

INHERITED MATRIX ENTRIES: PRINCIPAL SUBMATRICES
OF THE INVERSE

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DM-414-IR

JUNE 1986
REVISED: SEPTEMBER 1986

June 6, 1986 (revised September 10, 1986)

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¹AMS Subject Classification: 05C50, 15A09, 65F50, 68R10.

²The work of this author was partially supported by Brigham Young University College of Physical and Mathematical Sciences Summer Research Fellowships.

³The work of this author was partially supported by National Science Foundation Grant DMS8500372 and by the Office of Naval Research Contract N00014-86-K-0012. Some of the research was carried out while he was a Lansdowne Lecturer at the University of Victoria.

⁴The work of this author was partially supported by NSERC grant A-8214.

⁵The work of this author was partially supported by NSERC grant A-8965.

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ABSTRACT

For a nonsingular n -by- n matrix $A = [a_{ij}]$, let $\alpha \subseteq \{1, 2, \dots, n\}$ and let $A[\alpha]$ denote the principal submatrix of A lying in the rows and columns indicated by α . We determine the combinatorial circumstances under which the (i, j) entry of the Schur complement $[A^{-1}[\alpha]]^{-1}$ equals a_{ij} , and under which the graph of this Schur complement is contained in the graph of $A[\alpha]$.

1. INTRODUCTION.

Let $A = [a_{ij}]$ be an n -by- n nonsingular matrix and let $N = \{1, 2, \dots, n\}$. For index sets $\alpha, \beta \subseteq N$ we denote the submatrix of A lying in rows α and columns β by $A[\alpha|\beta]$; in case $\beta = \alpha$, the submatrix is principal and we abbreviate this to $A[\alpha]$. The set $N - \alpha$ is denoted by α^c . We index the entries of $A[\alpha]$ with their indices from α , so that each entry retains the indices associated with its position in A .

We are interested in questions of the following qualitative type. When do certain entries in a matrix derived from A coincide identically with the corresponding entries of A , or when is the zero pattern in A preserved in a matrix derived from A ? For example, provided the inverses exist, it is a familiar fact that

$$\left[A^{-1}[\alpha]\right]^{-1} = A[\alpha], \quad \text{or equivalently} \quad A^{-1}[\alpha] = (A[\alpha])^{-1}, \quad (1.1)$$

if A is triangular and α is a consecutive set of indices; i.e. inversion may be carried out "locally". The matrix $\left[A^{-1}[\alpha]\right]^{-1}$ is the Schur complement of $A[\alpha^c]$ in A (see, e.g. [3], [11]) and arises naturally in Gaussian elimination. It is obvious that (1.1) is valid for any nonempty index set α if A is a nonsingular diagonal matrix. If A is a restricted type of upper triangular matrix, namely

$$A = \begin{bmatrix} D_1 & A_{12} \\ 0 & D_2 \end{bmatrix} \quad (1.2)$$

where D_1 and D_2 are nonempty nonsingular diagonal matrices and A_{12} is an arbitrary rectangular matrix, then (1.1) still holds for all α . If A is tridiagonal then (1.1) does not in general hold; however, for α a consecutive set of indices, the graph of $\left[A^{-1}[\alpha]\right]^{-1}$ is contained in the graph of $A[\alpha]$. This follows from the often noted fact that if A is tridiagonal and nonsingular, then so is the inverse of any nonsingular principal submatrix of A^{-1} (see e.g. [1, Cor. 3.3]). So in this case there is preservation of the zero pattern rather than particular matrix entries. Our results here unify and generalize these familiar facts by determining the most general combinatorial circumstances under which there is entry preservation or graph containment.

We now state a list of general questions to be addressed, and then give some more notation and definitions which we need.

- QI. Under what combinatorial circumstances is the i, j entry of $\left[A^{-1}[\alpha]\right]^{-1}$ equal to a_{ij}
- (a) for a given α and particular $i, j \in \alpha$?
 - (b) for a given α and all $i, j \in \alpha$?
 - (c) for given i, j and all α with $i, j \in \alpha$?
 - (d) for all α and all $i, j \in \alpha$?
- QII. Under what combinatorial circumstances is the graph of $\left[A^{-1}[\alpha]\right]^{-1}$ contained in that of $A[\alpha]$
- (a) for a given α ?
 - (b) for all α ?
 - (c) for a given α , assuming A is combinatorially symmetric with all $a_{kk} \neq 0$?
 - (d) for all α , assuming A is combinatorially symmetric with all $a_{kk} \neq 0$?

We consider both the directed and undirected graphs of A . Given a matrix A , its directed graph, $D(A)$, has node set N and a directed edge (i, j) from i to j iff $a_{ij} \neq 0$. Given any directed graph, we say that a matrix B is consistent with that graph if $b_{ij} = 0$ whenever there is no edge between i and j in the graph. (Note that b_{ij} may be zero when such an edge exists.)

As our questions are combinatorial in nature, we neglect the possibility of accidental cancellations (see e.g. [2]). Given a directed graph D , we say that two numbers f and g computable from the entries of a matrix consistent with D are equal generically (written $f = g$ (generically)) if $f(A) = g(A)$ for all A consistent with D . With D given and for $i, j \in \alpha \subseteq N$, we say that j is reachable from i through α^C if there exists a path of length ≥ 2 in D , say $i \rightarrow p_1 \rightarrow p_2 \rightarrow \dots \rightarrow p_k \rightarrow j$, in which all the intermediate nodes $p_1, \dots, p_k \in \alpha^C$ and are distinct. This is similar to the definition in [8] but there a path can have length 1, i.e. be just the edge (i, j) . From our definition, a path has length $\leq |\alpha^C| + 1$, and in case $i = j$ it is a cycle.

When A is combinatorially symmetric ($a_{ij} \neq 0$ iff $a_{ji} \neq 0$) we also work with the undirected graph, $G(A)$, which has node set N and an undirected edge (i, j) between i and j iff $a_{ij} \neq 0$. In this case we assume that all a_{kk} are nonzero. Given an undirected graph G the definitions of " A is consistent with G " and " $f = g$ (generically)" are analogous to those for the directed graph. In questions QII the graph is directed in (a), (b), and undirected in (c), (d). The graph of $A[\alpha]$ is the subgraph of A generated by the nodes in α ; thus, the graph of $A[\alpha]$ is contained in the graph of A for all α .

We now state and prove our main results, which enable us in section 2 to answer questions QI and to make some observations concerning the relation of our results to Gaussian elimination for sparse matrices. Then, in section 3, we give

graphical interpretations to the results to answer questions QII. Finally, in section 4, we give examples to illustrate our results; a reader might like to consult these during the course of reading the next two sections.

2. SUBMATRIX RESULTS.

We begin with a relationship between minors of A and of the Schur complement of $A[\alpha^c]$ in A . We abbreviate the determinant of A to $\det A$, and if $\alpha = \emptyset$, then $\det A[\alpha] = 1$.

THEOREM 2.1. Let A be an n -by- n nonsingular matrix and suppose that $\alpha \subseteq N$ is an index set such that $A[\alpha^c]$ is nonsingular. Then for index sets $I, J \subseteq \alpha$ with $|I| = |J|$, we have

$$\det \left[\left[A^{-1}[\alpha] \right]^{-1} \right] [I|J] = (-1)^s \frac{\det A[\alpha^c \cup I | \alpha^c \cup J]}{\det A[\alpha^c]},$$

where $s = \sum_{i \in I} (r_i + i) + \sum_{j \in J} (c_j + j)$ and $r_i \langle c_j \rangle$ is the position of row i \langle column $j \rangle$ of A in $A[\alpha]$.

Proof. By repeated use of Jacobi's identity (see e.g. [11, p. 21]) we have, for $s_1 = \sum_{i \in I} r_i + \sum_{j \in J} c_j$, $s_2 = \sum_{i \in I} i + \sum_{j \in J} j$, and $s = s_1 + s_2$:

$$\begin{aligned} \det \left[\left[A^{-1}[\alpha] \right]^{-1} \right] [I|J] &= \frac{(-1)^{s_1} \det A^{-1}[\alpha - J | \alpha - I]}{\det A^{-1}[\alpha]} \\ &= \frac{(-1)^{s_1} (-1)^{s_2} \det A[\alpha^c \cup I | \alpha^c \cup J]}{\det A \det A^{-1}[\alpha]} \\ &= (-1)^s \frac{\det A[\alpha^c \cup I | \alpha^c \cup J]}{\det A[\alpha^c]}. \quad \blacksquare \end{aligned}$$

Note that if I and J have cardinality one, i.e. $I = \{i\}$ and $J = \{j\}$, with $i, j \in \alpha$, Theorem 2.1 specializes to a formula for the i, j entry of $[A^{-1}[\alpha]]^{-1}$ in terms of minors of A .

COROLLARY 2.2. Let A be an n -by- n nonsingular matrix and suppose that $\alpha \subseteq N$ is an index set such that $A[\alpha^c]$ is nonsingular. Then, for $i, j \in \alpha$ with $s = r_i + i + c_j + j$ we have

$$[A^{-1}[\alpha]]^{-1}_{ij} = (-1)^s \frac{\det A[\alpha^c \cup \{i\} | \alpha^c \cup \{j\}]}{\det A[\alpha^c]}. \quad \blacksquare$$

We note that this gives an expression for the i, j entry of the Schur complement, which is also the i, j entry of the reduced matrix of Gaussian elimination obtained after eliminating on the rows specified by α^c (provided $A[\alpha^c]$ has an LU factorization). The results of Theorem 2.1 and Corollary 2.2 are also given in [4] and [7, p. 26], respectively, for the case $\alpha^c = \{1, 2, \dots, p\}$.

To facilitate statements, we introduce the following hypothesis which specifies a class of matrices to which our results apply.

(H): Let $\alpha \subseteq N$, let D be a given directed graph on the node set N , let $A = [a_{ij}]$ be any nonsingular matrix consistent with D and with $A[\alpha^c]$ nonsingular.

We now state our main result, which provides a necessary and sufficient condition to answer QI(a). Although our question is about matrices, our characterization uses graph-theoretic ideas.

THEOREM 2.3. Assuming (H), given α and particular $i, j \in \alpha$, then

$$\left[A^{-1}[\alpha] \right]^{-1}_{ij} = a_{ij} \quad (\text{generically}) \quad (2.1)$$

iff either

(i) j is not reachable from i through $\alpha^c \equiv N - \alpha$;

or

(ii) if j is reachable from i through vertices $p_1, p_2, \dots, p_t \in \alpha^c$, then

$$\det A[\alpha^c - \{p_1, \dots, p_t\}] = 0 \quad (\text{generically}).$$

(Note that the "if" implication still holds if the equalities are not generic.)

Proof. From Corollary 2.2, we have $\left[A^{-1}[\alpha] \right]^{-1}_{ij} = a_{ij}$ iff

$$\det A[\alpha^c \cup \{i\} | \alpha^c \cup \{j\}] = (-1)^s a_{ij} \det A[\alpha^c].$$

Expanding the determinant about the i^{th} row,

$$\begin{aligned} \det A[\alpha^c \cup \{i\} | \alpha^c \cup \{j\}] &= (-1)^q a_{ij} \det A[\alpha^c] \\ &+ \sum \pm a_{ip_1} a_{p_1 p_2} \dots a_{p_t j} \det A[\alpha^c - \{p_1, \dots, p_t\}], \end{aligned} \quad (2.2)$$

where the summation is over all simple paths from i to j through nodes $p_1, p_2, \dots, p_t \in \alpha^c$, $t \geq 1$; $q = r_i + c_j$ where r_i $\langle c_j \rangle$ is now the position of row i \langle column $j \rangle$ of A in $A[\alpha^c \cup \{i\} | \alpha^c \cup \{j\}]$ and the \pm sign in the summation depends on i, j, α and the length of the path.

It can be shown that $s + q = 2(i+j+1)$, which is even. This can be seen by first taking $\beta = \{1, \dots, \ell\} \supseteq \alpha$, and deleting one index at a time until α is obtained.

Thus, if (2.1) holds, each term in the summation in (2.2) must be zero. So either there is no path in D from i to j through α^c (condition (i)), or if such a path exists, then the complementary minor must be zero generically (condition (ii)).

Conversely, if condition (i) is true, then there is no nonzero term $a_{i p_1} a_{p_1 p_2} \dots a_{p_t j}$ in the expansion (2.2), and so $\left[A^{-1}[\alpha] \right]_{ij}^{-1} = a_{ij}$. Alternatively if condition (ii) is true, then whenever $a_{i p_1} a_{p_1 p_2} \dots a_{p_t j}$ is nonzero, the complementary determinant $\det A[\alpha^c - \{p_1, \dots, p_t\}] = 0$, so in this case also $\left[A^{-1}[\alpha] \right]_{ij}^{-1} = a_{ij}$. ■

Note that if $A[\alpha^c \cup \{i\} | \alpha^c \cup \{j\}]$ is reducible with respect to $A[\alpha^c]$, then (2.1) holds, but the converse is not necessarily true. It is possible that $\left[A^{-1}[\alpha] \right]_{ij}^{-1} = a_{ij}$ for non-combinatorial reasons (i.e. this equality is not generic); see Example 4.1. The fact that the result of Theorem 2.3 holds for a whole class of matrices consistent with a given D is illustrated in Example 4.6.

If D contains a self loop for each node in α^c , then the determinant in (ii) is never generically zero, and so the characterization rests solely on (i).

COROLLARY 2.4. Assuming (H), given α and particular $i, j \in \alpha$, and assuming D contains a self loop on each node in α^c , then (2.1) holds iff j is not reachable from i through α^c . ■

This yields the following monotonicity result.

COROLLARY 2.5. Assuming (H), given α , particular $i, j \in \alpha$ and β such that $\alpha \subseteq \beta \subseteq N$ with $A[\beta^c]$ nonsingular, and assuming D contains a self loop on each node in α^c , then $\left[A^{-1}[\alpha]\right]_{ij}^{-1} = a_{ij}$ (generically) implies $\left[A^{-1}[\beta]\right]_{ij}^{-1} = a_{ij}$ (generically).

Proof. Since there is no path from i to j through α^c , there is none through β^c . ■

There is a vast literature concerning sparse matrix computation using graph-theoretic techniques to analyze fill-in during Gaussian elimination. Corollary 2.4 (for the case that $a_{ij} = 0$) is essentially the fundamental theoretical result upon which this sparse matrix analysis is based. Thus our main results in Theorems 2.1 and 2.3 may be viewed as part of the theoretical foundation of this analysis. The application of Corollary 2.4 to the modelling of Gaussian elimination using reachable sets may be found in [8], and indeed may be traced back to [12], but our more general theorems do not seem to be in the literature. Whereas the literature on Gaussian elimination for sparse matrices focuses on the preservation of zero entries in the Schur complement (and indeed in the LU factorization of A), our results characterize the preservation of both zero and nonzero entries.

Question QI(b) can now be answered by requiring the conditions of Theorem 2.3 to hold for all $i, j \in$ given α , giving necessary and sufficient conditions for the entire Schur complement of $A[\alpha^c]$ in A to be generically equal to $A[\alpha]$.

COROLLARY 2.6. Assuming (H) and given α , then (1.1) holds (generically) iff for each $i, j \in \alpha$ either (i) or (ii) of Theorem 2.3 holds. ■

From Corollary 2.6, we have the following special case.

COROLLARY 2.7. Assuming (H), given α and assuming D contains a self loop on each node in α^c , then (1.1) holds (generically) iff there exists a permutation matrix P such that

$$P^T A P = \left[\begin{array}{c|c|c} A[\gamma] & A_{12} & A_{13} \\ \hline 0 & A[\alpha] & A_{23} \\ \hline 0 & 0 & A[\beta] \end{array} \right], \quad (2.3)$$

where $\alpha \cup \beta \cup \gamma = N$ and either one of β, γ may be empty.

Proof: This form is obtained by noting that under the hypotheses (1.1) holds (generically) iff there is no path in D from any node in α to any node in α through α^c . It can be shown that this path condition is satisfied iff N can be written as the disjoint union of α and sets β, γ such that there is no edge in D from β to α , from β to γ and from α to γ . This in turn is equivalent to the existence of a permutation matrix P such that (2.3) holds. ■

Next suppose we are given particular $i, j \in N$ and want this entry to be inherited for all α containing $\{i, j\}$ (i.e., question QI(c)). Since we must now assume that $A[\alpha^c]$ is nonsingular for all such α , the determinant in condition (ii) of Theorem 2.3 can never be zero, so we have the following characterization.

COROLLARY 2.8. Assuming (H) for all α containing a given i, j , then (2.1) holds for all such α iff there exists a permutation matrix P such that

$$P^T A P = \left[\begin{array}{c|c} A_{11} & \begin{array}{c} \circ \\ a_{ij} \end{array} \\ \hline A_{21} & A_{22} \end{array} \right], \quad (2.4)$$

where A_{11} , A_{21} , A_{22} are arbitrary matrices consistent with (H), and a_{ij} is the only (possibly) nonzero entry in its off-diagonal block.

Proof: Condition (i) of Theorem 2.3 holds for all α containing $\{i, j\}$ iff there is no path from i to j through any intermediate nodes. Equivalently, removal of the edge (i, j) from D makes A reducible and this occurs iff (2.4) holds for some permutation matrix P . ■

If, in addition to the hypotheses of Corollary 2.8, A is assumed to be combinatorially symmetric, then there can be no path from j to i through any intermediate nodes; removal of the edge (i, j) in the undirected graph causes i and j to be in different connected components. (We note that such an edge is often called a bridge.) Thus there exists a permutation matrix P such that

$$P^T A P = \left[\begin{array}{c|c} A_{11} & \begin{array}{c} \circ \\ a_{ij} \end{array} \\ \hline \begin{array}{c} \circ \\ a_{ji} \end{array} & A_{22} \end{array} \right], \quad (2.5)$$

where A_{11} and A_{22} are arbitrary combinatorially symmetric matrices consistent with (H), and a_{ij} , a_{ji} are the only (possibly) nonzero entries in the off-diagonal blocks. Note that tridiagonal matrices with nonvanishing principal

minors are of this type; and the property of tridiagonal matrices that (2.1) holds for all $\alpha \subseteq N$, $i \in \alpha$, $j = i + 1 \in \alpha$ extends to general matrices of this type. Note that in this case when A is combinatorially symmetric and $i = j$, then node i can be connected to no other node.

Our fourth submatrix question, $QI(d)$, requires equality (generically) for the Schur complement of $A[\alpha^c]$ in A for every α .

COROLLARY 2.9. Assuming (H) for all α , then (1.1) holds (generically) for all α iff there exists a permutation matrix P such that $P^T A P$ has the form (1.2). ■

Equality in this case is equivalent to having no simple path of length ≥ 2 in D . If in addition A is combinatorially symmetric, then (1.1) holds (generically) for all α iff A is a diagonal matrix.

Clearly we can also use Theorem 2.3 to characterize generic equality of a particular set of submatrices. Upper triangular matrices (considered in the introduction) are an example where it is natural to consider α as any set of consecutive indices. As another example, there is a simple but useful sufficient condition for

$$\left[A^{-1}[\alpha] \right]_{ij}^{-1} = a_{ij} \tag{2.6}$$

to hold for an entire row or column of $A[\alpha]$.

COROLLARY 2.10. Let $\alpha \subseteq N$ and let A be a nonsingular matrix with $A[\alpha^c]$ nonsingular. If for some fixed $i \in \alpha$, $a_{ik} = 0$ for all $k \in \alpha^c$, then (2.6) holds for all $j \in \alpha$. (A similar result holds for a fixed $j \in \alpha$.)

Proof. This follows from the if part of Theorem 2.3 applied to $D(A)$, since there is no edge from i to any node in α^c . ■

Corollary 2.10 and its analogue show that a simple sufficient condition for (2.6) is that either $A[\{i\}|\alpha^c]$ or $A[\alpha^c|\{j\}]$ be a zero matrix. This is far from necessary in general. But in the case that $\alpha^c = \{k\}$, then the result of Corollary 2.2 reduces to $\left[A^{-1}[N-\{k\}]\right]_{ij}^{-1} = a_{ij} - a_{ik}a_{kj}/a_{kk}$, and thus the vanishing of a_{ik} or of a_{kj} is necessary and sufficient for (2.6) to hold.

3. GRAPH CONTAINMENT RESULTS.

We now consider the graph containment questions raised in QII. These focus on the zero-nonzero pattern in A , and require that the zero entries are inherited, that is no new edge is created. Given a nonsingular matrix A and a set $\alpha \subseteq N$ with $A[\alpha^c]$ nonsingular, we write $D\left[A^{-1}[\alpha]\right]^{-1} \subseteq D(A[\alpha])$ iff whenever $a_{ij} = 0$ for $i, j \in \alpha$ then $\left[A^{-1}[\alpha]\right]_{ij}^{-1} = 0$. We write $D\left[A^{-1}[\alpha]\right]^{-1} \subseteq D(A[\alpha])$ (generically) if for any nonsingular matrix B with $B[\alpha^c]$ nonsingular and $D(B) = D(A)$ we have $D\left[B^{-1}[\alpha]\right]^{-1} \subseteq D(B[\alpha])$. Graph containment for the undirected graph of a combinatorially symmetric matrix is defined similarly. On restricting $D = D(A)$ in Theorem 2.3, the following result answers question QII(a) and thereby specifies conditions for which the digraph of the Schur complement $\left[A^{-1}[\alpha]\right]^{-1}$ is a subgraph of the digraph of $A[\alpha]$.

COROLLARY 3.1. Given nonsingular A and $\alpha \subseteq N$ with $A[\alpha^c]$ nonsingular, then $D\left[A^{-1}[\alpha]\right]^{-1} \subseteq D(A[\alpha])$ (generically) iff for each $i, j \in \alpha$ such that $a_{ij} = 0$ either (i) or (ii) of Theorem 2.3 holds with respect to $D = D(A)$. ■

In case $\alpha^c = \{k\}$, we have digraph containment iff there is an edge (i, j) in $D(A)$ whenever there are edges (i, k) and (k, j) , and a self loop at node i whenever there are edges (i, k) and (k, i) . In the terminology of [13], this containment condition is equivalent to the deficiency of k equal to the empty set and node k not in any 2-cycle $i \rightarrow k \rightarrow i$ with $a_{ii} = 0$.

On restricting A to be combinatorially symmetric with all diagonal entries nonzero, condition (ii) is not required in Corollary 3.1 (as we are considering the "generic" inheritance of zeros); this provides an answer to QII(c). In the

case that α^c is a single node k , we obtain an answer to this question which involves a well-known concept from the study of Gaussian elimination on sparse matrices (see, e.g. [8]). The remark following Corollary 2.10 allows us to omit "generically" here.

COROLLARY 3.2. Let A be an n -by- n nonsingular combinatorially symmetric matrix having all $a_{kk} \neq 0$. Then for $k \in N$

$$G\left[A^{-1}[N-\{k\}]\right]^{-1} \subseteq G\left[A[N-\{k\}]\right]$$

iff every two neighboring nodes of k are connected by an edge in $G(A)$. ■

Equivalently, the above condition can be stated as node k is simplicial in $G(A)$ (see e.g. [10]), and Corollary 3.2 embodies the well-known fact that pivoting on a simplicial node causes no fill-in in Gaussian elimination (that is, no zero entries become nonzero). However, when $|\alpha^c| > 1$, it is possible that none of the nodes in α^c is simplicial (see Example 4.4), but that the undirected graph containment holds. In order to obtain a necessary and sufficient condition in this case, we introduce a new definition. A set of connected nodes V of an undirected graph G is called simplicial in G if the set of all nodes not in V and adjacent to any node in V induces a complete subgraph in G .

THEOREM 3.3. Let A be an n -by- n nonsingular combinatorially symmetric matrix having all $a_{kk} \neq 0$. For a given $\alpha \subseteq N$ such that $A[\alpha^c]$ is nonsingular, let $G_{\alpha^c}(A)$ denote the subgraph of $G(A)$ induced by the node set α^c . Suppose $\alpha^c = \bigcup_{k=1}^m \beta_k$ where β_k are mutually disjoint and the subgraphs

$G_{\beta_k}(A)$ are the connected components of $G_{\alpha^c}(A)$. Then

$$G\left[A^{-1}[\alpha]\right]^{-1} \subseteq G(A[\alpha]) \quad (\text{generically})$$

iff each set of nodes β_k ($1 \leq k \leq m$) is simplicial in $G(A)$.

Proof. Suppose first that each set of nodes β_k is simplicial in $G(A)$. If $i, j \in \alpha$ with $a_{ij} = 0$, then this implies that node j is not connected to node i by a path with nodes solely in α^c . Thus $G\left[A^{-1}[\alpha]\right]^{-1} \subseteq G(A[\alpha])$, by Theorem 2.3.

Conversely, if $G\left[A^{-1}[\alpha]\right]^{-1} \subseteq G(A[\alpha])$ (generically), then (as all $a_{kk} \neq 0$) condition (i) of Theorem 2.3 must hold for all $i, j \in \alpha$ such that $a_{ij} = 0$. Thus any two nodes in α which are adjacent to nodes in some set β_k must be connected by an edge, i.e. each set β_k is simplicial in $G(A)$. ■

Corollary 3.1 does not seem to have been stated in the literature, although the graph structure of the Schur complement is considered in [2]. We note that this graph structure is important in partitioned (or block) methods of solving sparse linear systems (see [2], [5]). Corollary 3.2 is also well-known (see e.g. [8]), but our generalization (Theorem 3.3) and the concept of simplicial sets of nodes is new.

Coming now to our final pair of questions QII(b), (d) we have to consider all index sets α . In the directed graph case, if digraph containment holds for every choice of $\alpha \subseteq N$, then $A[\alpha^c]$ must be nonsingular for all α , implying that all $a_{kk} \neq 0$. Thus $D\left[A^{-1}[\alpha]\right]^{-1} \subseteq D(A[\alpha])$ (generically) iff the deficiency of each node is empty. Note that the deficiency of each node being empty means that the graph $D(A)$ is transitively closed. In this event $D\left[A^{-1}\right] \subseteq D(A)$ (see

e.g. [9]). If $D(A)$ is transitively closed, then so is $D(A[\alpha])$, and thus $D[A^{-1}[\alpha]]^{-1} \subseteq$ the transitive closure of $D[A^{-1}[\alpha]] \subseteq D(A[\alpha])$. Similar reasoning via transitive closure verifies the converse, providing an alternate elementary verification of the answer to QII(b).

For the undirected case we have the following theorem.

THEOREM 3.4. Let A be an n -by- n nonsingular combinatorially symmetric matrix having all principal minors nonvanishing. Then

$$G[A^{-1}[\alpha]]^{-1} \subseteq G(A[\alpha]) \quad \text{for every } \alpha \subseteq N$$

iff each node is simplicial, that is iff $G(A)$ is a direct sum of complete graphs.

Proof. If the graph containment holds, then using Corollary 3.2 with $\alpha^c = \{k\}$ each node k must be simplicial. Conversely, if each node is simplicial, every connected set of nodes in $G(A)$ must form a complete subgraph. Therefore the subgraph induced by the neighbors of any connected subset V of nodes of $G(A)$ must also form a complete subgraph, and thus V is a simplicial set of nodes in $G(A)$. Theorem 3.3 gives the graph containment. ■

Finally, we mention a different approach to inherited zeros, one which utilizes the structure of A^{-1} corresponding to a given zero pattern in A . The main tool is the following rank result recently proved independently (see [6, Cor. 3]).

THEOREM 3.5. Let A be a nonsingular n -by- n matrix and let $\beta, \gamma \subseteq N$ with cardinalities p and q , respectively. Then

$$\text{rank } A^{-1}[\gamma^c | \beta^c] = \text{rank } A[\beta | \gamma] + n - p - q. \quad \blacksquare$$

Application of this identity gives a less direct method of ascertaining inherited zeros than the results above, but in some cases yields insights that the graph theoretic approach does not. It can also be used to give simpler and more informative proofs for Th. 2.1, 3.1 in [1]. We illustrate the use of Theorem 3.5 in Example 4.7.

4. EXAMPLES.

We now give examples to illustrate our results and answers to questions QI and QII.

EXAMPLE 4.1

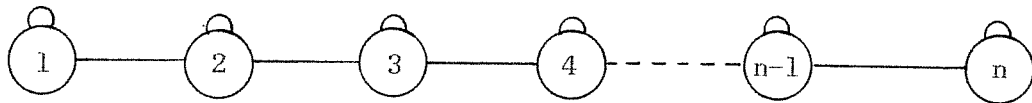
We note that if $\left[A^{-1}[\alpha]\right]_{ij}^{-1} = a_{ij}$ (in which the equality is not necessarily generic), then neither (i) nor (ii) of Theorem 2.3 necessarily holds, as the following example shows. Let $\alpha = \{3,4\}$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 6 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \text{so } A^{-1} = \begin{bmatrix} 1 & -5 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ -2 & 4 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

and $\left[A^{-1}[\alpha]\right]_{34}^{-1} = a_{34} = 0$. The preservation of this zero entry is due to the fact that $\det A[\{1,2,3\}|\{1,2,4\}] = 0$ because of the numerical values of the entries, not because of the graph $D = D(A)$. Relative to D this is an example of "chance cancellation", rather than a generic identity. ■

EXAMPLE 4.2.

Let G denote a "straight-chain" graph on n nodes with a self loop at each node:



Let A be any n -by- n nonsingular combinatorially symmetric matrix which has all $a_{kk} \neq 0$ and $G(A) = G$. Let $\alpha = \{p, p+1, \dots, q\}$ where $1 < p \leq q < n$.

If $A[\alpha^c]$ is nonsingular, then using Theorem 2.3 with the corresponding $D(A)$ implies that (1.1) holds except for a_{pp} and a_{qq} (as the only paths through α^c between two nodes in α are cycles from node p to node p , and from node q to node q).

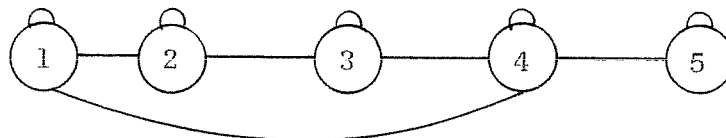
By Theorem 3.3, $G[A^{-1}[\alpha]]^{-1} \subseteq G(A[\alpha])$ (generically) since $G_{\alpha^c}(A)$ has two connected components (one with node set $\{1, 2, \dots, p-1\}$ and the other with node set $\{q+1, q+2, \dots, n\}$) and both of these node sets are simplicial sets in $G(A)$. ■

EXAMPLE 4.3.

Let G and A be as in Example 4.2, but now consider $\alpha^c = \{p, p+1, \dots, q\}$, where $1 < p \leq q < n$. Suppose $A[\alpha^c]$ is nonsingular. By Theorem 2.3, equality (1.1) holds except for the entries $a_{p-1, p-1}$, $a_{p-1, q+1}$, $a_{q+1, p-1}$ and $a_{q+1, q+1}$. As $a_{q+1, p-1} = a_{p-1, q+1} = 0$ and these entries become nonzero in $[A^{-1}[\alpha]]^{-1}$, clearly $G[A^{-1}[\alpha]]^{-1} \not\subseteq G(A[\alpha])$ (generically). This can also be seen from Theorem 3.3 since the set of nodes $\{p, p+1, \dots, q\}$ is not a simplicial set (as its adjacent nodes $p-1$ and $q+1$ are not connected). As noted in the introduction, $[A^{-1}[\alpha]]^{-1}$ is also tridiagonal; thus, the zero pattern is preserved despite the lack of graph containment. ■

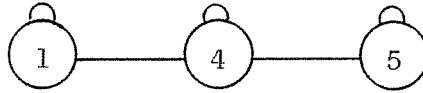
EXAMPLE 4.4.

Let A be a nonsingular combinatorially symmetric matrix with the following undirected graph.



Let $\alpha = \{1, 4, 5\}$. If $A[\alpha^c]$ is nonsingular, then $G[A^{-1}[\alpha]]^{-1} \subseteq G(A[\alpha])$ (generically) by Theorem 3.3 as the set of nodes $\{2, 3\}$ is a simplicial set in

$G(A)$. Specifically, the zero entries a_{15} and a_{51} are inherited by $[A^{-1}[\alpha]]^{-1}$, and $G[A^{-1}[\alpha]]^{-1}$ is as follows:

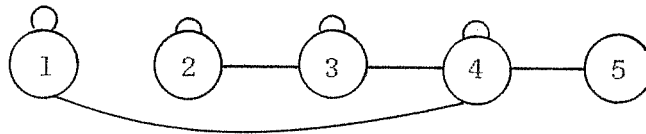


Note that neither node 2 nor 3 is simplicial in $G(A)$. ■

EXAMPLE 4.5.

In this example only, we consider the undirected graph of a matrix that has a zero entry on the diagonal. With respect to the result of Theorem 3.3, each set of vertices β_k simplicial in $G(A)$ implies $G[A^{-1}[\alpha]]^{-1} \subseteq G(A[\alpha])$ (generically) even when some $a_{kk} = 0$ for $k \in \alpha^c$. However, the converse of this result does not follow, as this example illustrates.

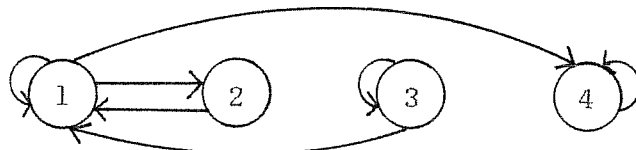
Let A be a nonsingular combinatorially symmetric matrix with the following undirected graph.



Letting $\alpha = \{1,2\}$, the set of nodes $\alpha^c = \{3,4,5\}$ is not a simplicial set, however $G[A^{-1}[\alpha]]^{-1} \subseteq G(A[\alpha])$ (generically) as the zero entries are inherited, using the fact that $a_{55} = 0$. (Compare with Example 4.4). ■

EXAMPLE 4.6.

Consider the following directed graph D :



Let $\alpha = \{3,4\}$. If A is a nonsingular matrix with $D(A) = D$ and $A[\alpha^c]$ is nonsingular, then either Theorem 2.3 or Corollary 2.6 implies that $\left[A^{-1}[\alpha]\right]^{-1} = A[\alpha]$ (generically) since the only path between two nodes in α passing through α^c is $3 \rightarrow 1 \rightarrow 4$; however, $\det A[\alpha^c - \{1\}] \equiv a_{22} = 0$. This remains true for all A consistent with D such that $A[\alpha^c]$ is nonsingular, specifically any or all of a_{11}, a_{14}, a_{31} can be set to zero.

Thus, the Schur complement of $A[\{1,2\}]$ in A is identically equal to the diagonal submatrix $A[\{3,4\}]$. It is interesting to note, however, that the (3,4) entry of the Schur complement of $A[\{1\}]$ in A is nonzero (as can be seen from the remark following Corollary 2.10). ■

EXAMPLE 4.7.

Let A be a 5-by-5 nonsingular matrix with $a_{13} = a_{15} = a_{43} = a_{45} = 0$ and remaining entries arbitrary. Let $\alpha = \{1,2,3\}$, $\beta = \{1,4\}$ and $\gamma = \{3,5\}$ and assume $A[\alpha^c]$ is nonsingular. Clearly $\text{rank } A[\beta|\gamma] = 0$, so $\text{rank } A^{-1}[\gamma^c|\beta^c] = 1$ by Theorem 3.5. Thus $\text{rank } \left[A^{-1}[\alpha]\right][\{1,2\}|\{2,3\}] \leq 1$ implying that $\text{rank } \left[A^{-1}[\alpha]\right]^{-1}[\{1\}|\{3\}] \leq 0$ by Theorem 3.5; thus the zero entry a_{13} is inherited. This also follows from Theorem 2.3 (i) with $D = D(A)$. ■

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