SOME GENERALIZATIONS OF A COMBINATORIAL IDENTITY OF L. VIETORIS

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where m, n, k are integers, with

$$0 \leqq k \leqq m - 1 \quad \text{and} \quad n \geqq 0. \tag{5}$$

Making use of the definition (3), Vietoris's identity (4) can readily be rewritten in its equivalent form:

$$\binom{m+n}{n} = \sum_{i=0}^{n} \binom{k+i}{i} \binom{m+n-k-i-1}{n-i}, \tag{6}$$

where, as before, m, n, k are integers constrained by (5).

A closer examination of the combinatorial identity (6) would suggest the existence of an interesting generalization of Vietoris's result (4) in the form:

$$\binom{\mu+n}{n} = \sum_{i=0}^{n} \binom{\lambda+i}{i} \binom{\mu-\lambda+n-i-1}{n-i}, \tag{7}$$

where λ and μ are arbitrary complex numbers, and $n = 0, 1, 2, \ldots$.

Formula (7) can indeed be rewritten in a form analogous to (4) by using Gamma functions.

2. Derivation of the Identity (7)

In view of the elementary relationship (2), we have

$$\binom{\lambda+i}{i} = (-1)^{i} \binom{-\lambda-1}{i}, \quad i = 0, 1, 2, \ldots, \tag{8}$$

and

$$\binom{\mu-\lambda+n-i-1}{n-i} = (-1)^{n-i} \binom{\lambda-\mu}{n-i}, \quad 0 \leqq i \leqq n, \tag{9}$$

$$\binom{\lambda+n-1}{n} = \frac{(\lambda)_n}{n!}, \quad n = 0, 1, 2, \ldots, \tag{15}$$

the right-hand side of the general identity (7) equals

$$(\mu-\lambda)_n \; F(-n,\lambda+1;\lambda-\mu-n+1;1), \tag{16}$$

where the hypergeometric series is finite because n is a nonnegative integer.

Now apply a special case of Gauss's summation theorem ([3], p. 19) in the form:

$$F(-n,b;c;1) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \ldots, \tag{17}$$

and note from (14) that

$$\frac{(-\mu-n)_n}{(\lambda-\mu-n+1)_n} = \frac{(\mu+1)_n}{(\mu-\lambda)_n}, \quad n = 0, 1, 2, \ldots, \tag{18}$$

$$\lambda - \mu + 1 \neq 1, 2, 3, \ldots,$$

and (16) immediately yields the left-hand side of the general identity (7) under the (easily removable) constraint that $\lambda - \mu + 1$ is not a positive integer.

3. A Basic (or *q*-) Extension of the Identity (7)

In terms of the basic (or *q*-) number $[\lambda]$ and basic (or *q*-) factorial $[n]!$ defined by

$$[\lambda] = \frac{1-q^\lambda}{1-q}; \quad [n]! = [1][2][3] \ldots [n], \; [0]! = 1, \tag{19}$$

let the basic (or *q*-) binomial coefficient be given by [*cf.* Equation (1) *et seq.*]

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