INEQUALITIES CONCERNING DOMINATING SETS IN GRAPHS

by

E.J. COCKAYNE, C.W. KO and F.B. SHEPHERD

DM-370-IR JUNE 1985

INEQUALITIES CONCERNING DOMINATING SETS IN GRAPHS

E.J. Cockayne, C.W. Ko and F.B. Shepherd

University of Victoria

Abstract. The n-domination number $\gamma_n(G)$ of a graph G is the smallest number of vertices in a set D such that each vertex of $V - D$ is adjacent to at least n vertices in D. Several inequalities concerning $\gamma_n(G)$ are established. For example, for a p vertex graph, we prove that $\gamma_1(G) \leq \lceil p/3 \rceil$ if G is free of certain subgraphs, $\gamma_2(G) + \gamma_2(\bar{G}) \leq p + 2$ and $\lfloor \gamma_1(G)/n \rfloor \gamma_n(\bar{G}) \leq p$.

1. Introduction

A set D of vertices is an n-dominating set for a graph G if each vertex in $V - D$ is adjacent to at least n vertices in D. We define $\gamma_n(G)$ to be the smallest cardinality of an n-dominating set for the graph G. A 1-dominating set is simply called a dominating set (or externally stable set) and we use the traditional notation $\gamma(G)$ for $\gamma_1(G)$. Dominating sets were first studied by Ore [9] and Berge [1]. The generalisation to n-domination was first considered by Fink and Jacobson [4]. In recent literature, over 100 papers have appeared on the topic of domination [6].

In this paper we establish a variety of inequalities concerning $\gamma_n(G)$ and other parameters. For example, for graphs G with p vertices, we prove $\gamma_1(G) \leq \lceil p/3 \rceil$ if G is free of certain subgraphs called claws and nets, that

$\gamma_2(G) + \gamma_2(\bar{G}) \leq p + 2$, $\lfloor \gamma(G)/n \rfloor \gamma_n(\bar{G}) \leq p$ and for claw-free graphs we show that $\gamma_{n+1}(G) \leq 3\gamma_n(G)$. Where the context is clear $\gamma_n(G)$, $\gamma_n(\bar{G})$, $\gamma(G)$ etc., will be abbreviated to γ_n, $\bar{\gamma}_n$, γ. We use the notation $G[u]$ for the subgraph induced by the vertex subset U in graph G and $d(u, G)$ for the degree of vertex u in graph G.

2. The Results

The first result concerns connected graphs which do not have a claw (Fig. 1a) or a net (Fig. 1b) as an induced subgraph. We exhibit a vertex partition of such graphs which shows that $\lceil p/3 \rceil$ vertices will dominate, thus improving Ore's result ([9, Page 206) for this class. Hammer, Mahedev and De Werra [5]

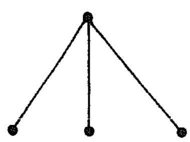

Fig. 1a

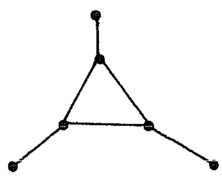

Fig. 1b

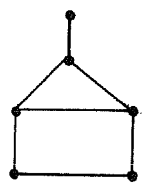

Fig. 1c

have given an algorithm for finding the stability number of graphs which do not contain a claw, a net or an antenna (Fig. 1c).

THEOREM 1. Suppose the p vertex graph G is connected, claw-free and net-free, then $\gamma_1(G) \leq \lceil p/3 \rceil$ and $\gamma_2 \leq \frac{2}{3}(p+1)$.

Proof: Let $p = 3k + r$ where $r = 0, 1$ or 2. It is sufficient to prove that $V(G)$ may be partitioned into $\{U_1, U_2, \ldots, U_k, R\}$ where $|U_i| = 3$, $|R| = r$, $\Delta(G[U_i]) \geq 2$ and $G[R] = K_r$.

If no such partition exists, then there is a connected graph G with $|V| > 3$ satisfying the hypohteses such that the removal of any 3-subset $U = \{u,v,w\}$ where $\Delta(G[U]) \geq 2$, disconnects G. Consider a longest path in G, $v_1 v_2 \ldots v_\ell$ (note $\ell \geq 3$). Let $S = \{v_1,\ldots,v_\ell\}$. Since $v_1 \ldots v_\ell$ is a longest path we have

$$N(v_1) \subseteq S. \tag{1}$$

If v_2 is adj to some $w \notin S$ then since G is $K_{1,3}$-free either $v_1v_3 \in E$, $wv_1 \in E$ or $wv_3 \in E$. In any case we can obtain a longer path, therefore

$$N(v_2) \subseteq S. \tag{2}$$

Since $G\left[V - \{v_1,v_2,v_3\}\right]$ is disconnected, v_3 is adjacent to some $w \notin S$, and so $\ell \geq 4$. By (1), (2) $wv_1 \notin E$ and $wv_2 \notin E$. If $wv_4 \in E$ then $v_1v_2v_3wv_4 \ldots v_\ell$ is a longer path, hence $wv_4 \notin E$. Hence G claw-free implies that

$$v_2v_4 \in E \tag{3}$$

(otherwise $G\left[\{v_2,v_3,w,v_4\}\right]$ is a claw) and so it follows that $\ell \geq 5$.

Suppose v_4 is adjacent to some $u \notin S$, then $uv_3 \notin E$ or else we can obtain a longer path. By (1) and (2) uv_1 and uv_2 are not in E. Also $v_1v_4 \notin E$ or else $v_\ell \ldots v_5v_4v_1v_2v_3w$ would be a path of length $\ell + 1$. But then $v_1v_3 \notin E$ since G is $K_{1,3}$-free and so $\{v_1,v_2,v_3,v_4,w,u\}$ forms a net, a contradiction. Hence

$$N(v_4) \subseteq S. \tag{4}$$

Finally since the longest path has length ℓ, we have

$$v_1v_3,\ v_1v_4,\ v_1v_5 \notin E. \tag{5}$$

Now since $G\left[\{v_1,\ldots,v_5,w\}\right]$ is not a net, we have by (1) - (5) that one or both of v_2v_5 and v_3v_5 is an edge. If $v_2v_5 \in E$ and $v_3v_5 \notin E$ then $G\left[\{v_1,v_2,v_3,v_5\}\right]$ is a claw. If $v_3v_5 \in E$ and $v_2v_5 \notin E$ then $G\left[\{w,v_3,v_2,v_5\}\right]$ is a claw. Thus

$$v_2v_5 \text{ and } v_3v_5 \text{ are edges.} \tag{6}$$

If there is some $w' \notin (S \cup \{w\})$ such that $w'v_3 \in E$ then

(a) $ww' \notin E \Rightarrow G\left[\{v_3,w,w',v_5\}\right]$ is a claw

(b) $ww' \in E \Rightarrow v_\ell \ldots v_5v_4v_2v_3ww'$ is a longer path.

Therefore

$$N(v_3) \cap (V-S) = \{w\}. \tag{7}$$

Also (b) implies

$$N(w) \subseteq S. \tag{8}$$

Hence by (4), (6), (7) and (8) $G\left[V - \{w,v_3,v_4\}\right]$ is connected. The result follows by contradiction.

We now establish a relationship between the degree sequence of a graph and the domination number of its complement.

<u>PROPOSITION 2</u>. Let $k \geq 2$ $(d_1,\ldots,d_p)$ be the degree sequence of G. If $\gamma(\bar{G}) > k$ then $\sum_{i=1}^{p} \binom{d_i}{k} \geq \binom{p}{k}$.

Proof: Suppose $\bar{\gamma} > k$. Then for each subset of k vertices, say $\{v_1,\ldots,v_k\}$, $\bigcap_{i=1}^{k} N(v_i) \neq \emptyset$. Let

$$t = \sum_{\{v_1,\ldots,v_k\}\subseteq V} \left|\bigcap N(v_i)\right|, \tag{9}$$

then $t \geq \binom{p}{k}$. For each vertex v_i of degree d_i, v_i is counted exactly $\binom{d_i}{k}$ times in the sum (9). Thus $\sum_{i=1}^{p} \binom{d_i}{k} = t \geq \binom{p}{k}$ as desired.

COROLLARY 1. If $k \geq 2$ and $\bar{\gamma} > k$, then $p\binom{\Delta}{k} \geq \binom{p}{k}$.

Proof: By proposition 2 it suffices to note that $p\binom{\Delta}{k} \geq \sum_{i=1}^{p} \binom{d_i}{k}$.

COROLLARY 2. If $\Delta \leq \sqrt{p}$, then $\bar{\gamma} \leq 2$.

Proof: If $\bar{\gamma} > 2$, then by corollary 1, $p\,\frac{\Delta(\Delta-1)}{2} \geq \frac{p(p-1)}{2}$ or $\Delta^2 - \Delta - (p-1) \geq 0$. This implies that $\Delta \geq \frac{1}{2} + \sqrt{p - \frac{3}{4}} > \sqrt{p}$.

COROLLARY 3. If $k \geq 2$ and $\Delta \leq (p-k+1)^{\frac{k-1}{k}} + \frac{k-1}{2}$, then $\bar{\gamma} \leq k$.

Proof: If $\bar{\gamma} > k$, then by corollary 1 we have,

$$p\binom{\Delta}{k} \geq \binom{p}{k}. \tag{10}$$

Thus

$$\left(\Delta - \left(\frac{k-1}{2}\right)\right)^k \geq \prod_{i=0}^{k-1}(\Delta - i) \quad \text{since} \quad \big(\Delta-(k-1)+i\big)(\Delta-i) \leq \left(\Delta - \left(\frac{k-1}{2}\right)\right)^2 \text{ for all } 0 \leq i \leq \frac{k-1}{2}$$

$$\geq (p-1)\ \ldots\ (p-k+1) \qquad \text{by (10)}$$

$$> (p-k+1)^{k-1}$$

This implies $\Delta > (p-k+1)^{\frac{k-1}{k}} + \frac{k-1}{2}$ and the proof is complete.

The next theorem shows that if the minimum degree δ is sufficiently large that any $k-1$ vertices are contained in a dominating set with k vertices.

<u>THEOREM 3</u>. If $\delta > p - \frac{k+1}{2} - \sqrt{p + \frac{(k-1)(k-5)}{4}}$ $(k \geq 2)$, then for any $\{v_1,\ldots,v_{k-1}\} \subseteq V$, there is a vertex u such that $\{u,v_1,\ldots,v_{k-1}\}$ is a dominating set.

<u>Proof</u>: Let $v_1,\ldots,v_{k-1}$ be any $k-1$ vertices of V and let $A = \left(\bigcup_{i=1}^{k-1} N(v_i)\right) \setminus \{v_1,\ldots,v_{k-1}\}$, $B = V - A - \{v_1,\ldots,v_{k-1}\}$ and $|A| = d \geq \delta - (k-2)$.
If there is a vertex $u \in B$ such that $d(u, B) = |B| - 1$, then we may choose u to complete the dominating set, so suppose for any vertex in B, the number of vertices in A to which it is adjacent is greater than $\delta - |B| + 1 < d$. Then each $b \in B$ is non-adjacent to at most $d - (\delta - |B| + 2) = p - \delta - (k+1) \geq 0$ vertices in A. Thus there is a v in A which dominates B, unless

$$\begin{aligned} \delta - (k-2) \leq d = |A| &\leq |B|\big(p - \delta - (k+1)\big) \\ &= \big(p - d - (k-1)\big)\big(p - \delta - (k+1)\big) \\ &\leq \big(p - \delta - 1\big)\big(p - \delta - (k+1)\big). \end{aligned}$$

The quadratic formula yields

$$\delta \leq p - \frac{k+1}{2} - \sqrt{p + \frac{(k-1)(k-5)}{4}} \quad \text{or} \quad \delta \geq p - \frac{k+1}{2} + \sqrt{p + \frac{(k-1)(k-5)}{4}} \geq p - 2.$$

Thus if $\delta > p - \frac{k+1}{2} - \sqrt{p + \frac{(k-1)(k-5)}{4}}$ the existence of u is assured.

THEOREM 4. If $\delta > p - k + \frac{1}{2} - \sqrt{p - k + \frac{5}{4}}$, then for any $k - 1$ independent vertices $v_1, \ldots, v_{k-1}$ there is a vertex u such that $\{u, v_1, \ldots, v_{k-1}\}$ is a dominating set for G $(k \geq 2)$.

Proof: The proof is similar to that of Theorem 3.

We now establish some results concerning $\gamma_n(G)$. A theorem of O. Favaron [3] will be used. A set U of vertices is n-dependent if and only if the maximum degree of $G[A]$ is less than n. The n-dependence number $\beta_n(G)$ is the maximum number of vertices in an n-dependent set.

THEOREM 5 (Favaron). For any simple graph G and $n \geq 1$, there is a set D which is both n-dependent, and n-dominating in G.

COROLLARY 4. For any graph G, $\gamma_n(G) \leq \beta_n(G)$.

PROPOSITION 6a. (i) If $\delta \geq n$, then $\gamma_n \leq p - \beta_{\delta-n+1}$

(ii) If $\delta \geq n$, then $\gamma_n + \gamma_{\delta-n+1} \leq p$

(iii) $\delta \geq 2n - 1$, then $\gamma_n \leq p/2$.

Proof: If S is a δ-n+1-dependent set, then each vertex in S is adjacent to at least n vertices in $V - S$. Thus (i) is proved. Corollary 5 and (i) together imply (ii). From (ii) it follows that $\min\{\gamma_n, \gamma_{\delta-n+1}\} \leq p/2$; if $\delta \geq 2n - 1$, then $n = \min\{n, \delta-n+1\}$ and so (iii) follows immediately.

PROPOSITION 6b. (i) If $\Delta \geq n$, then $\gamma_n \geq p - \beta_{\Delta-n+1}$

(ii) If $\Delta \geq n$, then $\beta_n + \beta_{\Delta-n+1} \geq p$

(iii) If $\Delta \leq 2n - 1$, then $\beta_n \geq p/2$.

Proof: Part (i) was shown in [5]; (ii) and (ii) are similar to proposition 6a.

We now study connections between various parameters for the graph G and its complement $\bar{G}$ (i.e. results of the Nordhaus-Gaddum type).

PROPOSITION 7. For any p-vertex graph G, $\beta_n + \bar{\beta}_m \leq p + m + n - 1$.

Proof: Let S be a maximal n-dependent set in G and T a maximal m-dependent set in $\bar{G}$. If $|S| + |T| \geq p + m + n$ then $|S \cap T| \geq m + n$. Let $v \in S \cap T$. It follows that $d(v, S \cap T) \leq n - 1$, but this implies that v is adjacent to at least m vertices in $S \cap T$ in $\bar{G}$ contradicting the m-dependence of T in $\bar{G}$.

Equality is obtained for an $(m-1)$-regular graph on $m+n-1$ vertices.

COROLLARY 5. For any p-vertex graph G,

$$\gamma_n + \bar{\gamma}_m \leq p + m + n - 1.$$

Proof: By proposition 7 and Theorem 5.

For the special case $m=n=2$, Corollary 5 yields $\gamma_2 + \bar{\gamma}_2 \leq p + 3$. This may be slightly improved.

THEOREM 8. For any p-vertex graph G,

$$\gamma_2 + \bar{\gamma}_2 \leq p + 2.$$

Proof: Let S be a 2-dependent, 2-dominating set of G (existence by Theorem 5). If V = S, then $\Delta(G) \leq 1$ and the result is clear. So W.L.O.G. $|V-S| \geq 1$.

Define $E = \{x \in S: x$ is non-adjacent to at most one vertex in $V - S\}$.

Case 1. $|E| \leq 2$

Then $(V-S) \cup E$ is 2-dominating in $\bar{G}$ since every vertex set in $S - E$ is non-adjacent to at least 2 vertices in $V - S$.

Therefore $\gamma_2 + \bar{\gamma}_2 \leq |S| + |V-S| + 2 = p + 2$.

Case 2. $|E| > 2$

If $\Delta(G[E]) = 0$ then let x,y be any two vertices in E, otherwise let x,y be two adjacent vertices in E. In either case neither x nor y is adjacent to any other vertex in E, and so $\{x,y\}$ 2-dominates $E - \{x,y\}$ in $\bar{G}$. But $S - E$ is 2-dominated by $V - S$ in $\bar{G}$, thus

$\gamma_2 + \bar{\gamma}_2 \leq |S| + |(V-S) \cup \{x,y\}| = p + 2$.

Theorem 8 cannot be generalised to n, in fact for $n > 2$,

$\gamma_n(C_{n+2}) + \gamma_n(\bar{C}_{n+2}) = 2n + 4 > p + n$.

THEOREM 9. For any graph G $\lfloor \gamma/n \rfloor \bar{\gamma}_2 \leq p$.

Proof: If $\gamma < 2n$, then $\lfloor \gamma/n \rfloor \leq 1$ and the result is clearly true. Otherwise let A be a minimum dominating set of G and choose a partition $\{B_1, \ldots, B_\gamma\}$ so that for each B_i there is a unique x in $A \cap B_i$ such that x is adjacent to all of $B_i - \{x\}$. Let $\gamma = kn + r$ and form k disjoint sets $C_1, \ldots, C_k$ such that $C_1, \ldots, C_{k-1}$ are each the union of n disjoint sets from the partition and C_k is the union of the remaining $n+r$ sets. We choose the partitions $\{B_i\}$ and $\{C_i\}$ so that

$$\sum_{i=1}^{k} \left|\{x \in V - C_i : x \text{ is not n-dominated by } C_i \text{ in } \bar{G}\}\right| \text{ is minimized.} \quad (11)$$

Suppose some C_j is not n-dominating in $\bar{G}$, then there is some $x \in B_k \subseteq C_\ell$ such that x is non-adjacent to at most $n-1$ vertices in C_j. Thus there is some $B_i \subseteq C_j$ such that x is adjacent to all of B_i. If there is a $B_m \subseteq C_\ell$ such that x is adjacent to all of B_m, then $(A - (B_m \cup B_i)) \cup \{x\}$ dominates G, contradicting the minimality of A. Thus C_ℓ n-dominates x in $\bar{G}$. But then if we let $B_k = B_k - \{x\}$, $B_i = B_i \cup \{x\}$ and let the C_i's be the unions over the same sets in $\{B_i\}$, then the sum of (10) is strictly smaller. Thus it must be that each C_i is n-dominating in $\bar{G}$. So

$$p = \sum_{i=1}^{k} |C_i| \geq \lfloor \gamma/n \rfloor \bar{\gamma}_n.$$

Corollary 5 and Theorem 9 generalise the results for domination given in Jaeger and Payan [8]. Fink and Jacobson [4] have considered relationships between $\gamma_{f(n)}$ and γ_n. Our next result is of this type.

PROPOSITION 10. If G is claw-free, then $\gamma_{n+1} \leq 3\gamma_n$.

Proof: Let $S = \{v_1, \ldots, v_{\gamma_n}\}$ be a minimum n-dominating set. If we can dominate $V - S$ with at most $2\gamma_n$ vertices, then clearly $\gamma_{n+1} \leq 3\gamma_n$. For each i, if $N(v_i)$ is complete (W.L.O.G. $N(v_i) \neq \emptyset$) then $N(v_i)$ can be dominated by a single vertex in $N(v_i)$, otherwise any two non-adjacent vertices in $N(v_i)$ will dominate $N(v_i)$ since G is claw-free. Hence

$$\gamma(G[V-S]) \le \sum_{i=1}^{\gamma_n} \gamma(G[N(v_i)]) \le 2\gamma_n \text{ as required.}$$

COROLLARY 6. If G is claw-free, then $\gamma_{n+1} \le 3^n\gamma$.

The n-domatic number $d_n(G)$ is the largest order of a partition of $V(G)$ into n-dominating sets. This is the obvious generalisation of the domatic number studied by Cockayne and Hedetniemi in [2]. Our final result lists some properties of the n-domatic number.

PROPOSITION 11. (i) If $\delta \ge 2n - 1$, then $d_n \ge 2$

(ii) $d_n \le p/\gamma_n$

(iii) $d_n \le \delta/n + 1$

(iv) $d_n + \bar{d}_n \le \frac{p-1}{n} + 2$.

Proof: (i) By Theorem 4 there is an n-dependent, n-dominating set S, but since $\delta \ge 2n - 1$, $V - S$ is also n-dominating. Hence the result.

(ii) and (iii) obvious.

(iv) $d_n + \bar{d}_n \le \frac{\delta}{n} + \frac{\bar{\delta}}{n} + 2$ by (iii)

$$= \frac{\delta}{n} + \frac{p-1-\Delta}{n} + 2$$

$$\le \frac{p-1}{n} + 2.$$

Acknowledgements

The authors gratefully acknowledge the support of the Canadian NSERC Grant A7544 and the Challenge '85 programme of the Government of British Columbia.

REFERENCES

[1] C. Berge. Theory of Graphs and its Applications. Methuen, London, 1962, pp. 40-51.

[2] E.J. Cockayne and S.T. Hedetniemi. Towards a Theory of Domination in Graphs. Networks. Fall (1977), pp. 247-261.

[3] O. Favaron. On a Conjecture of Fink and Jacobson Concerning the k-domination and the k-dependence.

[4] J.F. Fink and M.S. Jacobson. On n-domination, n-dependence and Forbidden Subgraphs (to appear, Proceedings of the Kalamazoo Conf., 1984).

[5] J.F. Fink and M.S. Jacobson. n-Domination in Graphs (to appear, Proceedings of the Kalamazoo Conf., 1984).

[6] P.L. Hammer, N.V.R. Mahadev, and D. De Werra. Stability in CAN-Free Graphs. JCT, Series B, Vol. 38 (1985), pp. 23-30.

[7] S.T. Hedetniemi. Domination Bibliography. (Private Communication)

[8] F. Jaeger and C. Payan. Relations du type Nordhaus-Gaddum pour le Nombre d'absorption d'un graph simple. C.R. Acad. Sc. Paris, Series A, t. 274 (1972), pp. 728-730.

[9] O. Ore. The Theory of Graphs. A.M.S. Colloquium Publ. Vol. 38 (1962).