ON THE VALIDITY OF MÜCKET-TREDER GRAVITATIONAL LAW

by

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DMS-621-IR November 1992

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[1] Research supported in part by NSERC Grant 3-48376

Abstract

We consider the Mücket-Treder post-Newtonian gravitational law with logharithmic correction term. This law explains, with a very good approximation, the perihelion advance of the inner planets. The 3-body problem has the potential function $W(\mathbf{q}) = \sum_{1\leq i<j\leq 3} m_i m_j (1+\alpha+\alpha \ln q_{ij}) q_{ij}^{-1}$, where q_{ij} are the mutual distances between particles i and j, $m_i, i = 1,2,3$, denote the masses and $|\alpha| \ll 1$. We show that for $\alpha > 0$, singularities (in particular collisions) can not occur, therefore this law doesn't make much sense under these circumstances. We further study the case $\alpha < 0$. Since distorsions do not usually arise far from singularities, it is important to study the qualitative behavior of orbits coming close to collisions. We use the technique of McGehee transformations and study the flow on the collision manifold. It is surprising to see that the rest points of the flow depend only on the Newtonian component of the force and not on α. This shows that Mücket-Treder law is very close to the classical Newtonian one in the neighborhood of collisions and asymptotic to it at the collision moment. Consequently, the use of this law, for $-10^{-8} < \alpha < 0$, is theoretically justified.

1 Introduction

The discrepancy between Mercury's observed perihelion advance and the theoretical value obtained in the Newtonian model, led astronomers during pre-relativistic years to attack this problem in two different ways. One is to consider the influence of perturbation forces, the other one is to change the attraction law. There were several attempts in both directions in the last century, none fully succesful. For example, Bertrand (1922) proposed in 1873 the use of the law of the inverse α-power of the distance with $\alpha > 2$. Both Hall (1895) and Newcomb (1895) found independently very close values for α. Hall computed $\alpha = 2.00000016$ by making the theory fit the observed perihelion advance of Mercury, while Newcomb obtained $\alpha = 2.0000001574$, taking into account the secular variations of the inner planets. However, Brown (1903) showed that the motion of the Moon's perigee is in accord to the Newtonian model and that necessarily $|\alpha - 2| < 4{\cdot}10^{-8}$, in order to make both, observation and theory, match together. As a consequence, Bertrand's proposal was considered an unsuccessful attempt. For a discussion of other gravitation laws one can consult Hagihara (1972).

Since relativistic mechanics encounters conceptual difficulties in tackling collision orbits of the n-body problem and leads, in general, to lengthy and cumbersome computations, there still exist post-relativistic attempts to improve Newton's model. Mücket and Treder (1977) proposed a logarithmic correction term to the Newtonian potential, trying to conciliate the two pre-relativistic directions of work mentioned at the beginning. Their reasoning was as follows. If δ is the distance between two particles, by keeping only the first two terms in the MacLaurin expansion $\delta^\alpha = 1 + \alpha \ln \delta + (\alpha \ln \delta)^2/2! + (\alpha \ln \delta)^3/3! + \ldots$, a force proportional to $\delta^{-2}(1 + \alpha \ln \delta)$ is suggested. If one further considers $|\alpha| < 10^{-8}$, then both, the perihelion and perigee motions of the inner planets and the Moon respectively, can be theoretically explained (see Mücket and Treder (1977)). However, this could very well be a coincidence and other phenomena may seem impossible to prove with this attraction law. Mioc and Blaga (1991) studied therefore orbital motion finding some advantages in using Mücket-Treder's potential. Ballinger and Diacu (1992) performed a qualitative study of the Kepler problem, justifying theoretically the use of this model for the case of two bodies, if $\alpha < 0$

The goal of this paper is to see how far the Mücket-Treder law can be used in the 3-body problem. Having a parametrized system of differential equations, small changes of the parameter can lead to drastic changes in the global behavior of solutions. Distortions usually start in the neighborhood of singular solutions, therefore these points should be first checked.

At the beginning we consider the case $\alpha > 0$ and see that all solutions are globally defined. In other words, singularities (in particular, collisions) can not appear in this case. Such a situation doesn't correspond to the astronomical reality, so it has to be rejected.

We therefore restrict our attention to the case $\alpha < 0$ and study orbits passing close to a total collapse. This restriction is not sharp and any particular colliding cluster in the n-body problem can be tackled with the method we use (see Diacu (1992a)). Anyway, the computations are easier to present and comprehend in the triple collision case of the 3-body problem. The technique of McGehee (1974) transformations is applied, that will allow to "blow-up" the collision singularity and to "paste" instead a, so called, *collision manifold*. Due to the property of continuity of solutions with respect to initial data, a study of the flow on the fictitious collision manifold gives pretious information on solutions passing close to triple collision. Our computations show that the rest points on the collision manifold depend only on the Newtonian component of the potential and not on the logarithmic correction term or on the value of α. This will imply not only the lack of distortions, but the fact that Mücket-Treder's law stays very close to the Newtonian one in the neighborhood of collisions and tends asymptotically to it when the time variable approaches the collision instant. This is not at all clear from the original equations of motion.

As a conclusion, the model proposed by Mücket and Treder, with $-10^{-8} < \alpha < 0$, seems suitable to substitute a relativistic approach for solutions of few-body problems with close encounters. Its applicability is not necessarily restricted to the solar system.

2 Equations of motion

Consider 3 particles of masses $m_i > 0, i = 1,2,3$, in the Euclidean space $\Re^3$ having coordinates $\mathbf{q}_i = (q_i^1, q_i^2, q_i^3) \in \Re^3, i = 1,2,3$. Denote by

$$M = diag(m_1, m_1, m_1, m_2, m_2, m_2, m_3, m_3, m_3)$$

a square matrix having the above elements on the diagonal and 0 in rest. The 9-dimensional vector $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ represents the *configuration* of the system and will be regarded as a column vector. The *momentum* of the system is thus defined as $\mathbf{p} = M\dot{\mathbf{q}}$. The equations of motion of the 3-body problem in the Mücket-Treder model will be given by the second order 9-dimensional system of ordinary differential equations

$$M\ddot{\mathbf{q}} = \nabla W(\mathbf{q}), \tag{1}$$

where ∇ is the gradient, $W(\mathbf{q}) = U(\mathbf{q}) + V(\mathbf{q})$,

$$U : \Re^9 \setminus \Delta \rightarrow \Re_+, \quad U(\mathbf{q}) = \sum_{1\le i<j\le 3} m_i m_j q_{ij}^{-1},$$

is the Newtonian potential, $\Delta = \bigcup_{1\le i,j\le 3}\{\mathbf{q} \,|\, \mathbf{q}_i = \mathbf{q}_j\}$ being the *collision set*, and

$$V : \Re^9 \setminus \Delta \rightarrow \Re_+, \quad V(\mathbf{q}) = \sum_{1\le i<j\le 3} \alpha m_i m_j (\ln q_{ij}) q_{ij}^{-1},$$

denotes the logarithmic correction term, α being a real constant and $q_{ij} = |q_i - q_j|$ denoting the Euclidian distance between particles i and j. The case $\alpha = 0$ corresponds to the classical Newtonian one. Remark that since only theoretical investigations are performed, we have considered the gravitational constant to be 1.

The equations (1) can be written as an 18-dimensional first order system

$$\begin{cases} \dot{\mathbf{q}} = M^{-1}\mathbf{p} \\ \dot{\mathbf{p}} = \nabla W(\mathbf{q}) \,. \end{cases} \tag{2}$$

The ten calssical *first integrals* of the Newtonian case remain true for the Mücket-Treder model (with obvious changes for the energy integral, only). Thus, from the *momentum* and *center of mass integrals*, one can draw the conclusion that the sets

$$Q = \{\mathbf{q} \,|\, \sum_{i=1}^{3} m_i \mathbf{q}_i = \mathbf{0}\}, \; P = \{\mathbf{p} \,|\, \sum_{i=1}^{3} \mathbf{p}_i = \mathbf{0}\}$$

are invariant for the equations (2) (i.e. if initial data are in the set then the whole orbit is in the set). By restricting equations (2) to the invariant set $Q \times P$ means, from the physical point of view, that one can translate the origin of the absolute frame to the center of mass of the system of particles, without to change the form of the equations of motion. The *integral of energy* will be

$$T(\mathbf{p}(t)) - W(\mathbf{q}(t)) = h,$$

where $T : \Re^9 \to [0, \infty)$, $T(\mathbf{p}) = \frac{1}{2}\sum_{i=1}^{3} m_i^{-1}|\mathbf{p}_i|^2$ is the kinetic energy and h is the *energy constant*.

Using standard results of differential equations theory one can show that for any initial conditions $(\mathbf{q}, \mathbf{p})(0) = (\mathbf{q}_0, \mathbf{p}_0) \in (\Re^9 \setminus \Delta) \times \Re^9$, there exists an analytic solution $(\mathbf{q},\mathbf{p})$ of the corresponding initial value problem, defined on some interval (t^-, t^+) containing 0. Without loss of generality we may study only one of the intervals $(t^-, 0]$ or $[0, t^+)$. Let's choose the second one. The solution can be further extended to a maximal interval $[0, t^*)$, where $0 < t^* \leq t^+ \leq \infty$. Recall that a function is called analytic at some point if there is an interval around that point where the function can be written as a Taylor series. There are several equivalent ways to define solutions with singularities. We adopt here the following one.

Definition 1 *A solution* $(\mathbf{q}, \mathbf{p})$ *of the equations (2) with initial conditions* $(\mathbf{q}_0, \mathbf{p}_0)$, *defined on the maximal interval* $[0, t^*)$ *is called regular if it is analytic and* $t^* = \infty$. *If the solution is analytic on* $[0, t^*)$ *but fails to be analytic on* $[0, t^*]$, *then we say that it has a singularity at* t^*.

Notice that, in particular, a singularity can be due to a collision between at least two particles. In the classical n-body problem there may appear noncollision singularities which correspond physically to a motion becoming unbounded in finite time. In the next section we will show that for $\alpha > 0$ this is not the case. In other words all solutions are regular.

3 The case $\alpha > 0$

In this section we will prove the following result

Theorem 1 *Any solution* $(\mathbf{q}, \mathbf{p})$ *of the equations (2) with* $\alpha > 0$, *having initial conditions* $(\mathbf{q}_0, \mathbf{p}_0) \in (\Re^9 \setminus \Delta) \times \Re^9$ *is defined on* $(-\infty, +\infty)$.

In other words the equations (3) with $\alpha > 0$ do not have solutions with singularities. In particular, collisions between particles can not occur.

Proof of Theorem 1. The idea of the proof is very simple. Supposing a solution has a singularity, we will show that the energy relation is not satisfied. For this, take a solution $(\mathbf{q}, \mathbf{p})$ of the equations (2), defined on some interval $[0, t^*)$ with $t^* < \infty$. Let's show first that

$$\lim_{t\to t^*} \min_{i<j} q_{ij}(t) = 0. \tag{3}$$

(This is analog to a result due to Painlevé in the Newtonian case.) For proving it, suppose relation (3) is not true, i.e. $\limsup_{t\to t^*} q_{ij}(t) \geq \gamma > 0$, where γ is a constant. Then, there will exist a sequence $(t_\nu)_{\nu\in\mathrm{N}}, t_\nu \to t^*$, and a constant $\beta > 0$, such that $W(\mathbf{q}(t_\nu)) \leq \beta$, for all $\nu \in \mathbf{N}$. By the integral of energy it follows that $T(\mathbf{p}(t_\nu)) \leq \beta + h$ for all $\nu \in \mathbf{N}$. Therefore there is a constant $\mu > 0$ such that $|\mathbf{p}(t_\nu)| \leq \mu$, for all $\nu \in \mathbf{N}$. Applying now the standard existence theorem to the equations (2) for some initial data $t_\nu, \mathbf{q}(t_\nu), \mathbf{p}(t_\nu)$, with t_ν sufficiently close to t^*, it follows that the solution is analytic in some interval $(t_\nu - a, t_\nu + a), a > 0$, that contains t^*. This contradicts the fact that t^* is a singularity, so relation (3) follows.

By (3) we can say there are indices i, j and a sequence $(t_m)_{m\in\mathrm{N}}, t_m \to t^*$, such that $q_{ij}(t_m) \to 0$ when $m \to \infty$. Notice that

$$\limsup_{t\to t^*}[-W(\mathbf{q})] \geq \lim_{\nu\to\infty}[-m_i m_j(1 + \alpha + \alpha \ln q_{ij})q_{ij}^{-1}] = \infty.$$

This means that along the sequence $(t_m)_{m\in\mathrm{N}}$ the left side of the energy relation becomes unbounded while the right side is constant, a contradiction. Thus the theorem is proved.

This result shows that, from astronomical point of view, Mücket-Treder model doesn't make much sense for $\alpha > 0$. Large scale collisions occur in the Univers, so no positive value of α corresponds to reality. We will further see that this is not at all the case for negative values of α.

4 The case $\alpha < 0$

Since the case $\alpha > 0$ doesn't lead anywhere, we will restrict our attention to negative values of α. The original McGehee (1974) transformations will be adapted to Mücket-Treder's gravitation law. This will allow to "blow-up" the singularity arising from the total collapse of all particles and paste instead a *collision manifold*. Orbits on the collision manifold are fictitious but their study will give us pretious information on the behavior of solutions coming close to a triple collapse.

Unless otherwise stated, all transformations we consider are analytic functions, bijective and have the inverse analytic. This will allow us to obtain equivalent equations of motion. Consequently, performing this kind of transformations, we can use new convenient equations instead of the original ones. Our goal will be to eliminate from the equations of motion, the singularity arising from the total collapse. Thus define

$$\begin{cases} r = (\mathbf{q}^T M \mathbf{q})^{1/2} \\ \mathbf{s} = r^{-1}\mathbf{q} \\ y = \mathbf{p}^T \mathbf{s} \\ \mathbf{x} = \mathbf{p} - yM\mathbf{s} \ . \end{cases} \tag{4}$$

These are the classical McGehee (1974) transformations, where ^{T} means the transpose of the column vector $\mathbf{q}$, i.e. the line vector $\mathbf{q}$. Notice that r, y are scalars while s, x are vectors. (As a rule, we denote scalars by plane letters and vectors with bold ones.) Also observe that

$$\mathbf{s}^T M\mathbf{s} = 1, \quad \mathbf{s}^T\mathbf{x} = 0$$

$$U(\mathbf{q}) = r^{-1}U(\mathbf{s}), \quad \nabla U(\mathbf{q}) = r^{-2}\nabla U(\mathbf{s})$$

$$V(\mathbf{q}) = r^{-1}V(\mathbf{s}) + \alpha r^{-1}(\ln r)U(\mathbf{s})$$

$$\nabla V(\mathbf{q}) = r^{-2}\nabla V(\mathbf{s}) + \alpha r^{-2}(\ln r)\nabla U(\mathbf{s}).$$

Since U is a homogeneous function of degree -1, Euler's relation yields

$$\mathbf{q}^T\nabla U(\mathbf{q}) = -U(\mathbf{q}).$$

Using the change of variables (4) and the above relations, equations (2) become

$$\begin{cases} \dot{r} = y \\ \dot{y} = r^{-1}\mathbf{x}^T M^{-1}\mathbf{x} + r^{-2}\mathbf{s}^T\nabla V(\mathbf{s}) - (1 + \alpha \ln r)r^{-2}U(\mathbf{s}) \\ \dot{\mathbf{s}} = r^{-1}M^{-1}\mathbf{x} \\ \dot{\mathbf{x}} = -r^{-1}y\mathbf{x} - r^{-1}(\mathbf{x}^T M^{-1}\mathbf{x})M\mathbf{s} + (1 + \alpha \ln r)r^{-2}[U(\mathbf{s})M\mathbf{s} + \nabla U(\mathbf{s})] + \\ \qquad r^{-2}\nabla V(\mathbf{s}) - r^{-2}[\mathbf{s}^T\nabla V(\mathbf{s})]M\mathbf{s} \ . \end{cases} \tag{5}$$

Notice that r is a measure of the distribution of particles in space. For r small, the bodies are close together. Since we are interested in the behavior of solutions coming close to a total collapse, we may assume that r is positive and small. This will allow us to consider further the transformations

$$\begin{cases} v = r^{1/2}(-\ln r)^{-1/2}y \\ \mathbf{u} = r^{1/2}(-\ln r)^{-1/2}\mathbf{x} \ . \end{cases} \tag{6}$$

This new change of variables will transform equations (5) into the following system of differential equations

$$\begin{cases} \dot{r} = r^{-1/2}(-\ln r)^{1/2} v \\ \dot{v} = r^{-3/2}(-\ln r)^{1/2}[\frac{v^2}{2}(1 - \frac{1}{\ln r}) + \mathbf{u}^T M^{-1}\mathbf{u} - \frac{\mathrm{s}^T \nabla V(\mathrm{s})}{\ln r} + (\alpha + \frac{1}{\ln r})U(\mathrm{s})] \\ \dot{\mathrm{s}} = r^{-3/2}(-\ln r)^{1/2} M^{-1}\mathbf{u} \\ \dot{\mathbf{u}} = -r^{-3/2}(-\ln r)^{1/2}\{(1 + \frac{1}{\ln r})\mathbf{u}v/2 + (\mathbf{u}^T M^{-1}\mathbf{u}M)\mathrm{s} + (\alpha + \frac{1}{\ln r})[U(\mathrm{s})M\mathrm{s} + \\ \qquad \nabla U(\mathrm{s})] + \frac{1}{\ln r}[\nabla V(\mathrm{s}) + (\mathrm{s}^T \nabla V(\mathrm{s}))M\mathrm{s}]\} \ . \end{cases} \tag{7}$$

In order to eliminate the cumbersome negative powers of r and $-\ln r$ at the beginning of each equation in system (7) we will consider the time transformation

$$d\tau = r^{-3/2}(-\ln r)^{1/2} dt. \tag{8}$$

This will introduce the fictitious time variable τ. We continue to denote (by abuse) the variables with the same letters and $'$ will mean derivation with respect to τ. Thus equations (7) become

$$\begin{cases} r' = rv \\ v' = (1 - \frac{1}{\ln r})\frac{v^2}{2} + \mathbf{u}^T M^{-1}\mathbf{u} - \frac{\mathrm{s}^T \nabla V(\mathrm{s})}{\ln r} + (\alpha + \frac{1}{\ln r})U(\mathrm{s}) \\ \mathrm{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = -[(1 + \frac{1}{\ln r})\mathbf{u}v/2 + (\mathbf{u}^T M^{-1}\mathbf{u})M\mathrm{s} + (\alpha + \frac{1}{\ln r})[U(\mathrm{s})M\mathrm{s} + \nabla U(\mathrm{s})] + \\ \qquad \frac{1}{\ln r}[\nabla V(\mathrm{s}) + (\mathrm{s}^T \nabla V(\mathrm{s}))M\mathrm{s}] \ . \end{cases} \tag{9}$$

Using all the above transformations, the energy relation becomes

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - \frac{U(\mathrm{s}) + V(\mathrm{s})}{-\ln r} + \alpha U(\mathrm{s}) = \frac{hr}{-\ln r}.$$

Let's finally introduce the transformation

$$\rho = \frac{1}{-\ln r}. \tag{10}$$

System (9) becomes

$$\begin{cases} \rho' = v\rho^2 \\ v' = (1 + \rho)\frac{v^2}{2} + \mathbf{u}^T M^{-1}\mathbf{u} + \rho \mathrm{s}^T \nabla V(\mathrm{s}) + (\alpha - \rho)U(\mathrm{s}) \\ \mathrm{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = (\rho - 1)\mathbf{u}v/2 - (\mathbf{u}^T M^{-1}\mathbf{u})M\mathrm{s} + (\alpha - \rho)[U(\mathrm{s})M\mathrm{s} + \nabla U(\mathrm{s})] + \\ \qquad \rho[\nabla V(\mathrm{s}) + (\mathrm{s}^T \nabla V(\mathrm{s}))M\mathrm{s}] \ . \end{cases} \tag{11}$$

and the energy relation changes to

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - \rho[U(\mathbf{s}) + V(\mathbf{s})] + \alpha U(\mathbf{s}) = h\rho e^{-\frac{1}{\rho}}. \tag{12}$$

Obviously, since a total collapse takes place for $r = 0$, in the new equations this happens for $\rho = 0$. Equations (11) extend naturally to $\rho = 0$, unfortunately this is not immediate for the energy relation (12) because of the term $e^{-\frac{1}{\rho}}$. Let's therefore write relation (12) as

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) - \rho[U(\mathbf{s}) + V(\mathbf{s})] + \alpha U(\mathbf{s}) = hg(\rho), \tag{13}$$

where

$$g(\rho) = \begin{cases} \rho e^{-\frac{1}{\rho}}, & if\ \rho \neq 0 \\ 0, & if\ \rho = 0\ . \end{cases}$$

The function g is analytic for $\rho \neq 0$ and infinitely many times differentiable at 0. Also notice that this is the only way to extend g to $\rho = 0$ and to maintain the infinite differentiability. This makes relation (12) also to extend to the value $\rho = 0$. Since the set $\{(\rho, y, \mathbf{s}, \mathbf{u}) \mid \rho = 0\}$ is invariant for the equations (11), we have pasted a manifold to the phase space of system (11). Notice that on this manifold, relation (13) becomes

$$\frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) + \alpha U(\mathbf{s}) = 0, \tag{14}$$

and that it doesn't depend on the value of h. It is also important to notice that while for the original equations of motion the total collision was reached at t^*, for the new ones this happens if $\tau = \infty$. The reason is that $\{(\rho, v, \mathbf{s}, \mathbf{u}) \mid \rho = 0\}$ is an invariant manifold for equations (11). The set

$$\mathbf{C} = \{(\rho, y, \mathbf{s}, \mathbf{u}) \mid \rho = 0,\ \frac{1}{2}(\mathbf{u}^T M^{-1}\mathbf{u} + v^2) + \alpha U(\mathbf{s}) = 0\}$$

is an infinitely differentiable manifold and will be called the *collision manifold* for the triple collapse. Equations (11) restricted to $\mathbf{C}$ take the form

$$\begin{cases} v' = \frac{v^2}{2} + \mathbf{u}^T M^{-1}\mathbf{u} + \alpha U(\mathbf{s}) \\ \mathbf{s}' = M^{-1}\mathbf{u} \\ \mathbf{u}' = -\mathbf{u}v/2 + (\mathbf{u}^T M^{-1}\mathbf{u})M\mathbf{s} + \alpha[U(\mathbf{s})M\mathbf{s} + \nabla U(\mathbf{s})]\ . \end{cases} \tag{15}$$

We have thus blown-up the singularity arising from the total collision and pasted to the phase space a fictitious collision manifold. The orbits of system (15) on $\mathbf{C}$, do not have any real meaning or correspondence in the physical space. However, due to the property of continuity of solutions with respect to initial data, a study of these orbits will offer us information on the physical behavior of orbits passing close to a triple collision. Obtaining such information is, in fact, our goal.

Any qualitative study of ordinary differential equations starts with the search for equilibrium solutions (i.e. orbits remaining constant in time). They are given by zeroes of the vector field. Let's solve therefore the equations arising by making the right parts of system (15) equal to zero. Looking to the first one and using the energy relation (14), one can see that this equation is equivalent to $\mathbf{u}^T M^{-1}\mathbf{u} = \mathbf{0}$. This obviously yields $\mathbf{u} = \mathbf{0}$. It's now easy to see that in order to fulfill these equations we need $v = \pm(-2\alpha U(\mathrm{s}))^{1/2}$ and $U(\mathrm{s})M\mathrm{s} + \nabla U(\mathrm{s}) = \mathbf{0}$. Consequently we have shown

Proposition 1 *Equations* (15) *have equilibrium orbits if and only if the following three conditions are fulfilled:*

(i) $\mathbf{u} = \mathbf{0}$

(ii) $v = \pm(-2\alpha U(\mathrm{s}))^{1/2}$

(iii) $U(\mathrm{s})M\mathrm{s} + \nabla U(\mathrm{s}) = \mathbf{0}$.

Before interpreting this result let's give the following

Definition 2 *A configuration* $\mathbf{q} \in \Re^9 \setminus \Delta$ *of the classical Newtonian 3-body problem is called central if there is some constant* σ *such that*

$$\nabla U(\mathbf{q}) = \sigma M\mathbf{q}. \qquad (16)$$

Central configurations play a very important role in the study of the n-body problem. It is known that for three bodies there exist five central configurations, three collinear (called Eulerian) and two equilateral (called Lagrangian). There are many important properties connected to central configurations and the literature on this topic is growing (see e.g. Diacu (1992b)). We are now interested in just one known asymptotic property, namely

Theorem 2 *Collision solutions of the Newtonian 3-body problem tend to form a central configuration in the neighborhood of collisions.*

A simple computation will show that using the above McGehee-type transformations, relation (16) takes, in the new variables, the form of relation (*iii*) in Proposition 1.

This means that equilibrium solutions on the collision manifold of Mücket-Treder's model are given by Newtonian central configurations.

Continuing the qualitative study of orbits on $\mathbf{C}$, one is looking for periodic orbits. It is easy to see that since (by (14) and (15)) $v' = \mathbf{u}^T M^{-1} \mathbf{u}$, it follows that v is an increasing function (except on equilibrium solutions). Thus we can conclude

Proposition 2 *There are no periodic orbits on the collision manifold.*

We will now prove the following result

Theorem 3 *Any triple collision solution of the three-body problem in the Mücket-Treder model, approaches one of the Newtonian central configurations, when the time variable tends to the collision instant.*

In order to prove this result we adapt the ideas of McGehee (1974) to our case. Denote by $\phi(\tau) = (\rho(\tau), v(\tau), \mathbf{s}(\tau), \mathbf{u}(\tau))$ an orbit of equations (11) and let $\phi_0(\tau)$ be an orbit having the initial conditions $\mathbf{w}_0 = (\rho_0, v_0, \mathbf{s}_0, \mathbf{u}_0)$. Recall that the ω-limit set $L_\omega(\mathbf{w}_0)$ of $\mathbf{w}_0$ is the set of all points $\mathbf{w} = (\rho, v, \mathbf{s}, \mathbf{u})$ such that there is a sequence $(\tau_n)_{n \in \mathrm{N}}, \tau_n \to \infty$, and $\lim_{n\to\infty} \phi_0(\tau_n) = \mathbf{w}$. In other words the ω-limit set of an orbit is the set of points where the orbit "dies". We will make use of the following result of McGehee (1974):

Lemma 1 *If $L_\omega(\mathbf{w}_0)$ is a nonempty compact set, then $L_\omega(\mathbf{w}_0)$ is a single point.*

Proof of Theorem 3. Consider $(\mathbf{q}, \mathbf{p})$ to be a triple collision solution of equations (2) and let the initial conditions be sufficiently close to the collision instant such that ρ is sufficiently small and the transformations (4), (6), (8) and (10) make sense. Thus solution $(\mathbf{q}, \mathbf{p})$ is transformed to a solution $\phi = (\rho, v, \mathbf{s}, \mathbf{u})$ of equations (11) and let $\phi_0(\tau)$ be the orbit going through the corresponding initial point $\mathbf{w}_0$. Using McGehee's lemma it is enough to show that $L_\omega(\mathbf{w}_0)$ is a nonempty compact set. Indeed, since there are exactly five rest points, ϕ_0 has to approach one of them, consequently the orbit will tend to one of the five Newtonian central configurations.

In order to see that $L_\omega(\mathbf{w}_0)$ is nonempty and compact, consider first the following notations:

$$S = S(h, \epsilon) = \{\phi \,|\, \rho \leq \epsilon \; and \; \frac{1}{2}(\mathbf{u}^T M^{-1} \mathbf{u} + v^2) - \rho[U(\mathbf{s}) + V(\mathbf{s})] + \alpha U(\mathbf{s}) = hg(\rho)\},$$

$\epsilon > 0$, constant,

$$\Gamma = \Gamma(h, \epsilon, \mu) = \{\phi \in S \,|\, |v| \leq \mu\}, \; \mu > 0,$$

$$\Gamma^+ = \Gamma^+(h, \epsilon, \mu) = \{\phi \in S \,|\, v \geq \mu\},$$

$$\Gamma^- = \Gamma^-(h, \epsilon, \mu) = \{\phi \in S \,|\, v \leq -\mu\},$$

$$\gamma^\pm = \gamma^\pm(h, \epsilon, \mu) = \{\phi \in \Gamma^\pm \,|\, \rho = \epsilon\},$$

$$\sigma^\pm = \sigma^\pm(h, \epsilon, \mu) = \{\phi \in S \,|\, v = \pm\mu\}.$$

If $[\tau_1, \tau_2]$, with $\tau_1 < \tau_2$, is a time interval in the new fictitious time variable, we call $\phi([\tau_1, \tau_2])$ an *orbit segment*. We say that the orbit segment $\phi([\tau_1, \tau_2])$ is *maximal* in a closed set K, if $\phi([\tau_1, \tau_2]) \subset K$ but $\phi(I)$ is not included in K, for any interval I containing $[\tau_1, \tau_2]$ but larger than $[\tau_1, \tau_2]$. The following two statements are then true:

(i) For ϵ and μ suitably chosen, if $\phi_0([\tau_1, \tau_2])$ is a maximal orbit segment in Γ^+, then $\phi_0(\tau_1) \in \sigma^+$ and $\phi_0(\tau_2) \in \gamma^+$.

(ii) For the same ϵ and μ in (1), if $\phi_0([\tau_1, \tau_2])$ is a maximal orbit segment in Γ^-, then $\phi_0(\tau_1) \in \gamma^-$ and $\phi_0(\tau_2) \in \sigma^-$.

The proof works similar for (i) and (ii), so we will do it only for (i). If $\mathbf{w}(\tau) \in \sigma^+$, then $v = \mu > 0$ and $\rho \leq \epsilon$. Using the energy relation (13) and the second equation in (11) we get

$$v' = (\frac{1}{4} + \frac{\rho}{2})v^2 + \frac{1}{2}\mathbf{u}^T M^{-1}\mathbf{u} + \rho(\mathbf{s}^T \nabla V(\mathbf{s}) + V(\mathbf{s}) + he^{-\frac{1}{\rho}}).$$

Thus for $\mathbf{w}(\tau) \in \sigma^+$ (i.e. for ϵ sufficiently small and a fixed positive μ), we have $v'(\tau) > 0$. From the first equation in (11) we also get $\rho'(\tau) > 0$ when $\mathbf{w}(\tau) \in \sigma^+$. These imply that points on σ^+ are entering Γ^+, so $\phi_0(\tau_2) \in \gamma^+$. Also, points on γ^+ are leaving Γ^+, so $\phi_0(\tau_1) \in \sigma^+$. Statement (i) is thus proved.

Finally, we would like to show that for the above choice of ϵ and μ, since $\rho(\tau) \to 0$ when $\tau \to \infty$, there is a τ_3 such that $\phi_0(\tau) \in \Gamma$ for all $\tau \geq \tau_3$. Suppose that this is not the case, i.e. there exists a $\tau_4 \geq \tau_3$ such that $\phi_0(\tau_4) \in \Gamma^+$. Then, since $\rho(\tau) < \epsilon$, it follows that ρ_0 can never reach γ^+. Thus, by (i) we obtain that $\phi_0(\tau) \in \Gamma^+$ for all $\tau \geq \tau_4$. But $v > 0$ in Γ^+, so $\rho' = v\rho^2 > 0$ for $\tau \geq \tau_4$. This contradicts the fact that $\rho(\tau) \to 0$, when $\tau \to \infty$ and consequently $\phi_0(\tau)$ can not be in Γ^+ for $\tau \geq \tau_3$. In the same way, by (ii), $\phi_0(\tau)$ can not be in Γ^- for $\tau \geq \tau_3$. Consequently $\phi_0(\tau) \in \Gamma$ for $\tau \geq \tau_3$.

Since Γ is compact and any ω-limit set is closed, it follows that $L_\omega(\mathbf{w}_0)$ is nonempty and compact. This completes the proof.

Theorem 3 implies that asymptotically, Mücket-Treder's law approaches the Newtonian one when the time variable tends to the triple collision instant. Consequently, as we have seen in our introduction, this result endorses astronomical investigations using a Mücket-Treder attraction law with $10^{-8} < \alpha < 0$.

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