q_1,q_2) = (q_1^2 + q_2^2)^{-1/4} \\ \theta = \theta(q_1,q_2) = \begin{cases} \arctan(q_2/q_1) & \text{if } q_1 > 0, q_2 \geq 0; \\ \pi + \arctan(q_2/q_1) & \text{if } q_1 < 0; \\ 2\pi + \arctan(q_2/q_1) & \text{if } q_1 > 0, q_2 < 0; \\ \pi/2 & \text{if } q_1 = 0, q_2 > 0; \\ 3\pi/2 & \text{if } q_1 = 0, q_2 < 0, \end{cases} \\ y = y(q_1,q_2,p_1,p_2) = \dfrac{q_1p_1 + q_2p_2}{\sqrt{q_1^2 + q_2^2}} \\ \rho = \rho(q_1,q_2,p_1,p_2) = q_1p_2 - q_2p_1. \end{cases} \tag{15}$$

Before we use Λ^{-1} to transform the equations (5) into (x, θ, y, ρ)-variables, we need to compute the transformations of (12) and (13) under Λ. Let $W_1 = \partial U/\partial q_1$ and $W_2 = \partial U/\partial q_2$. Let $\tilde{W}_1$ and $\tilde{W}_2$ be the transformations of W_1 and W_2 respectively under Λ. Note that W_1 and W_2 are functions of the variables $(q_1, q_2, t; \mu, e)$, and so their respective transformations under Λ are functions of $(x, \theta, t; \mu, e)$. Define the function $R_1 : S^1 \times \mathbf{R} \times [0, 1) \to \mathbf{R}$ by

$$R_1 = R_1(\theta, t; e) = r_1\cos\theta + r_2\sin\theta, \tag{16}$$

where r_1 and r_2 are given by (1) and (2) respectively.

Proposition 3: *Let $(\mu, e) \in [0, 1)^2$. The transformations of W_1 and W_2 under Λ*

are, to the first order approximation in μ,

$$\tilde{W}_1(x,\theta,t;\mu,e) = -\,x^4\cos\theta + \mu\Bigg(x^4\cos\theta + x^6 r_1 - 3x^6 R_1\cos\theta - \frac{x^4\cos\theta + x^6 r_1}{(1+2x^2R_1 + x^4(r_1^2+r_2^2))^{3/2}}\Bigg) + O(\mu^2), \tag{17}$$

and

$$\tilde{W}_2(x,\theta,t;\mu,e) = -\,x^4\sin\theta + \mu\Bigg(x^4\sin\theta + x^6 r_2 - 3x^6 R_1\sin\theta - \frac{x^4\sin\theta + x^6 r_2}{(1+2x^2R_1 + x^4(r_1^2+r_2^2))^{3/2}}\Bigg) + O(\mu^2), \tag{18}$$

where R_1 *is given by* (16).

Proof: From the definition of x given in (15) we have

$$(q_1^2+q_2^2)^{3/2} = ((q_1^2+q_2^2)^{-1/4})^{-6} = x^{-6}. \tag{19}$$

From (14) and (16), we have that

$$(q_1+r_1)^2 + (q_2+r_2)^2 = x^{-4}(1+2x^2R_1 + x^4(r_1^2+r_2^2)), \tag{20}$$

and

$$q_1r_1 + q_2r_2 = x^{-2}R_1. \tag{21}$$

Substituting the expressions for q_1 and q_2 from (14) into (12), and using (19), (20), and (21), we obtain (17). Substituting the expressions for q_1 and q_2 from (14) into (13), and using (19), (20), and (21), we obtain (18). This completes the proof.

Now we are ready to use Λ^{-1} to transform the equations of motion (5) into (x,θ,y,ρ)-variables. Define the function $R_2 : S^1\times\mathbf{R}\times[0,1)\to\mathbf{R}$ by

$$R_2 = R_2(\theta,t;e) = r_2\cos\theta - r_1\sin\theta, \tag{22}$$

where r_1 and r_2 are given by (1) and (2) respectively.

Theorem 4: *Let* $(\mu,e)\in[0,1)^2$ *be fixed. The transformation of the equations of motion* (5) *under* Λ^{-1} *is, to the first order approximation in* μ, *the two-parameter family of equations given by*

$$\begin{cases} x' = -\dfrac{1}{2}x^3 y \\ \theta' = x^4\rho \\ y' = -x^4 + x^6\rho^2 + \mu g_1(x,\theta,t;e) + O(\mu^2) \\ \rho' = \mu g_2(x,\theta,t;e) + O(\mu^2), \end{cases} \tag{23}$$

where

$$g_1(x,\theta,t;e) = x^4\left(1 - 2x^2R_1 - \frac{1+x^2R_1}{(1+2x^2R_1+x^4(r_1^2+r_1^2))^{3/2}}\right), \tag{24}$$

$$g_2(x,\theta,t;e) = x^4R_2\left(1 - \frac{1}{(1+2x^2R_1+x^4(r_1^2+r_2^2))^{3/2}}\right), \tag{25}$$

where R_1 *and* R_2 *are given by* (16) *and* (22) *respectively, and* r_1 *and* r_2 *are given by* (1) *and* (2) *respectively.*

Proof: Differentiating the expression $x = (q_1^2+q_2^2)^{-1/4}$ from (15) with respect to t, and noting that $q_1' = p_1$ and $q_2' = p_2$ from (5), we have that

$$\begin{aligned} x' &= -\frac{1}{4}(q_1^2+q_2^2)^{-5/4}(2q_1q_1' + 2q_2q_2') \\ &= -\frac{1}{2}(q_1^2+q_2^2)^{-5/4}(q_1p_1+q_2p_2) \\ &= -\frac{1}{2}(x^{-4}\cos^2\theta + x^{-4}\sin^2\theta)^{-5/4}(x^{-2}y) \\ &= -\frac{1}{2}x^3y. \end{aligned} \tag{26}$$

Differentiating the expression for θ in (15) with respect to t and noting the definition of ρ given in (15), we have

$$\begin{aligned} \theta' &= \frac{1}{1+(q_1^2/q_2^2)}\frac{q_2'q_1 - q_2q_1'}{q_1^2} \\ &= \frac{p_2q_1 - q_2p_1}{q_1^2+q_2^2} \\ &= x^4\rho. \end{aligned} \tag{27}$$

For convenience in the computations we will write the expression for y in (15) as the sum $y = y_1 + y_2$ where

$$y_1 = \frac{q_1p_1}{\sqrt{q_1^2+q_2^2}}, \tag{28}$$

$$y_2 = \frac{q_2p_2}{\sqrt{q_1^2+q_2^2}}. \tag{29}$$

Differentiating (28) with respect to t, and then using (5), the definition of y given in (15), and Proposition 3, we have that

$$\begin{aligned} y_1' &= \frac{(q_1'p_1+q_1p_1')(q_1^2+q_2^2)^{1/2} - q_1p_1(q_1^2+q_2^2)^{-1/2}(q_1q_1'+q_2q_2')}{q_1^2+q_2^2} \\ &= -x^4y\rho\cos\theta\sin\theta + x^6\rho^2\sin^2\theta - x^4\cos^2\theta + \mu x^4\cos^2\theta + \mu x^6 r_1\cos\theta \\ &\quad - 3\mu x^6R_1\cos^2\theta - \frac{\mu x^4\cos^2\theta + \mu x^6r_1\cos\theta}{(1+x^2R_1+x^4(r_1^2+r_2^2))^{3/2}} + O(\mu^2). \end{aligned} \tag{30}$$

Differentiating (29) with respect to t, using (5), the definition of y given in (15), and Proposition 3, we have that

$$\begin{aligned}
y_2' &= \frac{(q_2'p_2 + q_2p_2')(q_1^2+q_2^2)^{1/2} - q_2p_2(q_1^2+q_2^2)^{-1/2}(q_1q_1' + q_2q_2')}{q_1^2+q_2^2} \\
&= x^4 y\rho\cos\theta\sin\theta + x^6\rho^2\cos^2\theta - x^4\sin^2\theta + \mu x^4\sin^2\theta + \mu x^6 r_2\sin\theta \\
&\quad - 3\mu x^6 R_1\sin^2\theta - \frac{\mu x^4\sin^2\theta + \mu x^6 r_2\sin\theta}{(1+2x^2R_1 + x^4(r_1^2+r_2^2))^{3/2}} + O(\mu^2) \\
&\quad - x^2y^2\sin^2\theta - x^2y\rho\cos\theta\sin\theta. \qquad (31)
\end{aligned}$$

Adding (30) and (31) we have

$$\begin{aligned}
y' &= -x^4 + x^6\rho^2 + \mu x^4 + \mu x^6 R_1 \\
&\quad - 3\mu x^6 R_1 - \mu\frac{x^4 + x^6R_1}{(1+2x^2R_1 + x^4(r_1^2+r_2^2))^{3/2}} + O(\mu^2) \\
&= -x^4 + x^6\rho^2 + \mu\left(1 - 2x^2R_1 - \frac{1+2x^2R_1}{(1+2x^2R_1+x^4(r_1^2+r_2^2))^{3/2}}\right) + O(\mu^2) \\
&= -x^4 + x^6\rho^2 + \mu g_1(x,\theta,t;e) + O(\mu^2), \qquad (32)
\end{aligned}$$

where g_1 is given by (24). Differentiating the expression of ρ given in (15), using (5), Proposition 3, and (22), we have

$$\begin{aligned}
\rho' &= q_1'p_2 + q_1p_2' - (q_2'p_1 + q_2p_1') \\
&= \mu x^4 R_2\left(1 - \frac{1}{(1+2x^2R_1+x^4(r_1^2+r_2^2))^{3/2}}\right) + O(\mu^2) \\
&= \mu g_2(x,\theta,t;e) + O(\mu^2), \qquad (33)
\end{aligned}$$

where g_2 is given by (25). The equations in (23) are given by (26), (27), (32), and (33). This completes the proof.

We refer to (23) as the *transformed equations of motion* for the zero-mass P_3.

In the proof of Theorem 4, we computed $D(\Lambda^{-1}(q_1,q_2,p_1,p_2))$ for fixed parameters μ and e, using the first order approximations of the partials $\partial U/\partial q_1$ and $\partial U/\partial q_2$ in terms of μ, and obtained the vector field of (23). So, if $(\mathbf{q},\mathbf{p})$ is a non-collision solution of (5), then $\Lambda^{-1}(\mathbf{q},\mathbf{p})$ is a solution of (23) that is defined and real analytic for all $t\in\mathbf{R}$. If we compute $D(\Lambda(x,\theta,y,\rho))$ we obtain the vector field of (5) where the partials $\partial U/\partial q_1$ and $\partial U/\partial q_2$ are given by Proposition 2. So, if (x,θ,y,ρ) is a solution of (23) not in

$\Lambda^{-1}(\tilde{\mathcal{C}})$, then $\Lambda(x,\theta,y,\rho)$ is a solution of (5) that is defined and real analytic for all $t \in \mathbf{R}$. We say a solution not in $\Lambda^{-1}(\tilde{\mathcal{C}})$ is a non-collision solution of (23). Since Λ is a diffeomorphism we have a bijection between the the non-collision solutions of (5) and (23).

Definition 5: We say that the non-collision solutions $(\mathbf{q},\mathbf{p})$ and (x,θ,y,ρ) of (5) and (23) respectively are *corresponding solutions* if

$$(\mathbf{q}(t),\mathbf{p}(t)) = \Lambda(x(t),\theta(t),y(t),\rho(t))$$

for all $t \in \mathbf{R}$.

Since Λ is a diffeomorphism, then non-collision solutions $(\mathbf{q},\mathbf{p})$ and (x,θ,y,ρ) of (5) and (23) respectively are also corresponding solutions if $(x(t),\theta(t),y(t),\rho(t)) = \Lambda^{-1}(\mathbf{q}(t),\mathbf{p}(t))$ for all time.

To finish this section off we will compute the first order approximation of the potential in the (x,θ,y,ρ)-variables. Let $\tilde{U} : \mathbf{R}^+ \times S^1 \times \mathbf{R} \times [0,1)^2 \to \mathbf{R}$ be the transformation of the potential function (4) under Λ. Since U is a function of the variables $(q_1,q_2,t;\mu,e)$, then $\tilde{U}$ is a function of the variables $(x,\theta,t;\mu,e)$.

Proposition 6: *For fixed* $(\mu,e) \in [0,1)^2$, *the transformation of the potential function* (4) *under* Λ *is, to the first order approximation in* μ,

$$\tilde{U}(x,\theta,t;\mu,e) = x^2 + \mu x^2\left(-1 + x^2 R_1 + \frac{1}{\sqrt{1+2x^2R_1 + x^4(r_1^2+r_2^2)}}\right) + O(\mu^2), \quad (34)$$

where R_1 *is given by* (16), *and* r_1 *and* r_2 *are given by* (1) *and* (2) *respectively.*

Proof: Substituting the expressions for q_1 and q_2 from (14) into (6), and using (16) and (19), we obtain (34). This completes the proof.

3. Parabolic Solutions and the Unperturbed Problem

For $\mu = 0$ the transformed equations (23) describe the unperturbed problem in the variables (x,θ,y,ρ). We will reduce the transformed equations (23) of the unperturbed problem to a one-parameter family of autonomous planar equations, and with the aid of a first integral of these equations, we will explicitly solve the unperturbed problem for its *parabolic* solutions.

Proposition 7: *The transformed equations of motion* (23) *of the unperturbed problem are reducible to the autonomous planar system*

$$\begin{cases} x' = -\dfrac{1}{2}x^3 y \\ y' = -x^4 + x^6\rho^2, \end{cases} \tag{35}$$

where ρ is a constant. Furthermore, the variable θ is given by

$$\theta(t) = \rho \int_{t_0}^{t} x^4(\tau)d\tau + \theta_0 \pmod{2\pi}. \tag{36}$$

Proof: Setting $\mu = 0$ in (23) we have that the transformed equations of motion of the unperturbed problem are

$$\begin{cases} x' = -\dfrac{1}{2}x^3 y \\ \theta' = x^4\rho \\ y' = -x^4 + x^6\rho^2 \\ \rho' = 0. \end{cases} \tag{37}$$

Since $\rho' = 0$ on any interval, then ρ is a constant, and we can think of ρ as a parameter. Since the vector field of (37) does not depend on θ we can integrate $\theta' = x^4\rho$ from t_0 to t and obtain (36). Thus (37) is reducible to (35). This completes the proof.

We refer to the equations in (35) as the *reduced transformed unperturbed problem.*

Definition 8: For any fixed parameter values $(\mu, e) \in [0,1)^2$, we say a solution $(\mathbf{q}, \mathbf{p})$ of (5) is *parabolic* if $||\mathbf{q}(t)|| \to \infty$ and $||\mathbf{p}(t)|| \to 0$ as $t \to \pm\infty$.

From (14) and some simple computations we have that

$$||\mathbf{q}(t)|| = (x(t))^{-2} \quad \text{and} \quad ||\mathbf{p}(t)|| = \sqrt{(y(t))^2 + (x(t))^4(\rho(t))^2}.$$

For the unperturbed problem, ρ is a constant by Proposition 6. Then for this problem we have the following. If $(\mathbf{q}, \mathbf{p})$ is a non-collision solution of (5), then for the corresponding non-collision solution $(x, \theta, y, \rho) = \Lambda^{-1}(\mathbf{q}, \mathbf{p})$ of (37) we have $x(t) \to 0$ and $y(t) \to 0$ as $t \to \pm\infty$. Conversely, if (x, θ, y, ρ) is a non-collision solution of (37) and $x(t) \to 0$, $y(t) \to 0$ as $t \to \pm\infty$, then the corresponding non-collision solution $(\mathbf{q}, \mathbf{p}) = \Lambda(x, \theta, y, \rho)$ of (5) is parabolic. Therefore we call a non-collision solution of the unperturbed problem (37) *parabolic* if $x(t) \to 0,\ y(t) \to 0$ as $t \to \pm\infty$. Since the unperturbed problem reduces to (35) by Proposition 6, finding parabolic solutions of (5) is equivalent to finding solutions (x, y) of (35) such that $x(t) \to 0$ and $y(t) \to 0$ as $t \to \pm\infty$.

To solve (35) for its parabolic solutions we need the following. Define the functional $H : \mathbf{R}^3 \to \mathbf{R}$ by

$$H(x, y; \rho) = \frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - x^2. \tag{38}$$

Using (35), we have that (38) is a *first integral* of (35) since

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial y}\frac{dy}{dt} \\ &= (2x^3\rho^2 - 2x)(-\frac{1}{2}x^3 y) + y(-x^4 + x^6\rho^2) \\ &= -x^4 y\rho^2 + x^4 y - x^4 y + x^4 y\rho^2 \\ &= 0.\end{aligned}$$

We will now give the main result of this section.

Theorem 9: *For $\rho \neq 0$, the parabolic solutions of the reduced transformed unperturbed problem* (35) *are given by*

$$\begin{cases} x(t) = \xi(t; \rho) = \dfrac{\sqrt{2}}{\sqrt{(3t + \sqrt{9t^2 + \rho^6})^{2/3} + (3t - \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2}}, \\ y(t) = \eta(t; \rho) = \begin{cases} \sqrt{2\xi^2(t;\rho) - \xi^4(t;\rho)\rho^2} & t \geq 0, \\ -\sqrt{2\xi^2(t;\rho) - \xi^4(t;\rho)\rho^2} & t < 0. \end{cases} \end{cases} \tag{39}$$

Remark: Note that for $\rho = 0$, (39) is discontinuous at $t = 0$.

Proof: Fix $\rho \neq 0$. The phase space of the reduced unperturbed problem (35) is $\mathbf{R}^+ \times \mathbf{R}$. However, the equations (35) are, by themselves, naturally extendable to all of $x \in \mathbf{R}$, and it will be convenient to make this extension. For every point y_0 on the y-axis, $x \equiv 0, y \equiv y_0$ is an equilibrium solution. Thus solutions of (35) do not cross the y-axis. So if the intial condition of a solution of the extended equations (35) is in the half-plane $\{(x, y) : x > 0\}$ then that solution is in this half-plane for all time. Recall that non-collision solutions in this half-plane correspond to non-collision solutions of (5) under Λ.

The first integral (38) foliates the extended phase space of $\mathbf{R}^2$ into invariant one-dimensional "energy" manifolds. In particular, the trivial solution $x \equiv 0, y \equiv 0$ lies in the manifold defined by $H(x, y; \rho) = 0$. Thus we are looking for solutions of (35) which are forward and backward asymptotic to this trivial solution at the origin. Since no manifold defined by $H(x, y; \rho) = H_0 \neq 0$ contains the origin, the solutions we are looking for lie in the manifold $H(x, y; \rho) = 0$. Any solution (x, y) lying in this manifold must satisfy

$$(y(t))^2 = 2(x(t))^2 - (x(t))^4 \rho^2 \tag{40}$$

for all time. Note that the symmetry in the y variable implies that the manifold defined by $H(x, y; \rho) = 0$ is symmetric across the x-axis in extended phase space. So if we know the shape of the manifold in the quadrant $\{(x, y) : x > 0, y \geq 0\}$, we know its shape in the half plane $\{(x, y) : x > 0\}$. So we can suppose that $y \geq 0$. Hence, from (40),we have

$$y(t) = \sqrt{2(x(t))^2 - (x(t))^2 \rho^2}, \tag{41}$$

as long as $y(t) \geq 0$. Restricting the equations (35) to the invariant manifold $H(x, y; \rho) = 0$ by using (41), we further reduce (35) to

$$x' = -\frac{1}{2} x^4 \sqrt{2 - x^2 \rho^2}. \tag{42}$$

If we integrate (42), then from (41) and the direction of the flow on the manifold $H(x, y; \rho) = 0$ we can get $y(t)$. Since $(y(t))^2 \geq 0$, then from (40),

$$2(x(t))^2 - (x(t))^4 \rho^2 = (x(t))^2 (2 - (x(t))^2 \rho^2) \geq 0,$$

which implies that $x(t) \leq \sqrt{2}|\rho|^{-1}$. Since we are interested in solutions which exist only in the half-plane $\{(x, y) : x > 0\}$, then we have that $0 < x(t)$ for all $t \in \mathbf{R}$. Choose for an initial condition some $x_0 = x(t_0) \in (0, \sqrt{2}|\rho|^{-1}]$ where, for now, t_0 is arbitrary. The differential equation (42) is separable, and upon integration we have

$$-\frac{1}{2}(t - t_0) = -\frac{1}{2} \int_{t_0}^{t} d\tau = \int_{x_0}^{x} \frac{dx}{x^4 \sqrt{2 - x^2 \rho^2}}. \tag{43}$$

For the right-hand side of (43), let $x = \sqrt{2}\rho^{-1} \cos \alpha$. (This is where we need $\rho \neq 0$). Then $dx = -\sqrt{2}\rho^{-1} \sin \alpha d\alpha$, $x^2 = 2\rho^{-2} \cos^2 \alpha$ and $x^4 = 4\rho^{-4} \cos^4 \alpha$. As $\alpha =$

$\arccos(x\rho/\sqrt{2})$, then

$$\tan\alpha = \frac{\sqrt{2-x^2\rho^2}}{x\rho} \text{ and } \sec^2\alpha = \frac{2}{x^2\rho^2}. \tag{44}$$

Hence, by letting $\alpha_0 = \arccos(x_0\rho/\sqrt{2})$, and using (44), we have for the right-hand side of (43)

$$\begin{aligned}
\int_{x_0}^{x} \frac{dx}{x^4\sqrt{2-x^2\rho^2}} &= \int_{\alpha_0}^{\alpha} \frac{-\sqrt{2}\rho^{-1}\sin\alpha d\alpha}{4\rho^{-4}\cos^4\alpha(2-2\rho^{-2}\cos^2\alpha\rho^2)^{1/2}} \\
&= \frac{-\sqrt{2}\rho^3}{4}\int_{\alpha_0}^{\alpha} \frac{\sin\alpha d\alpha}{\cos^4\alpha\sqrt{2}\sqrt{1-\cos^2\alpha}} \\
&= \frac{-\rho^3}{4}\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\cos^4\alpha} \\
&= \frac{-\rho^3}{4}\int_{\alpha_0}^{\alpha} \sec^4\alpha d\alpha \\
&= \frac{-\rho^3}{4}\left[\frac{\tan\alpha\sec^2\alpha}{3} + \frac{2}{3}\tan\alpha\right]_{\alpha_0}^{\alpha} \\
&= \frac{-\rho^3}{4}\left[\frac{1}{3}\frac{\sqrt{2-x^2\rho^2}}{x\rho}\frac{2}{x^2\rho^2} + \frac{2}{3}\frac{\sqrt{2-x^2\rho^2}}{x\rho}\right] + K \\
&= \frac{-\rho^3}{6}\left[\frac{\sqrt{2-x^2\rho^2}}{x^3\rho^3} + \frac{x^2\rho^2\sqrt{2-x^2\rho^2}}{x^3\rho^3}\right] + K \\
&= \frac{-1}{6x^3}(1+x^2\rho^2)\sqrt{2-x^2\rho^2} + K,
\end{aligned}$$

where K is some constant. Since t_0 is arbitrary we choose t_0 so that $\frac{1}{2}t_0 = K$. Thus (43) yields

$$t = \frac{1}{3x^3}(1+x^2\rho^2)\sqrt{2-x^2\rho^2}, \tag{45}$$

which implicitly defines the solution x of (42) for the intial condition $x(t_0) = x_0$. Since $x \in (0, \sqrt{2}|\rho|^{-1}]$ then (45) implies that $t \geq 0$. From (42),

$$\frac{dt}{dx} = \frac{-2}{x^4\sqrt{2-x^2\rho^2}} \neq 0$$

for all $x \in (0, \sqrt{2}|\rho|^{-1})$ and so (45) is invertible.

From (45), we obtain

$$9t^2 + \rho^6 = 2x^{-6} + 3x^{-4}\rho^2. \tag{46}$$

Setting $v = x^{-2}$ we have $v^2 = x^{-4}$ and $v^3 = x^{-6}$. Thus (46), in terms of v, is

$$v^3 + \frac{3}{2}\rho^2 v^2 - \frac{1}{2}(9t^2 + \rho^6) = 0. \tag{47}$$

Equation (47) is a cubic polynomial in v, and to solve it we will use Cardano's cubic formula (see Hungerford [1974]). Let

$$a = -\frac{(\frac{3}{2}\rho^2)^2}{3} = \frac{-3\rho^4}{4} \tag{48}$$

$$b = \frac{2(\frac{3}{2}\rho^2)^2}{27} - \frac{1}{2}(9t^2 + \rho^6) = -\frac{1}{4}\left[18t^2 + \rho^6\right]. \tag{49}$$

Then from (48) and (49) we have

$$\frac{-a}{2} = \frac{1}{8}\left[18t^2 + \rho^6\right] \tag{50}$$

$$\frac{a^2}{4} = \frac{1}{64}\left[18t^2 + \rho^6\right]^2 = \frac{1}{64}\left[324t^4 + 36t^2\rho^6 + \rho^{12}\right] \tag{51}$$

$$\frac{b^3}{27} = \frac{-\rho^{12}}{64}. \tag{52}$$

Using (50),(51), and (52) let

$$P = \left[\frac{-a}{2} + \sqrt{\frac{b^3}{27} + \frac{a^2}{4}}\right]^{1/3} = \frac{1}{2}\left[3t + \sqrt{9t^2 + \rho^6}\right]^{2/3}, \tag{53}$$

and

$$Q = \left[\frac{-a}{2} - \sqrt{\frac{b^3}{27} + \frac{q^2}{4}}\right]^{1/3} = \frac{1}{2}\left[3t - \sqrt{9t^2 + \rho^6}\right]^{2/3}. \tag{54}$$

Let $\omega \in \mathbf{C}$ be a cubic root of unity such that $\omega^2 \neq 1$. Then the three roots of (47) are given by

$$P + Q - \frac{\rho^2}{2}, \tag{55}$$

$$\omega P + \omega^2 Q - \frac{\rho^2}{2}, \tag{56}$$

$$\omega^2 P + \omega Q - \frac{\rho^2}{2}, \tag{57}$$

where P and Q are given by (53) and (54). We want the root that is real for all $t \geq 0$. Clearly the root (55) satisfies this. We will rule out the other two roots.

Let $f(v) = v^3 + \frac{3}{2}\rho^2 v^2 - \frac{1}{2}(9t^2 + \rho^6)$. The roots of $f'(v) = 3v(v + \rho^2)$ are $v = 0$ and $v = -\rho^2$. Since $f''(v) = 6v + 3\rho^2$, then $f'' > 0$ when $v > \frac{-\rho^2}{2}$, and $f'' < 0$ when $v < \frac{-\rho^2}{2}$. Thus $v = -\rho^2$ is a local maximum of f. Now $f(-\rho^2) = \frac{-9t^2}{2}$, and as f is a cubic with $f(v) \to \infty$ as $v \to \infty$, then there is only one root of f that is real for all $t \geq 0$. Thus the roots (56) and (57) are in general complex valued.

From (53) and (54),

$$v = \frac{1}{2}\left[(3t + \sqrt{9t^2 + \rho^6})^{2/3} + (3t - \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2\right]. \tag{58}$$

Since $v = x^{-2}$, then the solution of (42) for the intial condition $x(t_0) = x_0$ is found from (58) and is

$$x(t) = \frac{\sqrt{2}}{\sqrt{(3t + \sqrt{9t^2 + \rho^6})^{2/3} + (3t - \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2}} \tag{59}$$

for $t \geq 0$. Note that $x(-t) = x(t)$ and so (59) is naturally extendable to all of $t \in \mathbf{R}$.

Now that we have x, let us get y. Note that

$$x(0) = \frac{\sqrt{2}}{\left[(\sqrt{\rho^6})^{2/3} + (\sqrt{\rho^6})^{2/3} - \rho^2\right]^{1/2}} = \sqrt{2}|\rho|^{-1},$$

and so by (41),

$$y(0) = \sqrt{2(x(0))^2 - (x(0))^4\rho^2} = \sqrt{4\rho^{-2} - 4\rho^{-2}} = 0.$$

Thus the manifold $H(x, y, \rho) = 0$ intersects the x-axis at $(\sqrt{2}|\rho|^{-1}, 0)$. Setting $y = 0$ in $H(x, y; \rho) = 0$ we find that this is the only intersection of y with the x-axis in the half-plane $\{(x, y) : x > 0\}$. Thus there are two cases: either $y(t) > 0$ for $t > 0$, or $y(t) < 0$ for $t > 0$. To decide which case it is, consider the vector field of (35). Since $x(0) = \sqrt{2}|\rho|^{-1}$, then $y'(0) = 4\rho^{-4} > 0$ since $\rho \neq 0$. So at the point $(\sqrt{2}|\rho|^{-1}, 0)$ the flow on the manifold $H(x, y; \rho) = 0$ is moving in the positive y-direction. So $y(t) > 0$ for $t > 0$. Hence $y(t)$ is given by (41) for $t \geq 0$. This can also be seen since we assumed $y(t) \geq 0$ to get (45), and (45) implies that $t \geq 0$. By the symmetry of the manifold $H(x, y; \rho) = 0$ across the x-axis, $y(t) = -\sqrt{2(x(t))^2 - (x(t))^4\rho^2}$ for $t < 0$. Define $\xi(t, \rho)$

to be (59) extended to all of $t \in \mathbf{R}$ and $\eta(t;\rho)$ to be $y(t)$ as just formulated. This completes the proof.

Note that for the parabolic solution (39), $\sqrt{2}|\rho|^{-1} > \xi(t;\rho) > 0$ for all $t \neq 0$, $\xi(0;\rho) = \sqrt{2}|\rho|^{-1}$, $\eta(t;\rho) > 0$ for all $t > 0$, $\eta(t;\rho) < 0$ for all $t < 0$, and $\eta(0;\rho) = 0$.

To find the corresponding parabolic solution in $(\mathbf{q},\mathbf{p})$-variables of (39), we compute θ from (36) with $x(t) = \xi(t;\rho)$ and $\rho \neq 0$ and apply Λ to (ξ,θ,η,ρ). This gives us an idea what these parabolic solutions actually look like in the physical plane.

4. The Jacobi Integral and Reduction of the Transformed Circular Problem

With $\mu \in (0,1)$ fixed and $e = 0$, the transformed equations (23) describe the circular problem in the variables (x,θ,y,ρ). We will simplify this one-parameter family of equations and prove that there is a first integral for these simplified equations. This first integral is known as the *Jacobi integral.* By using this first integral and by defining a new time variable s, we will reduce the transformed circular problem to a family of two-parameter nonautonomous planar equations.

Proposition 10: *The transformed equations of motion of the circular problem simplify to the one-parameter family of equations*

$$\begin{cases} x' = -\frac{1}{2}x^3 y \\ \theta' = x^4\rho \\ y' = -x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,t-\theta) + O(\mu^2) \\ \rho' = \mu\tilde{g}_2(x,t-\theta) + O(\mu^2), \end{cases} \tag{60}$$

where μ is the parameter, and

$$\tilde{g}_1(x,t-\theta) = x^4\left(1 - 2x^2\cos(t-\theta) - \frac{1 + x^2\cos(t-\theta)}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}\right), \tag{61}$$

$$\tilde{g}_2(x,t-\theta) = x^4\sin(t-\theta)\left(1 - \frac{1}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}\right). \tag{62}$$

The variables t and θ appear in the form $t-\theta$ in the higher-order μ-terms. Furthermore, the vector field of (60) is 2π-periodic in $t-\theta$.

Proof: Fix $\mu \in (0,1)$ and set $e = 0$. Then (3) takes the form

$$T(t;0) = t. \tag{63}$$

Substituting (63) into (1) and (2) we have

$$r_1 = \cos t, \tag{64}$$

$$r_2 = \sin t, \tag{65}$$

respectively. Thus $r_1^2 + r_2^2 = 1$. Substituting (64) and (65) into (16) and (22) we have

$$R_1 = \cos(t - \theta), \tag{66}$$

$$R_2 = \sin(t - \theta), \tag{67}$$

respectively. Thus (24) and (25) become

$$g_1(x, \theta, t; 0) = x^4 \left(1 - 2x^2 \cos(t - \theta) - \frac{1 + x^2 \cos(t - \theta)}{(1 + 2x^2 \cos(t - \theta) + x^4)^{3/2}}\right), \tag{68}$$

$$g_2(x, \theta, t; 0) = x^4 \sin(t - \theta) \left(1 - \frac{1}{(1 + 2x^2 \cos(t - \theta) + x^4)^{3/2}}\right), \tag{69}$$

respectively. Define $\tilde{g}_1(x, t - \theta)$ by (68) and $\tilde{g}_2(x, t - \theta)$ by (69). Then, by Theorem (4) with $\mu \in (0,1)$ and $e = 0$, the equations in (60) follows.

Now we will show that the variables t and θ appear in the form $t - \theta$ in the higher-order μ-terms, and that the vector field of (60) is 2π-periodic in $t - \theta$. Write the vector field of (60) as $f = (f_1, f_2, f_3, f_4)$ where

$$f_1(x, y) = -\frac{1}{2}x^3 y,$$

$$f_2(x, \rho) = x^4 \rho,$$

$$f_3(x, \theta, \rho, t; \mu) = -x^4 + x^6 \rho^2 + \mu \tilde{g}_1(x, t - \theta) + O(\mu^2),$$

$$f_4(x, \theta, t; \mu) = \mu \tilde{g}_2(x, t - \theta) + O(\mu^2).$$

Since f_1 and f_2 do not depend explicitly on $t - \theta$ they are 2π-periodic in $t - \theta$. However, the variables t and θ appear explicitly in f_3 and f_4. In the functions $\tilde{g}_1$ and $\tilde{g}_2$, the variables t and θ are in the form $t - \theta$. Furthermore, the functions $\tilde{g}_1$ and $\tilde{g}_2$ are 2π-periodic in $t - \theta$ since $t - \theta$ appears as an argument of cos or sin. However, we must show that in the $O(\mu^2)$ terms the variables t and θ appear in the form $t-\theta$, and that the $O(\mu^2)$ terms are 2π-periodic in $t-\theta$. To show both of these, we will derive f_3 and f_4 without using Proposition 3. (Recall that we used Proposition 3 in Theorem 4 wherein we derived the equations (23)). These functions we will show are 2π-periodic in $t-\theta$. Expanding these functions in terms of μ does not destroy the periodic property, and so the result will follow.

We begin with some necessary computations. With $e=0$, r_1 and r_2 are given by (64) and (65) respectively, and so (4) becomes

$$\begin{aligned} U(q_1,q_2,t;\mu,0) = &\frac{1-\mu}{\sqrt{(q_1-\mu\cos t)^2+(q_2-\mu\sin t)^2}} \\ &+\frac{\mu}{\sqrt{(q_1+(1-\mu)\cos t)^2+(q_2+(1-\mu)\sin t)^2}}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial U}{\partial q_1}(q_1,q_2,t;\mu,0) = &\frac{-(1-\mu)(q_1-\mu\cos t)}{((q_1-\mu\cos t)^2+(q_2-\mu\sin t)^2)^{3/2}} \\ &+\frac{-\mu(q_1+(1-\mu)\cos t)}{((q_1+(1-\mu)\cos t)^2+(q_2+(1-\mu)\sin t)^2)^{3/2}}, \end{aligned} \tag{70}$$

$$\begin{aligned} \frac{\partial U}{\partial q_2}(q_1,q_2,t;\mu,0) = &\frac{-(1-\mu)(q_2-\mu\sin t)}{((q_1-\mu\cos t)^2+(q_2-\mu\sin t)^2)^{3/2}} \\ &=\frac{-\mu(q_2+(1-\mu)\sin t)}{((q_1+(1-\mu)\cos t)^2+(q_2+(1-\mu)\sin t)^2)^{3/2}}. \end{aligned} \tag{71}$$

Recall that $W_1=\partial U/\partial q_1$ and $W_2=\partial U/\partial q_2$, and that $\tilde{W}_1$ and $\tilde{W}_2$ are the transformations of W_1 and W_2 respectively under Λ. Hence, substituting the expressions of q_1 and q_2 from (14) into (70) and (71), we have

$$\begin{aligned} \tilde{W}_1(x,\theta,t,\mu,0) = &\frac{-(1-\mu)(x^{-2}\cos\theta-\mu\cos t)}{(x^{-4}-2\mu x^{-2}\cos(t-\theta)+\mu^2)^{3/2}} \\ &+\frac{-\mu(x^{-2}\cos\theta+(1-\mu)\cos t)}{(x^{-4}+2(1-\mu)x^{-2}\cos(t-\theta)+(1-\mu)^2)^{3/2}}, \end{aligned} \tag{72}$$

$$\begin{aligned} \tilde{W}_2(x,\theta,t;\mu,0) = &\frac{-(1-\mu)(x^{-2}\sin\theta-\mu\sin t)}{(x^{-4}-2\mu x^{-2}\cos(t-\theta)+\mu^2)^{3/2}} \\ &+\frac{-\mu(x^{-2}\sin\theta+(1-\mu)\cos t)}{(x^{-4}+2(1-\mu)x^{-2}\cos(t-\theta)+(1-\mu)^2)^{3/2}}. \end{aligned} \tag{73}$$

First we show that the result holds for f_3. Thus, from (14), we have

$$q_1p_1+q_2p_2 = x^{-2}y, \tag{74}$$

$$p_1^2+p_2^2 = y^2+x^4\rho^2. \tag{75}$$

So, using (19), (74), and (75), the derivative of y (as defined in (15)) with respect to t is

$$f_3(x,\theta,\rho,t;\mu) = x^6\rho^2 + \tilde{W}_1(x,\theta,t;\mu,0)\cos\theta + \tilde{W}_2(x,\theta,t;\mu,0)\sin\theta. \qquad (76)$$

In (76), the variables t and θ appear only in the terms $\tilde{W}_1\cos\theta + \tilde{W}_2\sin\theta$. Now

$$\tilde{W}_1\cos\theta + \tilde{W}_2\sin\theta = \frac{-(1-\mu)(x^{-2} - \mu\cos(t-\theta))}{(x^{-4} - 2\mu x^{-2}\cos(t-\theta) + \mu^2)^{3/2}} + \frac{-\mu(x^{-2} + (1-\mu)\cos(t-\theta))}{(x^{-4} + 2(1-\mu)x^{-2}\cos(t-\theta) + (1-\mu)^2)^{3/2}}. \qquad (77)$$

In (77), the variables t and θ appear in the form $t-\theta$. So it follows from (77) that (76) is 2π-periodic in $t-\theta$.

We will show that the result holds for f_4. From the derivative of ρ (as given in (15)) with respect to t, we have

$$f_4(x,\theta,t;\mu) = x^{-2}\big(\tilde{W}_2(x,\theta,t;\mu,0)\cos\theta - \tilde{W}_1(x,\theta,t;\mu,0)\sin\theta\big),$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are given by (72) and (73) respectively. Here, the variables t and θ are found only in the terms $\tilde{W}_2\cos\theta - \tilde{W}_1\sin\theta$. By (72) and (73),

$$\tilde{W}_2\cos\theta - \tilde{W}_1\sin\theta = \frac{-\mu(1-\mu)\sin(t-\theta)}{\left(x^{-4} - 2\mu x^{-2}\cos(t-\theta) + \mu^2\right)^{3/2}} + \frac{-\mu(1-\mu)\sin(t-\theta)}{\left(x^{-4} + 2(1-\mu)x^2\cos(t-\theta) + (1-\mu)^2\right)^{3/2}}. \qquad (78)$$

In (78), the variables t and θ appear in the form $t-\theta$. Thus (78) implies that f_4 is 2π-periodic in $t-\theta$. This completes the proof.

We refer to (60) as the *transformed circular problem.* Before we can define the functional which we will prove to be a first integral of the transformed circular problem, we need the following. Recall that $\tilde{U}$ is the transformation of U under Λ.

Proposition 11: *Fix $\mu \in (0,1)$, and set $e = 0$. The transformation of the potential function* (4) *under Λ is, to the first order approximation in μ,*

$$\tilde{U}(x,\theta,t;\mu,0) = x^2 + \mu x^2\left(-1 + x^2\cos(t-\theta) + \frac{1}{(1+2x^2\cos(t-\theta)+x^4)^{1/2}}\right) + O(\mu^2), \qquad (79)$$

and the variables t and θ appear in the form $t - \theta$ in the higher-order μ-terms. Furthermore, (79) *is 2π-periodic in $t - \theta$.*

Proof: Fix $\mu \in (0,1)$ and let $e = 0$. Substituting (64), (65), and (66) into (34) yields (79). To see that t and θ appear in the form $t-\theta$ in the higher μ-terms, substitute q_1 and q_2 from (14), and (64), (65), (66) into (4). This yields

$$\tilde{U}(x,\theta,t;\mu,0) = (1-\mu)x^2\left(1 - 2\mu x^2\cos(t-\theta) + x^4\mu^2\right)^{-1/2}$$
$$+\mu x^2\left(1 + 2(1-\mu)x^2\cos(t-\theta) + x^4(1-\mu)^2\right)^{-1/2}. \quad (80)$$

Expanding (80) in terms of μ does not change the form of the variables t and θ. Thus these variables appear in the form $t-\theta$ in the higher order μ-terms. Since $t-\theta$ appears as the argument of the cos function, then (79) is 2π-periodic in $t-\theta$. This completes the proof.

Let

$$\hat{U}(x, t-\theta;\mu) = \tilde{U}(x,\theta,t;\mu,0). \quad (81)$$

Define the functional $J : \mathbf{R}^+ \times S^1 \times \mathbf{R}^3 \times [0,1) \to \mathbf{R}$ by

$$J(x,\theta,y,\rho,t;\mu) = \frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - \hat{U}(x,t-\theta;\mu) - \rho. \quad (82)$$

Note that J is 2π-periodic in $t-\theta$ by Proposition 11.

Theorem 12: *The functional* (82) *is a first integral of the transformed circular problem* (60).

Proof: Fix $\mu \in (0,1)$ and $e = 0$. The equations of motion (5) of the circular problem can be written as

$$q_i'' = \frac{\partial U}{\partial q_i} \quad \text{for } i = 1,2. \quad (83)$$

We will make a change of variables to bring (83) into a form which is independent of t. Recall that the partials $\partial U/\partial q_i$ are dependent on t since $\mu \in (0,1)$. To remove this time dependency, implicitly define the variables $(u,v) \in \mathrm{R}^2$ by

$$\begin{cases} q_1 = u\cos t - v\sin t, \\ q_2 = u\sin t + v\cos t. \end{cases} \quad (84)$$

Explicitly, (u, v) are given by

$$\begin{cases} u = q_1 \cos t + q_2 \sin t, \\ v = -q_1 \sin t + q_2 \cos t. \end{cases} \tag{85}$$

Let P be transformation of the potential function (4) under (84). Formally then, P is a function of $(u, v, t; \mu)$. However, we will see that P does not depend explicitly on t. From (84) we have that

$$q_1^2 + q_2^2 = u^2 + v^2. \tag{86}$$

Substituting (84) into (4), using (86) and the definition of u in (85), we have

$$P = \frac{1-\mu}{d_1} + \frac{\mu}{d_2}, \tag{87}$$

where $d_i : \mathbf{R}^2 \times [0,1) \to [0,\infty)$ for $i = 1,2$ are given by

$$d_1(u,v;\mu) = \sqrt{(u-\mu)^2 + v^2}, \tag{88}$$

$$d_2(u,v;\mu) = \sqrt{(u+(1-\mu))^2 + v^2}. \tag{89}$$

It is important to note that (88), (89) do not depend on t, and so P is a function of just (u, v, μ). The quantities d_1 and d_2 are the distances between P_1 and P_3, and P_2 and P_3 respectively. In the (u,v)-plane, P_1 and P_2 are at rest on the u-axis; P_1 is at $(\mu, 0)$ and P_2 is at $(-1+\mu, 0)$.

We assert that the equations of motion (83) transform under (85) to give

$$\begin{cases} u'' - 2v' - u = \dfrac{\partial P}{\partial u} \\ v'' + 2u' - v = \dfrac{\partial P}{\partial v}. \end{cases} \tag{90}$$

Let us prove this assertion. From (87), we have that

$$\frac{\partial P}{\partial u}(u,v;\mu) = -\frac{(1-\mu)(u-\mu)}{d_1^3} - \frac{\mu(u+(1-\mu))}{d_2^3}, \tag{91}$$

$$\frac{\partial P}{\partial v}(u,v;\mu) = -\frac{(1-\mu)v}{d_1^3} - \frac{\mu v}{d_2^3}, \tag{92}$$

where d_1 and d_2 are given by (88) and (89) respectively. Let V_1 and V_2 be the transformations of W_1 and W_2 under (84). So upon substituting (84) into (70) and (71) respectively, and using (88) and (89), we have

$$V_1(u,v,t;\mu) = -\frac{(1-\mu)(u\cos t - v\sin t - \mu\cos t)}{d_1^3} - \frac{\mu(u\cos t - v\sin t + (1-\mu)\cos t)}{d_2^3}, \tag{93}$$

$$V_2(u,v,t;\mu) = -\frac{(1-\mu)(u\sin t + v\cos t - \mu\sin t)}{d_1^3} - \frac{\mu(u\sin t + v\cos t + (1-\mu)\sin t)}{d_2^3}. \tag{94}$$

From (85) we have

$$\begin{aligned} u' &= q_1'\cos t - q_1\sin t + q_2'\sin t + q_2\cos t \\ &= q_1'\cos t + q_2'\sin t + v, \end{aligned} \tag{95}$$

$$\begin{aligned} u'' &= q_1''\cos t - q_1'\sin t + q_2''\sin t + q_2'\cos t + v' \\ &= W_1(q_1,q_2,t;\mu,0)\cos t + W_2(q_1,q_2,t;\mu,0)\sin t - q_1'\sin t + q_2'\cos t + v', \end{aligned} \tag{96}$$

$$\begin{aligned} v' &= -q_1'\sin t - q_1\cos t + q_2'\cos t - q_2\sin t \\ &= -q_1\sin t + q_2'\cos t - u, \end{aligned} \tag{97}$$

$$\begin{aligned} v'' &= -q_1''\sin t - q_1'\cos t + q_2''\cos t - q_2'\sin t - u' \\ &= -W_1(q_1,q_2,t;\mu,0)\sin t + W_2(q_1,q_2,t;\mu,0)\cos t - q_1'\cos t - q_2'\sin t - u'. \end{aligned} \tag{98}$$

Substituting (93) and (94) into the transformation of (96) under (84), and using (97) and (91), we obtain

$$\begin{aligned} u'' &= -\frac{(1-\mu)(u-\mu)}{d_1^3} - \frac{\mu(u+(1-\mu))}{d_2^3} + 2v' + u \\ &= \frac{\partial P}{\partial u} + 2v' + u. \end{aligned} \tag{99}$$

Substituting (93) and (94) into the transformation of (98) under (84), and using (95) and (92), we obtain

$$\begin{aligned} v'' &= -\frac{(1-\mu)v}{d_1^3} - \frac{\mu v}{d_2^3} - 2u' + v \\ &= \frac{\partial P}{\partial v} - 2u' + v. \end{aligned} \tag{100}$$

Thus (90) is established by (99) and (100).

Define $\Phi : \mathbf{R}^2 \times [0,1) \to \mathbf{R}$ by

$$\Phi(u,v;\mu) = \frac{1}{2}(u^2+v^2) + P(u,v;\mu), \tag{101}$$

where P is given by (87). Since

$$\frac{\partial \Phi}{\partial u}(u,v;\mu) = u + \frac{\partial P}{\partial u}(u,v;\mu),$$
$$\frac{\partial \Phi}{\partial v}(u,v;\mu) = v + \frac{\partial P}{\partial v}(u,v;\mu),$$

then we can write (90) as

$$\begin{cases} u'' - 2v' = \dfrac{\partial \Phi}{\partial u} \\ v'' + 2u' = \dfrac{\partial \Phi}{\partial v}. \end{cases} \tag{102}$$

Multiplying the first equation in (102) by u' and adding this to the second equation in (102) multiplied by v', we have

$$u''u' + v''v' = \frac{\partial \Phi}{\partial u}u' + \frac{\partial \Phi}{\partial v}v'. \tag{103}$$

Note that the right-hand side of (103) is $d\Phi/dt$. So integrating (103) with respect to t we obtain

$$(u')^2 + (v')^2 = 2\Phi(u,v;\mu) + 2C, \tag{104}$$

where C is the constant of integration. From (104) it follows that

$$\frac{d}{dt}\Big((u')^2 + (v')^2 - 2\Phi(u,v;\mu)\Big) = 0, \tag{105}$$

and so (104) is a first integral of (90). Since the transformation (85) is a real analytic diffeomorphism and the transformation Λ is also, then by the chain rule and (105), the transformation of (105) into the variables (x,θ,y,ρ,t) will be a first integral of the transformed circular problem (60). So all we need to do now to complete the proof is make all the necessary transformations.

Let us begin with the left-hand side of (104). Using (95), (97), (85), (5), (75), and the definition of ρ given in (15), we have

$$(u')^2 + (v')^2 = y^2 + x^4\rho^2 + q_1^2 + q_2^2 - 2\rho. \tag{106}$$

For the right-hand side of (104), using (86), the definition of P (given in (87), (88), and (89)), Proposition 11, and (81), we obtain

$$\begin{aligned} 2\Phi(u,v;\mu) + 2C &= u^2 + v^2 + 2P(u,v;\mu) + 2C \\ &= q_1^2 + q_2^2 + 2U(q_1, q_2, t; \mu, 0) + 2C \\ &= q_1^2 + q_2^2 + 2\tilde{U}(x, \theta, t; \mu, 0) + 2C \\ &= q_1^2 + q_2^2 + 2\hat{U}(x, t-\theta; \mu) + 2C. \end{aligned} \tag{107}$$

Bringing (106) and (107) together, and using (82), we have

$$\begin{aligned} 2C &= y^2 + x^4\rho^2 - 2\hat{U}(x, t-\theta; \mu) - 2\rho \\ &= 2J(x, \theta, y, \rho, t; \mu). \end{aligned}$$

This completes the proof.

Definition 13: The functional (82) is called the *Jacobi integral*, and the constant of integration C in (104) is called the *Jacobi constant.*

From Proposition 10 we know that the variables t and θ appear in the form $t - \theta$ in the vector field of (60). From Proposition 11 we know that these variables also appear in the same form in the Jacobi integral (82). This leads us to defining a new time variable s by

$$s = t - \theta. \tag{108}$$

Note that, since the vector field of (60) and the Jacobi integral are 2π-periodic in $t - \theta$ by Proposition 10 and 11 respectively, then we may consider t as belonging to S^1, even though we shall continue to think of t as a real time variable. Similarly, we may consider s as belonging to S^1, but shall think of s as a real time variable.

Lemma 14: *Fix $(\mu, e) \in [0,1)^2$. The time variables s and t satisfy the differential equation*

$$\frac{dt}{ds} = \frac{1}{1 - x^4\rho}. \tag{109}$$

Proof: Differentiating (108) with respect to s, using the chain rule and the equation for $d\theta/dt$ from (23), we have that

$$1 = \frac{dt}{ds} - \frac{d\theta}{ds}$$

$$= \frac{dt}{ds} - \frac{d\theta}{dt}\frac{dt}{ds}$$
$$= \frac{dt}{ds}(1 - x^4\rho). \tag{110}$$

Now (110) implies (109). This completes the proof.

Note that Lemma 14 holds for the transformations of the unperturbed, circular, and the elliptic problem. We finish this section with a reduction of the circular problem.

Theorem 15: *The transformed circular problem* (60) *is reducible to the two-parameter family of nonautonomous planar equations*

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{ds} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} + O(\mu^2), \end{cases} \tag{111}$$

where the two parameters are μ *and* C, $\tilde{g}_1$ *is given by* (61) *and* (108),

$$\rho(x,y,s;\mu,C) = x^{-4}\left(1 \pm \sqrt{1 - x^4(y^2 - 2\hat{U}(x,s;\mu) - 2C)}\right), \tag{112}$$

the $\pm$ *depending on the relative angular velocity of* P_3, *and* $\hat{U}$ *is given by* (81). *Also*

$$\theta(s) = \int_{s_0}^{s} \frac{x^4\rho}{1 - x^4\rho} ds + \theta_0 \ (\mathrm{mod}\ 2\pi), \tag{113}$$

where ρ *is given by* (112). *Furthermore, the vector field of* (111) *is* 2π*-periodic in* s.

Proof: Fix $\mu \in (0,1)$ and set $e = 0$. Fix a Jacobi constant C. The Jacobi integral (82) foliates the five-dimensional generalized phase space of the circular problem (60) into invariant four-dimensional manifolds, each such manifold corresponding to a particular Jacobi constant. On the manifold $J(x,\theta,y,\rho,t;\mu) = C$, we can solve the Jacobi integral for ρ. Note that the Jacobi integral is quadratic in ρ. So

$$\rho = \frac{1}{x^4}\left(1 \pm \sqrt{1 - x^4(y^2 - 2\hat{U}(x,t-\theta;\mu) - 2C)}\right). \tag{114}$$

By (108), (114) becomes (112). By solving the Jacobi integral for ρ, we have restricted the transformed circular problem (60) to the manifold $J(x,\theta,y,\rho,t;\mu) = C$. Under this restriction, the transformed circular problem reduces to

$$\begin{cases} x' = -\dfrac{1}{2}x^3 y \\ \theta' = x^4\rho \\ y' = -x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,t-\theta) + O(\mu^2). \end{cases} \tag{115}$$

Now we invoke Lemma 14. By the chain rule, (115) becomes

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{dx}{dt}\dfrac{dt}{ds} = \dfrac{-\frac{1}{2}x^3 y}{1-x^4\rho} \\ \dfrac{d\theta}{ds} = \dfrac{d\theta}{dt}\dfrac{dt}{ds} = \dfrac{x^4\rho}{1-x^4\rho} \\ \dfrac{dy}{ds} = \dfrac{dy}{dt}\dfrac{dt}{ds} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1-x^4\rho} + O(\mu^2). \end{cases} \tag{116}$$

Note that the vector field of (116) does not depend explicitly on θ. So we may integrate $d\theta/ds$ from s_0 to s to obtain (113), where ρ is given by (112). Hence we can drop the equation for $d\theta/ds$ from (116) to obtain (111). Since the Jacobi integral is 2π-periodic in $t - \theta$ by Proposition 11, then ρ, as given in (112), is 2π-periodic in s by (108). It then follows from Proposition 10 that the vector field of (111) is 2π-periodic in s. That the equations (111) are nonautonomous is clear since ρ and $\tilde{g}_1$ explicitly depend on s. Note that the vector field of (111) depends on the parameter μ and the constant C. We think of C as another parameter. This completes the proof.

We call the equations (111) the *reduced transformed circular problem.* The reduction done in the proof of Theorem 15 introduced another parameter, namely the Jacobi constant, into the transformed circular problem, but we reduced of the size of the problem by two dimensions. We reduced the phase space of the transformed circular problem (60) from $\mathbf{R}^+ \times S^1 \times \mathbf{R}^3$ to $\mathbf{R}^+ \times S^1 \times \mathbf{R}$.

5. Transverse Symmetric Homoclinic Orbits in the Transformed Circular Problem

In this section we continue the analysis of the circular problem, but our attention will now turn to the qualitative aspects of the phase space picture of the reduced transformed circular problem. The vector field of the equations of the reduced transformed circular problem (111) is naturally extendable to all of $x \in \mathbf{R}$. Reducing the three-dimensional flow to a *Poincaré map*, the existence of transverse "symmetric" orbits, which are homoclinic to an artificial *periodic orbit at infinity*, will be shown. The type of symmetry these orbits possess is linked to the *reversibility* of the reduced transformed circular problem. Furthermore, these transverse symmetric homoclinic orbits can be approximated in some sense by the parabolic solutions of the unperturbed problem. By the use of symbolic dynamics, the existence of a transverse symmetric homoclinic orbit implies the existence of periodic orbits of arbitrary period, and we will show that the stable and unstable manifolds of some of the periodic orbits, with very large periods, intersect transversely.

Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. The generalized phase space of the reduced transformed circular problem (111) is $\mathbf{R}^+ \times \mathbf{R} \times S^1$ since the vector field of (111) is defined only for $x > 0$ by virtue of the domain of the transformation Λ. However, the domain of the vector field of (111) is naturally extendable to all of $x \in \mathbf{R}$. Under this extension, the generalized phase space of the reduced transformed circular problem extends to $\mathbf{R}^2 \times S^1$. We call this extension of the equations (111) the *extended reduced transformed circular problem.* The domain of the transformed potential function $\tilde{U}$ given in (79) is also extendible in the natural manner to all of $x \in \mathbf{R}$, as is the domain of the Jacobi integral (82). The reason for making this extension is that the dynamics in the infinite cylinder $\{x = 0\} \equiv \{(0, y, s) : y \in \mathbf{R}, s \in S^1\}$ are easily understood. By looking then at orbits aymptotic to certain orbits in the infinite cylinder $\{x = 0\}$ we can deduce information about the structure of the orbits in the region $\{x > 0\} \equiv \{(x, y, s) \in \mathbf{R}^+ \times \mathbf{R} \times S^1\}$ of the phase space of the extended reduced transformed circular problem. Note that the region $\{x > 0\}$ is the phase space of the reduced transformed circular problem (60), and so non-collision solutions in this region correspond to non-collision solutions under Λ in the physical plane.

For the extended vector field of (111) we have the following result. Note that the vector field of (111) depends on the two parameters, μ and C.

Proposition 16: *Fix $(\mu, C) \in (0,1) \times \mathbf{R}$, and set $e = 0$. Then for any $y_0 \in \mathbf{R}$, $x \equiv 0, y \equiv y_0$ is a 2π-periodic solution of the extended reduced transformed circular problem.*

Proof: Write the extended vector field of (111) as (h_1, h_2) where

$$\begin{aligned} h_1(x, y, s; \mu, C) &= \frac{-\frac{1}{2}x^3 y}{1 - x^4 \rho(x, y, s; \mu, C)}, \\ h_2(x, y, s; \mu, C) &= \frac{-x^4 + x^6 + \mu \tilde{g}_1(x, s)}{1 - x^4 \rho(x, y, s; \mu, C)} + O(\mu^2). \end{aligned} \tag{117}$$

From (80), the extended potential function when evaluated at $x = 0$ is zero. Thus through (81), the extended Jacobi integral for $x = 0$ is linear in ρ, and so yields $\rho(0, y, s; \mu, C) = \frac{1}{2}y^2 - C$. Clearly then $h_1(0, y, s; \mu, C) = 0$. For h_2, given in (117) as an expansion in terms of μ, there are higher-order μ-terms to contend with. Note that h_2, in closed form, is simply the function f_3 given by (76) and (77) in conjunction with (108). Hence h_2, in closed form, is given by

$$\begin{aligned} h_2(x, y, s; \mu, C) = x^6\rho^2 &- \frac{(1-\mu)\big(x^4 - \mu x^6\cos(s)\big)}{\big(1 - 2\mu x^2\cos(s) + x^4\mu^2\big)^{3/2}} \\ &- \frac{\mu\big(x^4 + (1-\mu)x^6\cos(s)\big)}{\big(1 + 2(1-\mu)x^2\cos(s) + x^4(1-\mu)^2\big)^{3/2}}. \end{aligned} \tag{118}$$

Since $\rho(0, y, s; \mu, C) = \frac{1}{2}y^2 - C$, then from (118) it is easily seen that $h_2(0, y, s; \mu, C) = 0$. Let $y_0 \in \mathbf{R}$. Since the vector field of the extended reduced transformed circular problem is 2π-periodic in s, then the solution $x \equiv 0, y \equiv y_0$ is a 2π-periodic solution. This completes the proof.

By the uniqueness of solutions, it is an immediate consequence that solutions in $\{x > 0\}$ do not intersect the infinite cylinder $\{x = 0\}$.

It is important to note that the 2π-periodic solutions $x \equiv 0, y \equiv y_0$ do not exist in the physical plane since they are defined outside of the domain of Λ: these solutions are *artificial.* We call the particular solution $x \equiv 0, y \equiv 0$ the *periodic orbit at infinity.* This is the artificial periodic orbit "at infinity" mentioned at the beginning of this section. We will show that there exist transverse symmetric orbits in the generalized phase space of the reduced transformed circular problem homoclinic to this periodic orbit at infinity. Before doing this, we need to develop two other ideas first. The first of these is a concept given in the following

Definition 17: Let M be a Banach space over $\mathbf{R}$. A linear transformation $L : M \to M$ is a *reflection* of M if $L^2 = id$ on M. An autonomous system $w' = z(w)$, where $w \in M$ and $z : M \to M$, is *reversible* if there exists a reflection L of M such that $z(Lw) = -Lz(w)$. If $z(Lw) = -Lz(w)$ for some reflection L of M, we say the autonomous system $w' = z(w)$ is *reversible under L.*

Note that a reflection is also known as an involution.

Solutions of a reversible autonomous system have the following property.

Proposition 18: *Suppose the autonomous system $w' = z(w)$ is reversible under the reflection L of M, where $w \in M$ and $z : M \to M$. If $w(\tau)$ is a solution, then $Lw(-\tau)$ is a solution.*

Proof: Suppose $w(\tau)$ is a solution of the autonomous system $w' = z(w)$. Then, by the properties of the reflection L and the differential equation, we have that

$$\begin{aligned}\big(Lw(-\tau)\big)' &= L\big(w(-\tau)\big)' \\ &= L\big(-w'(-\tau)\big) \\ &= -Lz\big(w(-\tau)\big) \\ &= z\big(Lw(-\tau)\big).\end{aligned}$$

This completes the proof.

The extended reduced transformed circular problem is not in autonomous form. By defining s as a function of $\tau \in \mathbf{R}$ satisfying $s(0) = 0$ we can write the equations of the extended reduced transformed circular problem, using τ as a dummy time variable, in the autonomous form

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{d\tau} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} + O(\mu^2) \\ \dfrac{ds}{d\tau} = 1, \end{cases} \tag{119}$$

where $\tilde{g}_1$ is given by (61) and (108), and ρ is given by the Jacobi integral (82). Since $ds/d\tau = 1$ and $s(0) = 0$ then $s = \tau$. Since we can think of s as a S^1 variable, we can also think of τ as a S^1 variable.

We claim that for each $(\mu, C) \in (0,1) \times \mathbf{R}$, the system given by (119) is reversible. This will be useful in deducing certain symmetry results for the homoclinic orbits we are looking for. The following Lemma will give some clues as to possible reflections to consider. Let $h(x,y,s;\mu,C) = (h_1(x,y,s;\mu,C), h_2(x,y,s;\mu,C), 1)$ denote the vector field of (119), where h_1 and h_2 are given by (117).

Lemma 19: *Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. The vector field $h : \mathbf{R}^2 \times S^1 \times [0,1) \times \mathbf{R} \to$*

$\mathbf{R}^2 \times S^1$ *of* (119) *has the following properties:*

1) $h(x, y, s + 2\pi; \mu, C) = h(x, y, s; \mu, C)$,

2) $h(x, y, s + s_0; \mu, C) = h(x, y, -s + s_0; \mu, C)$ for $s_0 = 0, \pi$,

3) $h_1(x, y, s; \mu, C) = -h_1(x, -y, s; \mu, C)$,

4) $h_2(x, y, s; \mu, C) = h_2(x, -y, s; \mu, C)$.

Proof: Property 1 follows from Theorem 15. For property 2, we will show that the s appears always in the form $\cos(s)$. Property 2 then easily follows by the symmetry of the cosine function. For h_1, s appears only in ρ which is given by (112). From (80), (81), and (108) we see that s appears always in the form $\cos(s)$. For h_2, we observe that in the closed form (118), s appears always in the form $\cos(s)$. Property 3 follows directly from (119). By noting that in (118) the only place where y appears in h_2 is in ρ, then, from (112), property 4 is shown. This completes the proof.

For fixed $(\mu, C) \in (0,1) \times \mathbf{R}$, the properties listed in Lemma 19 suggest that the system given by the equations (119) is reversible. Two reflections of the phase space of (119), which is the same as the phase space of the extended reduced transformed circular problem, merit consideration. Here $M = \mathbf{R}^2 \times S^1$ is the phase space of (119). Define the linear transformations L_1 and L_2 of M by

$$L_1(x, y, \tau) = (x, -y, -\tau),$$
$$L_2(x, y, \tau) = (x, -y, 2\pi - \tau).$$

Note that these linear transformations easily satisfy the definition of a reflection on $M = \mathbf{R} \times S^1$. Since we can think of τ as a S^1 variable, then

$$L_2(x, y, \tau) = (x, -y, 2\pi - \tau) = (x, -y, -\tau) = L_1(x, y, \tau).$$

So it suffices to show that (119) is reversible under L_1.

Proposition 20: *For fixed* $(\mu, C) \in (0,1) \times \mathbf{R}$, *the autonomous system* (119) *is reversible under the reflection* L_1.

Proof: Recall that we denoted the vector field of (119) by

$$h(x, y, s; \mu, C) = (h_1(x, y, s; \mu, C), h_2(x, y, s; \mu, C), 1).$$

Suppressing the parameter μ and the Jacobi constant C, and writing $w = (x, y, s)$, we have that

$$\begin{aligned} h(L_1 w) &= h\big(L_1(x,y,s)\big) \\ &= h(x,-y,-s) \\ &= \big(h_1(x,-y,-s), h_2(x,-y,-s), 1\big) \\ &= \big(-h_1(x,y,s), h_2(x,y,s), 1\big) \\ &= -\big(h_1(x,y,s), -h_2(x,y,s), -1\big) \\ &= -L_1\big(h_1(x,y,s), h_2(x,y,s), 1\big) \\ &= -L_1 h(x,y,s) \\ &= -L_1 h(w), \end{aligned}$$

where we have used the properties 2), 3), and 4) from Lemma 19. Therefore the system (119) is reversible under L_1. This completes the proof.

We now develop the other idea we will need. Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. By Lemma 19, the vector field of (119) is 2π-periodic in s. Because of the equation $ds/d\tau = 1$, the phase space of (119) admits *global cross sections* of the form

$$\{s = s_0 : \Gamma\} \equiv \Gamma_C^{s=s_0} \times \{s_0\}$$

for each $s_0 \in S^1$, where

$$\Gamma_C^{s=s_0} = \left\{(x,y) \in \mathbf{R}^2\right\}.$$

Fix $s_0 \in S^1$. If we follow the solution $w(s) = \big(w_1(s), w_2(s), s + s_0\big)$ of the initial value problem (119) forward in time from the initial condition $w(0) = (x_0, y_0, s_0)$ we will intersect the section $\{s = s_0 : \Gamma\}$ at the point

$$\big(w_1(2\pi + s_0), w_2(2\pi + s_0), 2\pi + s_0\big)$$

at time $s = 2\pi + s_0 = s_0 \pmod{2\pi}$. We call this point the *first return* of the solution $w(s)$ to the section $\{s = s_0 : \Gamma\}$. We have thus described a way of reducing the flow in the three-dimensional phase space of (119) to a three-parameter, two-dimensional mapping of the $\Gamma_C^{s=s_0}$. Let

$$\psi_{s_0,\mu,C} : \Gamma_C^{s=s_0} \to \Gamma_C^{s=s_0}$$

be this mapping. This mapping is called a *Poincaré map.* We also refer to $\Gamma_C^{s=s_0}$ as a section, since it is simply the projection of the section $\{s = s_0 : \Gamma\}$ onto the plane.

Proposition 21: *For each* $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$, *the Poincaré map* $\psi_{s_0,\mu,C}$ *is a real analytic diffeomorphism of the section* $\Gamma_C^{s=s_0}$ *that leaves the origin* $(0,0) \in \Gamma_C^{s=s_0}$ *fixed.*

Remark: The real analyticity of $\psi_{s_0,\mu,C}$ is in the sense that the collision solutions have been analytically regularized.

Proof: Fix $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$. Since the vector field of (119) is real analytic, its solutions are real analytic in the time variable s, and in the intial conditions x_0, y_0, s_0. Let $w(s) = (w_1(s), w_2(s), s + s_0)$ be the solution for the intial value problem (119) with the initial condition $w(0) = (x_0, y_0, s_0)$. By the real analyticity of $w_i(s)$ in s,

$$w_i(s) = \sum_{n=0}^{\infty} \frac{(s - s_0)^n}{n!} \left(\frac{d^n w_i}{ds^n}\bigg|_{s=s_0} \right)$$

for $i = 1, 2$. Suppressing the constants s_0, μ, and C from $\psi_{s_0,\mu,C}$, and then writing $\psi(x,y) = (\psi_1(x,y), \psi_2(x,y))$, we have that the image (x_1, y_1) of (x_0, y_0) under ψ is given by

$$x_1 = \psi_1(x_0, y_0) = w_1(2\pi + s_0) = \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!} \left(\frac{d^n w_1}{ds^n}\bigg|_{s=s_0} \right),$$

$$y_1 = \psi_2(x_0, y_0) = w_2(2\pi + s_0) = \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!} \left(\frac{d^n w_2}{ds^n}\bigg|_{s=s_0} \right).$$

Since the solution $w(s)$ depends on x_0, y_0 in a real analytic manner, then ψ does too. That ψ is surjective follows since, for $(x_0, y_0) \in \Gamma_C^{s=s_0}$, the point given by $(w_1(-2\pi + s_0), w_2(-2\pi + s_0))$ is mapped by ψ to (x_0, y_0). To show that ψ is injective, suppose it is not. Then there exists $(x_0, y_0) \neq (\bar{x}_0, \bar{y}_0)$ such that $\psi(x_0, y_0) = \psi(\bar{x}_0, \bar{y}_0)$. This implies that the solutions $w(s; s_0, x_0, y_0)$ intersects the solution $w(s; s_0, \bar{x}_0, \bar{y}_0)$ at $(\psi(x_0, y_0), s_0)$ contradicting uniqueness of solutions. Thus ψ is a real analytic bijection, and so ψ^{-1} exists. Writing $\psi^{-1} = (\psi_1^{-1}, \psi_2^{-1})$, we have that the image (x_{-1}, y_{-1}) of the point (x_0, y_0) under ψ^{-1} is given by $x_{-1} = \psi_1^{-1}(x_0, y_0) = w_1(-2\pi + s_0)$ and $y_{-1} = \psi_2^{-1}(x_0, y_0) = w_2(-2\pi + s_0)$. Since $w_1(s)$ and $w_2(s)$ are real analytic in x_0, y_0, then it follows that ψ^{-1} is real analytic in x_0, y_0. Therefore ψ is a real analytic diffeomorphism of $\Gamma^{s=s_0}$. That ψ leaves the origin fixed follows since the solution $w(s; s_0, 0, 0)$

is the periodic orbit at infinity. This completes this proof.

A set $A \subset \Gamma_C^{s=s_0}$ is *invariant* under the map $\psi_{s_0,\mu,C}$ if for every $(x_0, y_0) \in A$, $\psi^j_{s_0,\mu,C}(x_0, y_0) \in A$ for every $j \in \mathbf{Z}$. In particular, the fixed point $(0,0)$ is an invariant set of $\psi_{s_0,\mu,C}$.

Definition 22: Fix $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$. The *stable set* of the fixed point $(0,0) \in \Gamma_C^{s=s_0}$ of the Poincaré map $\psi_{s_0,\mu,C}$, which we denote by $W^s_{\psi_{s_0,\mu,C}}(0,0)$, is defined by

$$W^s_{\psi_{s_0,\mu,C}}(0,0) = \left\{(x_0, y_0) \in \Gamma_C^{s=s_0} : \lim_{n\to\infty} \psi^n_{s_0,\mu,C}(x_0, y_0) = (0,0)\right\}.$$

The *unstable set* of the fixed point $(0,0) \in \Gamma_C^{s=s_0}$ of the Poincaré map $\psi_{s_0,\mu,C}$, which we denote by $W^u_{\psi_{s_0,\mu,C}}(0,0)$, is defined by

$$W^u_{\psi_{s_0,\mu,C}}(0,0) = \left\{(x_0, y_0) \in \Gamma_C^{s=s_0} : \lim_{n\to\infty} \psi^{-n}_{s_0,\mu,C}(x_0, y_0) = (0,0)\right\}.$$

In a similar manner we can define the stable and unstable sets of other invariant sets of $\psi_{s_0,\mu,C}$.

Fix $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$. In a trivial way the stable and unstable set of the fixed point $(0,0)$ of the Poincaré map $\psi_{s_0,\mu,C}$ exist since both contain the fixed point $(0,0)$. The existence of nontrivial stable and unstable sets of the fixed point $(0,0)$ is usually shown by the Stable Manifold Theorem (see Guckenheimer and Holmes [1983]). This requires showing that $D\psi_{s_0,\mu,C}(0,0)$ has no eigenvalues of modulus one. Direct computation of $D\psi_{s_0,\mu,C}(0,0)$ is rather difficult, but it turns out that the fixed point $(0,0)$ is degenerate (see McGehee [1973]). However we do have the following result about the nontrivial existence and smoothness of the stable and unstable sets of the fixed point $(0,0)$. Let $\{x > 0 : \Gamma\}$ be the set $\{(x,y) : x > 0, y \in \mathbf{R}\}$.

Theorem 23 (McGehee): *Fix* $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$. *The sets*

$$W^s_{\psi_{s_0,\mu,C}}(0,0) \subset \Gamma_C^{s=s_0} \quad \textit{and} \quad W^u_{\psi_{s_0,\mu,C}}(0,0) \subset \Gamma_C^{s=s_0}$$

of the fixed point $(0,0)$ *of the Poincaré map* $\psi_{s_0,\mu,C}$ *exist nontrivially, and are real analytic manifolds in* $\{x > 0 : \Gamma\}$.

Proof: See McGehee [1973].

We therefore call the sets $W^s_{\psi_{s_0,\mu,C}}(0,0) \cap \{x > 0 : \Gamma\}$ and $W^u_{\psi_{s_0,\mu,C}}(0,0) \cap \{x > 0 : \Gamma\}$ the *stable and unstable manifolds of the fixed point* $(0,0)$ of the Poincaré map $\psi_{s_0,\mu,C}$. We now look at some of the properties of these manifolds. By making use of the reversibility of (119) under the reflection L_1, we have the following result.

Proposition 24: *Fix* $(\mu, C) \in (0,1) \times \mathbf{R}$. *If* $s_0 = 0$ *or* π, *then, in the section* $\Gamma_C^{s=s_0}$,

$$W^s_{\psi_{s_0,\mu,C}}(0,0)$$

is the reflection of

$$W^u_{\psi_{s_0,\mu,C}}(0,0)$$

across the x*-axis.*

Proof: Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. Let $s_0 = 0$ or π. It suffices to take

$$(x_0, y_0) \in W^s_{\psi_{s_0,\mu,C}}(0,0)$$

and show that $(x_0, -y_0) \in W^u_{\psi_{s_0,\mu,C}}(0,0)$. So take $(x_0, y_0) \in W^s_{\psi_{s_0,\mu,C}}(0,0)$. Let

$$w(s) = (w_1(s+s_0), w_2(s+s_0), s+s_0)$$

be the solution $w(s; s_0, x_0, y_0)$ of (119). By Propositions 18 and 20,

$$\tilde{w}(s) = (w_1(-s+s_0), -w_2(-s+s_0), s-s_0)$$

is also a solution of (119), where $\tilde{w}(0) = (x_0, -y_0, s_0)$ since $-s_0 = s_0 (\text{mod } 2\pi)$. Suppress s_0, μ and C from the $\phi_{s_0,\mu,C}$. Now

$$\begin{aligned}\psi^{-1}(x_0, -y_0) &= (w_1(-(-2\pi)+s_0), -w_2(-(-2\pi)+s_0)) \\ &= (w_1(2\pi+s_0), -w_2(2\pi+s_0)),\end{aligned}$$

and so

$$\psi^{-n}(x_0, -y_0) = (w_1(2\pi n+s_0), -w_2(2\pi n+s_0)).$$

Since $(x_0, y_0) \in W^s_\psi(0,0)$, then

$$\lim_{n\to\infty} \psi^n(x_0, y_0) = \lim_{n\to\infty} (w_1(2\pi n+s_0), w_2(2\pi n+s_0)) = (0,0).$$

Hence

$$\lim_{n\to\infty} \psi^{-n}(x_0, -y_0) = \lim_{n\to\infty} (w_1(2\pi n + s_0), -w_2(2\pi n + s_0)) = (0,0),$$

and therefore $(x_0, y_0) \in W^u_\psi(0,0)$. This completes the proof.

Corollary 25: *Fix* $(\mu, C) \in (0,1)\times\mathbf{R}$. *Let* $s_0 = 0$ *or* π. *If* $(x_0, 0) \in W^s_{\psi_{s_0,\mu,C}}(0,0)$, *then* $(x_0, 0) \in W^u_{\psi_{s_0,\mu,C}}(0,0)$.

Proof: Take $y_0 = 0$ in the proof of Proposition 24. This completes the proof.

The corollary identifies the type of symmetry we are looking for. We use this to define the notion of a symmetric homoclinic point.

Definition 26: Fix $(s_0, \mu, C) \in S^1 \times (0,1) \times \mathbf{R}$. A point $(x_0, y_0) \in \Gamma_C^{s=s_0}$ with $x_0 \neq 0$ is *homoclinic to the fixed point* $(0,0) \in \Gamma_C^{s=s_0}$ of $\psi_{s_0,\mu,C}$ if

$$(x_0, y_0) \in W^s_{\psi_{s_0,\mu,C}}(0,0) \cap W^u_{\psi_{s_0,\mu,C}}(0,0).$$

A homoclinic point (x_0, y_0) is *transverse* if the tangent vector of $W^s_{\psi_{s_0,\mu,C}}(0,0)$ at (x_0, y_0) is not a scalar multiple of the tangent vector of $W^u_{\psi_{s_0,\mu,C}}(0,0)$ at (x_0, y_0). A homoclinic point (x_0, y_0) is *symmetric* if $(x_0, y_0) \in \Gamma_C^{s=s_0}$ for $s_0 = 0$ or π, and $y_0 = 0$.

Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. Let $s_0 = 0$ or π. A symmetric homoclinic point (x_0, y_0) of the fixed point $(0,0)$ of the Poincaré map $\psi_{s_0,\mu,C}$ corresponds to an orbit in the phase space of the extended reduced transformed circular problem which is homoclinic to the periodic orbit at infinity and has the symmetry associated with the reflection L_1. So to find a symmetric orbit homoclinic to the periodic orbit at infinity we simply need to find a point on the positive x-axis in $\Gamma_C^{s=s_0}$ for either $s_0 = 0$ or π that is homoclinic to the fixed point $(0,0)$. To show such a point exists, we proceed as follows, establishing a few lemmas before giving the proof.

If we set $\mu = 0$ in (111) we have the system

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{ds} = \dfrac{-x^4 + x^6\rho^2}{1 - x^4\rho}, \end{cases} \tag{120}$$

where ρ is a constant by Proposition 7. This is the extended reduced transformed unperturbed problem written in terms of the time variable s. We can bring (120) onto the same footing as (119), if we write (120) in the redundant form

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{d\tau} = \dfrac{-x^4 + x^6\rho^2}{1 - x^4\rho} \\ \dfrac{ds}{d\tau} = 1, \end{cases} \tag{121}$$

where τ is our dummy time variable from (119). Note that (119) is an autonomous perturbation of (121). The phase space of (121) is the same as the phase space of (119), which is $\mathbf{R}^2 \times S^1$. To "transfer" what we know about the phase space of the reduced transformed unperturbed problem (35) to the phase space of (121), we consider the orbit-to-set embedding map

$$\left\{(x(t), y(t)) : t \in \mathbf{R}\right\} \to \left\{(x(t(s+s_0)), y(t(s+s_0)), s + s_0 \pmod{2\pi}) : s \in \mathbf{R}\right\}, \tag{122}$$

where $s_0 \in S^1$ is fixed, and $t : \mathbf{R} \to \mathbf{R}$ is implicitly defined by

$$t(s) = \int_{s_0}^{s} \frac{1}{1 - x^4(t(s))\rho} ds. \tag{123}$$

For a fixed $s_0 \in S^1$, we call the image of the embedding map (122) the s_0*-embedding of the orbit* $\{(x(t), y(t)) : t \in \mathbf{R}\}$ into $\mathbf{R}^2 \times S^1$. Note that (123) is obtained by integrating (109) from s_0 to s requiring that $t(s_0) = 0$. It is not obvious if $t(s)$ even exists. We say the s_0-embedding (122) is *well-defined and smooth* if t, as a function of s, is well-defined at least C^1. We will show that under certain general conditions the s_0-embedding (122) is well-defined and smooth and has certain asymptotic properties.

Definition 27: Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous functions. We say f is *asymptotic* to g at plus infinity if

$$\lim_{s\to\infty} \frac{f(s)}{g(s)} \to 1,$$

and we write $f \sim g$ as $s \to \infty$. We say f is *asymptotic* to g at minus infinity if

$$\lim_{s\to-\infty} \frac{f(s)}{g(s)} \to 1,$$

and write $f \sim g$ as $s \to -\infty$. If $f \sim g$ as $s \to \infty$ and as $s \to -\infty$, we write $f \sim g$ as $s \to \pm\infty$.

Lemma 28: *Suppose $f \sim g$ as $s \to \infty$. If $g(s) \to \infty$ as $s \to \infty$, then $f(s) \to \infty$ as $s \to \infty$. If $g(s) \to -\infty$ as $s \to -\infty$, then $f(s) \to -\infty$ as $s \to -\infty$.*

Proof: We will show that $g(s) \to \infty$ as $s \to \infty$ implies that $f(s) \to \infty$ as $s \to \infty$. The other statement follows in a similar manner. Let $\epsilon > 0$. Since g is continuous and $g(s) \to \infty$ as $s \to \infty$, then there exists $\bar{s}$ such that

$$g(s) > 0 \quad \text{for all } s > \bar{s}. \tag{124}$$

Thus $1/g(s)$ is continuous for all $s > \bar{s}$. Since $g(s) \to \infty$ as $s \to \infty$, then there exists $\tilde{s} \geq \bar{s}$ such that

$$\left| \frac{1}{g(s)} \right| < \epsilon \quad \text{for all } s > \tilde{s}. \tag{125}$$

Suppose $f(s) \not\to \infty$ as $s \to \infty$. Then

$$\liminf_{s \to \infty} f(s) \equiv L_f < \infty,$$

which implies there is a sequence of $\{s_i\}_{i=1}^{\infty} \subset \mathbf{R}$ such that $s_i \to \infty$ as $i \to \infty$ and $\lim_{i \to \infty} f(s_i) = L_f$. There are two cases to consider.

Case 1. Suppose L_f is finite. Then $\{f(s_i)\}$ is a convergent sequence and hence is there is a $K \in \mathbf{R}^+$ such that

$$|f(s_i)| \leq K \quad \text{for all } i. \tag{126}$$

Since $s_i \to \infty$ as $i \to \infty$, there exists $\bar{i}$ such that $s_i > \tilde{s}$ for all $i \geq \bar{i}$. Thus, by (126) and (125),

$$\begin{aligned} \left| \frac{f(s_i)}{g(s_i)} \right| &= \frac{|f(s_i)|}{|g(s_i)|} \\ &\leq \frac{K}{|g(s_i)|} \\ &< K\epsilon \end{aligned}$$

for all $i \geq \bar{i}$. This implies that

$$\liminf_{s \to \infty} \frac{f(s)}{g(s)} \leq 0.$$

Hence it is impossible that

$$\lim_{s \to \infty} \frac{f(s)}{g(s)} \to 1,$$

and we have a contradiction with the hypothesis that $f \sim g$ as $s \to \infty$.

Case 2. Suppose $L_f = -\infty$. Then there exists $\tilde{i}$ such that

$$f(s_i) < 0 \quad \text{for all } i \geq \tilde{i}. \tag{127}$$

Since $s_i \to \infty$ as $i \to \infty$, there exists $\hat{i}$ such that $s_i > \bar{s}$ for all $i \geq \hat{i}$. Thus, by (127) and (124),

$$\frac{f(s_i)}{g(s_i)} < 0 \quad \text{for all } i \geq \min\{\tilde{i}, \hat{i}\}.$$

This implies that

$$\liminf_{s\to\infty} \frac{f(s)}{g(s)} \leq 0,$$

and we arrive at the same contradiction as in Case 1. This completes the proof.

Now we will establish some conditions under which the s_0-embedding (122) is well-defined and smooth, giving the asymptotic property that t, as a function of s, has under these conditions.

Lemma 29: *Fix $s_0 \in S^1, \mu = 0$. Let $\{(x(t), y(t)) : t \in \mathbf{R}\}$ be an orbit in the phase space of the extended reduced transformed unperturbed problem. If $\rho < 0$,*

$$\lim_{t\to\pm\infty} \int_0^t x^4(t)dt < \infty, \tag{128}$$

and $x(t) \to 0$ as $t \to \pm\infty$, then $t(s)$ exists, is smooth, and $t(s) \sim s$ as $s \to \pm\infty$.

Proof: Recall that ρ is a constant when $\mu = 0$ by Proposition 7. Differentiating (108) with respect to t, and using $d\theta/dt$ from (23), we have

$$\frac{ds}{dt} = 1 - \frac{d\theta}{dt} = 1 - x^4\rho. \tag{129}$$

Integrating (129) from 0 to t along the orbit $\{(x(t), y(t)) : t \in \mathbf{R}\}$ we have

$$s(t) = \int_0^t \left(1 - x^4(t)\rho\right) dt + s(0), \tag{130}$$

where $s(0) = s_0$. Since (129) is continuous, then s is at least a C^1 function of t. Since $\rho < 0$, then (129) never equals zero, and so s is a strictly increasing function of t. By (128), (130) implies

$$\lim_{t\to\pm\infty} s(t) = \pm\infty, \tag{131}$$

and so s is a bijective function of t. Therefore t exists as a strictly increasing smooth bijective function of s. Further,

$$\lim_{s \to \pm\infty} t(s) = \pm\infty \tag{132}$$

by the monotonicity of s as a function of t and (131). Since $x(t) \to 0$ as $t \to \pm\infty$, then $x^4(t) \to 0$ as $t \to \pm\infty$. Hence, by l'Hospital's rule (see Johnsonbaugh and Pfaffenberger [1981]), Lemma 14, and (132),

$$\begin{aligned} \lim_{s \to \pm\infty} \frac{t(s)}{s} &= \lim_{s \to \pm\infty} \frac{dt(s)}{ds} \\ &= \lim_{s \to \pm\infty} \frac{1}{1 - x^4(t(s))\rho} \\ &= 1. \end{aligned}$$

Therefore $t(s) \sim s$ as $s \to \pm\infty$. This completes the proof.

We have the following result regarding the integration of the parabolic solutions (39) over semi-infinite intervals.

Lemma 30: *If $\rho \neq 0$ and $x(t) = \xi(t; \rho)$ as given in* (39), *then*

$$\lim_{t \to \pm\infty} \int_0^t x^4(t) dt < \infty.$$

Proof: From (39),

$$\begin{aligned} \xi^2(t; \rho) &= \frac{2}{(3t + \sqrt{9t^2 + \rho^6})^{2/3} + (3t - \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2} \\ &\leq \frac{2}{(3t + \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2}, \end{aligned} \tag{133}$$

since $(3t - \sqrt{9t^2 + \rho^6})^{2/3} \geq 0$ for all $t \in \mathbf{R}$. As $9t^2 + \rho^6 \geq 9t^2$ for all $t \in \mathbf{R}$, then

$$\begin{aligned} (3t + \sqrt{9t^2 + \rho^6})^{2/3} - \rho^2 &\geq (3t + \sqrt{9t^2})^{2/3} - \rho^2 \\ &= (6t)^{2/3} - \rho^2. \end{aligned} \tag{134}$$

Hence, by (134), (133) becomes

$$\begin{aligned} \xi^4(t; \rho) &\leq \frac{4}{[(6t)^{2/3} - \rho^2]^2} \\ &= \frac{4}{(6t)^{4/3} - 2(6t)^{2/3}\rho^2 + \rho^4} \\ &< \frac{4}{(6t)^{4/3} - 2(6t)^{2/3}\rho^2} \\ &= \frac{4}{(6t)^{2/3}[(6t)^{2/3} - 2\rho^2]}. \end{aligned} \tag{135}$$

If $a > 0$, then

$$\left(6\frac{a^6}{6}\right)^{2/3} - 2\rho^2 - \left(6\frac{a^6}{6}\right)^{1/2} = a^4 - 2\rho^2 - a^3$$
$$= a^3(a-1) - 2\rho^2 > 0 \qquad (136)$$

for all sufficiently large a. Let $\bar{a}$ be sufficiently large so that (136) holds for all $a > \bar{a}$. Let $\bar{t} = \bar{a}^6/6$. Then (136) implies that for all $t > \bar{t}$, $(6t)^{2/3} - 2\rho^2 > (6t)^{1/2}$ Thus (135) yields

$$\xi^4(t;\rho) < \frac{4}{(6t)^{2/3}(6t)^{1/2}}$$
$$= \frac{4}{(6t)^{7/6}} \quad \text{for all } t \geq \bar{t}. \qquad (137)$$

Since $0 < \xi(t;\rho) \leq \sqrt{2}|\rho|^{-1}$, then

$$0 < \xi^4(t;\rho) \leq \frac{4}{|\rho|^4} \quad \text{for all } t \in [0, \bar{t}\,]. \qquad (138)$$

Therefore, by (137) and (138),

$$\begin{aligned}\lim_{t\to\infty}\int_0^t \xi^4(t;\rho)dt &= \int_0^{\bar{t}} \xi^4(t;\rho)dt + \lim_{t\to\infty}\int_{\bar{t}}^t \xi(t;\rho)dt \\ &\leq \int_0^{\bar{t}} \frac{4}{|\rho|^4}dt + \lim_{t\to\infty}\int_{\bar{t}}^t \frac{4}{(6t)^{7/6}}dt \\ &= \frac{4\bar{t}}{|\rho|^4} - 4\lim_{t\to\infty}\left[(6t)^{-1/6}\right]_{\bar{t}}^t \\ &= \frac{4\bar{t}}{|\rho|^4} - 4\left[\lim_{t\to\infty}(6t)^{-1/6} - (6\bar{t})^{-1/6}\right] \\ &= \frac{4\bar{t}}{|\rho|^4} + 4(6\bar{t})^{-1/6} \\ &< \infty \end{aligned} \qquad (139)$$

if $\rho \neq 0$. Since $\xi(t;\rho)$ is an even function in t, then

$$\begin{aligned}\lim_{t\to-\infty}\int_0^t \xi^4(t;\rho)dt &= \lim_{t\to\infty}\int_0^{-t} \xi^4(t;\rho)dt \\ &= \lim_{t\to\infty}\int_0^t \xi(-t;\rho)dt \\ &= \lim_{t\to\infty}\int_0^t \xi(t;\rho)dt \\ &< \infty\end{aligned}$$

by (139). This completes the proof.

Before we can prove the existence of symmetric homoclinic points, homoclinic to the fixed point $(0,0)$, we need just one more

Lemma 31: *If $\mu = 0$ and $\rho \neq 0$, then $C = -\rho$ on any solution in $H(x,y;\rho) = 0$.*

Proof: Recall that ρ is a constant when $\mu = 0$ by Proposition 7. Let (x,y) be a solution in $H(x,y;\rho) = 0$. Then from (38) and (82) with $\mu = 0$,

$$\begin{aligned} C = J(x,\theta,y,\rho,t;0) &= \frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - \hat{U}(x,t-\theta;0) - \rho \\ &= \frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - x^2 - \rho \\ &= H(x,y;\rho) - \rho \\ &= -\rho. \end{aligned}$$

This completes the proof.

Theorem 32: *Let $s_0 = 0$ or π. If $C > 0$, then for all sufficiently small $\mu > 0$ there exists a symmetric homoclinic point $(x_0, 0) \in \Gamma_C^{s=s_0}$, homoclinic to the fixed point $(0,0) \in \Gamma_C^{s=s_0}$ of the Poincaré map $\psi_{s_0,\mu,C}$.*

Proof: Let $C > 0$, and set $\mu = 0$. The equilibrium solution

$$x \equiv 0, \; y \equiv 0 \tag{140}$$

of the extended reduced transformed unperturbed problem is a solution contained in the manifold $H(x,y;\rho) = 0$. So by Lemma 30, $C = -\rho > 0$ on (140). By Lemma 29, the s_0-embedding of (140) is well-defined and smooth. The s_0-embedding of (140) is the 2π-periodic orbit

$$\{(0, 0, s + s_0 \; (\mathrm{mod}\; 2\pi)) : s \in \mathbf{R}\} \tag{141}$$

in the phase space of (121). For $s_1 \in S^1$ with $s_1 \neq s_0$, the s_1-embedding of (140) has the same orbit as (141). We call (141) the *unperturbed periodic orbit at infinity.* It has the same orbit as the periodic orbit at infinity. Recall the latter is for when $\mu \in (0,1)$ while the former is for $\mu = 0$. We will first determine what the stable manifold of the unperturbed periodic orbit at infinity (141) is in the region $\{x > 0\}$, showing that it intersects the infinite cylinder $\{y = 0\} = \{(x, 0, s) : x \in \mathbf{R}, s \in S^1\}$. Then we will consider what happens to this stable set when μ is perturbed.

In the extended reduced transformed unperturbed problem, the stable set of the equilibrium solution (140) is given by the parabolic solution (39). Since the parabolic solution is a solution in the manifold $H(x,y;\rho)=0$, then by Lemma 31, $C=-\rho>0$ on the parabolic solution. So by Lemma 30, and the fact that $\xi(t;\rho)\to 0$ as $t\to\pm\infty$, Lemma 29 implies that the s_0-embedding of the parabolic solution with $\rho=-C$ is well-defined and smooth, and given by

$$\left\{\left(\xi(t(s+s_0);-C),\eta(t(s+s_0);-C),s+s_0 \pmod{2\pi}\right):s\in\mathbf{R}\right\}. \qquad (142)$$

Since $t(s)\sim s$ as $s\to\infty$ by Lemma 29, then by Lemma 28, $t(s)\to\infty$ as $s\to\infty$. Thus the orbit (142) belongs to the stable set of the unperturbed periodic orbit at infinity. If we take $s_1\in S^1$ with $s_1\neq s_0$, then the s_1-embedding of the parabolic solution (39) with $\rho=-C<0$ belongs to the stable set of (141), and is disjoint from the orbit of (142). Thus the set

$$\bigcup_{s_0\in S^1}\left\{\left(\xi(t(s+s_0);-C),\eta(t(s+s_0);-C),s+s_0 \pmod{2\pi}\right):s\in\mathbf{R}\right\} \qquad (143)$$

is the stable set of the unperturbed periodic orbit at infinity. The set in (143) is simply the set

$$\left\{\left(\xi(t(s+s_0);-C),\eta(t(s+s_0);-C)\right):s\in\mathbf{R}\right\}\times S^1. \qquad (144)$$

By the smoothness of t as a function of s and the smoothness of the parabolic solution (39) with $\rho=-C$, the set (144) is a smooth two-dimensional manifold in the phase space of (121), smooth in the sense that it is at least C^1. Actually, the stable set of (141) is a real analytic manifold in the region $\{x>0\}$. Since $t(s_0)=0$, then

$$\begin{aligned}\left(\xi(t(s_0);-C),\eta(t(s_0);-C)\right) &= \left(\xi(0;-C),\eta(0;-C)\right)\\ &= (\sqrt{2}|C|^{-1},0)\end{aligned}$$

and so the stable manifold (144) intersects the infinite cylinder $\{y=0\}$ in the circle

$$\left\{(\sqrt{2}|C|^{-1},0)\right\}\times S^1. \qquad (145)$$

Now perturb $\mu>0$. The unperturbed periodic orbit at infinity becomes the periodic orbit at infinity. By invariant manifold theory (see Fenichel [1971]), for small enough $\mu>0$, the set

$$\left\{\left(\xi(t(s+s_0);-C),\eta(t(s+s_0);-C)\right):s\in[-3\pi,\infty)\right\}\times S^1\cup\{(0,0)\} \qquad (146)$$

is C^1-close to the stable manifold of the periodic orbit at infinity. From Theorem 23, we know that the stable manifold of the periodic orbit at infinity is given by

$$\bigcup_{s_0 \in S^1} W^s_{\psi_{s_0,\mu,C}}(0,0) \times \{s_0\}. \tag{147}$$

By the C^1-closeness, (147) intersects with the infinite cylinder $\{y = 0\}$, and the intersection is C^1-close to the circle (145). Therefore, in the section $\Gamma_C^{s=s_0}$, $W^s_{\psi_{s_0,\mu,C}}(0,0)$ intersects the x-axis at some point $(x_0, 0)$, where x_0 is close to $\sqrt{2}|C|^{-1}$. Since $s_0 = 0$ or π, then Corollary 25 implies $(x_0, 0) \in W^u_{\psi_{s_0,\mu,C}}(0,0)$. The point $(x_0, 0)$ is a symmetric homoclinic point of the fixed point $(0,0)$. This completes the proof.

Since $s_0 = 0$ or π, there exists two distinct families of symmetric homoclinic points, homoclinic to the fixed point of the fixed point $(0,0)$, by Theorem 31. We denote the ones in $\Gamma_C^{s=0}$ by $\bar{p}(\mu, C)$, and denote the ones in $\Gamma_C^{s=\pi}$ by $p(\mu, C)$. Now that we have established the existence of symmetric homoclinic points of the fixed point $(0,0)$ of the Poincaré map $\psi_{s_0,\mu,C}$, for small enough μ, we consider the transversality of the symmetric homoclinic points $p(\mu, C)$ and $\bar{p}(\mu, C)$. The usual way in which to show that homoclinic points are transverse is by the method of Melnikov (see Guckenheimer and Holmes [1983]). Xia used the method of Melnikov to prove that the symmetric homoclinic points $p(\mu, C)$ and $\bar{p}(\mu, C)$ are transverse.

Theorem 33 (Xia): *Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. For all sufficiently small $\mu > 0$ the symmetric homoclinic points $p(\mu, C)$ and $\bar{p}(\mu, C)$ of the fixed point $(0,0)$ of the Poincaré map $\psi_{s_0,\mu,C}$, where $s_0 = 0$ or π, are transverse.*

Proof: See Xia [1992], and Xia [1993].

Definition 34: Fix $(\mu, C) \in (0,1) \times \mathbf{R}$. Let $s_0 \in S^1$ and $(x_0, y_0) \in \Gamma_C^{s=s_0}$. The *orbit operator* $\gamma_{s_0} : \Gamma_C^{s=s_0} \to \mathcal{P}(\mathbf{R}^2 \times S^1)$ is given by $\gamma_{s_0}(x_0, y_0) = \{w(\tau + s_0) : \tau \in \mathbf{R}\}$, where $w(\tau + s_0)$ is the solution of (119), with parameters μ and C, that satisfies the intial condition $w(s_0) = (x_0, y_0, s_0)$, and where $\mathcal{P}$ is the power set operator.

Recall that the Poincaré map $\psi_{s_0,\mu,C}$ reduced the three-dimensional flow in the phase space $\mathbf{R}^2 \times S^1$ of (119) to a two-dimensional mapping of points in the section $\Gamma_C^{s=s_0}$. The orbit operator $\Gamma_C^{s=s_0}$ takes a point $(x_0, y_0) \in \Gamma_C^{s=s_0}$ and gives the solution curve in the three-dimensional flow of the phase space of (119) passing through the point (x_0, y_0, s_0). The relationship between the Poincaré map $\psi_{s_0,\mu,C}$ and the orbit operator γ_{s_0} is

$$\begin{aligned}\gamma_{s_0}(x_0, y_0) \cap \{s = s_0 : \Gamma\} &= \gamma_{s_0}(x_0, y_0) \cap (\Gamma_C^{s=s_0} \times \{s_0\}) \\ &= \left\{ \left(\psi^j_{s_0,\mu,C}(x_0, y_0), s_0\right) : j \in \mathbf{Z} \right\}.\end{aligned}$$

The periodic orbit at infinity can be written as $\gamma_{s_0}(0,0)$. Note that $\gamma_{s_1}(0,0) = \gamma_{s_0}(0,0)$ for all $s_1 \in S^1$. The stable set $W^s(\gamma_{s_0}(0,0))$ is given by (147), and the unstable set of $\gamma_{s_0}(0,0)$ is

$$W^u(\gamma_{s_0}(0,0)) = \bigcup_{s_1 \in S^1} W^u_{\psi_{s_1,\mu,C}}(0,0) \times \{s_1\}. \tag{148}$$

The transverse symmetric orbits, in the phase space of (119), homoclinic to the periodic orbit at infinity given by Theorem 33 can be written as $\gamma_0(\bar{p}(\mu, C))$, $\gamma_\pi(p(\mu, C))$. Note that these orbits are subsets of $W^s(\gamma_{s_0}(0,0)) \cap W^u(\gamma_{s_0}(0,0))$.

We will now consider approximating solutions in the stable and unstable manifold of the periodic orbit at infinity by the s_0-embedding of a parabolic solution (39) by applying some perturbation theory (see Yosida [1991]). Fix $C > 0$. Let $(x^s_\mu(s + s_0), y^s_\mu(s+s_0), s+s_0)$ be a solution in $W^s(\gamma_{s_0}(0,0))$. Then $(x^s_\mu(s_0), y^s_\mu(s_0)) \in \Gamma_C^{s=s_0}$. Let $(x^u_\mu(s + s_0), y^u_\mu(s + s_0), s + s_0)$ be a solution in $W^u(\gamma_{s_0}(0,0))$. Then $(x^u_\mu(s_0), y^u_\mu(s_0)) \in \Gamma_C^{s=s_0}$. The extended reduced transformed circular problem can be written as

$$\frac{dv}{ds} = z_0(v) + \mu z_1(v, s) + O(\mu^2), \tag{149}$$

where $v = (x, y)$, z_0 is the vector field of (120), and

$$z_1(v, s) = \left(0, \frac{\tilde{g}_1(x, s)}{1 - x^4\rho}\right). \tag{150}$$

Since the s_0-embedding of the parabolic solution for $\rho = -C < 0$ is real analytic in μ, then we can write

$$(x^s_\mu(s + s_0), y^s_\mu(s + s_0)) \equiv v^s_\mu(s + s_0) = v_0(s + s_0) + \mu v^s_1(s + s_0) + O(\mu^2), \tag{151}$$

where

$$\begin{aligned}v_0(s + s_0) &= (x_0(s + s_0), y_0(s + s_0)) \\ &= (\xi(t(s + s_0); -C), \eta(t(s + s_0); -C)).\end{aligned} \tag{152}$$

We will derive a differential equation that v_1^s satisfies. By differentiating v_μ^s with respect to μ, and then setting $\mu = 0$, we have

$$v_1^s(s+s_0) = \left.\frac{\partial v_\mu^s(s+s_0)}{\partial \mu}\right|_{\mu=0}. \tag{153}$$

Substituting v_μ^s, given in (151), into (149), differentiating both sides with respect to μ, then setting $\mu = 0$, and using (153), we have

$$\begin{aligned}
\frac{d}{ds}\left(v_1^s(s+s_0)\right) &= \frac{d}{ds}\left[\frac{\partial v_\mu^s(s+s_0)}{\partial \mu}\right]_{\mu=0} \\
&= \frac{\partial}{\partial \mu}\left[\frac{d v_\mu^s(s+s_0)}{ds}\right]_{\mu=0} \\
&= \frac{\partial}{\partial \mu}\left[z_0\left(v_\mu^s(s+s_0)\right) + \mu z_1\left(v_\mu^s(s+s_0), s+s_0\right) + O(\mu^2)\right]_{\mu=0} \\
&= \left[D_v z_0\left(v_\mu^s(s+s_0)\right)\frac{\partial v_\mu^s(s+s_0)}{\partial \mu}\right]_{\mu=0} + z_1\left(v_\mu^s(s+s_0), s+s_0\right)|_{\mu=0} \\
&= D_v z_0\left(v_0(s+s_0)\right)v_1^s(s+s_0) + z_1\left(v_0(s+s_0), s+s_0\right),
\end{aligned} \tag{154}$$

where $v_0(s+s_0)$ is given by (152), z_0 is the vector field of (120), and z_1 is given by (150). Equation (154) is a first order nonlinear differential equation which v_1^s satisfies, and is called the *first variational equation of* (119). Similarly,

$$\frac{d}{ds}\left(v_1^u(s+s_0)\right) = D_v z_0\left(v_0(s+s_0)\right)v_1^u(s+s_0) + z_1\left(v_0(s+s_0), s\right)$$

is the first variational equation that v_1^u satisfies, where

$$v_\mu^u(s+s_0) = v_0(s+s_0) + \mu v_1^u(s+s_0) + O(\mu^2).$$

Note that

$$x_\mu^s(s+s_0) = x_0(s+s_0) + \mu x_1^s(s+s_0) + O(\mu^2), \tag{155}$$

$$y_\mu^s(s+s_0) = y_0(s+s_0) + \mu y_1^s(s+s_0) + O(\mu^2), \tag{156}$$

$$x_\mu^u(s+s_0) = x_0(s+s_0) + \mu x_1^u(s+s_0) + O(\mu^2), \tag{157}$$

$$y_\mu^u(s+s_0) = y_0(s+s_0) + \mu y_1^u(s+s_0) + O(\mu^2), \tag{158}$$

where $\left(x_1^s(s+s_0), y_1^s(s+s_0)\right) \equiv v_1^s(s+s_0)$, and $\left(x_1^u(s+s_0), y_1^u(s+s_0)\right) \equiv v_1^u(s+s_0)$.

An approximation $v(s) = v_0(s) + \mu v_1(s) + O(\mu^2)$ is said to be *uniformly valid* on an interval $A \subset \mathbf{R}$ if there exists a function $\tilde{v} : A \to \mathbf{R}$ satisfying $\tilde{v}(s) = O(\mu)$ such that $v(s) = v_0(s) + \tilde{v}(s)$ for all $s \in A$.

Theorem 35: *Fix $C > \sqrt{2}$. The approximations* (155) *and* (156) *hold with uniform validity on the time interval* $[s_0, \infty)$, *and the approximations* (157) *and* (158) *hold with uniform validity on the time interval* $(-\infty, s_0]$.

Proof: The reason for the condition $C > \sqrt{2}$ is as follows. Recall that the parabolic solution (39) intersects the x-axis at the point $\sqrt{2}C^{-1}$, and recall that $x^{-2} = \|\mathbf{q}\|$ where $\mathbf{q}$ is the position vector of the zero-mass. For $\mu > 0$, the primary P_2 has mass μ and its orbit is uniformly approximated by the unit circle for small enough μ. As we showed in the proof of Theorem 32, the perturbation of the parabolic solution is close to the parabolic solution on a semi-infinite time interval. So if $0 < C \leq \sqrt{2}$, then the perturbed parabolic solution will intersect the unit circle. It is therefore possible that the perturbed parabolic solution will collide with P_2, and so there may be binary collisions in the stable and unstable manifolds of the periodic point at infinity. At the collision points the potential function experiences a infinite discontinuity, and so we avoid that problem by requiring $C > \sqrt{2}$.

Fix $C > \sqrt{2}$. Recall that for the unperturbed problem (120), the stable and unstable manifold of the unperturbed periodic orbit at infinity are identical and given by (144). Let V be a neighbourhood of $\gamma_{s_0}(0,0)$ such that inside V the perturbed local stable and unstable manifolds are μ C^1-close to the unperturbed stable manifold. Let $(x_0(s_0), y_0(s_0), s_0)$ be a point in the unperturbed stable manifold (144) of the unperturbed periodic orbit at infinity which is outside of V. Since solutions depend continuously on the initial conditions, then perturbed orbits starting near $(x_0(s_0), y_0(s_0), s_0)$ remains within $O(\mu)$ of $(x_0(s + s_0), y_0(s + s_0), s + s_0)$ for finite times. So we can follow the perturbed orbit $(x^s_\mu(s + s_0), y^s_\mu(s + s_0), s + s_0)$ starting near $(x_0(s_0), y_0(s_0), s_0)$ to the boundary of V. Once in V, since the perturbed orbit $(x^s_\mu(s + s_0), y^s_\mu(s + s_0), s + s_0)$ lies in the perturbed stable manifold $W^s(\gamma_{s_0}(0,0))$, then

$$\left|(x^s_\mu(s + s_0), y^s_\mu(s + s_0)) - (x_0(s + s_0), y_0(s + s_0))\right| = O(\mu)$$

for all $s \in [s_0, \infty)$. Similarly, one obtains the result for $(x^u_\mu(s + s_0), y^u_\mu(s + s_0), s + s_0)$.

This completes the proof.

This Theorem says the orbit $\gamma_\pi(p(\mu, C))$ in the stable manifold $W^s(\gamma_\pi(0,0))$ is approximated on the semi-infinite time interval $[\pi, \infty)$, with uniform validity, by the π-embedding of the parabolic solution (39) with $\rho = -C$. And the orbit $\gamma_\pi(p(\mu, C))$ in the unstable manifold $W^u(\gamma_\pi(0,0))$ is approximated on the semi-infinite time interval $(-\infty, \pi]$, with uniform validity, by the π-embedding of the parabolic solution (39) with $\rho = -C$. Similar results hold for the orbits $\gamma_0(\bar{p}(\mu, C))$.

We will now introduce symbolic dynamics into the discussion. We will deal just with the transverse symmetric homoclinic points $p(\mu, C)$, the results going through verbatim for the points $\bar{p}(\mu, C)$. Let O be $(\sqrt{2}, \sqrt{2}+\delta)$ for $\delta > 0$ sufficiently small. Fix $\mu > 0$ sufficiently small so that $p(\mu, C)$ is a transverse symmetric homoclinic point. Near the point $p(\mu, C)$ in the section $\Gamma_C^{s=\pi}$, construct a small *quadrilateral* Q, the sides adjacent to $p(\mu, C)$ consisting of parts of $W^s_{\psi_{\pi,\mu,C}}(0,0)$ and $W^u_{\psi_{\pi,\mu,C}}(0,0)$, and the opposite sides consisting of straight lines segments parallel to the tangent vectors of $W^s_{\psi_{\pi,\mu,C}}(0,0)$ and $W^u_{\psi_{\pi,\mu,C}}(0,0)$ at $p(\mu, C)$. Consider the function

$$k_{Q,\pi,\mu} : O \times Q \to \mathbf{N} \cup \{\infty\},$$

where for $q \in Q$, $k_{Q,\pi,\mu}(C, q)$ is the smallest postive integer for which $\psi_{\pi,\mu,C}^{k_{Q,\pi,\mu}(C,q)}(q) \in Q$, if it exists; otherwise $k_{Q,\pi,\mu}(C, q) = \infty$. Let $D(Q, \pi, \mu, C)$ be the set of points $q \in Q$ for which $k_{Q,\pi,\mu}(C, q) < \infty$. Define $\tilde{\psi}_{\pi,\mu,C} : D(Q, \pi, \mu, C) \to Q$ by

$$\tilde{\psi}_{Q,\pi,\mu,C}(q) = \psi_{\pi,\mu,C}^{k_{Q,\pi,\mu}(C,q)}(q). \tag{159}$$

We call $\tilde{\psi}_{Q,\pi,\mu,C}$ the *transverse map* of $\psi_{\pi,\mu,C}$ on Q. It is not apparent if the set $D(Q, \pi, \mu, C)$ is nonempty. The space of symbolic sequences that we will consider is a set of bi-infinite sequences on infinitely-many symbols. In particular, we will consider the sequence space $S = \mathbf{N}^{\mathbf{Z}}$. The shift operator $\sigma : S \to S$ is the usual one. Let $s^* = (\ldots s^*_{-1}, s^*_0; s^*_1, s^*_2, \ldots) \in S$. The topology of S is generated from the basic sets

$$O_j = \{s \in S : s_i = s^*_i \text{ for } |i| < j\}$$

for $j \in \mathbf{N}$.

Definition 36: Let A_1 and A_2 each be a subset of some topological space. Two maps $f : A_1 \to A_1$ and $g : A_2 \to A_2$ are *topologically conjugate* if there exists a homeomorphism $\omega : A_1 \to A_2$ such that $f \circ \omega = \omega \circ g$.

Theorem 37 (Moser): *Fix $C > \sqrt{2}$ such that $C - \sqrt{2}$ is sufficiently small. Let Q be a quadrilateral as defined above. If $\mu > 0$ is sufficiently small, then there exists an invariant subset $I \subset Q \subset \Gamma_C^{s=\pi}$ such that the restriction of $\tilde{\psi}_{Q,\pi,\mu,C}$ to I and σ are topologically conjugate.*

Remark: Note that the set I given in Theorem 37 is a subset of $D(Q, \pi, \mu, C)$ defined above, and is nonempty since it is the homeomorphic image of the sequence space S.

Proof: The Poincaré map $\psi_{\pi,\mu,C}$ is real analytic by Proposition 21, and the stable and unstable manifolds of the fixed point $(0,0)$ exist and are real analytic in $\{x > 0 : \Gamma\}$ by Theorem 23. For small enough $\mu > 0$, the stable and unstable manifolds of the fixed point $(0,0)$ intersect transversely near $p(\mu, C)$ by Theorem 33. The proof now follows that in Moser [1973].

We will now use this result of Moser to show the existence of periodic points of the Poincaré map $\psi_{\pi,\mu,C}$ as close as we want to the transverse symmetric homoclinic point $p(\mu, C)$ and with arbitrarily large periods.

Proposition 38: *Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. For all $\mu > 0$ sufficiently small, there exists a sequence $\{v_m(\mu, C)\}_{m=1}^{\infty}$ of periodic points of the Poincaré map $\psi_{\pi,\mu,C}$ such that the period of $v_{m+1}(\mu, C)$ is strictly greater than that of $v_m(\mu, C)$, and*

$$\lim_{m \to \infty} v_m(\mu, C) = p(\mu, C).$$

Proof: Fix $\mu > 0$ sufficiently small. In what follows, we will suppress the constants π, μ, and C from the notation. Let $\{Q_m\}_{m=1}^{\infty}$ be a sequence of quadrilaterals such that $Q_{m+1} \subset Q_m$ and the Lebesgue measure of Q_m goes to zero as $m \to \infty$. For each Q_m we can define a transverse map $\hat{\psi}_m$ as we did in the discussion preceding Theorem 37. By Theorem 37, there exists an invariant subset $I_1 \subset Q_1$ and a homeomorphism $\omega_1 : S \to I_1$ such that $\hat{\psi}_1 \circ \omega_1 = \omega_1 \circ \sigma$, where $\hat{\psi}_1$ is the restriction of $\tilde{\psi}_{Q_1}$ to I_1. Let

$u_1 \in S$ be a fixed point of σ. Let $v_1 = \omega_1(u_1)$. Since $v_1 \in D(Q_1)$, then there exists a $k_1 \in \mathbf{N}$ such that

$$\psi^{k_1}(v_1) = \left(\omega_1 \circ \sigma \circ \omega_1^{-1}\right)(v_1) = (\omega_1 \circ \sigma)(u_1) = \omega_1(u_1) = v_1. \tag{160}$$

As k_1 is the smallest positive integer for which (160) is true, v_1 is a k_1-periodic point of ψ. Now we will find a v_2 such that the period of v_2 is strictly greater than that of v_1.

By Theorem 37, there exists an invariant subset $I_2 \subset Q_2$ and a homeomorphism $\omega_2 : S \to I_2$ such that $\hat{\psi}_2 \circ \omega_2 = \omega_2 \circ \sigma$, where $\hat{\psi}_2$ is the restriction of $\tilde{\psi}_{Q_2}$ to I_2. Let $u_2 \in S$ be a (k_1+1)-periodic point of σ. Let $v_2 = \omega_2(u_2)$. Since $v_2 \in D(Q_2)$, then there exists $k_2 \in \mathbf{N}$ such that

$$\begin{aligned} \psi^{k_2(k_1+1)} &= \left(\omega_2 \circ \sigma \circ \omega_2^{-1}\right)^{k_1+1}(v_2) \\ &= \left(\omega_2 \circ \sigma^{k_1+1} \circ \omega_2^{-1}\right)(v_2) \\ &= (\omega_2 \circ \sigma^{k_1+1})(u_2) \\ &= \omega_2(u_2) \\ &= v_2. \end{aligned} \tag{161}$$

As k_2 is the smallest positive integer for which (161) is true, v_2 is a $k_2(k_1+1)$-periodic point of ψ. Clearly the period of v_2 is strictly greater than that of v_1.

Continuing with Q_3, Q_4, etc, in a like manner, we generate a sequence $\{v_m\}_{m=1}^{\infty}$ of periodic points of ψ such that the period of v_{m+1} is strictly greater than that of v_m. The choice of the sequence of quadrilaterals $\{Q_m\}_{m=1}^{\infty}$ with the Lebesgue measure of the Q_m going to zero as $m \to \infty$ gives the result that $v_m(\mu, C) \to p(\mu, C)$ as $m \to \infty$. This completes the proof.

Corollary 39: *Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. If $\mu \in (0,1)$ is sufficiently small, then for any $\tilde{n} \in \mathbf{N}$ there exists $n > \tilde{n}$ such that there is a periodic point of $\psi_{\pi,\mu,C}$ of period n.*

Proof: By the construction of the sequence $\{v_m(\mu, C)\}$ of periodic points of ψ in Proposition 38, the period of a given $v_m(\mu, C)$ is some positive integer $l_m(\mu, C)$. The sequence $\{l_m(\mu, C)\}_{m=1}^{\infty}$ is strictly increasing, but in general will not contain every positive integer. If we are given a $\tilde{n} \in \mathbf{N}$ we can always find some m such that $l_m(\mu, C) > \tilde{n}$. Let $n = l_m(\mu, C)$. Then the periodic point $v_m(\mu, C)$ of $\psi_{\pi,\mu,C}$ has period n. This completes the proof.

Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. For each $\mu \in (0,1)$ with μ sufficiently small, fix a sequence $\{v_m(\mu, C)\}_{m=1}^{\infty}$ of perioidic points of $\psi_{\pi,\mu,C}$. Let

$$\tilde{\mathrm{N}}(\pi, \mu, C) = \{l_m(\mu, C) : m \in \mathbf{N}\}, \tag{162}$$

where $\{l_m(\mu, C)\}_{m=1}^{\infty}$ is the sequence of the periods of the periodic points $\{v_m(\mu, C)\}$. Thus for each $n \in \tilde{\mathrm{N}}(\pi, \mu, C)$, there is a periodic point $p_n(\mu, C)$ of $\psi_{\pi,\mu,C}$ which has period n. There may not be periodic points of a given period, but by Corollary 39, there are periodic points of arbitrarily large period. By Proposition 38, the periodic points $p_n(\mu, C)$ which have arbitrarily large period are very close to $p(\mu, C)$. Each $p_n(\mu, C)$ is hyperbolic (see Moser [1973]), and so applying the stable manifold theorem we know there exists a smooth one-dimensional stable manifold $W^s_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$, and a smooth one-dimensional unstable manifold $W^u_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$. We finish this section with the following result about the stable and unstable manifolds of $p_n(\mu, C)$.

Proposition 40: *Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small, and fix $\mu \in (0,1)$ sufficiently small. For all sufficiently large $n \in \tilde{\mathrm{N}}(\pi, \mu, C)$,*

$$W^s_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$$

intersects

$$W^u_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$$

transversely.

Proof: For $n \in \tilde{\mathrm{N}}(\pi, \mu, C)$ large enough,

$$W^s_{\psi_{\pi,\mu,C}}(p_n(\mu, C)) \quad \text{and} \quad W^u_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$$

are locally C^1-close to

$$W^s_{\psi_{\pi,\mu,C}}(0,0) \quad \text{and} \quad W^u_{\psi_{\pi,\mu,C}}(0,0)$$

respectively about $p(\mu, C)$ by the smoothness of the Poincaré map $\psi_{\pi,\mu,C}$ and by the closeness of $p_n(\mu, C)$ to $p(\mu, C)$. This implies that $W^s_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$ intersects $W^u_{\psi_{\pi,\mu,C}}(0,0)$ transversely, and that $W^u_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$ intersects $W^s_{\psi_{\pi,\mu,C}}(0,0)$ transversely since $W^s_{\psi_{\pi,\mu,C}}(0,0)$ intersects $W^u_{\psi_{\pi,\mu,C}}(0,0)$ transversely at $p(\mu, C)$ by Theorem 33. Therefore, by the λ-Lemma (see Palis and de Melo [1982]), $W^s_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$ intersects $W^u_{\psi_{\pi,\mu,C}}(p_n(\mu, C))$ transversely.

6. A Twist Map in the Reduced Transformed Circular Problem

In this section we continue the analysis of the reduced transformed circular problem. By the reductions of the equations, certain of the variables dependent on s are explicitly or implicitly solved for. By including the derivatives of certain of these dependent variables with the equations of reduced transformed circular problem, we produce a system which has a five-dimensional (generalized) phase space. The flow in this phase space is reducible to a four-dimensional Poincaré map $\phi_{\pi,\mu}$ which is related to the two-dimensional Poincaré map $\psi_{\pi,\mu,C}$ from the last section. We will show that "most" of the periodic points of $\psi_{\pi,\mu,C}$ found in the last section are, upon transfer to $\phi_{\pi,\mu}$, no longer periodic, but *quasiperiodic*. By showing this we will establish suitable restrictions on μ and C so that $\phi_{\pi,\mu}$ admits a twist map.

The variables depending on s in the extended reduced transformed circular problem are x, y, θ, ρ, C, and t. Knowing one of the variables θ or t, (108) gives us the other one. It will be advantageous to keep t, and so we drop θ. We also drop ρ since we can solve for it by the extended Jacobi integral. Therefore, the dependent variables essential to the dynamics of the extended transformed circular problem are x, y, C, and t. Now dx/ds and dy/ds are given by Theorem 15, dC/ds is given by Theorem 12 and Lemma 14, and dt/ds is given by Lemma 14. By including C in the equations, we no longer think of C as a parameter, and thus the equations are the one-parameter family

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{ds} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} + O(\mu^2) \\ \dfrac{dC}{ds} = \dfrac{dC}{dt}\dfrac{dt}{ds} = 0 \\ \dfrac{dt}{ds} = \dfrac{1}{1 - x^4\rho}, \end{cases} \tag{163}$$

where ρ is given by the extended Jacobi integral. The constant of motion C is included in the equations because when we perturb the eccentricity e of the primaries, C will no longer be a constant. We call (163) the *full equations* of the extended reduced transformed circular problem, full in the sense that once we know what happens with the dependent variables x, y, C, and t, we know what happens with all the dependent variables. For the sake of brevity, we refer to the equations (163) as the full equations. The generalized phase space of the full equations is $\mathbf{R}^3 \times S^1 \times S^1$ since x, y, C are real variables, and t, s can be thought of as S^1 variables. Using τ as a dummy time variable, we can write the full equations (163) as the one-parameter family of autonomous equations

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{d\tau} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} + O(\mu^2) \\ \dfrac{dC}{d\tau} = 0 \\ \dfrac{dt}{d\tau} = \dfrac{1}{1 - x^4\rho} \\ \dfrac{ds}{d\tau} = 1, \end{cases} \tag{164}$$

with $s(0) = 0$, where ρ is given by the Jacobi integral.

A set $A \subset \mathbf{R}^3 \times S^1 \times S^1$ is invariant under the flow of the autonomous full equations (164) if for each $(x_0, y_0, C_0, t_0, s_0) \in A$, the solution $w(s_0 + \tau)$ with initial condition $w(s_0) = (x_0, y_0, C_0, t_0, s_0)$ remains in A for all $\tau \in \mathbf{R}$. In a similar manner, we can define invariant sets of other flows.

Proposition 41: *Fix $\mu \in (0,1)$, and set $e = 0$. The set*

$$\{(0,0,C,t,s) : C \in \mathbf{R}, t, s \in S^1\} \tag{165}$$

is invariant under the flow of the autonomous full equations (164).

Proof: Since the vector field of (164) equals $(0,0,0,1,1)$ whenever it is evaluated at $(0,0,C,t_0,s_0)$, any solution with initial condition $(0,0,C,t_0,s_0)$ remains in the set (165) for all $\tau \in \mathbf{R}$. Hence the set (165) is invariant. This completes the proof.

Note that the solutions in (165) are artificial since they lie outside the domain of Λ. Since C is a constant of motion by Theorem 12, the flow in the invariant set (165) occurs on the 1-torus $T^1 = S^1 \times S^1$. Since $dt/d\tau = 1$ and $ds/d\tau = 1$, the flow on the invariant set (165) is the $1:1$ resonant motion on the 1-torus.

As we did in the last section, we reduce the flow of (164) to a Poincaré map. The vector field of (164) is 2π-periodic by Theorem 15 and Proposition 11. Thus the flow of the autonomous full equations admits global cross sections of the form

$$\{s = s_0 : \Sigma\} \equiv \Sigma^{s=s_0} \times \{s_0\}$$

for each fixed $s_0 \in S^1$, where

$$\Sigma^{s=s_0} = \left\{(x,y,C,t) : x,y,C \in \mathbf{R},\ t \in S^1\right\}.$$

Associated with each global cross section, there is a two-parameter four-dimensional Poincaré map

$$\phi_{s_0,\mu} : \Sigma^{s=s_0} \to \Sigma^{s=s_0}.$$

Since $\Sigma^{s=s_0}$ is simply the projection of $\{s = s_0 : \Sigma\}$ onto the four-dimensional space $\mathbf{R}^3 \times S^1$, we refer to $\Sigma^{s=s_0}$ also as a section. From Proposition 41, it follows that the infinite cylinder

$$\{x = 0, y = 0\} \equiv \left\{(0,0,C,t) : C \in \mathbf{R}, t \in S^1\right\} \subset \Sigma^{s=s_0} \tag{166}$$

is an invariant set of the Poincaré map $\phi_{s_0,\mu}$.

To "transfer" what we know about the Poincaré map $\psi_{s_0,\mu,C}$ to the Poincaré map $\phi_{s_0,\mu}$, we consider the orbit-to-set embedding map

$$\left\{\psi^j_{s_0,\mu,C}(x_0,y_0) : j \in \mathbf{Z}\right\} \to \left\{\left((\psi^j_{s_0,\mu,C}(x_0,y_0)), C, t_j\right) : j \in \mathbf{Z}\right\} \tag{167}$$

for $(x_0, y_0) \in \Gamma_C^{s=s_0}$, where

$$t_j = \int_{s_0}^{s_0+2\pi j} \frac{1}{1 - x^4\rho} d\tau + t_0 \tag{168}$$

is obtained by integrating the equation for $dt/d\tau$ from (164) over the indicated time interval. The variable ρ is given by the extended Jacobi integral, and x being the first component of the solution of the full equations (164) with the initial condition (x_0, y_0, C, t_0, s_0). We refer to the image of the embedding (167) as the t_0-*embedding* of the orbit $\{\psi^j_{s_0,\mu,C}(x_0, y_0) : j \in \mathbf{Z}\}$.

Proposition 42: *Fix* $(s_0, \mu) \in S^1 \times (0,1)$. *If* $(x_0, y_0, C, t_0) \in \Sigma^{s=s_0}$, *then*

$$\phi^j_{s_0,\mu}(x_0, y_0, C, t_0) = (\psi^j_{s_0,\mu,C}(x_0, y_0), C, t_j),$$

where t_j *is given by* (168).

Proof: Let

$$w(s_0 + \tau) = \big(w_1(s_0 + \tau), w_2(s_0 + \tau), w_3(s_0 + \tau), w_4(s_0 + \tau), s_0 + \tau (\text{ mod } 2\pi)\big)$$

be the solution of (164) with the initial condition $w(s_0) = (x_0, y_0, C, t_0, s_0)$. Let $j \in \mathbf{Z}$. Then $w(s_0 + 2\pi j)$ intersects the section $\Sigma^{s=s_0}$ at the point

$$\begin{aligned}\phi^j_{s_0,\mu}(x_0, y_0, C, t_0) &= \big(w_1(s_0 + 2\pi j), w_2(s_0 + 2\pi j), w_3(s_0 + 2\pi j), w_4(s_0 + 2\pi j)\big) \\ &= (x_j, y_j, C_j, t_j).\end{aligned}$$

Since the autonomous full equations (164) reduce to (119),

$$(x_j, y_j) = \psi^j_{s_0,\mu,C}(x_0, y_0).$$

Since $dC/d\tau = 0$, then $C_j = C_0$. The reduction of (164) to (119) implies that we can integrate $dt/d\tau$ from s_0 to $s_0 + 2\pi j$ to obtain t_j, which is (168). This completes the proof.

It follows from Proposition 42 that the t_0-embedding of the orbit $\{\psi^j_{s_0,\mu,C}(x_0, y_0) : j \in \mathbf{Z}\} \subset \Gamma_C^{s=s_0}$ is the orbit $\{\phi^j_{s_0,\mu}(x_0, y_0, C, t_0) : j \in \mathbf{Z}\} \subset \Sigma^{s=s_0}$. Thus, the embedding map (167) is an orbit-to-orbit map. Using the embedding map (167) we can "transfer" the information we have about the Poincaré map $\psi_{\pi,\mu,C}$ to the Poincaré map $\phi_{\pi,\mu}$.

Fix $s_0 = \pi$, $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small, $t_0 \in S^1$, and $\mu > 0$ sufficiently small. For $n \in \tilde{\mathbf{N}}(\pi, \mu, C)$, the t_0-embedding

$$\{\phi^j_{\pi,\mu}(p_n(\mu, C), C, t_0) : j \in \mathbf{Z}\},$$

of the periodic point $p_n(\mu, C)$ of the map $\psi_{\pi,\mu,C}$ is not necessarily periodic because of the presence of the t_j terms. We will show that there exists an open connected bounded interval of C-values such that the t_0-embeddings of "most" of the periodic points $p_n(\mu, C)$, with arbitrarily large period, are *quasiperiodic* in the sense that there exists a $\beta(C) \in \mathbf{R} \setminus \{\pi a : a \in \mathbf{R}$ and rational $\}$ such that $t_n = t_0 + \beta(C) \pmod{2\pi}$. Numbers contained in $\mathbf{R} \setminus \{\pi a : a \in \mathbf{R}$ and rational $\}$ are called *incommensurate with* π.

We "transfer" the information we have about the periodic points $p_n(\mu, C)$ of the Poincaré map $\psi_{\pi,\mu,C}$ to the Poincaré map $\phi_{\pi,\mu}$ as follows. Fix $\mu > 0$ sufficiently small. For the open connected interval $Y \subset (\sqrt{2}, \infty)$ with $\sup Y - \sqrt{2}$ sufficiently small, let $\mathcal{Q}_{Y,\pi,\mu}$ be the class of integer-valued functions given by

$$\mathcal{Q}_{Y,\pi,\mu} = \left\{ f : Y \to \mathbf{N} | \text{ for each } C \in Y,\ f(C) \in \tilde{\mathbf{N}}(\pi, \mu, C) \right\}, \tag{169}$$

where $\tilde{\mathbf{N}}(\pi, \mu, C)$ is given by (162). Define $min_Y : \mathcal{Q}_{Y,\pi,\mu} \to \mathbf{R}^+$ by

$$min_Y f = \inf\{f(C) : C \in Y\}. \tag{170}$$

Since $\{f(C) : C \in Y\} \subset \mathbf{N}$, then, by the Well-Ordering Principle (see Hungerford [1974]), $min_Y f$ exists and is a positive integer. For $f \in \mathcal{Q}_{Y,\pi,\mu}$, define the set $\Omega_{Y,f,\pi,\mu}$ by

$$\Omega_{Y,f,\pi,\mu} = \left\{ \left(p_{f(C)}(\mu, C), C, t\right) : C \in Y,\ t \in S^1 \right\} \subset \Sigma^{s=\pi}. \tag{171}$$

Note that the projection of $\Omega_{Y,f,\pi,\mu}$ onto the infinite cylinder $\{x = 0, y = 0\}$ given by (166), is diffeomorphic to an annulus. So, we refer to (171) as an *annulus*.

Proposition 43: *Fix $\mu > 0$ sufficiently small. Let Y be an open interval in $(\sqrt{2}, \infty)$ with $\sup Y - \sqrt{2}$ sufficiently small. If $f \in \mathcal{Q}_{Y,\pi,\mu}$, then the annulus $\Omega_{Y,f,\pi,\mu}$ given in (171) is an invariant set of $\phi^{f(C)}_{\pi,\mu}$.*

Proof: Let $C \in Y$ and let $t_0 \in S^1$. Let $f \in \mathcal{Q}_{Y,\pi,\mu}$. By the definition of $\mathcal{Q}_{Y,\pi,\mu}$ given in (169), $f(C)$ is the period of periodic point $p_{f(C)}(\mu, C)$. Then by Proposition 42,

$$\begin{aligned}\phi^{f(C)}_{\pi,\mu}(p_{f(C)}(\mu,C), C, t_0) &= (\psi^{f(C)}_{\pi,\mu,C}(p_{f(C)}(\mu,C)), C, t_{f(C)}) \\ &= (p_{f(C)}(\mu,C), C, t_{f(C)}) \\ &\in \Omega_{Y,f,\pi,\mu},\end{aligned}$$

where $t_{f(C)}$ is given by (168) with $j = f(C)$. Thus each circle $\{(p_{f(C)}(\mu,C), C, t) : t \in S^1\}$ is invariant under $\phi^{f(C)}_{\pi,\mu}$. Since $C \in Y$ is arbitrary, the annulus (171) is invariant under $\phi^{f(C)}_{\pi,\mu}$. This completes the proof.

Let Y be an open connected subset of $(\sqrt{2}, \infty)$ with $\sup Y - \sqrt{2}$ sufficiently small. For $f \in \mathcal{Q}_{Y,\pi,\mu}$, we define the (Y, f)-*transverse map* $\tilde{\phi}_{Y,f,\pi,\mu} : \Omega_{Y,f,\pi,\mu} \to \Omega_{Y,f,\pi,\mu}$ by

$$\tilde{\phi}_{Y,f,\pi,\mu}(x_0, y_0, C, t_0) = \phi^{f(C)}_{\pi,\mu}(x_0, y_0, C, t_0). \tag{172}$$

We may write (172) as

$$\tilde{\phi}_{Y,f,\pi,\mu}(x_0, y_0, C, t_0) = (x_0, y_0, C, t_{f(C)}), \tag{173}$$

since $(x_0, y_0, C, t_0) \in \Omega_{Y,f,\pi,\mu}$. Thus the dynamics on the annulus (171) occur only in the t-component, the other three components being fixed under the (Y, f)-transverse map $\tilde{\phi}_{Y,f,\pi,\mu}$. We will show for sufficiently small $\mu \in (0, 1)$ that there exists an open connected bounded $Y \subset (\sqrt{2}, \infty)$, with $\sup Y - \sqrt{2}$ sufficiently small, such that if $min_Y f$ (defined by (170)) is sufficiently large, the (Y, f)-transverse map is a twist map. This will be achieved by an analysis of the behavior of the t-component of (173).

A *smooth simple curve* is the diffeomorphic image of some interval of $\mathbf{R}$.

Definition 44: Let $g : M \to \mathbf{R}$ be a C^1 function, where M is Banach space over $\mathbf{R}$. The *variation of g along a smooth simple curve $c \subset M$* is denoted by $\Delta_c g$ and is defined by the line integral

$$\Delta_c g = \int_c \left(\frac{dg}{ds}\right) ds.$$

Lemma 45: *Fix $\mu \in (0, 1)$ sufficiently small. For an open connected $Y \subset (\sqrt{2}, \infty)$ with $\sup Y - \sqrt{2}$ sufficiently small, and for $f \in \mathcal{Q}_{Y,\pi,\mu}$, the t-component of the (Y, f)-transverse map* (173) *has the form*

$$t_{f(C)} = \Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t + t_0,$$

for all $C \in Y$*, and is a function of* μ*,* C *and* t_0 *only.*

Proof: From (168) we have

$$\begin{aligned} t_{f(C)} &= \int_{\pi}^{\pi+2\pi f(C)} \frac{1}{1-x^4\rho} d\tau + t_0 \\ &= \int_{\pi}^{\pi+2\pi f(C)} \left(\frac{dt}{d\tau}\right) d\tau + t_0, \end{aligned} \tag{174}$$

where $dt/d\tau$ is evaluated over the solution of (164) with the initial condition

$$(p_{f(C)}(\mu, C), C, t_0, \pi).$$

Since (164) reduces to (119) and $dt/d\tau$ depends only on $(x, y, \tau; \mu, C)$, then $dt/d\tau$ is evaluated over the solution of (119) with the intial condition $(p_{f(C)}(\mu, C), \pi)$. Since the solution of (119) with the initial condition $(p_{f(C)}(\mu, C), \pi)$ is periodic with period $2\pi f(C)$, and as $ds/d\tau = 1$ with $s(0) = 0$, then (174) becomes

$$\begin{aligned} t_{f(C)} &= \int_{\pi}^{\pi+2\pi f(C)} \left(\frac{dt}{ds}\right) ds + t_0 \\ &= \int_{\gamma_\pi(p_{f(C)}(\mu,C))} \left(\frac{dt}{ds}\right) ds + t_0 \\ &= \Delta_{\gamma_\pi(p_{f(C)}(\mu,C)} t + t_0. \end{aligned} \tag{175}$$

That (175) is a function of μ, C and t_0 only follows since the variables (x, y, s) are integrated out. This completes the proof.

Lemma 46: *Fix* $\mu \in (0,1)$ *sufficiently small. For an open connected* $Y \subset (\sqrt{2}, \infty)$ *with* $\sup Y - \sqrt{2}$ *sufficiently small, and* $f \in Q_{Y,\pi,\mu}$*,*

$$\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t = \Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} \theta \pmod{2\pi}$$

for all $C \in Y$.

Proof: From (108), we know that $t = s + \theta$. Then

$$\begin{aligned}
\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t &= \int_{\gamma_\pi(p_{f(C)}(\mu,C))} \left(\frac{dt}{ds}\right) ds \\
&= \int_{\pi}^{\pi+2\pi f(C)} \left(\frac{dt}{ds}\right) ds \\
&= \int_{\pi}^{\pi+2\pi f(C)} \left(\frac{d(s+\theta)}{ds}\right) ds \\
&= \int_{\pi}^{\pi+2\pi f(C)} \left(1 + \frac{d\theta}{ds}\right) ds \\
&= 2\pi f(C) + \int_{\pi}^{\pi+2\pi f(C)} \left(\frac{d\theta}{ds}\right) ds \\
&= 2\pi f(C) + \int_{\gamma_\pi(p_{f(C)}(\mu,C))} \left(\frac{d\theta}{ds}\right) ds \\
&= 2\pi f(C) + \Delta_{\gamma_\pi(p_{f(C)}(\mu,C))}\theta \\
&= \Delta_{\gamma_\pi(p_{f(C)}(\mu,C))}\theta \pmod{2\pi}.
\end{aligned}$$

This completes the proof.

It is difficult to evaluate $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))}\theta$ directly, and so we proceed as follows. Let $\mathbf{u} : \mathbf{R} \times S^1 \times \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$\mathbf{u}(x,\theta,y,\rho) = \left((1 - x^2\rho^2)\cos\theta - y\rho\sin\theta, (1 - x^2\rho^2)\sin\theta + y\rho\cos\theta\right). \tag{176}$$

Taking the usual euclidean norm of (176), we have

$$||\mathbf{u}|| = \sqrt{(1 - x^2\rho^2)^2 + y^2\rho^2}. \tag{177}$$

Note that $||\mathbf{u}||$ is a function of x, y, and ρ only. The following result will greatly simplify some of the calculations we will be doing later on.

Lemma 47: *If $\rho = -C$ and (x,y) is a solution in $H(x,y;\rho) = 0$, then*

$$||\mathbf{u}(x(t),y(t);\rho)|| = 1$$

for all $t \in \mathbf{R}$.

Proof: From (177), and by making use of (38), we have

$$\begin{aligned}
||\mathbf{u}(x,y;\rho)|| &= \sqrt{1 + 2\rho^2 H(x,y;\rho)} \\
&= 1
\end{aligned}$$

if (x,y) is a solution in $H(x,y;\rho)=0$. This completes the proof.

We use $\mathbf{u}$ to define a functional. Let $\alpha : \mathbf{R}\times S^1\times\mathbf{R}^2 \to \mathbf{R}$ be defined by

$$\mathbf{u} = ||\mathbf{u}||e^{i\alpha} = ||\mathbf{u}||(\cos\alpha + i\sin\alpha). \tag{178}$$

We will show that there is a simple relationship between $\Delta_{\gamma_\pi(p(\mu,C))}\theta$ and $\Delta_{\gamma_\pi(p(\mu,C))}\alpha$. If we write $\mathbf{u}=(u_1,u_2)$, then by looking at the components of (178) we see that

$$\cos\alpha = \frac{u_1}{||\mathbf{u}||}, \tag{179}$$

and

$$\sin\alpha = \frac{u_2}{||\mathbf{u}||}. \tag{180}$$

To compute the variation of α along $\gamma_\pi(p(\mu,C))$ we will need the following

Lemma 48: *Fix $\mu\in[0,1)$ and set $e=0$. The derivative of α with respect to s is given by*

$$\begin{aligned}\frac{d\alpha}{ds} = &\frac{\mu(1+x^2\rho^2)x^4y\sin(s)+\mu(1-x^2\rho^2)x^4\rho(1-2x^2\cos(s))}{\left((1-x^2\rho^2)^2+y^2\rho^2\right)(1-x^4\rho)}\\ &-\frac{\mu(1+x^2\rho^2)x^4y\sin(s)+\mu(1-x^2\rho^2)x^4\rho(1+x^2\cos(s))}{((1-x^2\rho^2)^2+y^2\rho^2)(1+2x^2\cos(s)+x^4)^{3/2}(1-x^4\rho)}\\ &+O(\mu^2).\end{aligned} \tag{181}$$

Proof: First we will take the derivative of α with respect to t, and after we have done that we will make the change of variable necessary to transform the derivative of α into a function of s. Implicitly differentiating (179) with respect to t and making use of (179) and (180), we have

$$\alpha' = \frac{-1}{||\mathbf{u}||^2u_2}\left(||\mathbf{u}||^2u_1' - \frac{1}{2}(||\mathbf{u}||^2)'u_1\right). \tag{182}$$

From (178) and Proposition 10 we have that

$$u_1' = -2\mu x^2\rho\tilde{g}_2(x,t-\theta)\cos\theta - \mu\left(\rho\tilde{g}_1(x,t-\theta)+y\tilde{g}_2(x,t-\theta)\right)+O(\mu^2), \tag{183}$$

and

$$\begin{aligned}(||\mathbf{u}||^2)' = &-4\mu x^2\rho\tilde{g}_2(x,t-\theta)+4\mu x^4\rho^3\tilde{g}_2(x,t-\theta)+2\mu y\rho^2\tilde{g}_1(x,t-\theta)\\ &+2\mu y^2\rho\tilde{g}_2(x,t-\theta)+O(\mu^2).\end{aligned} \tag{184}$$

Multiplying (183) by $||\mathbf{u}||^2$ we have

$$\begin{aligned}||\mathbf{u}||^2 u_1' = &-2\mu x^2 \rho \tilde{g}_2(x,t-\theta)\cos\theta - \mu\rho\tilde{g}_1(x,t-\theta)\sin\theta - \mu y\tilde{g}_2(x,t-\theta)\sin\theta \\ &+ 4\mu x^4\rho^3\tilde{g}_2(x,t-\theta)\cos\theta + 2\mu x^2\rho^3\tilde{g}_1(x,t-\theta)\sin\theta + 2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta)\sin\theta \\ &- 2\mu x^6\rho^5\tilde{g}_2(x,t-\theta)\cos\theta - \mu x^4\rho^5\tilde{g}_1(x,t-\theta)\sin\theta - \mu x^4 y\rho^4\tilde{g}_2(x,t-\theta)\sin\theta \\ &- 2\mu x^2 y^2\rho^3\tilde{g}_2(x,t-\theta)\cos\theta - \mu y^2\rho^3\tilde{g}_1(x,t-\theta)\sin\theta - \mu y^3\rho^2\tilde{g}_2(x,t-\theta)\sin\theta \\ &+ O(\mu^2). \end{aligned} \tag{185}$$

Multiplying (184) by $\frac{1}{2}u_1$ we have

$$\begin{aligned}\frac{1}{2}(||\mathbf{u}||^2)' u_1 = &-2\mu x^2\rho\tilde{g}_2(x,t-\theta)\cos\theta + 2\mu x^4\rho^3\tilde{g}_2(x,t-\theta)\cos\theta + \mu y\rho^2\tilde{g}_1(x,t-\theta)\cos\theta \\ &+ \mu y^2\rho\tilde{g}_2(x,t-\theta)\cos\theta + 2\mu x^2\rho^3\tilde{g}_2(x,t-\theta)\cos\theta - 2\mu x^6\rho^5\tilde{g}_2(x,t-\theta)\cos\theta \\ &- \mu x^2 y\rho^4\tilde{g}_1(x,t-\theta)\cos\theta - \mu x^2 y^2\rho^4\tilde{g}_2(x,t-\theta)\cos\theta + 2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta)\sin\theta \\ &- 2\mu x^4 y\rho^4\tilde{g}_2(x,t-\theta)\sin\theta - \mu y^2\rho^3\tilde{g}_1(x,t-\theta)\sin\theta - \mu y^3\rho^2\tilde{g}_2(x,t-\theta)\sin\theta \\ &+ O(\mu^2). \end{aligned} \tag{186}$$

Adding (185) to the negative of (186) we have

$$\begin{aligned}&||\mathbf{u}||^2 u_1' - \frac{1}{2}(||\mathbf{u}||^2)' u_1 \\ &\quad = -\left[2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta) + \mu(1-x^2\rho^2)\big(\rho\tilde{g}_1(x,t-\theta) + y\tilde{g}_2(x,t-\theta)\big)\right] u_2 \\ &\quad\quad + O(\mu^2). \end{aligned} \tag{187}$$

Substituting (187) into (182), and using (177), we have

$$\begin{aligned}\alpha' &= \frac{1}{||\mathbf{u}||^2}\left[2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta) + \mu(1-x^2\rho^2)\big(\rho\tilde{g}_1(x,t-\theta) + y\tilde{g}_2(x,t-\theta)\big)\right] + O(\mu^2) \\ &= \frac{2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta) + \mu(1-x^2\rho^2)\big(\rho\tilde{g}_1(x,t-\theta) + y\tilde{g}_2(x,t-\theta)\big)}{(1-x^2\rho^2)^2 + y^2\rho^2} + O(\mu^2). \end{aligned} \tag{188}$$

Using (61) and (62) we have

$$\begin{aligned}2\mu x^2 y\rho^2\tilde{g}_2(x,t-\theta) &= 2\mu x^2 y\rho^2\left[x^4\sin(t-\theta)\left(1 - \frac{1}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}\right)\right] \\ &= 2\mu x^6 y\rho^2\sin(t-\theta) - \frac{2\mu x^6 y\rho^2\sin(t-\theta)}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}, \end{aligned} \tag{189}$$

and

$$\rho\tilde{g}_1(x,t-\theta)+y\tilde{g}_2(x,t-\theta)=x^4\rho-2x^6\rho\cos(t-\theta)-\frac{x^4\rho+x^6\cos(t-\theta)}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}$$
$$+x^4y\sin(t-\theta)-\frac{x^4y\sin(t-\theta)}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}. \quad (190)$$

Adding (189) to (190) multiplied by $\mu(1-x^2\rho^2)$ we have

$$2\mu x^2y\rho^2\tilde{g}_2(x,t-\theta)+\mu(1-x^2\rho^2)\big(\rho\tilde{g}_1(x,t-\theta)+y\tilde{g}_2(x,t-\theta)\big)$$
$$=\mu x^4y(1+x^2\rho^2)\sin(t-\theta)+\mu(1-x^2\rho^2)x^4\rho\big(1-2x^2\cos(t-\theta)\big)$$
$$-\frac{\mu x^4y(1+x^2\rho^2)\sin(t-\theta)+\mu(1-x^2\rho^2)x^4\rho\big(1+x^2\cos(t-\theta)\big)}{(1+2x^2\cos(t-\theta)+x^4)^{3/2}}. \quad (191)$$

Substituting (191) into (188) and using (108) and Lemma 14 we obtain (181). This completes the proof.

Note that if $\mu=0$, then α is a first integral. From the calculation in the proof of Lemma 47 we see that α is related to the first integral $H(x,y;\rho)$.

For the remainder of this section we shall think of θ as given by

$$\theta(s)=\int_{s_0}^{s}\left(\frac{x^4\rho}{1-x^4\rho}\right)ds.$$

This is an abuse of notation in view of (113). The absence of the initial condition θ_0 will be made clear in what follows. By Lemma 14, we can also think of θ as

$$\theta(t)=\int_0^t x^4\rho dt. \quad (192)$$

We now proceed to obtain some information about the behavior of the variation of α along $\gamma_\pi(p(\mu,C))$. In the denominators of the terms in $d\alpha/dt$, as found in (191), the expression $1+2x^2\cos(t-\theta)+x^4$ is common.

Lemma 49: *If $C\neq 0$, then for all $t\in\mathbf{R}$,*

$$1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C)\geq 0.$$

If $|C|>\sqrt{2}$, then for all $t\in\mathbf{R}$,

$$1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C)>0.$$

If $|C| = \sqrt{2}$, *then*

$$1 + 2\xi^2(t; -C)\cos(t - \theta + \pi) + \xi^4(t; -C) = 0$$

if and only if $t = 0$.

Proof: Let $C \neq 0$. Since $\infty > \xi(t; -C) > 0$ for all $t \in \mathbf{R}$, then

$$\begin{aligned} 1 + &2\xi^2(t; -C)\cos(t - \theta + \pi) + \xi^4(t; -C) \\ &= \xi^4(t; -C)\|(\xi^{-2}(t; -C)\cos\theta, \xi^{-2}(t; -C)\sin\theta) - (\cos t, \sin t)\|^2 \end{aligned}$$

is nonnegative for all $t \in \mathbf{R}$. Now $(\cos t, \sin t)$ is on the unit circle for all $t \in \mathbf{R}$. Since

$$\|(\xi^{-2}(t; -C)\cos\theta, \xi^{-2}(t; -C)\sin\theta)\| = \xi^{-2}(t; -C) \geq \frac{C^2}{2} \tag{193}$$

for all $t \in \mathbf{R}$, then if $|C| > \sqrt{2}$, then $(\xi^{-2}(t; -C)\cos\theta, \xi^{-2}(t; -C)\sin\theta)$ is outside of the unit circle. So

$$\|(\xi^{-2}(t; -C)\cos\theta, \xi^{-2}(t; -C)\sin\theta) - (\cos t, \sin t)\| \neq 0$$

for all $t \in \mathbf{R}$.

Now let $|C| = \sqrt{2}$. Without loss of generality, suppose $C = \sqrt{2}$. Since $\theta(t)$ is given by (192), then $\theta(0) = 0$. Thus if $t = 0$, then

$$1 + 2\xi^2(0; -\sqrt{2})\cos(\pi) + \xi^4(0; -\sqrt{2}) = 0.$$

Now suppose that $t \neq 0$. Since the strict inequality in (193) holds when $t \neq 0$, then (193) implies that

$$1 + 2\xi^2(t; -C)\cos(t - \theta + \pi) + \xi^4(t; -C) > 0$$

for all $t \neq 0$. This completes the proof.

This Lemma gives us an approximation as to where and when, along the transverse symmetric homoclinic orbit $\gamma_\pi(p(\mu, C))$, the zero-mass and the primary P_2 come close to a binary collision. The π in $\cos(t - \theta + \pi)$ can be considered as θ_0. (We shall see how π gets into the argument of cos in Theorem 52.) Thus a *near-collision* occurs at time $t = 0$ near the point $(-1, 0)$ in the physical plane of motion.

For the transformed unperturbed problem we had that along the parabolic solution (39), $\rho = -C$. For the full equations we have the following result about the behavior of ρ (which is no longer a constant) along the transverse symmetric homoclinic orbit $\gamma_\pi(p(\mu, C))$.

Lemma 50: *Let $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. If $\mu \in (0,1)$ is sufficiently small, then along $\gamma_\pi(p(\mu, C))$, the approximation $\rho(s) = -C + O(\mu)$ is uniformly valid on $s \in \mathbf{R}$.*

Proof: Write $\rho = \rho_0 + O(\mu)$. Substituting this into the Jacobi integral we have

$$C = \frac{1}{2}y^2 + \frac{1}{2}x^4\rho_0^2 - x^2 - \rho_0 + O(\mu) \tag{194}$$

for all $s \in \mathbf{R}$, provided that $C > \sqrt{2}$. This previous condition is necessary since a first-order μ-term in $\hat{U}(x, s; \mu)$ has an infinite discontinuity when $C = \sqrt{2}$ by Proposition 11 and Lemma 49. By Theorem 35, $\gamma_\pi(p(\mu, C))$ can be approximated with uniform validity on $s \in \mathbf{R}$ by the π-embedding of the parabolic solution with $\rho = -C$. Thus (194) becomes

$$C = \frac{1}{2}\eta^2(t(s+\pi); -C) + \frac{1}{2}\xi^4(t(s+\pi); -C)\rho_0^2 - \xi^2(t(s+\pi); -C) - \rho_0 + O(\mu). \tag{195}$$

To solve for ρ_0 we set $\mu = 0$ in (195). This yields

$$0 = \frac{1}{2}\eta^2(t(s+\pi); -C) + \frac{1}{2}\xi^4(t(s+\pi); -C)\rho_0^2 - \xi^2(t(s+\pi); -C) - \rho_0 - C. \tag{196}$$

Since ρ_0 is given by the unperturbed problem, $\rho_0(s) = \rho_0(0)$ for all $s \in \mathbf{R}$. Hence for $s = 0$, (196) yields

$$(\rho_0 + C)\left(\frac{2}{C^4}\rho_0 - 1 - \frac{2}{C^3}\right) = 0.$$

So, there are two possible choices for ρ_0. In keeping with Lemma 31, we choose $\rho_0 = -C$. That $\rho(s) = -C + O(\mu)$ holds with uniform validity on $s \in \mathbf{R}$ is as follows. On any compact connected set $A \subset \mathbf{R}$, $\rho(s) = -C + O(\mu)$ is uniformly valid by the analyticity of ρ with respect to μ and s. If we let $s \to \pm\infty$, the Jacobi integral implies that $\rho(s) \to -C$. So the uniform validity on all of $s \in \mathbf{R}$ is verified. This completes the proof.

The expression $t-\theta$ appears in (191) and we will need the following local result about it in the calculations that we will do.

Lemma 51: *Let $\mu \in (0,1)$, $C > \sqrt{2}$ with $C-\sqrt{2}$ sufficiently small, and $x(t) = \xi(t;-C)$. If μ is sufficiently small, then locally about $t=0$, $t-\theta$ is a strictly increasing function of t.*

Proof: From (192), θ, as a function of t, is given by

$$\theta(t) = \int_0^t \xi^4(t;\rho)\rho dt \ (\text{mod } 2\pi). \tag{197}$$

With $x(t) = \xi(t;-C)$, $\rho = -C + O(\mu)$ provided $C > \sqrt{2}$ and $C - \sqrt{2}$ is sufficiently small. (This is shown by writing $\rho = \rho_0 + O(\mu)$ and solving the Jacobi integral for ρ_0 in a manner similar to that used in the proof of Lemma 50.) Thus (197) becomes

$$\theta(t) = -C\int_0^t \xi^4(t;-C+O(\mu))dt + O(\mu) \ (\text{mod } 2\pi). \tag{198}$$

By expanding $\xi^4(t;-C+O(\mu))$ in terms of μ, (198) becomes

$$\theta(t) = -C\int_0^t \xi^4(t;-C)dt + O(\mu) \ (\text{mod } 2\pi).$$

Thus

$$t-\theta(t) = t + C\int_0^t \xi^4(t;-C)dt + O(\mu) \ (\text{mod } 2\pi). \tag{199}$$

Taking the derivative of (199) with respect to t, we have

$$\frac{d}{dt}(t-\theta(t)) = 1 + C\xi^4(t;-C) + O(\mu) > 0$$

locally about $t=0$, provided μ is small enough. This completes the proof.

Theorem 52: *If $\mu \in (0,1)$ is small enough, then there exists $\delta_2 > \delta_1 > 0$ such that for all $C \in (\sqrt{2}+\delta_1, \sqrt{2}+\delta_2)$, $\Delta_{\gamma_x(p(\mu,C))}\alpha \neq 0$, and*

$$\frac{d}{dC}\Delta_{\gamma_x(p(\mu,C))}\alpha \neq 0.$$

Proof: Fix $\mu \in (0,1)$. Let $C > \sqrt{2}$. By definition,

$$\Delta_{\gamma_x(p(\mu,C))}\alpha = \int_{\gamma_x(p(\mu,C))} \left(\frac{d\alpha}{ds}(x,y,\rho,s;\mu,C)\right) ds. \tag{200}$$

By Theorem 35, $\gamma_\pi(p(\mu, C))$ can be approximated with uniform validity on $s \in \mathbf{R}$ by the π-embedding of the parabolic solution (39) with $\rho = -C$. By Lemma 50, ρ can be approximated with uniform validilty on $s \in \mathbf{R}$ by $-C$. Thus an approximation of (200) is

$$\int_{-\infty}^{\infty} \frac{d\alpha}{ds} (\xi(t(s+\pi); -C + O(\mu)), \eta(t(s+\pi); -C + O(\mu)), -C + O(\mu), s+\pi; \mu, C)\, ds \\ \equiv \tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha. \tag{201}$$

By expanding the integrand of (201) in terms of μ, and using the form of $d\alpha/ds$ given by Lemma 48, we can write (201) as

$$\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha = \int_{-\infty}^{\infty} \frac{d\alpha}{ds} (\xi(t(s+\pi); -C), \eta(t(s+\pi); -C), -C, s+\pi)\, ds + O(\mu^2). \tag{202}$$

By Lemma 14 and (108), we can change the variable of integration in (202) from s to t. The limits of integration will still be plus and minus infinity by Lemmas 30, 29, and 28. By Lemma 48, and Lemma 47 in conjunction with (177), (202) becomes

$$\begin{aligned}
\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha &= \int_{-\infty}^{\infty} \frac{d\alpha}{dt} (\xi(t; -C), \eta(t; -C), -C, t - \theta + \pi)\, dt \\
&= \mu \int_{-\infty}^{\infty} (1 + \xi^2(t; -C)C^2)\xi^4(t; -C)\eta(t; -C)\sin(t - \theta + \pi) dt \\
&\quad - \mu C \int_{-\infty}^{\infty} (1 - \xi^2(t; -C)C^2)\xi^4(t; -C)(1 - 2\xi^2(t; -C)\cos(t - \theta + \pi)) dt \\
&\quad - \mu \int_{-\infty}^{\infty} \frac{(1 + \xi^2(t; -C)C^2)\xi^4(t; -C)\eta(t; -C)\sin(t - \theta + \pi)}{(1 + 2\xi^2(t; -C)\cos(t - \theta + \pi) + \xi^4(t; -C))^{3/2}} dt \\
&\quad + \mu C \int_{-\infty}^{\infty} \frac{(1 - \xi^2(t; -C)C^2)\xi^4(t; -C)(1 + \xi^2(t; -C)\cos(t - \theta + \pi))}{(1 + 2\xi^2(t; -C)\cos(t - \theta + \pi) + \xi^4(t; -C))^{3/2}} dt \\
&\quad + O(\mu^2) \\
&\equiv \mu \int_{-\infty}^{\infty} F_1(t; C) dt + \mu C \int_{-\infty}^{\infty} F_2(t; C) dt + \mu \int_{-\infty}^{\infty} F_3(t; C) dt \\
&\quad + \mu C \int_{-\infty}^{\infty} F_4(t; C) dt + O(\mu^2).
\end{aligned} \tag{203}$$

If μ is small enough, $\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha$ is a good approximation of $\Delta_{\gamma_\pi(p(\mu,C))}\alpha$. We will show that as $C \to \sqrt{2}$, $\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha \to -\infty$ by showing that as $C \to \sqrt{2}$, the first three integrals in (203) are finite, and the last integral in (203) is unbounded with limit tending to $-\infty$. Recall that $C > \sqrt{2}$. Thus $C \to \sqrt{2}$ means $C \to \sqrt{2}$ with $C > \sqrt{2}$.

We begin by showing that

$$\lim_{C\to\sqrt{2}} \int_{-\infty}^{\infty} F_1(t;C)dt \tag{204}$$

exists and is finite. Since $2|C|^{-2} \geq \xi^2(t;-C) > 0$ for all $t \in \mathbf{R}$ and $\eta^2(t;-C) \leq 8|C|^{-2}$ for all $t \in \mathbf{R}$, there exists a $K_1 \in \mathbf{R}^+$ such that $|F_1(t;-C)| \leq K_1\xi^4(t;-C)$ provided that $C \neq 0$. Lemma 30 implies that

$$\lim_{t\to\infty} \int_{-t}^{t} \xi^4(t;-C)dt < \infty \tag{205}$$

for all $C \neq 0$. Thus (204) exists and is finite.

Next we show that

$$\lim_{C\to\sqrt{2}} \int_{-\infty}^{\infty} F_2(t;C)dt \tag{206}$$

exists and is finite. By the boundedness of $\xi(t;-C)$ when $C \neq 0$, there exists $K_2 \in \mathbf{R}^+$ such that $|F_2(t;C)| \leq K_2\xi^4(t;-C)$ for all $t \in \mathbf{R}$. Hence by (205), (206) exists and is finite.

Next we show that

$$\lim_{C\to\sqrt{2}} \int_{-\infty}^{\infty} F_3(t;C)dt \tag{207}$$

exists and is finite. Since $\xi(t;-C) \to 0$ as $t \to \pm\infty$, then

$$1 + 2\xi^2(t;-C)\cos(t-\theta+\pi) + \xi^4(t;-C) \to 1$$

as $t \to \pm\infty$. Hence there exists t_1 and $K_3 \in \mathbf{R}^+$ such that

$$\begin{aligned} |F_3(t;C)| &\leq K_3|(1 + \xi^2(t;-C)C^2\xi^4(t;-C)\eta(t;-C)\sin(t-\theta+\pi)| \\ &\leq K_1K_3\xi^4(t;-C) \end{aligned}$$

for all $|t| \geq t_1$ provided that $C \neq 0$. This implies that for all $C > 0$,

$$-\infty < \int_{t_1}^{\infty} F_3(t;C)dt < \infty, \tag{208}$$

and

$$-\infty < \int_{-\infty}^{-t_1} F_3(t;C)dt < \infty. \tag{209}$$

It follows then that in the limit as $C \to \sqrt{2}$, the integrals in (208) and (209) exists and are finite. Let $t_2 \in (0, t_1)$ be arbitrary. By Lemma 49 there exists $a_1 > 0$ such that

$$1 + 2\xi^2(t;-C)\cos(t-\theta+\pi) + \xi^4(t;-C) \geq a_1 > 0$$

for all $C \in [\sqrt{2}, 2]$ and for all $t \in [-t_1, -t_2] \cup [t_2, t_1]$. Thus

$$-\infty < \int_{t_2}^{t_1} F_3(t; C) dt < \infty, \tag{210}$$

and

$$-\infty < \int_{-t_1}^{-t_2} F_3(t; C) dt < \infty \tag{211}$$

for all $C \in [\sqrt{2}, 2]$. It follows then that in the limit as $C \to \sqrt{2}$, the integrals in (210) and (211) exist and are finite. Note that this holds for arbitrary $t_2 \in (0, t_1)$. Now $\eta(t; -C) > 0$ for $t \in (0, t_2]$ and $\eta(t; -C) < 0$ for $t \in [-t_2, 0)$. Since $t - \theta + \pi$ is strictly increasing locally about $t = 0$ by Lemma 51, then for small enough $t_2 \in (0, t_1)$, $\sin(t - \theta + \pi) > 0$ for all $t \in (0, t_2]$ and $\sin(t - \theta + \pi) < 0$ for all $t \in [-t_2, 0)$. Hence, by Lemma 49, $F_3(t; C) \leq 0$ for all $t \in [-t_3, t_3]$. Suppose that

$$\lim_{C \to \sqrt{2}} \int_{-t_2}^{t_2} F_3(t, C) dt = -\infty.$$

Then there exists a $t_3 \in (-t_2, t_2)$ such that

$$\lim_{C \to \sqrt{2}} F_3(t_3; C) = -\infty.$$

This implies that

$$\lim_{C \to \sqrt{2}} \left[1 + 2\xi^2(t; -C) \cos(t - \theta + \pi) + \xi^4(t; -C)\right]_{t = t_3} = 0.$$

By Lemma 49, $t_3 = 0$. But, since $\eta(0; -C) = 0$ for all $C \neq 0$,

$$\lim_{C \to \sqrt{2}} F_3(0; C) = 0,$$

and so we have a contradiction. Therefore

$$-\infty < \lim_{C \to \sqrt{2}} \int_{-t_3}^{t_3} F_3(t; C) dt < \infty. \tag{212}$$

So, by (208), (209), (210), (211), and (212) we have that in the limit as $C \to \sqrt{2}$, (207) exists and is finite.

Now we will show that

$$\lim_{C \to \sqrt{2}} \int_{-\infty}^{\infty} F_4(t; C) dt = -\infty. \tag{213}$$

Arguing as we did above, we have that for all $t_5 > 0$, both

$$\lim_{C\to\sqrt{2}} \int_{t_5}^{\infty} F_4(t;C)dt \tag{214}$$

and

$$\lim_{C\to\sqrt{2}} \int_{-\infty}^{-t_5} F_4(t;C)dt \tag{215}$$

exist and are finite. Now, since $\xi(0;-C) = \sqrt{2}|C|^{-1}$, and $0 < \xi(t;-C) < \sqrt{2}|C|^{-1}$ for all $t \neq 0$, then for small enough t_5, $1 - \xi^2(t;-C)C^2 \leq 0$ for all $t \in [-t_5, t_5]$ provided that $C \neq 0$. Since $\theta = 0$ when $t = 0$ by (192), then $\cos(0 - 0 + \pi) = -1$. Hence, for small enough t_5, $1 + \xi^2(t;-C)\cos(t - \theta + \pi) \geq 0$ for all $t \in [-t_5, t_5]$ by the properties of $\xi(t;-C)$ and $\cos(t - \theta + \pi)$ near $t = 0$. Thus by Lemma 49, $F_4(t;-C) \leq 0$ for all $t \in [-t_5, t_5]$. In particular,

$$F_4(0;C) = \frac{-4}{(C^2-2)^2}. \tag{216}$$

Since F_4 is real analytic in t and C for all $t \in \mathbf{R}$ and for all $C > \sqrt{2}$ by Lemma 49, then we can write

$$F_4(t;C) = \sum_{n=0}^{\infty} c_n(C)t^n$$

for $t \in (-t_5, t_5)$, where $c_0(C)$ is given by (216). If $t_6 \in (0, t_5)$, then by Taylor's Theorem there exists $t^*(t_6) \in (0, t_5)$ such that

$$F_4(t_6;C) = c_0(C) + \left[\frac{d}{dt}F_4(t;C)\right]_{t=t^*(t_6)}. \tag{217}$$

To integrate the right-hand side of (217), it is sufficient show that t^*, as a function of $t_6 \in (0, t_5)$, is continuous almost everywhere. Let $G : (0, t_5)^2 \times (\sqrt{2}, 2] \to \mathbf{R}$ be defined by

$$G(t;t_6,C) = F_4(t;C) - c_0(C) - \left(\frac{F_4(t_6;C) - c_0(C)}{t_6}\right)t.$$

From the proof of Taylor's Theorem (see Johnsonbaugh and Pfaffenberger [1981]), $t^*(t_6)$ satisfies

$$\frac{d}{dt}G(t^*(t_6);t_6,C) = 0. \tag{218}$$

Now for $\hat{t}_1,\ \hat{t}_2 \in (0, t_5)$,

$$\left|\frac{d}{dt}G(t;\hat{t}_1,C) - \frac{d}{dt}G(t;\hat{t}_2,C)\right|$$

$$= \left| \frac{d}{dt} F_4(t;C) - \frac{F_4(\hat{t}_1;C) - c_0(C)}{\hat{t}_1} - \frac{d}{dt} F_4(t;C) + \frac{F_4(\hat{t}_2;C) - c_0(C)}{\hat{t}_2} \right|$$
$$= \left| \frac{F_4(\hat{t}_1;C) - c_0(C)}{\hat{t}_1} - \frac{F_4(\hat{t}_2;C) - c_0(C)}{\hat{t}_2} \right|, \tag{219}$$

and

$$\left| \frac{d^2}{dt^2} G(t;\hat{t}_1, C) - \frac{d^2}{dt^2} G(t;\hat{t}_2, C) \right| = \left| \frac{d^2}{dt^2} F_4(t;C) - \frac{d^2}{dt^2} F_4(t;C) \right| = 0. \tag{220}$$

Note that the right-hand sides of (219) and (220) do not depend on t. By the continuity of G with respect to t, (219) can be made small for all $t \in (0, t_5)$ provided $|\hat{t}_1 - \hat{t}_2|$ is small enough. Thus by (219) and (220),

$$\frac{d}{dt} G(t;\hat{t}_1;C) \quad \text{and} \quad \frac{d}{dt} G(t;\hat{t}_2, C)$$

are C^1-close. Now we know that (218) holds by Taylor's Theorem for $t_6 = \hat{t}_1$. If

$$\frac{d^2}{dt^2} G(t^*(\hat{t}_1);\hat{t}_1, C) \neq 0 \tag{221}$$

and $|\hat{t}_1 - \hat{t}_2|$ is small enough, then we can choose, by the C^1-closeness, $t^*(\hat{t}_2) \in (0, \hat{t}_2)$ close to $t^*(\hat{t}_1)$ such that

$$\frac{d}{dt} G(t^*(\hat{t}_2);\hat{t}_2, C) = 0.$$

So, provided that (221) holds almost everywhere on $(0, t_5)$, then t^*, as a function of t_6, is continuous almost everywhere on $(0, t_5)$. To show that (221) holds almost everywhere on $(0, t_5)$, suppose

$$\frac{d^2}{dt^2} G(t^*(t_6);t_6, C) = 0$$

for all $t_6 \in (0, t_5)$. Then

$$\frac{d^2}{dt^2} F_4(t;C) \equiv 0$$

on $(0, t_5)$, which implies that F_4 is a linear function of t. But that is impossible since $F_4(0, C) = -4(C^2 - 2)^{-2} < 0$ for all $C \in (\sqrt{2}, 2]$ and

$$\lim_{t \to \pm\infty} F_4(t;C) = 0.$$

Thus (221) holds everywhere except on a discrete set, and hence holds almost everwhere on $(0, t_5)$. Since $t^*(t_6) \in (0, t_6)$ by Taylor's Theorem, then we can continuously extend the definition of t^* to 0 by $t^*(0) = 0$.

Integrating (217) from 0 to t_5 we have

$$\int_0^{t_5} F_4(t;C)dt = c_0(C)t_5 + \int_0^{t_5} \left(\frac{d}{dt_6}F_4(t^*(t_6);C)\right) dt_6. \tag{222}$$

For any $C \in (\sqrt{2}, 2]$, the integral on the right-hand side of (222) is finite by Lemma 49. We claim that

$$\lim_{C\to\sqrt{2}} \int_0^{t_5} \frac{d}{dt}F_4(t^*(t_6);C)dt_6 < \infty. \tag{223}$$

To show this suppose that

$$\lim_{C\to\sqrt{2}} \int_0^{t_5} \frac{d}{dt}F_4(t^*(t_6);C)dt_6 = \infty.$$

Then there exists a $t_7 \in [0, t_5)$ such that

$$\lim_{C\to\sqrt{2}} \frac{d}{dt}F_4(t^*(t_7);C) = \infty.$$

By Lemma 49, $t^*(t_7) = 0$, and by the definition of t^*, $t_7 = 0$. So

$$\lim_{C\to\sqrt{2}} \frac{d}{dt}F_4(0;C) = \infty. \tag{224}$$

But

$$\begin{aligned}
\frac{d}{dt}F_4(0;C) = {} & \frac{1}{(1+2\xi(0;-C)\cos(\pi)+\xi^4(0;-C))^3} \\
& \times \Bigg[\Bigg(-2\xi^6(0;-C)\eta(0;-C) \\
& - 3\xi^8(0;-C)\eta(0;-C)\cos(\pi) \\
& + C\xi^{10}(0;-C)\sin(\pi) + 3C^2\xi^8(0;-C)\eta(0;-C) \\
& + 4C^2\xi^{10}(0;C)\eta(0;-C)\cos(\pi) - C^3\xi^{12}(0;-C)\sin(\pi)\Bigg) \\
& \times \left(1+2\xi^2(0;-C)\cos(\pi)+\xi^4(0;-C)\right)^{3/2} \\
& - \frac{3}{2}(1-\xi^2(0;-C)C^2)\xi^4(0;-C)(1+\xi^2(0;-C)\cos(\pi)) \\
& \times \left(1+2\xi^2(0;-C)\cos(\pi)+\xi^4(0;-C)\right)^{1/2} \\
& \times \Bigg(-\xi^4(0;-C)\eta(0;-C)\cos(\pi) + 2C\xi^6(0;-C)\eta(0;-C)\sin(\pi) \\
& - 2\xi^6(0;-C)\eta(0;-C)\Bigg)\Bigg] \\
= {} & 0
\end{aligned}$$

for all $C \in (\sqrt{2}, 2]$, since $\eta(0; -C) = 0$ for all $C \neq 0$ and since $\sin(\pi) = 0$. Hence

$$\lim_{C \to \sqrt{2}} \frac{d}{dt} F_4(0; C) = 0,$$

a contradiction of (224). Thus the claim is established, and so from (223), we have that

$$\begin{aligned} \lim_{C \to \sqrt{2}} \int_0^{t_5} F_4(t; C) dt &= \lim_{C \to \sqrt{2}} c_0(C) + \lim_{C \to \sqrt{2}} \int_0^{t_5} \frac{d}{dt} F_4(t^*(t_6); C) dt_6 \\ &= -\infty. \end{aligned} \tag{225}$$

Since $F_4(t; C) \leq 0$ for all $t \in (-t_5, t_5)$, then (225) yields

$$\lim_{C \to \sqrt{2}} \int_{t_5}^{t_5} F_4(t; C) dt = -\infty. \tag{226}$$

Therefore, by (214), (215), and (226), we have that (213) holds.

Recall that we are showing that

$$\lim_{C \to \sqrt{2}} \tilde{\Delta}_{\gamma_\pi(p(\mu, C))} \alpha = -\infty. \tag{227}$$

As $C \to \sqrt{2}$, we showed that (204), (206), and (207) are finite, while (213) has limit $-\infty$. Thus (227) holds, and since $\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\alpha$ is a good approximation of $\Delta_{\gamma_\pi(p(\mu,C))}\alpha$ for small enough $\mu \in (0, 1)$, then

$$\lim_{C \to \sqrt{2}} \Delta_{\gamma_\pi(p(\mu, C))} \alpha = -\infty. \tag{228}$$

By Lemma 49, $\Delta_{\gamma_\pi(p(\mu,C))}\alpha$ is real analytic in C for $C > \sqrt{2}$, and so (228) implies that there exists $\delta_2 > \delta_1 > 0$ such that for all $C \in (\sqrt{2} + \delta_1, \sqrt{2} + \delta_2)$, $\Delta_{\gamma_\pi(p(\mu,C))}\alpha \neq 0$, and

$$\frac{d}{dC} \Delta_{\gamma_\pi(p(\mu, C))} \alpha \neq 0.$$

This completes the proof.

Now that we established the behavior of the variation of α along $\gamma_\pi(p(\mu, C))$, we turn our attention to the relationship between the variation of α along $\gamma_\pi(p(\mu, C))$ and the variation of θ along the same orbit.

Lemma 53: *The variables α and θ are related by*

$$\theta = \arccos\left(\frac{1 - x^2\rho^2}{\sqrt{(1 - x^2\rho^2)^2 + y^2\rho^2}}\right) + \alpha. \tag{229}$$

Proof: Multiplying (179) by $\cos\theta$ we have

$$\cos\theta\cos\alpha = \frac{(1-x^2\rho^2)\cos^2\theta - y\rho\sin\theta\cos\theta}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}. \tag{230}$$

Multiplying (180) by $\sin\theta$ we have

$$\sin\theta\sin\alpha = \frac{(1-x^2\rho^2)\sin^2\theta + y\rho\sin\theta\cos\theta}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}. \tag{231}$$

Adding (230) to (231) we have

$$\cos(\theta-\alpha) = \frac{1-x^2\rho^2}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}, \tag{232}$$

and (229) is obtained by applying arccos to both sides of (232). This completes the proof.

Let

$$v(x,y,\rho) \equiv \arccos\left(\frac{1-x^2\rho^2}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}\right). \tag{233}$$

Lemma 54: *The derivative of v with respect to s is given by*

$$\begin{aligned}\frac{dv}{ds} = {} & \frac{-1}{(1-x^4\rho)((1-x^2\rho^2)^2+y^2\rho^2)\left(\sqrt{1-\frac{1-\rho^2(2x^2-x^4\rho^2)}{(1-x^2\rho^2)^2+y^2\rho^2}}\right)} \\ & \times\Bigg[x^4y\rho^2\sqrt{(1-x^2\rho^2)^2+y^2\rho^2} - 2\mu x^2\rho\tilde{g}_2(x,s)\sqrt{(1-x^2\rho^2)^2+y^2\rho^2} \\ & - \frac{1}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}\Big(-2\mu x^2\rho\tilde{g}_2(x,s) + 4\mu x^4\rho^3\tilde{g}_2(x,s) - 2\mu x^6\rho^5\tilde{g}_2(x,s) \\ & + \mu y\rho^2\tilde{g}_1(x,s) + \mu y^2\rho\tilde{g}_2(x,s) - \mu x^2y\rho^4\tilde{g}_1(x,s) - \mu x^2y^2\rho^3\tilde{g}_2(x,s)\Big)\Bigg] \\ & + O(\mu^2).\end{aligned} \tag{234}$$

Proof: Taking the derivative of v (defined in (233)) with respect to t, and using Proposition 10, we have

$$\begin{aligned}\frac{dv}{dt} = {} & \frac{-1}{\sqrt{1-\frac{1-2x^2\rho^2+x^4\rho^4}{(1-x^2\rho^2)^2+y^2\rho^2}}} \times \frac{1}{(1-x^2\rho^2)^2+y^2\rho^2} \\ & \times\Bigg[\left(x^4y\rho^2 - 2\mu x^2\rho\tilde{g}_2(x,t-\theta)\right)\sqrt{(1-x^2\rho^2)^2+y^2\rho^2} \\ & - \frac{1}{\sqrt{(1-x^2\rho^2)^2+y^2\rho^2}}\Big(-2\mu x^2\rho\tilde{g}_2(x,t-\theta) + 4\mu x^4\rho^3\tilde{g}_2(x,t-\theta) \\ & - 2\mu x^6\rho^5\tilde{g}_2(x,t-\theta) + \mu y\rho^2\tilde{g}_1(x,t-\theta) + \mu y^2\rho\tilde{g}_2(x,t-\theta) \\ & - \mu x^2y\rho^4\tilde{g}_1(x,t-\theta) - \mu x^2y^2\rho^3\tilde{g}_2(x,t-\theta)\Big)\Bigg] + O(\mu^2).\end{aligned} \tag{235}$$

By using (108) and applying Lemma 14 to (235), we obtain (234). This completes the proof.

Theorem 55: *Fix $C > \sqrt{2}$ with $C - \sqrt{2}$ sufficiently small. If $\mu \in (0,1)$ is sufficiently small, then*

$$2\Delta_{\gamma_\pi(p(\mu,C))}\theta = \Delta_{\gamma_\pi(p(\mu,C))}\alpha + O(\mu).$$

Proof: By Lemma 53, we have

$$\Delta_{\gamma_\pi(p(\mu,C))}\theta = \Delta_{\gamma_\pi(p(\mu,C))}v + \Delta_{\gamma_\pi(p(\mu,C))}\alpha. \tag{236}$$

As in the proof of Theorem 52, we can approximate $\Delta_{\gamma_\pi(p(\mu,C))}\theta$ and $\Delta_{\gamma_\pi(p(\mu,C))}v$. By using Theorem 35, Lemma 50, and expanding the resultant integrand in terms of μ, we have that

$$\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta \equiv -C\int_{-\infty}^{\infty}\frac{\xi^4(t(s+\pi);-C)}{1+C\xi(t(s+\pi);-C)}ds + O(\mu)$$

is an approximation of $\Delta_{\gamma_\pi(p(\mu,C))}\theta$. By Lemmas 14, 30, 29, and 28, we can change the variable of integration from s to t, the limits of integration remaining the same. Thus

$$\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta = -C\int_{-\infty}^{\infty}\xi^4(t;-C)dt + O(\mu). \tag{237}$$

We will now approximate $\Delta_{\gamma_\pi(p(\mu,C))}v$. Proceeding as before, using Lemma 54, Lemma 47, the definition of η given in (39), and (237), we have

$$\begin{aligned}
&\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}v \\
&\equiv \int_{-\infty}^{\infty}\frac{dv}{ds}\Big(\xi(t(s+\pi);-C+O(\mu)),\eta(t(s+\pi);-C+O(\mu)),-C+O(\mu),s+\pi\Big)ds \\
&= \int_{-\infty}^{\infty}\frac{dv}{dt}\Big(\xi(t;-C+O(\mu)),\eta(t;-C+O(\mu)),-C+O(\mu),t-\theta+\pi\Big)dt \\
&= \int_{-\infty}^{\infty}\frac{1}{\eta(t;-C)(-C+O(\mu))}\Big[\xi^4(t;-C)\eta(t;-C)(-C+O(\mu))^2 \\
&\quad -4\mu\xi^4(t;-C)(-C+O(\mu))^3\tilde{g}_2(\xi(t;-C),t-\theta+\pi) \\
&\quad +2\mu\xi^6(t;-C)(-C+O(\mu))^5\tilde{g}_2(\xi(t;-C);t-\theta+\pi) \\
&\quad -\mu\eta(t;-C)(-C+O(\mu))^2\tilde{g}_1(\xi(t;-C),t-\theta+\pi) \\
&\quad -\mu\eta^2(t;-C)(-C+O(\mu))\tilde{g}_2(\xi(t;-C),t-\theta+\pi)
\end{aligned}$$

$$+ \mu\xi^2(t;-C)\eta(t;-C)(-C+O(\mu))^4\tilde{g}_1(\xi(t;-C),t-\theta+\pi)$$

$$+ \mu\xi^2(t;-C)\eta(t;-C)(-C+O(\mu))^3\tilde{g}_2(\xi(t;-C),t-\theta+\pi)\Big]dt + O(\mu^2)$$

$$= C\int_{-\infty}^{\infty}\xi^4(t;-C)dt + O(\mu)$$

$$+ 4\mu C^2\int_{-\infty}^{\infty}\frac{\xi^4(t;-C)\tilde{g}_2(\xi(t;-C),t-\theta+\pi)}{\eta(t;-C)}dt$$

$$- 2\mu C^4\int_{-\infty}^{\infty}\frac{\xi^6(t;-C)\tilde{g}_2(\xi(t;-C),t-\theta+\pi)}{\eta(t;-C)}dt$$

$$- \mu C\int_{-\infty}^{\infty}\tilde{g}_1(\xi(t;-C);t-\theta+\pi)dt$$

$$+ \mu\int_{-\infty}^{\infty}\eta(t;-C)\tilde{g}_2(\xi(t;-C),t-\theta+\pi)dt$$

$$+ \mu C^3\int_{-\infty}^{\infty}\xi^2(t;-C)\tilde{g}_1(\xi(t;-C),t-\theta+\pi)dt$$

$$- \mu C^2\int_{-\infty}^{\infty}\xi^2(t;-C)\eta(t;-C)\tilde{g}_2(\xi(t;-C),t-\theta+\pi)dt$$

$$+ O(\mu^2)$$

$$= -\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta + 4\mu C^2\int_{-\infty}^{\infty}G_1(t;C)dt - 2\mu C^4\int_{-\infty}^{\infty}G_2(t;C)dt$$

$$- \mu C\int_{-\infty}^{\infty}G_3(t;C)dt + \mu\int_{-\infty}^{\infty}G_4(t;C)dt + \mu C^3\int_{-\infty}^{\infty}G_5(t;C)dt$$

$$- \mu C^2\int_{-\infty}^{\infty}G_6(t;C)dt + O(\mu^2). \tag{238}$$

We assert that each of the six integrals in (238) is finite. First we show that

$$-\infty < \int_{-\infty}^{\infty}G_1(t;C)dt < \infty. \tag{239}$$

By (62), and the definition of η,

$$G_1(t;C) = \frac{\xi^7(t;-C)\sin(t-\theta+\pi)}{\sqrt{2-\xi^2(t;-C)C^2}} \times (1-(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{-3/2}). \tag{240}$$

The denominator in (240) has limit $\sqrt{2}$ as $t\to\pm\infty$ since $\xi(t;-C)\to 0$ as $t\to\pm\infty$. And since the term $(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{-3/2}\to 1$ as $t\to\pm\infty$, then there exist t_1 and $K_1\in\mathbf{R}^+$ such that $|G_1(t;C)|\le K_1\xi^4(t;-C)$ for all $|t|>t_1$. Thus, by Lemma 30,

$$-\infty < \int_{t_1}^{\infty}G_1(t;C)dt < \infty, \tag{241}$$

and

$$-\infty < \int_{-\infty}^{-t_1} G_1(t;C) < \infty. \tag{242}$$

Recall that $\eta(t;-C) > 0$ if $t > 0$ and $\eta(t;-C) < 0$ if $t < 0$. By Lemma 51, there exists $t_2 > 0$ such that $\sin(t - \theta + \pi) < 0$ if $t \in (0, t_2)$ and $\sin(t - \theta + \pi) > 0$ if $t \in (-t_2, 0)$. Since $0 < (C^2 - 2)C^{-2} < 1$ provided $C > \sqrt{2}$, then

$$1 - \frac{1}{(1 + 2\xi^2(0;C)\cos(\pi) + \xi^4(0;-C))^{3/2}} = 1 - \left(\frac{C^2 - 2}{C^2}\right)^{-3} < 0,$$

and so there exists $t_3 \in (0, t_2)$ such that

$$1 - \frac{1}{(1 + 2\xi(t;-C)\cos(t - \theta + \pi) + \xi^4(t;-C))^{3/2}} < 0$$

for all $|t| < t_3$. Thus $G_1(t;C) \geq 0$ for all $t \in (-t_3, 0) \cup (0, t_3)$. By the analyticity of G_1 with respect to t on $[-t_1, -t_3] \cup [t_3, t_1]$,

$$-\infty < \int_{t_3}^{t_1} G_1(t;C)dt < \infty, \tag{243}$$

and

$$-\infty < \int_{-t_1}^{-t_3} G_1(t;C)dt < \infty. \tag{244}$$

Now suppose that

$$\int_{-t_3}^{t_3} G_1(t;C)dt = \infty.$$

Then there exists $t_4 \in (-t_3, t_3)$ such that

$$\lim_{t \to t_4} G_1(t;C) = \infty.$$

Since $C > \sqrt{2}$, the term $(1 + 2\xi(t;-C)\cos(t - \theta + \pi) + \xi^4(t;-C))$ is bounded away from zero on $[-t_3, t_3]$ by Lemma 49. However, $\eta(0;-C) = 0$ if and only if $t = 0$. Thus, since $\eta(t;-C)$ is the denominator in G_1, t_4=0. Hence

$$\lim_{t \to 0} G_1(t;C) = \infty. \tag{245}$$

But, by l'Hospital's rule (see Johnsonbaugh and Pfaffenberger [1981]),

$$\begin{aligned}
&\lim_{t\to 0} G_1(t;C) \\
&= \lim_{t\to 0}\Bigg[\frac{\xi^8(t;-C)\sin(t-\theta+\pi)}{\eta(t;-C)} \\
&\quad \times \left(1-\frac{1}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}}\right)\Bigg] \\
&= \lim_{t\to 0}\Bigg[\frac{1}{-\xi^4(t;-C)+\xi^6(t;-C)C^2} \\
&\quad \times \Bigg(-4\xi^{10}(t;-C)\eta(t;-C)\sin(t-\theta+\pi) \\
&\qquad \times (1-\frac{1}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}}) \\
&\qquad + C\xi^{12}(t;-C)\cos(t-\theta+\pi) \\
&\qquad \times (1-\frac{1}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}}) \\
&\qquad - \xi^8(t;-C)\sin(t-\theta+\pi) \\
&\qquad \times (1-\frac{1}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}})'\Bigg)\Bigg] \\
&= \lim_{t\to 0}\frac{C\xi^{12}(t;-C)\cos(t-\theta+\pi)}{-\xi^4(t;-C)+\xi^6(t;-C)C^2} \\
&\qquad \times \left(1-\frac{1}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}}\right) \\
&= \frac{-64C^{-11}(1-(1-4C^{-2}+4C^{-4})^{-3/2})}{-4C^{-4}+8C^{-6}C^2} \\
&= \frac{-64C^{-11}(1-(1-2C^{-2})^{-3})}{4C^{-4}} \\
&= -16C^{-7}\left(1-\left(\frac{C^2-2}{C^2}\right)^{-3}\right) \\
&\neq \infty,
\end{aligned}$$

since $C > \sqrt{2}$. So we have a contradiction with (245). Thus

$$-\infty < \int_{-t_3}^{t_3} G_1(t;C)dt < \infty. \tag{246}$$

Therefore, by (241), (242), (243), (244), and (246) we have that (239) holds. Since

$0 < \xi(t;-C) \leq \sqrt{2}|C|^{-1}$ for all $t \in \mathbf{R}$, and $G_2(t;C) = \xi^2(t;-C)G_1(t;C)$, then

$$\begin{aligned}\int_{-\infty}^{\infty} |G_2(t;C)|dt &\leq \int_{-\infty}^{\infty} |\xi^2(t;-C)G_1(t;C)|dt \\ &\leq \frac{2}{C^2}\int_{-\infty}^{\infty} |G_1(t;C)|dt \\ &< \infty.\end{aligned}$$

Hence

$$-\infty < \int_{-\infty}^{\infty} G_2(t;C)dt < \infty. \tag{247}$$

Next we show that

$$-\infty < \int_{-\infty}^{\infty} G_3(t;C)dt < \infty. \tag{248}$$

By (61),

$$\begin{aligned}G_3(t;C) =&\xi^4(t;-C)\Bigg(1 - 2\xi^2(t;-C)\cos(t-\theta+\pi) \\ &- \frac{1+\xi^2\cos(t-\theta+\pi)}{(1+2\xi^2(t;-C)\cos(t-\theta+\pi)+\xi^4(t;-C))^{3/2}}\Bigg).\end{aligned}$$

Since $C > \sqrt{2}$, $1 + 2\xi^2(t;-C)\cos(t-\theta+\pi) + \xi^4(t;-C)$ is bounded away from zero on all of $\mathbf{R}$. Since $1 + 2\xi^2(t;-C)\cos(t-\theta+\pi) + \xi^4(t;-C) \to 1$ as $t \to \pm\infty$, then there exists $K_2 \in \mathbf{R}^+$ such that $|G_3(t;C)| \leq K_2\xi^4(t;-C)$. So, by Lemma 30, (248) holds. Since η is a bounded function of t, then

$$-\infty < \int_{-\infty}^{\infty} G_4(t;C)dt < \infty \tag{249}$$

follows in a manner similar to the above argument.

For the remaining two integrals, notice that $G_5(t;C) = \xi^2(t;-C)G_3(t;C)$, and $G_6(t;C) = \xi^2(t;-C)G_4(t;C)$. Since ξ is a bounded function of t, we have that

$$-\infty < \int_{-\infty}^{\infty} G_5(t;C)dt < \infty, \tag{250}$$

and

$$-\infty < \int_{-\infty}^{\infty} G_6(t;C)dt < \infty, \tag{251}$$

and thus the assertion is proved.

It follows from (239), (247), (248), (249), (250), and (251), that there exists a $K(C) \in \mathbf{R}^+$ such that we can write (238) as

$$\tilde{\Delta}_{\gamma_\pi(p(\mu,C))} v = -\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta + \mu K(C) + O(\mu^2). \tag{252}$$

If we choose μ small enough, then (252) implies that

$$\tilde{\Delta}_{\gamma_\pi(p(\mu,C))} v = -\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta + O(\mu). \tag{253}$$

By approximating the variation of θ and v along $\gamma_\pi(p(\mu,C))$, we can use (253) to write (236) as

$$\begin{aligned}\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta &= \tilde{\Delta}_{\gamma_\pi(p(\mu,C))} v + \Delta_{\gamma_\pi(p(\mu,C))}\alpha + O(\mu) \\ &= -\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta + \Delta_{\gamma_\pi(p(\mu,C))}\alpha + O(\mu).\end{aligned}$$

This implies that

$$2\tilde{\Delta}_{\gamma_\pi(p(\mu,C))}\theta = \Delta_{\gamma_\pi(p(\mu,C))}\alpha + O(\mu).$$

So if μ is small enough, then

$$2\Delta_{\gamma_\pi(p(\mu,C))}\theta = \Delta_{\gamma_\pi(p(\mu,C))}\alpha + O(\mu),$$

This completes the proof.

Theorem 56: *If $\mu \in (0,1)$ is sufficiently small, then there exists an open connected bounded $Y \subset (\sqrt{2},\infty)$, with $\sup Y - \sqrt{2}$ small, such that if $min_Y f$ is sufficiently large (for $f \in \mathcal{Q}_{Y,\pi,\mu}$), then for all $C \in Y$, $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0$ and*

$$\frac{d}{dC}\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0.$$

Proof: If $\mu \in (0,1)$ is sufficiently small, by Theorem 52 there exists $\delta_2 > \delta_1 > 0$ such that for all $C \in (\sqrt{2}+\delta_1, \sqrt{2}+\delta_2)$, $\Delta_{\gamma_\pi(p(\mu,C))}\alpha \neq 0$ and $(d/dC)\Delta_{\gamma_\pi(p(\mu,C))}\alpha \neq 0$. By Theorem 55 we can approximate $\Delta_{\gamma_\pi(p(\mu,C))}\theta$ by $\frac{1}{2}\Delta_{\gamma_\pi(p(\mu,C))}\alpha$ for μ sufficiently small. So we can find $\hat{\delta}_2 > \hat{\delta}_1 > 0$ such that for all $C \in (\sqrt{2}+\hat{\delta}_1, \sqrt{2}+\hat{\delta}_2)$, $\Delta_{\gamma_\pi(p(\mu,C))}\theta \neq 0$ and $(d/dC)\Delta_{\gamma_\pi(p(\mu,C)}\theta \neq 0$. Let $Y = (\sqrt{2}+\hat{\delta}_1, \sqrt{2}+\hat{\delta}_2)$. If $min_Y f$ is large enough, where $f \in \mathcal{Q}_{Y,\pi,\mu}$, then $\Delta_{\gamma_\pi(p(\mu,C))}\theta$ is approximated by $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))}\theta$. By Lemma 46, $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t$ is approximated by $\frac{1}{2}\Delta_{\gamma_\pi(p(\mu,C))}\alpha$. Thus, $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0$ and

$$\frac{d}{dC}\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0$$

for all $C \in Y$ provided that $min_Y f$ is sufficiently large. This completes the proof.

Corollary 57 (Xia): *If $\mu \in (0,1)$ is sufficiently small, there exists an open connected bounded $Y \subset (\sqrt{2}, \infty)$, with $\sup Y - \sqrt{2}$ small, such that if $min_Y f$ is sufficiently large (for $f \in \mathcal{Q}_{Y,\pi,\mu}$), there exists a real analytic function $\beta_f : Y \to \mathbf{R}$ such that for all $C \in Y$, $\beta_f(C) \neq 0$, $(d/dC)\beta_f(C) \neq 0$, and*

$$t_{f(C)} = \beta_f(C) + t_0.$$

Proof: Let $\mu \in (0,1)$ be sufficiently small. By Theorem 56 there is an open connected $Y \subset (\sqrt{2}, \infty)$, with $\sup Y - \sqrt{2}$ small, such that if $min_Y f$ is large enough, $\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0$ and $(d/dC)\Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t \neq 0$ for all $C \in Y$. Let $\beta_f(C) \equiv \Delta_{\gamma_\pi(p_{f(C)}(\mu,C))} t$. That $\beta_f(C)$ is real analytic in C follows by the real analyticity of the full equations and its solutions. By Lemma 45, the prove is complete.

Recall that we set out to show that there exists an open bounded interval of C-values such that the t_0-embedding of "most" of the periodic points $p_n(\mu, C)$, for $n \in \tilde{\mathbf{N}}(\pi, \mu, C)$, with arbitrarily large period are quasiperiodic in that there exists an $\beta(C) \in \mathbf{R}$, incommensurate with π, such that $t_n = t_0 + \beta(C) \pmod{2\pi}$. By Corollary 57 we have, for sufficiently small $\mu \in (0,1)$, the existence of an open connected bounded $Y \subset \mathbf{R}^+$ such that if $min_Y f$ is sufficiently large, there exists $\beta_f : Y \to \mathbf{R}$, such that

$$\frac{d}{dC}\beta_f(C) \neq 0 \tag{253}$$

for all $C \in Y$. Condition (253) implies that β_f is strictly monotone. So the Lebesgue measure of $\beta_f(Y)$ intersected with the set of real numbers incommensurate with π equals the Lebesgue measure of Y. Thus, for almost all $C \in Y$, $\beta_f(C)$ is incommensurate with π. This gives the existence of quasiperiodic points for the (Y, f)-transverse map.

Definition 58: Let a be contained in some open connected interval $A \subset \mathbf{R}$. Let τ be an S^1 variable. A map

$$\begin{cases} a \to a \\ \tau \to \tau + F(a) \end{cases}$$

is a called a *twist map* if F is C^1 and

$$\frac{d}{da}F(a) \neq 0$$

for all $a \in A$.

The projection of the (Y, f)-transverse map onto $\{(0,0,C,t) : C \in Y, t \in S^1\}$ satisfies the definition of a twist map. Since we know that the (Y, f)-transverse map fixes the x and y coordinates, then $\tilde{\phi}_{Y,f,\pi,\mu}$ is a twist map on the annulus $\Omega_{Y,f,\pi,\mu}$ given that μ is sufficiently small, and $min_Y f$ is sufficiently large.

7. Arnold Diffusion in the Elliptic Problem

In this section, we perturb the parameter e and study the effects of this perturbation on the dynamics of the autonomous full equations (164). The elliptic problem, as given in (23), is easily written in terms of the variables (x, y, C, t), and in these variables it is called the *perturbed autonomous full equations.* These equations admit a Poincaré map which is a perturbation of the Poincaré map $\phi_{s_0,\mu}$. Under a small perturbation of e, most of the invariant circles in $\Omega_{Y,\pi,\mu}$ *persist.* By studying the stable and unstable manifolds of the persistent circles, we will show that the chaotic phenomenon known as *Arnold Diffusion* exists in the elliptic problem.

Lemma 59: *Fix* $(\mu, e) \in (0,1)^2$. *Then the vector field of* (23) *is* 2π*-periodic in* t*, and* 2π*-periodic in* θ.

Proof: Since x' and θ' do not depend explicitly on t and θ, then x' and θ' are each 2π-periodic in t and in θ. With y' and ρ', there are higher order μ-terms to contend with. First we show that y' is 2π-periodic in t and in θ.

From the definition of y, as given in (15), we have that

$$y' = \frac{1}{q_1^2 + q_2^2}\left[\left(p_1^2 + p_2^2 + q_1\frac{\partial U}{\partial q_1} + q_2\frac{\partial U}{\partial q_2}\right)\sqrt{q_1^2 + q_2^2} - \frac{(q_1 p_1 + q_2 p_2)^2}{\sqrt{q_1^2 + q_2^2}}\right], \tag{254}$$

where we have made use of (5). Now

$$\begin{aligned}\frac{\partial U}{\partial q_1} = &- \frac{(1-\mu)(q_1 - \mu r_1)}{\left((q_1 - \mu r_1)^2 + (q_2 - \mu r_2)^2\right)^{3/2}} \\ &- \frac{\mu(q_1 + (1-\mu) r_1)}{\left((q_1 + (1-\mu) r_1)^2 + (q_2 + (1-\mu) r_2)^2\right)^{3/2}},\end{aligned} \tag{255}$$

and

$$\frac{\partial U}{\partial q_2} = -\frac{(1-\mu)(q_2-\mu r_2)}{\left((q_1-\mu r_1)^2+(q_2-\mu r_2)^2\right)^{3/2}} - \frac{\mu(q_2+(1-\mu)r_2)}{\left((q_1+(1-\mu)r_1)^2+(q_2+(1-\mu)r_2)^2\right)^{3/2}}. \tag{256}$$

Adding (255) multiplied by q_1 to (256) multiplied by q_2, we have

$$q_1\frac{\partial U}{\partial q_1} + q_2\frac{\partial U}{\partial q_2} = -\frac{(1-\mu)\left(q_1^2+q_2^2-\mu(q_1r_1+q_2r_2)\right)}{\left((q_1-\mu r_1)^2+(q_2-\mu r_2)^2\right)^{3/2}} - \frac{\mu\left(q_1^2+q_2^2+(1-\mu)(q_1r_1+q_2r_2)\right)}{\left((q_1+(1-\mu)r_1)^2+(q_2+(1-\mu)r_2)^2\right)^{3/2}}. \tag{257}$$

The transformation of (257) under Λ is

$$-\frac{(1-\mu)(x^2-\mu x^4R_1)}{\left(1-2\mu x^2R_1+\mu^2x^4(r_1^2+r_2^2)\right)^{3/2}} - \frac{\mu(x^2+(1-\mu)x^4R_1)}{\left(1+2(1-\mu)x^2R_1+(1-\mu)^2x^4(r_1^2+r_2^2)\right)^{3/2}}, \tag{258}$$

where we have used (16). By (19), (74), (75), and (258), the transformation of (254) under Λ is

$$y' = x^6\rho^2 - \frac{(1-\mu)(x^4-\mu x^6R_1)}{\left(1-2\mu x^2R_1+\mu^2x^4(r_1^2+r_2^2)\right)^{3/2}} - \frac{\mu(x^4+(1-\mu)x^6R_1)}{\left(1+2(1-\mu)x^2R_1+(1-\mu)^2x^4(r_1^2+r_2^2)\right)^{3/2}}. \tag{259}$$

In (259), the variable t is found only in r_1, r_2, and R_1. Each of these is 2π-periodic in t (see appendix and (16)). In (259), the variable θ is found only in R_1, which is 2π-periodic in θ by (16).

Now we show that ρ' is 2π-periodic in t and in θ. From the definition of ρ, as given in (15), we have that

$$\rho' = q_1\frac{\partial U}{\partial q_2} - q_2\frac{\partial U}{\partial q_1}, \tag{260}$$

where we have used (5). If we subtract (255) multiplied by q_2 from (256) multiplied by q_1, we have

$$q_1\frac{\partial U}{\partial q_2} - q_2\frac{\partial U}{\partial q_1} = -\frac{\mu(1-\mu)(q_2r_1-q_1r_2)}{\left((q_1-\mu r_1)^2+(q_2-\mu r_2)^2\right)^{3/2}} - \frac{\mu(1-\mu)(q_2r_1-q_1r_2)}{\left((q_1+(1-\mu)r_1)^2+(q_2+(1-\mu)r_2)^2\right)^{3/2}}. \tag{261}$$

The transformation of (261) under Λ is

$$\rho' = -\frac{\mu(1-\mu)x^6 R_2}{\left(1 - 2\mu x^2 R_1 + \mu^2 x^4 (r_1^2 + r_2^2)\right)^{3/2}} - \frac{\mu(1-\mu)x^6 R_2}{\left(1 + 2(1-\mu)x^2 R_1 + (1-\mu)^2 x^4 (r_1^2 + r_2^2)\right)^{3/2}}. \tag{262}$$

In (262), the variable t is found only in r_1, r_2, R_1, and R_2, each of which is 2π-periodic in t (see appendix, (16) and (22)). In (262), the variable θ is found only in R_1 and R_2, each of which is 2π- periodic in θ by (16) and (22). This completes the proof.

Define $\tilde{J} : \mathbf{R}^+ \times S^1 \times \mathbf{R}^3 \times [0,1)^2 \to \mathbf{R}$ by

$$\tilde{J}(x, \theta, y, \rho, t; \mu, e) = \frac{1}{2}y^2 + \frac{1}{2}x^4\rho^2 - \tilde{U}(x, \theta, t; \mu, e) - \rho, \tag{263}$$

where $\tilde{U}$ is the transformation of the potential function under Λ. Note that setting $e = 0$ in $\tilde{J}$ gives the Jacobi integral (82).

Lemma 60: *Fix $(\mu, e) \in (0,1)^2$. Then $\tilde{J}$ is 2π-periodic in t, and 2π-periodic in θ. Furthermore, $d\tilde{J}/dt$ is 2π-periodic in t, and 2π-periodic in θ.*

Proof: To show that $\tilde{J}$ is 2π-periodic in t and in θ, it is sufficient, by (263), to show that $\tilde{U}$ is 2π-periodic in t and in θ. The transformation of the potential function (4) under Λ is

$$\tilde{U}(x, \theta, t; \mu, e) = \frac{1-\mu}{\sqrt{x^{-4} - 2\mu x^{-2} R_1 + \mu^2 (r_1^2 + r_2^2)}} + \frac{\mu}{\sqrt{x^{-4} + 2(1-\mu)x^{-2} R_1 + (1-\mu)^2 (r_1^2 + r_2^2)}}. \tag{264}$$

In (264), the variable t is found only in r_1, r_2, and R_1, each of which is 2π-periodic in t (see appendix and (16)). In (264), the variable θ is found only in R_1 which is 2π-periodic in θ by (16).

To show that $d\tilde{J}/dt$ is 2π-periodic in t and in θ, differentiate (264) with respect to t. This gives

$$\frac{d\tilde{J}}{dt} = yy' + 2x^3\rho^2 x' + x^4\rho\rho' - \frac{d\tilde{U}}{dt} - \rho'. \tag{265}$$

By Lemma 59, the terms in (265) involving y', x', and ρ' are 2π-periodic in t and in θ.

From (264),

$$\begin{aligned}\frac{d\tilde{U}}{dt} = &-\frac{1}{2}(1-\mu)\left(x^{-4}-2\mu x^{-2}R_1+\mu^2(r_1^2+r_2^2)\right)^{-3/2}\\ &\quad\times\left(-4x^{-5}x'+4\mu x^{-3}R_1x'-2\mu x^{-2}R_1'\right.\\ &\qquad\left.+\mu^2(2r_1r_1'+2r_2r_2')\right)\\ &-\frac{1}{2}\left(x^{-4}+2(1-\mu)x^{-2}R_1+(1-\mu)^2(r_1^2+r_2^2)\right)^{-3/2}\\ &\quad\times\left(-4x^{-5}x'-4(1-\mu)x^{-3}R_1x'+2(1-\mu)x^{-2}R_1'\right.\\ &\qquad\left.+(1-\mu)^2(2r_1r_1'+2r_2r_2')\right).\end{aligned} \tag{266}$$

In (266), the variable t is found in the terms r_1, r_2, r_1', r_2', R_1, and R_1'. We know that r_1, r_2, and R_1 are 2π-periodic in t. Since, by (16),

$$R_1' = \frac{dr_1}{dT}\frac{dT}{dt}\cos\theta - r_1x^4\rho\sin\theta + \frac{dr_2}{dT}\frac{dT}{dt}\sin\theta + r_2x^4\rho\cos\theta, \tag{267}$$

and as dr_i/dt, $i=1,2$, are 2π-periodic in T (see appendix), and dT/dt is 2π-periodic in t (see appendix), then r_1', r_2', and R_1' are 2π-periodic in t. In (266), the variable θ is found in the terms R_1 and R_1'. By (16), R_1 is 2π-periodic in θ. By (267), R_1' is 2π-periodic in θ. This completes the proof.

With $e \in (0,1)$, C is no longer a constant of motion. However, we can still use

$$C = \tilde{J}(x,\theta,y,\rho,t;\mu,e) \tag{268}$$

to solve for $\rho = \rho(x,\theta,y,C,t;\mu,e)$. By Lemma 60, ρ is 2π-periodic in t and in θ.

Theorem 61: *Let $(\mu,e)\in(0,1)^2$. If e is sufficiently small, then the equations of motion of the elliptic problem in (x,y,C,t)- variables are given by the two-parameter family*

$$\begin{cases}\dfrac{dx}{ds} = \dfrac{-\frac{1}{2}x^3y}{1-x^4\rho}\\ \dfrac{dy}{ds} = \dfrac{-x^4+x^6\rho^2+\mu\tilde{g}_1(x,s)}{1-x^4\rho}+O(\mu^2)\\ \dfrac{dC}{ds} = \dfrac{d\tilde{J}}{ds}(x,t-s,y,\rho,t;\mu,e)\\ \dfrac{dt}{ds} = \dfrac{1}{1-x^4\rho},\end{cases} \tag{269}$$

where $\tilde{J}$ is given by (263), and ρ is given by (268). Furthermore, the vector field of (269) is 2π-periodic in t, and 2π-periodic in s.

Proof: Let $(\mu, e) \in (0,1)^2$. Since g_1 and g_2, as given by (24) and (25) respectively, are real analytic in e, then we can write

$$g_1(x, \theta, t; e) = \tilde{g}_1(x, t-\theta) + O(e), \tag{270}$$

and

$$g_2(x, \theta, t; e) = \tilde{g}_2(x, t-\theta) + O(e), \tag{271}$$

where $\tilde{g}_1$ and $\tilde{g}_2$ are given by (61) and (62) respectively. Substituting (270) and (271) into (23), we obtain

$$\begin{cases} x' = -\dfrac{1}{2}x^3 y \\ \theta' = x^4 \rho \\ y' = -x^4 + x^6\rho^2 + \mu\tilde{g}_1(x, t-\theta) + O(\mu e) + O(\mu^2) \\ \rho' = \mu\tilde{g}_2(x, t-\theta) + O(\mu e) + O(\mu^2). \end{cases} \tag{272}$$

If e is small enough, then $O(\mu e) + O(\mu^2) = O(\mu^2)$. If we replace θ by $t - s$, then by Lemma 14, (272) becomes

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{d\theta}{ds} = \dfrac{x^4\rho}{1 - x^4\rho} \\ \dfrac{dy}{ds} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} + O(\mu^2) \\ \dfrac{d\rho}{ds} = \dfrac{\mu\tilde{g}_2(x,s)}{1 - x^4\rho} + O(\mu^2), \end{cases} \tag{273}$$

Since the vector field of (273) does not explicitly depend on θ, we can drop $d\theta/dt$ from (273) and replace it with dt/ds as given by Lemma 14. And since we can solve for ρ by (268), we can drop $d\rho/ds$ from (273), and in its place put

$$\frac{dC}{ds} = \frac{d\tilde{J}}{ds}(x, t-s, y, \rho, t; \mu, e),$$

and we obtain (269). That the vector field of (269) is 2π-periodic in t and in $s = t - \theta$ follows by Lemmas 59 and 60. This completes the proof.

Note that the vector field of (269) is naturally extendable to all $x \in \mathbf{R}$. Since the vector field of (269) is 2π-periodic in t and in s, then the generalized phase space of the equations (269) is $\mathbf{R}^3 \times S^2$. By expanding $d\tilde{J}/ds$ in terms of e, it follows that the vector field of (269) is a perturbation of the vector field of the full equations (163). For this reason we call the equations (269), with the vector field of (269) extended to all $x \in \mathbf{R}$, the *perturbed full equations.* In autonomous form, the perturbed full equations are, using τ as a dummy time variable,

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{-\frac{1}{2}x^3 y}{1 - x^4\rho} \\ \dfrac{dy}{d\tau} = \dfrac{-x^4 + x^6\rho^2 + \mu\tilde{g}_1(x,s)}{1 - x^4\rho} \\ \dfrac{dC}{d\tau} = \dfrac{d\tilde{J}}{ds}(x, t-s, y, \rho, t; \mu, e) \\ \dfrac{dt}{d\tau} = \dfrac{1}{1 - x^4\rho} \\ \dfrac{ds}{d\tau} = 1, \end{cases} \tag{274}$$

where s(0)=0. We begin our analysis of the phase space of the autonomous perturbed full equations by looking for invariant sets in $\{(0, y, C, t, s) : y, C \in \mathbf{R}, t, s \in S^1\}$.

Proposition 62: *Let* $(\mu, e) \in (0,1)^2$. *Then the set*

$$\{(0,0,C,t,s) : C \in \mathbf{R}, t, s \in S^1\}$$

is invariant under the flow of the autonomous perturbed full equations (274).

Remark: The set given in the above Proposition is the same as the one given in (165), and is invariant under the flow of the autonomous full equations (164) by Proposition 41. Thus the set (165) is invariant for all $e \in [0, 1)$.

Proof: Let $(C_0, t_0, s_0) \in \mathbf{R} \times S^2$ be arbitrary. Let

$$w(s_0 + \tau) = \big(w_1(s_0+\tau), w_2(s_0+\tau), w_3(s_0+\tau), w_4(s_0+\tau), s_0 + \tau \pmod{2\pi}\big) \tag{275}$$

be the solution of (274) that satisfies the initial condition $w(s_0) = (0, 0, C_0, t_0, s_0)$. What we need to show is that $w_1(s_0 + \tau) = 0$ and $w_2(s_0 + \tau) = 0$ for all $\tau \in \mathbf{R}$. By observation of (274), we see that

$$\left.\frac{dx}{d\tau}\right|_{x=0} = 0.$$

Thus it follows that $w_1(s_0 + \tau) = 0$ for all $\tau \in \mathbf{R}$. By the chain rule

$$\frac{dy}{d\tau} = \frac{dy}{dt}\frac{dt}{ds}\frac{ds}{d\tau}. \tag{276}$$

From (259), we have that $y' = 0$ when $x = 0$. Thus (276) implies that $w_2(s_0 + \tau) = 0$ for all $\tau \in \mathbf{R}$. This completes the proof.

By considering what $dC/d\tau$ is when $x = 0$, we can discern something more about the solutions in the set (165). By the chain rule, and the definition of C, we have that

$$\begin{aligned}\frac{dC}{d\tau} &= \frac{dC}{dt}\frac{dt}{ds}\frac{ds}{d\tau} \\ &= \frac{d\tilde{J}}{dt}\frac{dt}{ds}\frac{ds}{d\tau}.\end{aligned} \tag{277}$$

From (265), we know that

$$\begin{aligned}\left.\frac{d\tilde{J}}{dt}\right|_{x=0} &= \left[yy' + 2x^3\rho^2 x' + x^4\rho\rho' - \frac{d\tilde{U}}{dt} - \rho'\right]_{x=0} \\ &= \left[-\frac{d\tilde{U}}{dt} - \rho'\right]_{x=0},\end{aligned}$$

since $x' = 0$ and $y' = 0$ when $x = 0$. From (266) and (262),

$$\left.\frac{d\tilde{U}}{dt}\right|_{x=0} = 0 \quad \text{and} \quad \rho'\big|_{x=0} = 0,$$

and so from (277), $dC/d\tau = 0$ when $x = 0$. Thus, for the solution (275) satisfying the initial condition $w(s_0) = (0, 0, C_0, t_0, s_0)$, $w_3(s_0 + \tau) = C_0$ for all $\tau \in \mathbf{R}$. By observation of the vector field of (274), we see that $dt/d\tau = 0$ when $x = 0$. Thus $w_4(s_0 + \tau) = t_0 + \tau \pmod{2\pi}$ for all $\tau \in \mathbf{R}$. Therefore, the solutions (275) with the initial conditions $w(s_0) = (0, 0, C_0, t_0, s_0)$ for $C_0 \in \mathbf{R}$ fixed and $(t_0, s_0) \in S^2$ form the resonant $1:1$ flow on the 1-torus $T^1 = S^1 \times S^1$.

Let $s_0 \in S^1$. By Theorem 61, the vector field of the autonomous perturbed full equations is 2π-periodic in s. So we can reduce the flow in the phase space of (274) to a three-parameter, four-dimensional Poincaré map $\varphi_{s_0,\mu,e} : \Sigma^{s=s_0} \to \Sigma^{s=s_0}$. Note that $\varphi_{s_0,\mu,0} = \phi_{s_0,\mu}$. By Proposition 41 and 62, the set $\{x = 0, y = 0\}$ (as defined by (166)) is invariant under $\varphi_{s_0,\mu,e}$ for all $e \in [0, 1)$. Moreover, by the discussion in the previous paragraph, each point of $\{x = 0, y = 0\}$ is a fixed point of $\varphi_{s_0,\mu,e}$ for all $e \in [0, 1)$. Let $\{x > 0 : \Sigma\} \equiv \{(x, y, C, t) : x > 0, y, C \in \mathbf{R}, t \in S^1\}$.

Theorem 63 (Robinson): *Let* $(s_0, \mu, e) \in S^1 \times (0,1) \times [0,1)$. *Then* $W^s_{\varphi_{s_0,\mu,e}}(\{x = 0, y = 0\})$ *and* $W^u_{\varphi_{s_0,\mu,e}}(\{x = 0, y = 0\})$ *exist and are real analytic in* $\{x > 0 : \Sigma\}$.

Proof: See Robinson [1984].

Note that in Theorem 63, $e \in [0, 1)$. For $e = 0$, we have the following relation between the stable (unstable) manifold of $\{x = 0, y = 0\}$ with respect to $\varphi_{s_0,\mu,0}$ and the stable (unstable) manifold of the fixed point $(0,0)$ of ψ_{s_0,μ,C^*}. Let $\{C = C^*\} \equiv \{(C^*, t) : t \in S^1\}$.

Proposition 64: *Let* $(s_0, \mu) \in S^1 \times (0,1)$, *and set* $e = 0$. *Then*

$$W^s_{\varphi_{s_0,\mu,0}}(\{x = 0, y = 0\}) = \bigcup_{C^* \in \mathbf{R}} \left(W^s_{\psi_{s_0,\mu,C^*}}(0,0) \times \{C = C^*\} \right),$$

and

$$W^u_{\varphi_{s_0,\mu,0}}(\{x = 0, y = 0\}) = \bigcup_{C^* \in \mathbf{R}} \left(W^u_{\psi_{s_0,\mu,C^*}}(0,0) \times \{C = C^*\} \right).$$

Proof: Let $(x_0, y_0, C^*, t_0) \in \Sigma^{s=s_0}$. Note that with $e = 0$, C^* is a constant of motion by Theorem 12. Recalling that $\varphi_{s_0,\mu,0} = \phi_{s_0,\mu}$, and applying Proposition 42, we have

$$\begin{aligned} \varphi^j_{s_0,\mu,0}(x_0, y_0, C^*, t_0) &= \phi^j_{s_0,\mu}(x_0, y_0, C^*, t_0) \\ &= (\psi^j_{s_0,\mu,C^*}, C^*, t_j). \end{aligned} \tag{278}$$

If $(x_0, y_0) \in W^s_{\psi_{s_0,\mu,C^*}}(0,0) \subset \Gamma^{s=s_0}_{C^*}$, then (278) implies that

$$(x_0, y_0) \times (C^*, t_0) = (x_0, y_0, C^*, t_0) \in W^s_{\varphi_{s_0,\mu,0}}(\{x = 0, y = 0\}).$$

On the other hand, if $(x_0, y_0, C^*, t_0) \in W^s_{\varphi_{s_0,\mu,0}}(\{x = 0, y = 0\})$, where $t_0 \in S^1$ is arbitrary, then (278) implies that

$$(x_0, y_0) \in W^s_{\psi_{s_0,\mu,C^*}}(0,0).$$

Hence

$$(x_0, y_0, C^*, t_0) = (x_0, y_0) \times (C^*, t_0) \in W^s_{\psi_{s_0,\mu,C^*}}(0,0) \times \{C = C^*\}.$$

The other statement regarding the unstable manifolds is proved in a similar manner. This completes the proof.

It follows from Propostion 64 that the manifolds $W^s_{\varphi_{s_0,\mu,0}}(\{x=0,y=0\})$ and $W^u_{\varphi_{s_0,\mu,0}}(\{x=0,y=0\})$ are three-dimensional manifolds in $\Sigma^{s=s_0}$.

Using the fact that C^* is a constant of motion when $e=0$ we can gain some insight into the set formulas given in Proposition 64. Let $(x_0,y_0,C^*,t_0) \in W^s_{\psi_{s_0,\mu,C^*}}(0,0) \times \{C=C^*\}$. Since C^* is a constant of motion when $e=0$, then

$$\varphi^j_{s_0,\mu,0}(x_0,y_0,C^*,t_0) \in W^s_{\psi_{s_0,\mu,C^*}}(0,0) \times \{C=C^*\}$$

for all $j \in \mathbf{Z}$. Thus $W^s_{\varphi_{s_0,\mu,0}}(\{x=0,y=0\})$ is the union of invariant sets. A similar result holds for $W^u_{\varphi_{s_0,\mu,0}}(\{x=0,y=0\})$.

From here on in, fix $s_0=\pi$, and fix $\mu \in (0,1)$ sufficiently small. Let Y be the open connected bounded subset of $(\sqrt{2},\infty)$ given by Theorem 56. Fix $f \in \mathcal{Q}_{Y,\pi,\mu}$ with $min_Y f$ sufficiently large. For $C^* \in Y$, define

$$T_{C^*} = \{p_{f(C^*)}(\mu,C^*)\} \times \{C=C^*\}.$$

Note that T_{C^*} is a subset of the annulus $\Omega_{Y,f,\pi,\mu}$. Also note that T_{C^*} is diffeomorphic to a circle. For this reason we refer to T_{C^*} as a circle. In the proof of Proposition 43, we showed that each circle T_{C^*} is invariant under $\phi^{f(C^*)}_{\pi,\mu}$. Thus each circle T_{C^*} is invariant under $\varphi^{f(C^*)}_{\pi,\mu,0}$.

Proposition 65: *Let $\mu \in (0,1)$ be sufficiently small, $f \in \mathcal{Q}_{Y,\pi,\mu}$, and set $e=0$. For each $C^* \in Y$,*

$$W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}) = W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu,C^*)) \times \{C=C^*\},$$

and

$$W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}) = W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu,C^*)) \times \{C=C^*\}.$$

Proof: Let $(C^*,t_0) \in \{C=C^*\} \subset Y \times S^1$. Note that C^* is a constant of motion by Theorem 12. If $(x_0,y_0) \in W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu,C^*))$, then (278) implies that

$$(x_0,y_0) \times (C^*,t_0) = (x_0,y_0,C^*,t_0) = \in W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}).$$

On the other hand, if $(x_0, y_0, C^*, t_0) \in W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$, then (278) implies that

$$(x_0, y_0) \in W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*)).$$

Hence

$$(x_0, y_0, C^*, t_0) = (x_0, y_0) \times (C^*, t_0) \in W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*)) \times \{C = C^*\}.$$

The result regarding the unstable manifolds is shown in a similar manner. This completes the proof.

There are two consequences of Proposition 65. The sets

$$W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}) \quad \text{and} \quad W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$$

are two dimensional manifolds. And they are real analytic manifolds in $\Sigma^{s=\pi}$ since

$$W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*)) \quad \text{and} \quad W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*))$$

are real analytic manifolds in $\Gamma^{s=\pi}_{C^*}$.

Definition 66: Let V be a Banach space over $\mathbf{R}$. Let M and N be differentiable manifolds in V. The *tangent space* of M at a point $p \in M$ is the linear span of the tangents vectors at p of differentiable curves in M passing through p. We denote the tangent space of M at p by $\mathcal{T}_p M$. The manifolds M and N *intersect transversally at a point* $p \in (M \cap N)$ if $\mathcal{T}_p M + \mathcal{T}_p N = \mathcal{T}_p V$. The manifolds M and N *intersect transversally on a set* $A \subset (M \cap N)$ if M and N intersect transversally at each point $p \in A$.

We are dealing with the Poincaré map $\varphi_{\pi,\mu,e}$ defined on the Banach space $\Sigma^{s=\pi}$. In particular, the tangent space of $\Sigma^{s=\pi}$ at each point $p \in \Sigma^{s=\pi}$ is $\mathcal{T}_p \Sigma^{s=\pi} = \mathbf{R}^4$. For the Banach space $\Gamma^{s=\pi}_{C^*}$, the tangent space at each point $p \in \Gamma^{s=\pi}_{C^*}$ is $\mathcal{T}_p \Gamma^{s=\pi}_{C^*} = \mathbf{R}^2 = \Gamma^{s=\pi}_{C^*}$.

A *smooth simple closed curve* is the diffeomorphic image of S^1.

Proposition 67: *Let* $\mu \in (0,1)$ *be sufficiently small, and set* $e = 0$. *If* $C^* \in Y$ *and* $\min_Y f$ *is sufficiently large, then there exist smooth simple closed curves* $c_1^0(C^*)$ *and* $c_2^0(C^*)$ *in* $\Sigma^{s=\pi}$ *such that* $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ *intersects* $W^s_{\varphi^{f(C^*)}_{\pi,\mu,C^*}}(\{x = 0, y = 0\})$ *transversally on* $c_1^0(C^*)$, *and* $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ *intersects* $W^u_{\varphi^{f(C^*)}_{\pi,\mu,C^*}}(\{x = 0, y = 0\})$ *transversally on* $c_2^0(C^*)$.

Proof: Let $C^* \in Y$, and $f \in \mathcal{Q}_{Y,\pi,\mu}$. If $min_Y f$ is sufficiently large, then from the proof of Proposition 40, $W^u_{\psi_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*))$ intersect $W^s_{\psi_{\pi,\mu,C^*}}(0,0)$ transversally at some point $(x_0, y_0) \in \Gamma^{s=\pi}_{C^*}$. Thus $W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*))$ intersect $W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(0,0)$ transversally at the point $(x_0, y_0) \in \Gamma^{s=\pi}_{C^*}$. Hence there are one-dimensional tangent spaces l^u and l^s in $\Gamma^{s=\pi}_{C^*}$ such that

$$l^u = \mathcal{T}_{(x_0,y_0)} W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*)),$$

$$l^s = \mathcal{T}_{(x_0,y_0)} W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(0,0),$$

and $l^u + l^s = \Gamma^{s=\pi}_{C^*}$. By Proposition 65,

$$W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}) = W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*)) \times \{C = C^*\}. \tag{279}$$

By a slight variation of the proof of Proposition 64,

$$W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\}) = \bigcup_{C_0 \in \mathbf{R}} \left(W^s_{\psi^{f(C^*)}_{\pi,\mu,C_0}}(0,0) \times \{C = C_0\} \right). \tag{280}$$

Since $W^u_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(p_{f(C^*)}(\mu, C^*))$ intersect $W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(0,0)$ at $(x_0, y_0) \in \Gamma^{s=\pi}_{C^*}$, then (279) and (280) imply that $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{s_0,\mu,0}}(\{x = 0, y = 0\})$ on the set $A = \{(x_0, y_0)\} \times \{C = C^*\}$. The set A is diffeomorphic to a circle, and so A is a smooth simple closed curve in $\Sigma^{s=\pi}$. Let $c^0_1(C^*) = A$. Now we need to show that $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{s_0,\mu,0}}(\{x = 0, y = 0\})$ transversally on the set $c^0_1(C^*)$. Let $(x_0, y_0, C^*, t_0) \in c^0_1(C^*)$. From (279),

$$\mathcal{T}_{(x_0,y_0,C^*,t_0)} W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*}) = l^u \times \{C^*\} \times \mathbf{R}. \tag{281}$$

From (280),

$$\mathcal{T}_{(x_0,y_0,C^*,t_0)} W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\}) = l^s \times \mathbf{R} \times \mathbf{R}. \tag{282}$$

Since $l^u + l^s = \Gamma^{s=\pi}_{C^*}$, then the sum of (281) and (282) is $\mathbf{R}^4$. As $t_0 \in S^1$ is arbitrary, then $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\})$ transversally on $c^0_1(C^*)$. That $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\})$ transversally on some circle $c^0_2(C^*)$ in $\Sigma^{s=\pi}$ follows in a similar manner. This completes the proof.

Definition 68: Let $C^* \in Y$, $f \in \mathcal{Q}_{Y,\pi,\mu}$, and $e \in [0,1)$. A set $A \subset \Sigma^{s=\pi}$ is *locally invariant* under $\varphi^{f(C^*)}_{\pi,\mu,e}$ if for each point $(x_0, y_0, C_0, t_0) \in A$, there exists $n_1, n_2 \in \mathbf{N} \cup \{0\}$, not both zero, such that

$$\left(\varphi^{f(C^*)}_{\pi,\mu,e}\right)^j (x_0, y_0, C_0, t_0) \in A$$

for all $j \in \{-n_1, -n_1+1, ..., n_2-1, n_2\}$.

Note that any set $A \subset \Sigma^{s=\pi}$ invariant under $\varphi^{f(C^*)}_{\pi,\mu,e}$ is also locally invariant under $\varphi^{f(C^*)}_{\pi,\mu,e}$.

Definition 69: Let $C^* \in Y$, $f \in \mathcal{Q}_{Y,\pi,\mu}$, and $r \in \mathbf{N}$. A compact C^r manifold $A \subset \Sigma^{s=\pi}$, (locally) invariant under $\varphi^{f(C^*)}_{\pi,\mu,0}$, is (locally) *persistent* if for all sufficiently small $e \in (0,1)$ there exists a C^r manifold $A^e \subset \Sigma^{s=\pi}$ (locally) invariant under $\varphi^{f(C^*)}_{\pi,\mu,e}$, C^r diffeomorphic to A, and C^1 close to A. If A is (locally) persistent, we say A (locally) persists and perturbs to A^e when e is perturbed.

Theorem 70: *Let $\mu \in (0,1)$ be sufficiently small. Fix $f \in \mathcal{Q}_{Y,\pi,\mu}$ so that $min_Y f$ is sufficiently large. Then most of the invariant circles T_{C^*} in $\Omega_{Y,f,\pi,\mu}$ persist when e is perturbed.*

Remark: By "most" we mean the following. Let $\hat{\Omega}_{Y,f,\pi,\mu}$ and $\hat{T}_{C^*}$ be the projections of $\Omega_{Y,f,\pi,\mu}$ and T_{C^*}, respectively, to $\{x=0, y=0\}$. Then, as $e \to 0$, with $e \in (0,1)$, the Lebesgue measure of the persistent circles $\hat{T}^e_{C^*}$ tends to the Lebesgue measure of $\hat{\Omega}_{Y,f,\pi,\mu}$.

Proof: In the last section, we showed that the (Y,f)-transverse map $\tilde{\phi}_{Y,f,\pi\mu}$ is a twist map on the invariant $\Omega_{Y,f,\pi,\mu}$. Using (172), we have that

$$\tilde{\phi}_{Y,f,\pi,\mu}(x_0, y_0, C^*, t_0) = \phi^{f(C^*)}_{\pi,\mu}(x_0, y_0, C^*, t_0) = \varphi^{f(C^*)}_{\pi,\mu,0}(x_0, y_0, C^*, t_0)$$

provided that $(x_0, y_0, C^*, t_0) \in \Omega_{Y,f,\pi\mu}$. Let $\tilde{\varphi}^{f(C^*)}_{\pi,\mu,0}$ be the restriction of $\varphi^{f(C^*)}_{\pi,\mu,0}$ to $\Omega_{Y,f,\pi,\mu}$. Then, $\tilde{\varphi}^{f(C^*)}_{\pi,\mu,0}$ is a twist map. Let A be a compact invariant submanifold of $\Omega_{Y,f,\pi,\mu}$ such that the projection $\tilde{A}$ of A to $\{x=0, y=0\}$ is connected and has Lebesgue measure greater than zero. Xia [1993] showed that A persists, and that $\tilde{\varphi}^{f(C^*)}_{\pi,\mu,e}$ restricted to A^e is area-preserving for all $e \in [0,1)$ sufficiently small. Appyling KAM Theory (see Guckenheimer and Holmes [1983]) gives the persistence of most of the invariant circles

T_{C^*} in A. By choosing A so that the Lebesgue measure of $\tilde{\Omega}_{Y,f,\pi,\mu} \setminus \tilde{A}$ is close to zero, the proof is complete.

Proposition 71: *Let $\mu \in (0,1)$ be sufficiently small, and let $C^* \in Y$. If T_{C^*} persists, then there exist smooth simple closed curves $c_1^e(C^*)$ and $c_2^e(C^*)$ in $\Sigma^{s=\pi}$ such that $c_i^e(C^*)$ is C^1-close to $c_i^0(C^*)$, $i = 1,2$, and $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x = 0, y = 0\})$ transversally on $c_1^e(C^*)$, and $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ intersects $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x = 0, y = 0\})$ transversally on $c_2^e(C^*)$.*

Proof: By Proposition 67, $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\})$ transversally on $c_1^0(C^*)$. Let A be a compact connected subset of $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ that contains T_{C^*} and $c_1^0(C^*)$ in its interior. The set A is a locally invariant manifold. Since T_{C^*} persists, perturbing to $T^e_{C^*}$, the set A locally persists, perturbing to some compact connected locally invariant set $A^e \subset W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ by Invariant Manifold Theory (see Fenichel [1971] where the connectedness of A is required). Note that A^e contains $T^e_{C^*}$. By Theorem 63, we know that $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x = 0, y = 0\})$ exists and is real analytic in $\{x > 0 : \Sigma\}$ for all $e \in [0,1)$. Since $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\{x = 0, y = 0\})$ transversally on $c_1^0(C^*)$, then $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T_{C^*})$ intersects $W^s_{\varphi^{f(C^*)}_{s0,\mu,e}}(\{x = 0, y = 0\})$ transversally on some smooth simple closed curve $c_1^e(C^*) \subset \Sigma^{s=\pi}$ C^1-close to $c_1^0(C^*)$, provided that $e \in (0,1)$ is sufficiently small. The other statement is proved in a similar manner. This completes the proof.

Definition 72: Let $\mu \in (0,1)$, $e \in [0,1)$, and $A \subset \Sigma^{s=\pi}$. A point $(x,y,C,t) \in \Sigma^{s=\pi}$ belongs to the ω-limit set of A, denoted by $\omega(A)$, if and only if there exists a point $(x_0, y_0, C_0, t_0) \in A$ such that

$$\liminf_{j \to \infty} \left\{ \|(\varphi^{f(C^*)}_{\pi,\mu,e})^j (x_0, y_0, C_0, t_0) - (x,y,C,t)\| \right\} = 0.$$

A point $(x,y,C,t) \in \Sigma^{s=\pi}$ belongs to the α-limit set of A, denoted by $\alpha(A)$, if and only if there exists a point $(x_0, y_0, C_0, t_0) \in A$ such that

$$\liminf_{j \to -\infty} \left\{ \|(\varphi^{f(C^*)}_{\pi,\mu,e})^j (x_0, y_0, C_0, t_0) - (x,y,C,t)\| \right\} = 0.$$

Proposition 73: *Let $\mu \in (0,1)$ be sufficiently small, $e \in [0,1)$, and $C^* \in Y$. If $e = 0$ or $e > 0$ is sufficiently small, then $\omega(c_1^e(C^*))$ and $\alpha(c_2^e(C^*))$ are circles in $\{x = 0, y = 0\}$. Furthermore, $\omega(c_1^e(C^*)) = \alpha(c_2^e(C^*))$ for $e = 0$ or $e \in (0,1)$ small.*

Proof: Suppose $\omega(c_1^0(C^*))$ is a circle in $\{x=0,y=0\}$, and let A be this circle. Since $A \subset \{x=0,y=0\}$, then A persist and perturbs to A^e. However, since $\{x = 0, y = 0\}$ is invariant for all $e \in [0,1)$, and as each point of $\{x=0,y=0\}$ is a fixed point (which does not move when e is perturbed) for all $e \in [0,1]$, then $A^e = A$. Thus $\omega(c_1^e(C^*))$ is also a circle, given by A^e, for all small $e \in (0,1)$. So, it is sufficient to show that $\omega(c_1^0(C^*))$ is a circle in $\{x=0,y=0\}$.

Recall that $c_1^0(C^*))$ is given by $\{(x_0,y_0)\} \times \{C=C^*\}$ for some

$$(x_0,y_0) \in W^s_{\psi^{f(C^*)}_{\pi,\mu,C^*}}(0,0).$$

Thus, $(x_0,y_0) \to (0,0)$ under iterations of $\varphi^{f(C^*)}_{\pi,\mu,0}$. Since $e=0$, then the vector field of the autonomous full equations (164) does not explicitly depend on t. Thus, for each $j \in \mathbf{Z}$,

$$t_{j\times f(C^*)} = \int_{\pi}^{\pi+2\pi\times j\times f(C^*)} \left(\frac{dt}{d\tau}\right) d\tau + t_0. \tag{283}$$

Note that the integral in (283) does not depend on t_0. Thus, each iterate of the projection of $\varphi^{f(C^*)}_{\pi,\mu,0}$ onto $\{x=0,y=0\}$ rotates the circle $\{C=C^*\}$. Therefore, for any $(C^*,t) \in \{C=C^*\}$, we can find $(x_0,y_0,C^*,t_0) \in c_1^0(C^*)$ such that

$$\liminf_{j\to\infty} \left\{ \|(\varphi^{f(C^*)}_{\pi,\mu,0})^j(x_0,y_0,C^*,t_0) - (0,0,C^*,t)\| \right\} = 0.$$

Hence $\omega(c_1^0(C^*))$ is a circle in $\{x=0,y=0\}$. The result for $\alpha(c_2^0(C^*))$ is shown similarly.

Now we show that $\omega(c_1^e(C^*)) = \alpha(c_2^e(C^*))$ for $e=0$ or $e \in (0,1)$ small. It is sufficient just to consider the case when $e=0$, since for small $e \in (0,1)$, $\omega(c_1^e(C^*)) = \omega(c_1^0(C^*))$ and $\alpha(c_2^e(C^*)) = \alpha(c_2^0(C^*))$ by the nature of the invariant set $\{x=0,y=0\}$. When $e=0$, then C^* is a constant of motion by Theorem 12. Thus $\omega(c_1^0(C^*)) = \alpha(c_2^0(C^*))$. This completes the proof.

Note that $c_1^0(C^*)$ belongs to the stable manifold of, and $c_2^0(C^*)$ belongs to the unstable manifold of, the circle $\{(0,0)\} \times \{C=C^*\}$. Also note that the circle $\{p(\mu,C^*)\} \times \{C=C^*\}$ is contained in the intersection of the stable and unstable manifold of the circle $\{(0,0)\} \times \{C=C^*\}$. Since $\omega(c_1^0(C^*)) = \alpha(c_2^0(C^*))$ by Proposition 73, then $W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\omega(c_1^0(C^*))$ intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\alpha(c_1^0(C^*))$ on the set $\{p(\mu,C^*)\} \times \{C=C^*\}$.

Now,

$$\mathcal{T}_{(p(\mu,C^*),C^*,t_0)} W^u_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\omega(c^0_1(C^*)) + \mathcal{T}_{(p(\mu,C^*),C^*,t_0)} W^s_{\varphi^{f(C^*)}_{\pi,\mu,0}}(\alpha(c^0_1(C^*)) = \mathbf{R}^3,$$

and so the intersection is not transversal. Recall that there are two families of tranverse symmetric homoclinic orbits. Since the later part of Section 5, we have concentrated on the family $p(\mu, C)$ in $\Gamma^{s=\pi}_C$. The proof of the next lemma requires this and the family $\bar{p}(\mu, C)$ in $\Gamma^{s=0}_C$.

Lemma 74 (Xia): *Let $\mu \in (0,1)$ be sufficiently small, and let $f \in \mathcal{Q}_{y,s_0,\mu}$ be such that $min_Y f$ is sufficiently large. If T_{C^*} persists when e is perturbed, then $W^u_{\varphi^{f(C^*)}_{s_0,\mu,e}}(\omega(c^e_1(C^*)))$ intersects $W^s_{\varphi^{f(C^*)}_{s_0,\mu,e}}(\alpha(c^e_1(C^*)))$ transversally at some point either in $\Sigma^{s=s_0}$ for either $s_0 = 0$ or $s_0 = \pi$.*

Proof: See Xia [1993].

We will need to apply the λ-lemma, but the fixed points in $\{x = 0, y = 0\}$ are not hyperbolic. However we do have the following nonstandard version of the λ-lemma.

Lemma 75: *Let c be a smooth simple closed curve in $\{x = 0, y = 0\}$. Let P be a smooth surface that intersects $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x = 0, y = 0\})$ in a curve l such that the intersection with P is transversal on l, and $\omega(l) = c$. Then, for any $p \in W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(c)$ and a small neighbourhood $N \subset W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(c)$ of p, there exists a small disk $\tilde{N} \subset P$ such that some image of $\tilde{N}$ under iteration of $\varphi^{f(C^*)}_{\pi,\mu,e}$ is C^1-close to N at the point p.*

Remark: By switching the superscripts 'u' and 's' and requiring $\alpha(l) = c$ in Lemma 75, or, by replacing π by 0, the resulting statements also holds.

Proof: The proof follows from a local behavior of the the map $\varphi^{f(C^*)}_{\pi,\mu,e}$ near the invariant set $\{x = 0, y = 0\}$, and is similar to the proof of the λ-lemma (see Palis and de Melo [1982]).

The combination of Proposition 71, Lemma 74, and Lemma 75 gives us the main result of Xia [1993].

Theorem 76 (Xia): *Let $\mu \in (0,1)$, $e \in (0,1)$ each be sufficiently small, and let $f \in \mathcal{Q}_{y,\pi,\mu}$ be so that $min_Y f$ is sufficiently large. If T_{C^*} persists and perturbs to $T^e_{C^*}$, then $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ intersects $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ transversally. Let $C_1, C_2 \in Y$ such that*

$C_1 \neq C_2$. If $|C_1 - C_2|$ is small enough and T_{C_1} and T_{C_2} persist and perturb to $T^e_{C_1}$ and $T^e_{C_2}$ respectively, then $W^s_{\varphi^{f(C_1)}_{\pi,\mu,e}}(T^e_{C_1})$ intersects $W^u_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ transervally.

Remark: By switching the superscripts 'u' and 's', we have that $W^u_{\varphi^{f(C_1)}_{\pi,\mu,e}}(T^e_{C_1})$ intersects $W^s_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ transversally provided that $|C_1 - C_2|$ is small enough.

Proof: First, we will show that for a persistent circle T_{C^*}, $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ and $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ intersect transversally at some point in $\Sigma^{s=\pi}$. By Proposition 71, there exist smooth simple closed curves $c^e_1(C^*)$ and $c^e_2(C^*)$ in $\Sigma^{s=\pi}$ such that $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ and $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x=0, y=0\})$ intersect tranversally on $c^e_1(C^*)$, and $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ and $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\{x=0, y=0\})$ intersect tranversally on $c^e_2(C^*)$. By Lemma 74, there exists a point $p \in \Sigma^{s=\pi}$ such that $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\omega(c^e_1(C^*)))$ and $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\alpha(c^e_2(C^*)))$ intersect transversally at p. (Here we are assuming the point p is in $\Sigma^{s=\pi}$. If it is in $\Sigma^{s=0}$ then replace π by 0 and continue on.) By Lemma 75, there are pieces of $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ C^1-close to $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\omega(c^e_1(C^*)))$ near p, and also there are pieces of $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ C^1-close to $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\alpha(c^e_2(C^*)))$ near p. Since $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\omega(c^e_1(C^*)))$ and $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(\alpha(c^e_2(C^*)))$ intersect transversally at p, then $W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ and $W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$ intersect transversally at some point in $\Sigma^{s=\pi}$.

Now we will show that for $C_1 \neq C_2$, with $|C_1 - C_2|$ small enough, $W^s_{\varphi^{f(C_1)}_{\pi,\mu,e}}(T^e_{C_1})$ intersects $W^s_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ transversally. If $|C_1 - C_2|$ is small enough, then pieces of $W^s_{\varphi^{f(C_1)}_{\pi,\mu,e}}(T^e_{C_1})$ are C^1-close to $W^s_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ near a point where $W^s_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ intersects $W^u_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ transversally. Thus, $W^s_{\varphi^{f(C_1)}_{\pi,\mu,e}}(T^e_{C_1})$ intersects $W^u_{\varphi^{f(C_2)}_{\pi,\mu,e}}(T^e_{C_2})$ transversally. This completes the proof.

Now we will define the main mechanism of Arnold Diffusion as it applies to our system.

Definition 77: For a persistent circle T_{C^*} with

$$W^s_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$$

intersecting

$$W^u_{\varphi^{f(C^*)}_{\pi,\mu,e}}(T^e_{C^*})$$

transversally at some point in $\Sigma^{s=\pi}$, we say $T^e_{C^*}$ is a *transition circle.* A finite sequence of transition circles $T^e_{C_i}$ for $i = 1, 2, ..., k$ satisfies a *chain condition* if for each $1 \leq i < k$,

$W^s_{\varphi_{\pi,\mu,e}^{f(C^*)}}(T^e_{C_i})$ intersects $W^u_{\varphi_{\pi,\mu,e}^{f(C^*)}}(T^e_{C_{i+1}})$ and $W^u_{\varphi_{\pi,\mu,e}^{f(C^*)}}(T^e_{C_i})$ intersects $W^s_{\varphi_{\pi,\mu,e}^{f(C^*)}}(T^e_{C_{i+1}})$. If a finite sequence of transition circles $T^e_{C_i}$ for $i = 1, 2, ..., k$ satisfies a chain condition, it is called a *transition chain.*

Theorem 78: *Let $(\mu, e) \in (0,1)^2$ both be sufficiently small. Let $f \in \mathcal{Q}_{Y,\pi,\mu}$ such that $\min_Y f$ is sufficiently large. Then there exists a transition chain in the elliptic problem.*

Proof: Take a circle T_{C_1} such that it is persistent. Then by Theorem 76, $T^e_{C_1}$ is a transition circle. By Theorem 70, we can find a persistent circle T_{C_2} with $C_2 > C_1$ and $|C_1 - C_2|$ very small. Thus, by Theorem 76, the transition circles $T^e_{C_i}$ for $i = 1, 2$ satisfy the chain condition. By Theorem 70, we can find a persistent circle T_{C_3} with $C_3 > C_2$ and $|C_3 - C_2|$ very small. Thus, by Theorem 76, the transition circles $T^e_{C_i}$ for $i = 1, 2, 3$ satisfy the chain condition. Proceeding in this way we generate a transition chain for the elliptic problem. This completes the proof.

Let $T^e_{C_i}$, $i = 1, 2, ..., k$, be a transition chain in the elliptic problem. Let $1 \leq j < k - 1$. A corollary of the λ-lemma (see Palis and de Melo [1982]) states that if stable (unstable) manifold of $T^e_{C_j}$ intersects transversally with the unstable (stable) manifold of $T^e_{C_{j+1}}$, and the stable (unstable) manifold of $T^e_{C_{j+1}}$ intersects transversally with the unstable (stable) manifold of $T^e_{C_{j+2}}$, then the stable (unstable) manifold of $T^e_{C_j}$ intersects transversally with the unstable (stable) manifold of $T^e_{C_{j+2}}$. Applying this to the transition chain, we have that for all $1 \leq i < j \leq k$, the stable (unstable) manifold of $T^e_{C_i}$ intersects transversally with the unstable (stable) manifold of $T^e_{C_j}$. These transversal intersections imply that all the stable (unstable) manifolds of the transition circles in the transition chain accumulate on each other, thus leading to very complicated dynamics in the neighbourhood of the transition chain. Thus, there are orbits (with respect to the Poincaré map) that "wander chaotically and slowly" among the transition circles in the transition chain. This phenomenon of slow chaotic wandering is called *Arnold Diffusion*, and was first described in Arnold [1964]. By Theorem 78, Arnold Diffusion exists in the elliptic problem.

Conclusion

This thesis has attempted to give a rigorous mathematical treatment of Xia's proof of the existence of Arnold Diffusion in the elliptic restricted planar three-body problem. We have filled in many gaps, discovered and corrected some minor errors, and given in rigorous detail many of the calculations that needed to be done. The constraints of time have prevented a complete mathematical treatment of all the lemmas and theorems used. For example, the proof of Lemma 74, as given in Xia [1993], would merit a thesis in itself. What has been included in this thesis is nevertheless a substantial contribution to the understanding of Xia's proof of the existence of Arnold Diffusion in the elliptic problem, and is by no means easily understood by a perfunctory glance.

There are two noteworthy consequences of the existence of Arnold Diffusion in the elliptic problem. The first is that there are no real analytic integrals of the elliptic restricted problem (see Xia [1993]). This extends a result of Poincaré [1899] regarding the existence of real analytic integrals of the circular restricted problem. The nonexistence of integrals for the elliptic restricted problem means that the problem is non-integrable. The second consequence is that there exists solutions of the elliptic restricted problem which are unstable in the sense of Liapunov (see Arnold [1964] and Xia [1992a]). An example of an unstable solution in the elliptic restricted problem is easily found by considering the Poincaré map of Section 7. For two persistent circles $T^e_{C_1}$ and $T^e_{C_2}$, the stable manifold of the first intersects with the unstable manifold of the second provided that $|C_1 - C_2|$ is small enough. By KAM Theory (see Guckenheimer and Holmes [1983]) the motion on the persistent circle $T^e_{C_2}$ is quasiperiodic. Since the stable manifold of $T^e_{C_1}$ intersects the unstable manifold of $T^e_{C_2}$, then the λ-lemma (see Palis and de Melo [1982]) implies that there are orbits near $T^e_{C_2}$ which prevent the quasiperiodic orbits in $T^e_{C_2}$ from being stable in the sense of Liapunov.

Bakker and Diacu [1993] have sought to bring to the attention of the astronomical community Xia's results. In particular, they investigated the existence of small celestial bodies with unpredictable motion in the solar system. Further, they speculated on the connection between the formation of the Kirkwood gaps and Arnold Diffusion. Their speculation is, however, without scientific support, but offers a direction for further studies in numerical and analytic methods. We note that the application of Xia's result to the Kirkwood gaps is ill-founded since the existence of Arnold Diffusion in the elliptic problem, given by Xia [1993], exists near parabolic-type orbits.

Studies have been done concerning the formation of the Kirkwood gaps. Brjuno [1970] gave a qualitative explanation of some of the low-order gaps. His approach gave the existence of zones of instability near families of periodic solutions. Froeschlé and Scholl [1976], through numerical studies of the averaged equations of motion, concluded that chaotic behavior is not involved in the formation of Kirkwood gaps. However, Wisdom [1983] with his numerical studies of an algebraic mapping he derived from the equations of motion, concluded that "... it is impossible to discount the importance of chaotic behavior in the formation ..." of the gaps. Wisdom [1982], [1985] support the possibility that Arnold Diffusion might play a role in the formation of the Kirkwood gaps. However, it has not been shown that in the restricted three-body problem Arnold Diffusion exists where it would affect asteroidal motion. This is an area in which further work could be done. Other reasons for the formation of the Kirkwood gaps have been proposed. Henrard and Lemaître [1983], while acknowledging that "... a kind of 'Arnold's diffusion' working over a time span of the order of the age of the solar system can produce the gaps," support the hypothesis that "... a displacement of the Jovian resonances in the asteroid belt can also produce gaps." Their hypothesis assumes the existence of an accretion disk in the early stage of the solar system and its subsequent removal.

We have pointed out some of the variety of the above conjectures (see Wisdom [1982] for some others) to bring to mind that there might not be one overall reason for the formation for the Kirkwood gaps. Each gap might possibly have its own set of reasons for existing, independent of the others. By modelling the motion of an asteroid by the three-body problem of the Sun-Jupiter-asteroid, we receive only an approximation of the true dynamics of the asteroid. Nevertheless, the model has proved itself useful in the past.

Mathematically, some of the techniques of Xia [1992], [1993] may prove useful in further analytic studies of the restricted three-body problem. This will lead to either confirming for rejecting the intuition we have or will gain through numerical studies. Knowing that Arnold Diffusion exists in the elliptic restricted planar three-body problem (as well as the planar three-body problem (see Xia [1992a])) should influence our intuition.

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Appendix: On the Formulation of the Equations of Motion

Under the assumption that the primaries P_1 and P_2 move along elliptic orbits, the distance between P_1 and P_2 is given by

$$r = \frac{1-e^2}{1+e\cos T},$$

where T is the *true anomaly*, and e is the eccentricity (see Szebehely [1967] where we have assumed that the semi-major axis a equals 1). In polar coordinates, the vector from the position of P_1 to the position of P_2 is given by (r, T). In rectangular coordinates, the vector from the position of P_1 to the position of P_2 is given by $(r\cos T, r\sin T) = (r_1, r_2)$. To obtain r_1 as given by (1), we expand $r\cos T$ in terms of e. Now

$$r\cos T\big|_{e=0} = \cos T,$$

and

$$\begin{aligned}\left.\frac{d(r\cos T)}{de}\right|_{e=0} &= \left[\frac{1}{(1+e\cos T)^2}\Big(-2e\cos T(1+e\cos T) - (1-e^2)\cos^2 T\Big)\right]_{e=0}\\ &= -\cos^2 T.\end{aligned}$$

Thus

$$\begin{aligned}r_1(T;e) &= \cos T - e\cos^2 T + O(e^2)\\ &= (1-e\cos T)\cos T + O(e^2),\end{aligned}$$

which is (1). To obtain r_2, as given by (2), we proceed likewise. Now

$$r\sin T\big|_{e=0} = \sin T,$$

and

$$\begin{aligned}\frac{d(r\sin T)}{de} &= \left[\frac{1}{(1+e\cos T)^2}\Big(-2e\sin T(1+e\cos T) - (1-e^2)\sin T\cos T\Big)\right]_{e=0}\\ &= -\sin T\cos T.\end{aligned}$$

Thus

$$\begin{aligned}r_2(T;e) &= \sin T - e\sin T\cos T + O(e^2)\\ &= (1-e\cos T)\sin T + O(e^2),\end{aligned}$$

which is (2). It follows from the definitions of r_1 and r_2, that each is 2π-periodic in T, and since

$$\frac{dr_1}{dT} = \frac{1}{(1+e\cos T)^2}\Big[-(1-e^2)\sin T(1+e\cos T) + e(1-e^2)\cos T\sin T\Big],$$

and

$$\frac{dr_2}{dT} = \frac{1}{(1+e\cos T)^2}\Big[(1-e^2)\cos T(1+e\cos T) + e(1-e^2)\sin^2 T\Big],$$

it follows that each of dr_1/dT and dr_2/dT are 2π-periodic in T.

The true anomaly T is given by

$$T(t;e) = 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan(t/2)\right],$$

where t is the *eccentric anomaly* (see Collins [1989]). To obtain T as given by (3), we expand T in terms of e. Now

$$\begin{aligned} T(t;0) &= 2\arctan(\tan(t/2)) \\ &= t, \end{aligned}$$

and

$$\begin{aligned} \left.\frac{dT(t;e)}{de}\right|_{e=0} &= \Bigg[\frac{1}{1+\left(\frac{1+e}{1-e}\right)\tan^2(t/2)} \\ &\quad \times \left(\frac{1+e}{1-e}\right)^{-1/2}\tan(t/2)\frac{2}{(1-e)^2}\Bigg]_{e=0} \\ &= \frac{2\tan(t/2)}{1+\tan^2(t/2)} \\ &= \frac{2\tan(t/2)}{\sec^2(t/2)} \\ &= 2\sin(t/2)\cos(t/2) \\ &= \sin t. \end{aligned}$$

Thus

$$T(t;e) = t + e\sin t + O(e^2),$$

which is (3). From the definition of T, it follows that T is 2π-periodic in t, and since

$$\frac{dT(t;e)}{dt} = \frac{1}{1+\left(\frac{1+e}{1-e}\right)\tan^2(t/2)}\left[\left(\frac{1+e}{1-e}\right)^{1/2}\sec^2(t/2)\right],$$

then dT/dt is 2π-periodic in t.

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