
Faculty of Science

Faculty Publications

Coefficient estimates of new families of analytic functions associated with q -Hermite polynomials

Al-Shbeil, I., Cătaş, A., Srivastava, H. M., & Aloraini, N.

2023

© 2023 Isra Al-Shbeil et al. This is an open access article distributed under the terms of the Creative Commons Attribution License.

<http://creativecommons.org/licenses/by/4.0/>

This article was originally published at:
<https://doi.org/10.3390/axioms12010052>

Citation for this paper:

Al-Shbeil, I., Cătaş, A., Srivastava, H. M., & Aloraini, N. (2023). "Coefficient estimates of new families of analytic functions associated with q -Hermite polynomials." *Axioms*, 12(1), 52. <https://doi.org/10.3390/axioms12010052>

Article

Coefficient Estimates of New Families of Analytic Functions Associated with q -Hermite Polynomials

Isra Al-Shbeil ^{1,*} , Adriana Cătaș ^{2,t} , Hari Mohan Srivastava ^{3,4,5,6,t}  and Najla Aloraini ^{7,t} 

¹ Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan

² Department of Mathematics and Computer Science, University of Oradea, 1 University Street, 410087 Oradea, Romania

³ Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada

⁴ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

⁵ Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

⁶ Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

⁷ Department of Mathematics, College of Sciences and Arts Onaizah, Qassim University, P.O. Box 6640, Buraydah 51452, Saudi Arabia

* Correspondence: i.shbeil@ju.edu.jo

† These authors contributed equally to this work.

Abstract: In this paper, we introduce two new subclasses of bi-univalent functions using the q -Hermite polynomials. Furthermore, we establish the bounds of the initial coefficients v_2 , v_3 , and v_4 of the Taylor–Maclaurin series and that of the Fekete–Szegő functional associated with the new classes, and we give the many consequences of our findings.

Keywords: q -convolution operator; coefficient estimates; q -Hermite; bi-univalent functions; Babalola operator

MSC: 05A30; 30C45; 11B65; 47B38



Citation: Al-Shbeil, I.; Cătaș A.; Srivastava, H.M.; Aloraini, N. Coefficient Results of Family of Analytic Functions Associated with q -Hermite Polynomials. *Axioms* **2023**, *12*, 52. <https://doi.org/10.3390/axioms12010052>

Academic Editor: Natália Martins

Received: 20 November 2022

Accepted: 27 December 2022

Published: 3 January 2023



Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction and Preliminaries

Let $\mathcal{M}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let \mathcal{A} be the subclass of $\mathcal{M}(\mathbb{U})$ whose functions f satisfy the normalization condition given by

$$f(0) = f'(0) - 1 = 0,$$

that is, each function f in \mathcal{A} can be represented by the following Taylor–Maclaurin series expansion:

$$f(z) = z + \sum_{k=2}^{\infty} v_k z^k \quad (z \in \mathbb{U}). \quad (1)$$

Moreover, let \mathcal{S} be the subclass of \mathcal{A} whose functions are univalent in \mathbb{U} . The Koebe one-quarter theorem ensures that the image of \mathbb{U} under every $f \in \mathcal{S}$ contains a disk of radius $1/4$.

It is known that every function $f \in \mathcal{S}$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f^{-1}(f(\omega)) = \omega \quad \left(|\omega| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(\omega) = g(\omega) = \omega - v_2\omega^2 + (2v_2^2 - v_3)\omega^3 - (5v_2^3 - 5v_2v_3 + v_4)\omega^4 + \dots \tag{2}$$

A function is said to be bi-univalent in \mathbb{U} if both f and f^{-1} are univalent in \mathbb{U} . Let Σ denote the class of bi-univalent functions in \mathbb{U} given by (1).

Lewin [1] investigated the class Σ and showed that $|v_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|v_2| < \sqrt{2}$. Netanyahu [3], on the other hand, showed that

$$\max_{f \in \Sigma} |v_2| = \frac{4}{3}.$$

The Taylor–Maclaurin coefficients $|v_n|$ ($n \geq 3, n \in \mathbb{N}$) in (1) are still unknown, and it is an open problem.

Similar to the subclasses $\mathcal{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ of the starlike and convex functions of the order ζ ($0 \leq \zeta < 1$), respectively, that we are familiar with, Brannan and Taha [4] gave two subclasses of Σ , which are called $\mathcal{S}_\Sigma^*(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$ of the bi-starlike functions and bi-convex functions of the order ζ ($0 \leq \zeta < 1$), respectively. It should be remarked here that, in their pioneering work, Srivastava et al. [5] actually revived the study of analytic and bi-univalent functions in recent years.

Moreover, for two analytic functions s_1 and s_2 , the function s_1 is called subordinated to the function s_2 , denoted as

$$s_1(z) \prec s_2(z) \quad (z \in \mathbb{U}),$$

if there is an analytic function w in \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1,$$

such that

$$s_1(z) = s_2(w(z)).$$

If the function $s_2 \in \mathcal{S}$, then

$$s_1(z) \prec s_2(z) \Leftrightarrow s_1(0) = s_2(0) \quad \text{and} \quad s_1(\mathbb{U}) \subset s_2(\mathbb{U}).$$

In 2008, Babalola [6] defined the operator $\mathcal{J}_n^\sigma : \mathcal{A} \rightarrow \mathcal{A}$ as

$$\mathcal{J}_m^\tau f(z) = (v_\tau * v_{\tau,m}^{-1} * f)(z) \tag{3}$$

where

$$v_{\tau,m}(z) = \frac{z}{(1-z)^{\tau-(m-1)}}, \quad \tau - (m-1) > 0, \quad v_\tau = v_{\tau,0}$$

and $v_{\tau,m}^{-1}$ is such that

$$(v_{\tau,m} * v_{\tau,m}^{-1})(z) = \frac{z}{1-z} \quad (\tau, m \in \mathcal{N}).$$

Let $f \in \mathcal{A}$, then (3) is equivalent to

$$\mathcal{J}_m^\tau f(z) = z + \sum_{j=2}^\infty \left[\frac{[\tau+j-1]!}{\tau!} \frac{[\tau-m]!}{[\tau+j-m-1]!} \right] v_j z^j. \tag{4}$$

The q -derivative operator \mathcal{D}_q of a function was introduced and researched by Jackson [7,8].

$$\mathcal{D}_q f(z) = \frac{f(qz) - f(z)}{z(q-1)} = z^{-1} \left\{ z + \sum_{k=2}^\infty [k]_q v_k z^k \right\} \tag{5}$$

and $\mathfrak{D}_q f(0) = f'(0)$. In particular, $f(z) = z^k$ for k is a positive integer, the q -derivative of $f(z)$ is given by

$$\mathfrak{D}_q z^k = \frac{(zq)^k - z^k}{z(q-1)} = [k]_q z^{k-1}, \tag{6}$$

$$\lim_{q \rightarrow 1} [k]_q = \lim_{q \rightarrow 1} \frac{q^k - 1}{q - 1} = k. \tag{7}$$

For function $f(z)$ given by (1) and $g(z)$ given by

$$g(z) = z + \sum_{k=2}^{\infty} c_k z^k$$

the convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} v_k c_k z^k = (g * f)(z).$$

Let

$$\mathfrak{D}_q(z) = \frac{z}{(1-qz)(1-z)} = z + (1+e_1)z^2 + (1+e_1+e_2)z^3 + \dots = z + \sum_{k=2}^{\infty} [k]_e z^k \tag{8}$$

where

$$[k]_e = 1 + e_1 + e_2 + \dots + e_{k-1}, \quad e_k = q^k. \tag{9}$$

See [9,10] for additional information on q -derivative theories.

Quantum (or q -) calculus is a strong instrument for investigating a wide range of analytic functions, and it has sparked new research in mathematics and other fields. The first time it was used in the context of univalent functions was by Srivastava [11]. Many academics have studied q -calculus and its many applications due to the usefulness of q -analysis in mathematics and other areas. With the help of certain higher-order q -derivative operators, Khan et al. [12] constructed and analyzed a number of subclasses of q -starlike functions. Shi et al. (see also [13]) created a novel subclass of multivalent q -starlike Janowski functions using the q -differential operator. A variety of adequate requirements as well as some other noteworthy characteristics were investigated in both articles [12,14].

Because of the large range of applications and the usefulness of q -operators above fundamental operators, many scholars have looked into q -calculus in depth. Furthermore, Srivastava's recently published survey-cum-expository review study [15–17] is useful for academics and scholars studying these topics.

The q -Hermite polynomial was first introduced by Rogers [18] (see also [19,20]) and is usually defined by means of their generating function as follows

$$B_k(s|q) = \sum_{k=0}^{\infty} H_k(x; q) \frac{t^k}{(q; q)_k} = \prod_{k=0}^{\infty} \frac{1}{1 - 2xtq^k + t^2q^{2k}} \quad (0 < q < 1).$$

The q -derivative of the q -Hermite polynomial is

$$\mathfrak{D}_q \{B_{k+1}(s|q)\} = [k]_q B_k(s|q). \tag{10}$$

Moreover, Ismail et al. [18] were able to define the recursion relation as

$$tB_k(s|q) = B_{k+1}(s|q) + [k]_q B_{k-1}(s|q) \tag{11}$$

with

$$B_0(s|q) = 1 \quad \text{and} \quad B_{-1}(s|q) = 0.$$

Also from (11), we have

$$\begin{aligned}
 B_1(s|q) &= s \\
 B_2(s|q) &= s^2 - 1 \\
 B_3(s|q) &= s^3 - (2 + q)s \\
 B_4(s|q) &= s^4 - (3 + 2q + q^2)s^2 + (1 + q + q^2).
 \end{aligned}$$

Remark 1. It is clear that

$$B_k(s|q = 1) = B_{c_k}(s)$$

is the Hermite polynomials. Moreover, when

$$B_k(s|q = 0) = U_k(s/2),$$

we have Chebyshev polynomials of the first kind, and they are defined by the recursion relation,

$$2sU_k(s) = U_{k-1}(s) + U_{k+1}(s) \tag{12}$$

with

$$U_0(s) = 1 \quad \text{and} \quad U_{-1}(s) = 0.$$

Next, we define the q -Babalola convolution operator which will be used throughout this paper.

Definition 1. Let $f \in A$. Denote by $\mathfrak{J}_{\gamma,q}f(z)$ the q -Babalola convolution operator defined by

$$\mathfrak{J}_{\gamma,q}f(z) = (v_{\tau,q} * v_{\gamma,q}^{(-1)} * f)(z) \tag{13}$$

where

$$v_{\gamma,q} = \frac{z}{(1 - qz)^\gamma(1 - z)}, \quad \gamma > -1 \quad \text{and} \quad v_{\gamma,q}^{(-1)}$$

is such that

$$(v_{\gamma,q} * v_{\gamma,q}^{(-1)})(z) = \frac{z}{1 - z}.$$

Hence,

$$\mathfrak{J}_{\gamma,q}f(z) = z + \sum_{k=2}^{\infty} \frac{[k]_e^\sigma}{[k]_e^\gamma} v_k z^k = z + \sum_{k=2}^{\infty} (k)_e^\gamma v_k z^k. \tag{14}$$

where

$$(k)_e^\gamma = \frac{1 + e_1(\tau) + e_2(\tau) \cdots e_{k-1}(\tau)}{1 + e_1(\gamma) + e_2(\gamma) \cdots e_{k-1}(\gamma)}$$

and

$$e_{k-1}(\tau) = \frac{(\tau + k - 2)!}{(\tau - 1)!} \frac{q^{k-1}}{(k - 1)!}, \quad e_{k-1}(\gamma) = \frac{(\gamma + k - 2)!}{(\gamma - 1)!} \frac{q^{k-1}}{(k - 1)!}.$$

Remark 2. It is easily seen that, upon setting $q \rightarrow 1-$, the extended Babalola convolution operator $\mathfrak{J}_{\gamma,q}f(z)$ reduces to the Babalola convolution operator $\mathfrak{J}_\sigma^m f(z)$ which was introduced and studied by Babalola [6]. For $m = \tau = 1$, the extended Babalola convolution operator $\mathfrak{J}_{\gamma,q}f(z)$ reduces to the q -derivative operator introduced and studied by Jackson [7,8]. Moreover, if $m = \tau$ and $q \rightarrow 1-$, we have the Ruscheweyh’s operator [21].

Consider the univalent normalized functions of the kind (1); the Fekete–Szegő functional $|v_3 - \phi v_2^2|$ has a long history in geometric function theory. The authors in [22] disproved Paley’s conjecture and Littlewood’s that the coefficients of odd univalent functions are confined by unity in 1933. Since then, the functional has gotten much attention,

especially in subclasses of the family of univalent functions. This problem appears to have piqued the interest of scholars in recent years (see, for example, [23,24]).

We know that the q -Hermite polynomials and q -convolution operators still have not been studied with bi-univalent functions. The main goal of this paper is to start looking at the properties of the bi-univalent functions that are connected to q -Hermite polynomials and the q -convolution operator. In this study, the initial coefficient estimates for the Fekete–Szegő problem of analytic and bi-univalent functions are determined using the q -Hermite polynomial expansions and the q -convolution operator.

In Definition 2, we describe a class of convex bi-univalent functions that are defined by the q -convolution operator and linked to the q -Hermite polynomial.

Definition 2. Let $N(z, s, q)$ be defined as follows:

$$N(z, s, q) = \sum_{k=2}^{\infty} B_k(s|q)z^k. \tag{15}$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\Gamma_{\Sigma}^q(s, \gamma, \tau)$, if the following conditions are satisfied:

$$1 + \frac{z\mathfrak{D}_q^2(\mathfrak{I}_{\gamma,q}f(z))}{\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f(z))} \prec N(z, s, q) \tag{16}$$

and

$$1 + \frac{\omega\mathfrak{D}_q^2(\mathfrak{I}_{\gamma,q}f^{-1}(\omega))}{\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f^{-1}(\omega))} \prec N(\omega, s, q). \tag{17}$$

Where $s \in (\frac{1}{2}, 1), 0 < q < 1, z \in \mathbb{U}, \omega \in \mathbb{U}, \gamma = \tau - m > -1$.

In Definition 3, we describe a class of starlike bi-univalent functions that are defined by the q -convolution operator and linked to the q -Hermite polynomial.

Definition 3. Let $N(z, s, q)$ be defined as follows:

$$N(z, s, q) = \sum_{k=2}^{\infty} B_k(s|q)z^k. \tag{18}$$

A function $f \in \Sigma$ given by (1) is said to be in the class $\Pi_{\Sigma}^q(s, \gamma, \tau)$, if the following conditions are satisfied:

$$\frac{z\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f(z))}{\mathfrak{I}_{\gamma,q}f(z)} \prec N(z, s, q) \tag{19}$$

and

$$\frac{\omega\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f^{-1}(\omega))}{\mathfrak{I}_{\gamma,q}f^{-1}(\omega)} \prec N(\omega, s, q). \tag{20}$$

We must recall the following lemma in order to arrive at our primary conclusions.

As is usually the case, we let \mathcal{P} be the family of functions $p(z) = 1 + p_1z + p_2z^2 + \dots$ regular with positive real part, for $z \in \mathbb{U}$.

Lemma 1 ([25]). Let $\varphi(z) \in \mathcal{P}$, then

$$|p_j| \leq 2 \quad (j \in \mathcal{N}).$$

2. Coefficient Estimates for the Class $\Gamma_{\Sigma}^q(s, \gamma, \tau)$

The initial coefficient bounds of the class $\Gamma_{\Sigma}^q(s, \gamma, \tau)$ of bi-univalent functions are investigated in this section.

Theorem 1. Let $f \in \Gamma_{\Sigma}^q(s, \gamma, \tau)$. Then,

$$|v_2| \leq \sqrt{\Psi_1(s, q, \gamma)}, \tag{21}$$

$$|v_3| \leq \frac{s^2}{[2]_b^2((2)_e^\gamma)^2} + \frac{s}{4[2]_b[3]_b(3)_e^\gamma}, \tag{22}$$

and

$$|v_4| \leq \frac{s^3(2(1 + e_1)[3]_e(3)_e^\gamma - [2]_e^2((2)_e^\gamma)^2)}{[2]_e^3[4]_e(4)_e^\gamma((2)_e^\gamma)^2} - \frac{s^2(2e_1[2]_e(2)_e^\gamma(3)_e^\gamma + 5[4]_e(4)_e^\gamma)}{2[2]_e^3[4]_e(2)_e^\gamma(3)_e^\gamma(4)_e^\gamma} + \frac{s^3 - 2s - 2qs - 4}{[2]_e[4]_e(4)_e^\gamma}$$

where

$$\Psi_1(s, q, \gamma) = \frac{s^3}{|s^2([2]_e[3]_e(3)_e^\gamma - [2]_e^2((2)_e^\gamma)^2) - [2]_e^2((2)_e^\gamma)^2(s^2 - s - 1)|}. \tag{23}$$

Proof. Let $f \in \Sigma$ be given by (1) be in the class $\Gamma_{\Sigma}^q(s, \gamma, \tau)$. Then,

$$1 + \frac{z\mathfrak{D}_q^2(\mathfrak{J}_{\gamma,q}f(z))}{\mathfrak{D}_q(\mathfrak{J}_{\gamma,q}f(z))} = N(d(z), s, q) \tag{24}$$

and

$$1 + \frac{z\mathfrak{D}_q^2(\mathfrak{J}_{\gamma,q}f^{-1}(\omega))}{\mathfrak{D}_q(\mathfrak{J}_{\gamma,q}f^{-1}(\omega))} = N(\omega(\omega), s, q), \tag{25}$$

Let $\varrho, \eta \in \mathcal{P}$ be defined as

$$\varrho(z) = \frac{1 + d(z)}{1 - d(z)} = 1 + \varrho_1z + \varrho_2z^2 + \varrho_3z^3 + \dots \Rightarrow d(z) = \frac{\varrho(z) - 1}{\varrho(z) + 1}, \quad (z \in \mathbb{U}) \tag{26}$$

and

$$\eta(\omega) = \frac{1 + \omega(\omega)}{1 - \omega(\omega)} = 1 + \eta_1\omega + \eta_2\omega^2 + \eta_3\omega^3 + \dots \Rightarrow \omega(\omega) = \frac{\eta(\omega) - 1}{\eta(\omega) + 1}, \quad (\omega \in \mathbb{U}). \tag{27}$$

It follows that from (26) and (27) that

$$d(z) = \frac{1}{2} \left[\varrho_1z + \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) z^2 + \left(\varrho_3 - \varrho_1\varrho_2 + \frac{\varrho_1^3}{4} \right) z^3 + \dots \right] \tag{28}$$

and

$$\omega(\omega) = \frac{1}{2} \left[\eta_1\omega + \left(\eta_2 - \frac{\eta_1^2}{2} \right) \omega^2 + \left(\eta_3 - \eta_1\eta_2 + \frac{\eta_1^3}{4} \right) \omega^3 + \dots \right]. \tag{29}$$

From (28) and (29), applying $N(z, s, q)$ as given in (18), we see that

$$N(d(z), s, q) = 1 + \frac{B_1(s|q)}{2} \varrho_1z + \left[\frac{B_1(s|q)}{2} \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_2(s|q)}{4} \varrho_1^2 \right] z^2 + \left[\frac{B_1(s|q)}{2} \left(\varrho_3 - \varrho_1\varrho_2 + \frac{\varrho_1^3}{4} \right) + \frac{B_2(s|q)}{2} \varrho_1 \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_3(s|q)}{8} \varrho_1^3 \right] z^3 + \dots$$

and

$$\begin{aligned}
 N(\omega(\omega), s, q) &= 1 + \frac{B_1(s|q)}{2} \eta_1 \omega + \left[\frac{B_1(s|q)}{2} \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_2(s|q)}{4} \eta_1^2 \right] \omega^2 \\
 &+ \left[\frac{B_1(s|q)}{2} \left(\eta_3 - \eta_1 \eta_2 + \frac{\eta_1^3}{4} \right) + \frac{B_2(s|q)}{2} \eta_1 \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_3(s|q)}{8} \eta_1^3 \right] \omega^3 + \dots
 \end{aligned} \tag{30}$$

It follows from (24), (30), and (25), we have

$$[2]_e(2]_e^\gamma v_2 = \frac{B_1(s|q)}{2} \varrho_1, \tag{31}$$

$$[2]_e[3]_e(3]_e^\gamma v_3 - [2]_e^2((2]_e^\gamma)^2 v_2^2 = \frac{B_1(s|q)}{2} \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_2(s|q)}{4} \varrho_1^2, \tag{32}$$

$$\begin{aligned}
 &[3]_e[4]_e(4]_e^\gamma v_4 - [2]_e[3]_e(2]_e^\gamma(3]_e^\gamma e_1 v_2 v_3 + [2]_e^3((2]_e^\gamma)^3 v_2^3 \\
 &= \frac{B_1(s|q)}{2} \left(\varrho_3 - \varrho_1 \varrho_2 + \frac{\varrho_1^3}{4} \right) + \frac{B_2(s|q)}{2} \varrho_1 \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_3(s|q)}{8} \varrho_1^3,
 \end{aligned} \tag{33}$$

$$- [2]_e(2]_e^\gamma v_2 = \frac{B_1(s|q)}{2} \eta_1, \tag{34}$$

$$(2[2]_e[3]_e(3]_e^\gamma - [2]_e^2((2]_e^\gamma)^2) v_2^2 - [2]_e[3]_e(3]_e^\gamma v_3 = \frac{B_1(s|q)}{2} \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_2(s|q)}{4} \eta_1^2, \tag{35}$$

$$\begin{aligned}
 &(2(2 + e_1)[2]_e[3]_e(2]_e^\gamma(3]_e^\gamma - 5[3]_e[4]_e(4]_e^\gamma - [2]_e^3((2]_e^\gamma)^2) v_2^3 + (5[3]_e[4]_e(4]_e^\gamma \\
 &+ [2]_e[3]_e(2]_e^\gamma(3]_e^\gamma e_1) v_2 v_3 - [3]_e[4]_e(4]_e^\gamma v_4 = \frac{B_1(s|q)}{2} \left(\eta_3 - \eta_1 \eta_2 + \frac{\eta_1^3}{4} \right) \\
 &+ \frac{B_2(s|q)}{2} \eta_1 \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_3(s|q)}{8} \eta_1^3.
 \end{aligned} \tag{36}$$

Adding (31) and (34), we have

$$\varrho_1 = -\eta_1, \quad \varrho_1^2 = \eta_1^2 \quad \text{and} \quad \varrho_1^3 = -\eta_1^3 \tag{37}$$

and

$$v_2^2 = \frac{(B_1(s|q))^2(\varrho_1^2 + \eta_1^2)}{8[2]_e^2((2]_e^\gamma)^2}. \tag{38}$$

Moreover, adding (32) and (35) and applying (37) yields

$$4v_2^2[[2]_e[3]_e(3]_e^\gamma - [2]_e^2((2]_e^\gamma)^2)] = B_1(s|q)(\varrho_2 + \eta_2) - \eta_1^2(B_1(s|q) - B_2(s|q)). \tag{39}$$

Applying (37) in (38) gives

$$\eta_1^2 = \frac{4[2]_e^2((2]_e^\gamma)^2 v_2^2}{(B_1(s|q))^2}. \tag{40}$$

Putting (40) into (39) and with some calculations, we have

$$|v_2|^2 = \left| \frac{(B_1(s|q))^3(\varrho_2 + \eta_2)}{4[[2]_e[3]_e(3]_e^\gamma - [2]_e^2((2]_e^\gamma)^2)](B_1(s|q))^2 + 4[2]_e^2((2]_e^\gamma)^2(B_1(s|q) - B_2(s|q))} \right|.$$

Applying triangular inequality and Lemma 1, we have

$$|v_2| \leq \sqrt{\Psi_1(s, q, \gamma)}. \tag{41}$$

Subtracting (35) from (32) and with some calculations, we have

$$v_3 = v_2^2 + \frac{B_1(s|q)[\varrho_2 - \eta_2]}{4[2]_e[3]_e(3)_e^\gamma} \tag{42}$$

$$v_3 = \frac{(B_1(s|q))^2 \varrho_1^2}{4[2]_e^2((2)_e^\gamma)^2} + \frac{B_1(s|q)[\varrho_2 - \eta_2]}{4[2]_e[3]_e(3)_e^\gamma}. \tag{43}$$

Applying triangular inequality and Lemma 1, we have

$$|v_3| \leq \frac{s^2}{[2]_b^2((2)_e^\gamma)^2} + \frac{s}{4[2]_b[3]_b(3)_e^\gamma}. \tag{44}$$

Subtracting (36) from (33), we have

$$\begin{aligned} 2[2]_e[4]_e(4)_e^\gamma v_4 = & \frac{(2(1 + e_1)[3]_e(3)_e^\gamma - [2]_e^2((2)_e^\gamma)^2)(B_1(s|q))^3 \varrho_1^3}{4[2]_e^2((2)_e^\gamma)^2} \\ & - \frac{(2e_1[2]_e(2)_e^\gamma(3)_e^\gamma + 5[4]_e(4)_e^\gamma)(B_1(s|q))^2 \varrho_1(\varrho_2 - \eta_2)}{8[2]_e^2(2)_e^\gamma(3)_e^\gamma} \\ & + \frac{B_1(s|q)(\varrho_3 - \eta_3)}{2} + \frac{[B_2(s|q) - B_1(s|q)]\varrho_1(\varrho_2 + \eta_2)}{2} \\ & + \frac{(B_1(s|q) - 2B_2(s|q) + B_3(s|q))\varrho_1^3}{4}. \end{aligned} \tag{45}$$

Applying triangular inequality and Lemma 1, we have

$$\begin{aligned} |v_4| \leq & \frac{s^3(2(1 + e_1)[3]_e(3)_e^\gamma - [2]_e^2((2)_e^\gamma)^2)}{[2]_e^3[4]_e(4)_e^\gamma((2)_e^\gamma)^2} - \frac{s^2(2e_1[2]_e(2)_e^\gamma(3)_e^\gamma + 5[4]_e(4)_e^\gamma)}{2[2]_e^3[4]_e(2)_e^\gamma(3)_e^\gamma(4)_e^\gamma} \\ & + \frac{s^3 - 2s - 2qs - 4}{[2]_e[4]_e(4)_e^\gamma}. \end{aligned}$$

□

3. Coefficient Estimates for the Class $\Pi_\Sigma^q(s, \gamma, \tau)$

The initial coefficient bounds of the class $\Pi_\Sigma^q(s, \gamma, \tau)$ of bi-univalent functions are investigated in this section.

Theorem 2. Let $f \in \Pi_\Sigma^q(s, \gamma, \tau)$. Then,

$$|v_2| \leq \sqrt{X_1(s, q, \gamma)}, \tag{46}$$

$$|v_3| \leq \frac{s^2}{e_1^2((2)_e^\gamma)^2} + \frac{s}{(e_1 + e_2)(3)_e^\gamma}, \tag{47}$$

and

$$\begin{aligned} |v_4| \leq & \frac{s^3((2)_e^\gamma(3)_e^\gamma(4e_1 + 2e_2) - 2((2)_e^\gamma)^3e_1 - 10(e_1 + e_2 + e_3)(4)_e^\gamma)}{2(e_1 + e_2 + e_3)((2)_e^\gamma)^3(4)_e^\gamma e_1^3} \\ & - \frac{5s^2}{2(2)_e^\gamma(3)_e^\gamma e_1(e_1 + e_2)} + \frac{s^3 - 2s - 2qs - 4}{(e_1 + e_2 + e_3)(4)_e^\gamma} \end{aligned}$$

where

$$X_1(s, q, \gamma) = \frac{2s^3}{|s^2\{2(e_1 + e_2)(3]_e^\gamma + (2]_e^\gamma - ((2]_e^\gamma)^2(1 + 2e_1)\} - 2((2]_e^\gamma)^2e_1^2(s^2 - s - 1)|}. \tag{48}$$

Proof. Let $f \in \Sigma$ be given by (1) be in the class $\Pi_\Sigma^q(s, \gamma, \tau)$. Then,

$$\frac{z\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f(z))}{\mathfrak{I}_{\gamma,q}f(z)} = N(d(z), s, q) \tag{49}$$

and

$$\frac{\omega\mathfrak{D}_q(\mathfrak{I}_{\gamma,q}f^{-1}(\omega))}{\mathfrak{I}_{\gamma,q}f^{-1}(\omega)} = N(\omega(\omega), s, q). \tag{50}$$

Let $\varrho, \eta \in \mathcal{P}$ be defined by

$$\varrho(z) = \frac{1 + d(z)}{1 - d(z)} = 1 + \varrho_1(z) + \varrho_2z^2 + \varrho_3z^3 + \dots \Rightarrow d(z) = \frac{\varrho(z) - 1}{\varrho(z) + 1}, \quad (z \in \mathbb{U}) \tag{51}$$

and

$$\eta(\omega) = \frac{1 + \omega(\omega)}{1 - \omega(\omega)} = 1 + \eta_1(\omega) + \eta_2\omega^2 + \eta_3\omega^3 + \dots \Rightarrow \omega(\omega) = \frac{\eta(\omega) - 1}{\eta(\omega) + 1}, \quad (\omega \in \mathbb{U}). \tag{52}$$

It follows that from (51) and (52) that

$$d(z) = \frac{1}{2} \left[\varrho_1z + \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) z^2 + \left(\varrho_3 - \varrho_1\varrho_2 + \frac{\varrho_1^3}{4} \right) z^3 + \dots \right] \tag{53}$$

and

$$\omega(\omega) = \frac{1}{2} \left[\eta_1\omega + \left(\eta_2 - \frac{\eta_1^2}{2} \right) \omega^2 + \left(\eta_3 - \eta_1\eta_2 + \frac{\eta_1^3}{4} \right) \omega^3 + \dots \right]. \tag{54}$$

From (53) and (54), applying $N(z, s, q)$ as given in (18), we see that

$$\begin{aligned} N(d(z), s, q) &= 1 + \frac{B_1(s|q)}{2} \varrho_1z + \left[\frac{B_1(s|q)}{2} \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_2(s|q)}{4} \varrho_1^2 \right] z^2 \\ &+ \left[\frac{B_1(s|q)}{2} \left(\varrho_3 - \varrho_1\varrho_2 + \frac{\varrho_1^3}{4} \right) + \frac{B_2(s|q)}{2} \varrho_1 \left(\varrho_2 - \frac{\varrho_1^2}{2} \right) + \frac{B_3(s|q)}{8} \varrho_1^3 \right] z^3 + \dots \end{aligned}$$

and

$$\begin{aligned} N(\omega(\omega), s, q) &= 1 + \frac{B_1(s|q)}{2} \eta_1\omega + \left[\frac{B_1(s|q)}{2} \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_2(s|q)}{4} \eta_1^2 \right] \omega^2 \\ &+ \left[\frac{B_1(s|q)}{2} \left(\eta_3 - \eta_1\eta_2 + \frac{\eta_1^3}{4} \right) + \frac{B_2(s|q)}{2} \eta_1 \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_3(s|q)}{8} \eta_1^3 \right] \omega^3 + \dots \end{aligned} \tag{55}$$

It follows from (49), (55), and (50), we have

$$(2]_e^\gamma e_1 v_2 = \frac{B_1(s|q)}{2} \varrho_1, \tag{56}$$

$$(e_1 + e_2)(3]_e^\gamma v_3 - ((2]_e^\gamma)^2 e_1 v_2^2 = \frac{B_1(s|q)}{2} \left(q_2 - \frac{q_1^2}{2} \right) + \frac{B_2(s|q)}{4} q_1^2, \tag{57}$$

$$(e_1 + e_2 + e_3)(4]_e^\gamma v_4 - (2e_1 + e_2)(2]_e^\gamma (3]_e^\gamma v_2 v_3 + e_1 ((2]_e^\gamma)^3 v_2^3) \\ = \frac{B_1(s|q)}{2} \left(q_3 - q_1 q_2 + \frac{q_1^3}{4} \right) + \frac{B_2(s|q)}{2} q_1 \left(q_2 - \frac{q_1^2}{2} \right) + \frac{B_3(s|q)}{8} q_1^3, \tag{58}$$

$$- (2]_e^\gamma e_1 v_2 = \frac{B_1(s|q)}{2} \eta_1, \tag{59}$$

$$2(3]_e^\gamma (e_1 + e_2) v_2^2 - e_1 ((2]_e^\gamma)^2 v_2^2 - (e_1 + e_2)(3]_e^\gamma v_3 = \frac{B_1(s|q)}{2} \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_2(s|q)}{4} \eta_1^2, \tag{60}$$

$$((2]_e^\gamma (3]_e^\gamma (4e_1 + 2e_2) - 5(4]_e^\gamma (e_1 + e_2 + e_3) - ((2]_e^\gamma)^2 e_1) v_2^3 - ((2]_e^\gamma (3]_e^\gamma (2e_1 + e_2) \\ + 5(4]_e^\gamma (e_1 + e_2 + e_3)) v_2 v_3 - (4]_e^\gamma (e_1 + e_2 + e_3) v_4 = \frac{B_1(s|q)}{2} \left(\eta_3 - \eta_1 \eta_2 + \frac{\eta_1^3}{4} \right) \\ + \frac{B_2(s|q)}{2} \eta_1 \left(\eta_2 - \frac{\eta_1^2}{2} \right) + \frac{B_3(s|q)}{8} \eta_1^3. \tag{61}$$

Adding (56) and (59), we have

$$q_1 = -\eta_1, \quad q_1^2 = \eta_1^2 \quad \text{and} \quad q_1^3 = -\eta_1^3 \tag{62}$$

and

$$v_2^2 = \frac{(B_1(s|q))^2 (q_1^2 + \eta_1^2)}{8((2]_e^\gamma)^2 e_1^2}. \tag{63}$$

Moreover, adding (57) and (60) and applying (62) yields

$$2v_2^2 \{ 2(e_1 + e_2)(3]_e^\gamma + (2]_e^\gamma - ((2]_e^\gamma)^2 (1 + 2e_1)) \} = B_1(s|q)(q_2 + \eta_2) - \eta_1^2 (B_1(s|q) - B_2(s|q)). \tag{64}$$

Applying (62) in (63) gives

$$\eta_1^2 = \frac{4((2]_e^\gamma)^2 e_1^2 v_2^2}{(B_1(s|q))^2}. \tag{65}$$

Putting (65) into (64) and with some calculations, we have

$$|v_2|^2 = \left| \frac{(B_1(s|q))^3 (q_2 + \eta_2)}{2[2(e_1 + e_2)(3]_e^\gamma + (2]_e^\gamma - ((2]_e^\gamma)^2 (1 + 2e_1))(B_1(s|q))^2 + 4((2]_e^\gamma)^2 e_1^2 (B_1(s|q) - B_2(s|q))]} \right|.$$

Applying triangular inequality and Lemma 1, we have

$$|v_2| \leq \sqrt{X_1(s, q, \gamma)}. \tag{66}$$

Subtracting (60) from (57) and with some calculations, we have

$$v_3 = v_2^2 + \frac{B_1(s|q)[q_2 - \eta_2]}{4(e_1 + e_2)(3]_e^\gamma} \tag{67}$$

$$v_3 = \frac{(B_1(s|q))^2 q_1^2}{4e_1^2((2)_e^\gamma)^2} + \frac{B_1(s|q)[q_2 - \eta_2]}{4(e_1 + e_2)(3)_e^\gamma}. \tag{68}$$

Applying triangular inequality and Lemma 1, we have

$$|v_3| \leq \frac{s^2}{e_1^2((2)_e^\gamma)^2} + \frac{s}{(e_1 + e_2)(3)_e^\gamma}. \tag{69}$$

Subtracting (61) from (58), we have

$$2(e_1 + e_2 + e_3)(4)_e^\gamma v_4 = \frac{(3)_e^\gamma(4e_1 + 2e_2)(B_1(s|q))^3 q_1^3}{8e_1^2((2)_e^\gamma)^2} - \frac{(B_1(s|q))^3 q_1^3}{4e_1^2} - \frac{5(4)_e^\gamma(e_1 + e_2 + e_3)(B_1(s|q))^3 q_1^3}{4((2)_e^\gamma)^3 e_1^3} \tag{70}$$

$$- \frac{5(4)_e^\gamma(e_1 + e_2 + e_3)(B_1(s|q))^2 q_1(q_2 - \eta_2)}{8(2)_e^\gamma(3)_e^\gamma e_1(e_1 + e_2)} + \frac{B_1(s|q)(q_3 - \eta_3)}{2} + \frac{[B_2(s|q) - B_1(s|q)]q_1(q_2 + \eta_2)}{2} + \frac{(B_1(s|q) - 2B_2(s|q) + B_3(s|q))q_1^3}{4}. \tag{71}$$

Applying triangular inequality and Lemma 1, we have

$$|v_4| \leq \frac{s^3((2)_e^\gamma(3)_e^\gamma(4e_1 + 2e_2) - 2((2)_e^\gamma)^3 e_1 - 10(e_1 + e_2 + e_3)(4)_e^\gamma)}{2(e_1 + e_2 + e_3)((2)_e^\gamma)^3(4)_e^\gamma e_1^3} - \frac{5s^2}{2(2)_e^\gamma(3)_e^\gamma e_1(e_1 + e_2)} + \frac{s^3 - 2s - 2qs - 4}{(e_1 + e_2 + e_3)(4)_e^\gamma}.$$

□

4. Fekete–Szegő Inequalities for the Function Class $\Gamma_\Sigma^q(s, \gamma, \tau)$

Theorem 3. Let $f \in \Gamma_\Sigma^q(s, \gamma, \tau)$. Then, for some $\varphi \in \mathbb{R}$,

$$|v_3 - \varphi v_2^2| \leq \begin{cases} 2|1 - \varphi|\Psi_1(s, q, \gamma) & \left(|1 - \varphi| \geq \frac{s}{[2]_b[3]_b(3)_e^\gamma \Psi_1(s, q, \gamma)} \right) \\ \frac{2s}{[2]_b[3]_b(3)_e^\gamma} & \left(|1 - \varphi| \leq \frac{s}{[2]_b[3]_b(3)_e^\gamma \Psi_1(s, q, \gamma)} \right), \end{cases}$$

where

$$\Psi_1(s, q, \gamma) = \frac{s^3}{|s^2([2]_e[3]_e(3)_e^\gamma - [2]_e^2((2)_e^\gamma)^2) - [2]_e^2((2)_e^\gamma)^2(s^2 - s - 1)|}. \tag{72}$$

Proof. From (42), we have

$$v_3 - \varphi v_2^2 = v_2^2 + \frac{B_1(s|q)[q_2 - \eta_2]}{4[2]_e[3]_e(3)_e^\gamma} - \varphi v_2^2.$$

By triangular inequality, we have

$$|v_3 - \varphi v_2^2| \leq \frac{s}{[2]_e[3]_e(3)_e^\gamma} + |1 - \varphi|\Psi_1(s, q, \gamma). \tag{73}$$

Suppose

$$|1 - \varphi|\Psi_1(s, q, \gamma) \geq \frac{s}{[2]_e[3]_e(3)_e^\gamma}$$

then we have

$$|v_3 - \varphi v_2^2| \leq 2|1 - \varphi| \Psi_1(s, q, \gamma) \tag{74}$$

where

$$|1 - \varphi| \geq \frac{s}{[2]_e [3]_e (3)_e^\gamma \Psi_1(s, q, \gamma)}$$

and suppose

$$|1 - \varphi| \Psi_1(s, q, \gamma) \leq \frac{s}{[2]_e [3]_e (3)_e^\gamma}$$

then we have

$$|v_3 - \delta v_2^2| \leq \frac{2s}{[2]_e [3]_e (3)_e^\gamma}$$

where

$$|1 - \varphi| \leq \frac{s}{[2]_e [3]_e (3)_e^\gamma \Psi_1(s, q, \gamma)}$$

and $\Psi_1(s, q, \gamma)$ is given in (72). \square

5. Fekete–Szegő Inequalities for the Function Class $\Pi_\Sigma^q(s, \gamma, \tau)$

Theorem 4. Let $f \in \Pi_\Sigma^q(s, \gamma, \tau)$. Then, for some $\varphi \in \mathbb{R}$,

$$|v_3 - \varphi v_2^2| \leq \begin{cases} 2|1 - \varphi| X_1(s, q, \gamma) & \left(|1 - \varphi| \geq \frac{s}{(e_1 + e_2)(3)_e^\gamma X_1(s, q, \gamma)}\right) \\ \frac{2s}{(e_1 + e_2)(3)_e^\gamma} & \left(|1 - \varphi| \leq \frac{s}{(e_1 + e_2)(3)_e^\gamma X_1(s, q, \gamma)}\right), \end{cases}$$

where

$$X_1(s, q, \gamma) = \frac{2s^3}{|s^2\{2(e_1 + e_2)(3)_e^\gamma + (2)_e^\gamma - ((2)_e^\gamma)^2(1 + 2e_1)\} - 2((2)_e^\gamma)^2 e_1^2 (s^2 - s - 1)|}. \tag{75}$$

Proof. From (67), we have

$$v_3 - \varphi v_2^2 = v_2^2 + \frac{B_1(s|q)[\varrho_2 - \eta_2]}{4(e_1 + e_2)(3)_e^\gamma} - \varphi v_2^2.$$

By triangular inequality, we have

$$|v_3 - \varphi v_2^2| \leq \frac{s}{(e_1 + e_2)(3)_e^\gamma} + |1 - \varphi| X_1(s, q, \gamma). \tag{76}$$

Suppose

$$|1 - \varphi| X_1(s, q, \gamma) \geq \frac{s}{(e_1 + e_2)(3)_e^\gamma}$$

then we have

$$|v_3 - \varphi v_2^2| \leq 2|1 - \varphi| X_1(s, q, \gamma) \tag{77}$$

where

$$|1 - \varphi| \geq \frac{s}{(e_1 + e_2)(3)_e^\gamma X_1(s, q, \gamma)}$$

and suppose

$$|1 - \varphi| \Psi_1(s, q, \gamma) \leq \frac{s}{(e_1 + e_2)(3)_e^\gamma}$$

then we have

$$|v_3 - \delta v_2^2| \leq \frac{2s}{(e_1 + e_2)(3)_e^\gamma}$$

where

$$|1 - \varphi| \leq \frac{s}{(e_1 + e_2)(3]^\gamma X_1(s, q, \gamma)}$$

and $X_1(s, q, \gamma)$ is given in (75). \square

6. Conclusions

As we mentioned earlier, q -calculus is a vital tool for understanding a large class of analytic functions and its applications. Several useful results related to the q -version of the starlike function and the q -derivative, bi-univalent functions, for instance, were provided in [26–31]. In recent decades, the orthogonal polynomials and special functions have played an essential role in mathematics, physics, engineering, and other research disciplines. In our current analysis, we used q -Hermite polynomials and q -convolution operators and systematically defined two new subclasses of bi-univalent functions, which was primarily prompted by the recent research cited in this paper. We then obtained several significant findings, such as bounds for the initial coefficients of v_2 , v_3 , and v_4 of the Taylor–Maclaurin series and the Fekete–Szegő functional results for our established function classes.

Moreover, to have more new theorems under the present examinations, new generalizations and applications can be explored with some positive and novel outcomes in various fields of science, mainly in geometric function theory. These recent surveys will be presented in the future research work being processed by the authors of the present paper.

However, the purported trivial (p, q) -calculus extension was clearly demonstrated to be a relatively insignificant variation of the classical q -calculus, the extra parameter p being redundant or superfluous (see, for details, [17], p. 340, and [32], pp. 1511–1512). This observation by Srivastava (see [17,32]) will indeed also apply to any future attempt to produce the rather straightforward (p, q) -variants of the results which we have presented in this paper.

Author Contributions: Conceptualization, I.A.-S.; Formal analysis, A.C.; Investigation, I.A.-S.; Resources, H.M.S. and N.A.; Data curation, H.M.S.; Writing—original draft, A.C.; Writing—review & editing, N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: The last author thanks the Deanship of Scientific Research at Qassim University for supporting her work.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Lewin, M. On a coefficient problem for bi-univalent functions. *Proc. Am. Math. Soc.* **1967**, *18*, 63–68 [[CrossRef](#)]
- Brannan, D.A.; Clunie, J.G. Aspect of Contemporary Complex Analysis. In Proceedings of the NATO Advanced Study Institute Held at the University of Durham, Durham, UK, 1–20 July 1979; Academic Press: New York, NY, USA; London, UK, 1980.
- Netanyahu, E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Arch. Ration. Mech. Anal.* **1969**, *32*, 100–112.
- Brannan, D.A.; Taha, T. On some classes of bi-univalent functions. *Babes-Bolyai Math.* **1986**, *31*, 70–77.
- Srivastava, H.M.; Mishra, A.K.; Gochhayat, P. Certain subclasses of analytic and bi-univalent functions. *Appl. Math. Lett.* **2010**, *23*, 1188–1192. [[CrossRef](#)]
- Babalola, K.O. New subclasses of analytic and univalent functions involving certain convolution operator. *Math. Tome* **2008**, *50*, 3–12.
- Jackson, F.H. On q -definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
- Jackson, F.H. On q -definite integrals on q -functions and a certain difference operator. *Trans. R. Soc. Edinb.* **1908**, *46*, 253–281. [[CrossRef](#)]
- Khan, B.; Liu, Z.G.; Srivastava, H.M.; Khan, N.; Darus, M.; Tahir, M. A study of some families of multivalent q -starlike functions involving higher-order q -derivatives. *Mathematics* **2020**, *8*, 1490 [[CrossRef](#)]
- Khan, B.; Liu, Z.G.; Srivastava, H.M.; Khan, N.; Tahir, M. Applications of higher-order derivatives to subclasses of multivalent q -starlike functions. *Maejo Int. J. Sci. Technol.* **2021**, *15*, 61–72.
- Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; John Wiley & Sons: New York, NY, USA, 1989.

12. Khan, B.; Liu, Z.G.; Srivastava, H.M.; Araci, S.; Khan, N.; Ahmad, Z. Higher-order q -derivatives and their applications to subclasses of multivalent Janowski type q -starlike functions. *Adv. Diff. Equ.* **2021**, *440*, 1–15. [[CrossRef](#)]
13. Hu, Q.-X.; Srivastava, H.M.; Ahmad, B.; Khan, N.; Khan, M.G.; Mashwani, W.K.; Khan, B. A subclass of multivalent Janowski type q -starlike functions and its consequences. *Symmetry* **2021**, *13*, 1275. [[CrossRef](#)]
14. Shi, L.; Ahmad, B.; Khan, N.; Khan, M.G.; Araci, S.; Mashwani, W.K.; Khan, B. Coefficient estimates for a subclass of meromorphic multivalent q -close-to-convex functions. *Symmetry* **2021**, *13*, 1840. [[CrossRef](#)]
15. Shi, L.; Khan, M.G.; Ahmad, B. Some geometric properties of a family of analytic functions involving a generalized q -operator. *Symmetry* **2020**, *12*, 291. [[CrossRef](#)]
16. Islam, S.; Khan, M.G.; Ahmad, B.; Arif, M.; Chinram, R. q -extension of starlike functions subordinated with a trigonometric sine function. *Mathematics* **2020**, *8*, 1676. [[CrossRef](#)]
17. Srivastava, H.M. Operators of basic (or q -) calculus and fractional q -calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. A Sci.* **2020**, *44*, 327–344. [[CrossRef](#)]
18. Ismail, E.H.; Stanton, D.; Viennot, G. The combinatorics of q -Hermite polynomial and the Askey-Wilson Integral. *Eur. J. Combinatorics* **1987**, *8*, 379–392. [[CrossRef](#)]
19. Chavda, N.D. Average-fluctuation separation in energy levels in quantum many-particle systems with k -body interactions using q -Hermite polynomials. *arXiv* **2021**, arXiv:2111.12087v1.
20. Rao, P.; Vyas, M.; Chavda, N.D. Eigenstate structure in many-body bosonic system: Analysis using random matrices and q -Hermite polynomial. *arXiv* **1933**, arXiv:2111.08820v1.
21. Ruscheweyh, S.T. New criteria for univalent functions. *Proc. Am. Math. Soc.* **1975**, *49*, 109–115. [[CrossRef](#)]
22. Fekete, M.; Szego, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [[CrossRef](#)]
23. Magesh, N.; Yamini, J. Fekete-Szego problem and second Hankel determinant for a class of bi-univalent functions. *arXiv* **2015**, arXiv:1508.07462v2
24. Tang, H.; Srivastava, H.M.; Sivasubramanian, S.; Gurusamy, P. The Fekete-Szego functional problems for some classes of m -fold symmetric bi-univalent functions. *J. Math. Inequal.* **2016**, *10*, 1063–1092. [[CrossRef](#)]
25. Duren, P.L. *Univalent Functions, Grundlehrender Mathematischer Wissenschaftler*; Springer: New York, NY, USA, 1983; Volume 259.
26. Al-Shbeil, I.; Shaba, T.G.; Catas, A. Second Hankel Determinant for the Subclass of Bi-Univalent Functions Using q -Chebyshev Polynomial and Hohlov Operator. *Fractal Fract.* **2022**, *6*, 186. [[CrossRef](#)]
27. Saliu, A.; Al-Shbeil, I.; Gong, J.; Malik, S.N.; Aloraini, N. Properties of q -Symmetric Starlike Functions of Janowski Type. *Symmetry* **2022**, *14*, 1907. [[CrossRef](#)]
28. Al-Shbeil, I.; Wanas, A.K.; Saliu, A.; Catas, A. Applications of Beta Negative Binomial Distribution and Laguerre Polynomials on Ozaki Bi-Close-to-Convex Functions. *Axioms* **2022**, *11*, 451. [[CrossRef](#)]
29. Khan, M.F.; Al-Shbeil, I.; Aloraini, N.; Khan, N.; Khan, S. Applications of Symmetric Quantum Calculus to the Class of Harmonic Functions. *Symmetry* **2022**, *14*, 2188. [[CrossRef](#)]
30. A Saliu, A.; Jabeen, K.; Al-shbeil, I.; Oladejo, S.O.; Cătaș, A. Radius and Differential Subordination Results for Starlikeness Associated with Limaçon Class. *J. Funct. Spaces* **2022**, *2022*, 8264693. [[CrossRef](#)]
31. Ur Rehman, M.S.; Ahmad, Q.Z.; Al-Shbeil, I.; Ahmad, S.; Khan, A.; Khan, B.; Gong, J. Coefficient Inequalities for Multivalent Janowski Type q -Starlike Functions Involving Certain Conic Domains. *Axioms* **2022**, *11*, 494. [[CrossRef](#)]
32. Srivastava, H.E. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.