A Minimal Subsystem of the Kari-Culik Tilings

by

Jason Siefken
M.Sc., University of Victoria, 2010
H.B.Sc., Oregon State University, 2008

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in the Department of Mathematics and Statistics

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ABSTRACT

The Kari-Culik tilings are formed from a set of 13 Wang tiles that tile the plane only aperiodically. They are the smallest known set of Wang tiles to do so and are not as well understood as other examples of aperiodic Wang tiles. We show that a certain subset of the Kari-Culik tilings, namely those whose rows can be interpreted as Sturmian sequences (rotation sequences), is minimal with respect to the $\mathbb{Z}^2$ action of translation. We give a characterization of this space as a skew product as well as explicit bounds on the waiting time between occurrences of $m \times n$ configurations.

Contents

List of Figures

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The University of Victoria, for providing funding and a house of learning.

Paul Strauss, Dr. Bob Burton, for keeping the graduate school spirit, along with my spirit, alive and in good condition.

DEDICATION

To my friends

—You've been a distraction, but a welcome one.

蝸牛そろそろ登れ富士の山

一茶小林

Chapter 1

Introduction

Before presenting precise, foundational definitions for the work in this dissertation, we will give a brief overview and motivation for some of the ideas. Section 1.1 will be fairly informal, since the definitions and concepts from Section 1.1 will be reintroduced formally later in the dissertation.

1.1 Background and Results

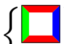
Of principal concern to us are tilings of the plane by square tiles with colored edges. That is, given the plane $\mathbb{R}^2$, we cover each point in $\mathbb{R}^2$ by non-overlapping translated copies of the unit square $[0, 1]^2$. To make things more interesting (and non-trivial), we decorate each copy of the unit square by coloring each of its four edges and then insisting that two squares may lie adjacent only if the colors on their shared edge match. With this restriction, a tiling of the plane by square tiles is called a *Wang tiling.*

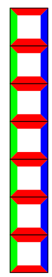
Definition 1.1 (Wang Tiling System)**.** *A* Wang tiling system *is a tiling by a set of square tiles (of identical size) with colored edges satisfying the following properties:*

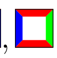
1. *two tiles may lie adjacent only if the colors on their shared edge match, and*

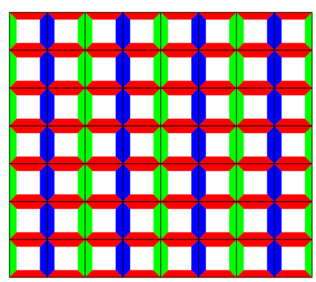
2. *tiles may be translated but not rotated, reflected, or otherwise transformed.*

Since each tile in a Wang tiling system is square and of the same size, instead of tiling the plane, we may think of Wang tilings as tiling the two-dimensional lattice of integers, $\mathbb{Z}^2$. To see this, notice that a tiling of the plane using Wang tiles can be represented by a tiling of $\mathbb{Z}^2$ by first ensuring that each tile is of unit width and has as its center a point (x, y) where $x, y \in \mathbb{Z}$. We then associate each point in $\mathbb{Z}^2$ with the tile whose center is at that point. Similarly, Wang tilings may sometimes be thought of as tilings of $\mathbb{Z}$. Some examples will make this clear.

Example 1.2 (Tiling of $\mathbb{Z}$)**.** Consider the set of Wang tiles $\mathfrak{T} = \{$  $\}$ consisting of a single tile whose left edge is green, right edge is blue, and top and bottom edges are red. The set $\mathfrak{T}$ can tile $\mathbb{Z}$, as illustrated, but cannot tile $\mathbb{Z}^2$ since only translated copies of tiles in $\mathfrak{T}$ are allowed and not rotations or reflections.



Example 1.3 (Tiling of $\mathbb{Z}^2$)**.** Consider the set of Wang tiles $\mathfrak{T} = \{$ ,  $\}$. This set of two tiles can tile the whole plane in the pattern illustrated.



It should be noted that given a set of Wang tiles $\mathfrak{T}$, it is entirely possible that $\mathfrak{T}$ cannot tile $\mathbb{Z}^2$ or even $\mathbb{Z}$. For example, if $\mathfrak{T}$ consists of a single tile where every edge is a different color, since this tile cannot be rotated, $\mathfrak{T}$ cannot tile any region larger than 1×1. In general, the problem of deciding whether a set of Wang tiles admits a tiling of the plane, introduced by Hao Wang in [4], is undecidable [3].

In working with Wang tilings, we are already translating individual tiles around, so it seems natural to introduce the action of translation onto an entire tiling of the plane and to turn a Wang tiling system into a dynamical system.

Definition 1.4. *Given a tiling of the plane x, Tx is the translation of x left by one unit and Sx is the translation of x down by one unit.*

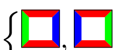
We may iterate the maps T and S as many times as we please (including applying the maps T^{-1}, translation right by one unit, and S^{-1}, translation up by one unit). Further, T and S commute, meaning $TSx = STx$.

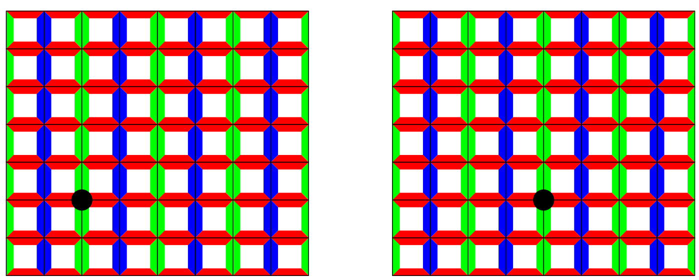
We may now define what it means for a tiling of the plane to be *periodic*.

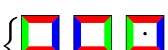
Definition 1.5. *A Wang tiling, x, of $\mathbb{Z}^2$ is* periodic *if some non-trivial translate of x equals x. That is, x is periodic if*

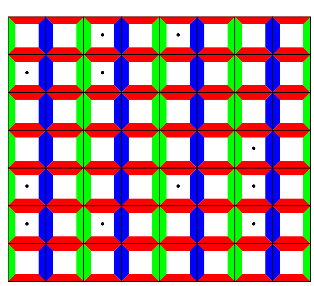
$$T^a S^b x = x$$

for some $(a, b) \neq (0, 0)$. If a tiling is not periodic, it is called aperiodic.

Example 1.6 (Periodic Tiling)**.** Consider the set of Wang tiles $\mathfrak{T} = \Big\{$$\Big\}$. This set of two tiles tile $\mathbb{Z}^2$ periodically as illustrated. The tiling on the left is a translation of the tiling on the right leftwards by two units. A black dot has been added for reference (since this tiling is periodic, you would not be able to tell if it were translated by one period unless some additional reference point were introduced).



Example 1.7 (Aperiodic Tiling)**.** Consider the set of Wang tiles $\mathfrak{T} = \Big\{$$\Big\}$ consisting of three tiles. The last tile is a copy of the first tile with a distinguishing dot placed in the center. The tiling illustrated was obtained by randomly choosing between the first and last tile where allowed. Since this choice was random, (with probability one) there is no translate of this tiling that equals itself, making it aperiodic.



Related to aperiodicity is a concept called *recurrence*.

Definition 1.8 (Recurrent)**.** *A tiling x is called* recurrent *if every $n \times n$ configuration that appears in x reoccurs infinitely many times.*

Periodicity implies recurrence (since every $n \times n$ configuration reoccurs every period), but recurrence does not imply periodicity. This distinction is subtle because if x is recurrent, it indeed means that every pattern seen in x will repeat, but there is no restriction on the spacing between repetitions of an $n \times n$ pattern, whereas if x were periodic, every $n \times n$ pattern must repeat with regular spacing.

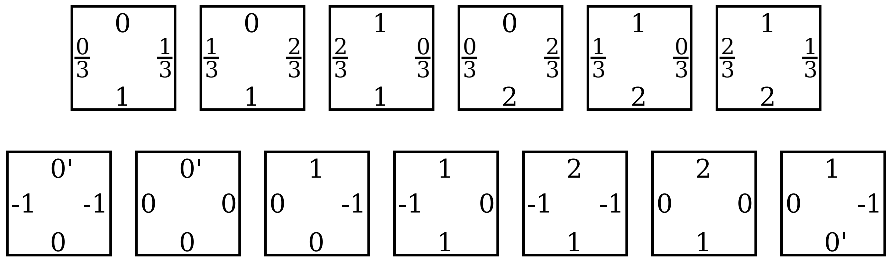
Considering now the tiling in Example 1.7, in some sense the aperiodicity is not intrinsic. We introduced a third tile and placed it randomly, but if we used only the first two tiles in $\mathfrak{T}$, we could have produced a periodic tiling. That is, $\mathfrak{T}$ can tile the plane in both periodic and aperiodic ways.

Definition 1.9. *A set of Wang tiles $\mathfrak{T}$ is called* aperiodic *if it admits a tiling of the plane and only admits aperiodic tilings of the plane.*

It was unknown whether aperiodic sets of Wang tiles existed until Berger, a student of Wang, produced an example in 1966 of 20,426 tiles that tiled the plane only aperiodically [2]. Berger later reduced his tile set to one containing only 104 tiles, and since then, many more tile sets that tile the plane only aperiodically have been produced. Of note, in 1971, Raphael Robinson produced a set of 56 tiles that tile the plane only aperiodically [17], and in 1995, Kari and Culik produced a set of 13 tiles that tile the plane only aperiodically. Currently, the Kari-Culik tile set is the smallest set of tiles known to only tile the plane aperiodically.

All known examples before the Kari-Culik tilings forced aperiodicity by exploiting a hierarchical structure. For example, the Robinson tilings force the formation of square patterns of sizes 3×3, 5×5, 9×9, ... (every size of the form $(2^n+1)\times(2^n+1)$). If the Robinson tilings ever tiled in a periodic way, there would be a largest one of these square patterns. Since there is no bound on the size of these square patterns, the tiles cannot tile periodically.

However, the Kari-Culik tiles, shown below, have no known hierarchical description.



Instead, the Kari-Culik tilings rely on number-theoretic properties to force aperiodicity (essentially relying on the fact that $2^a \neq 3^b$ unless $a = b = 0$). Though the proof of the existence and aperiodicity of tilings by the Kari-Culik tile set is straightforward, not much else about the structure of the Kari-Culik tilings is known. Durand, Gamard, and Grandjean showed in 2013 that the set of all Kari-Culik tilings has positive topological entropy [5], and this, along with some results by Arthur Robinson in [16], gives the state of knowledge about the Kari-Culik tilings circa 2014.

This dissertation recasts a subset of the Kari-Culik tilings, KC, as a generalization of rotation sequences, and exploits this rotation-sequence-like framework to produce several dynamical system-related results about the subset KC and its associated dynamical system. The main theorems pertaining to the Kari-Culik tilings presented in this dissertation are as follows.

Theorem (3.23)**.** *The function* $\Phi : KC \to \{0, 1, 2\}^{\mathbb{Z}^2}$ *given by projection onto the top label of each tile is one-to-one almost everywhere and is at most sixteen-to-one.*

Theorem (3.34)**.** *The set* KC *can be parameterized by the set* $[1/3, 2] \times \varprojlim \mathbb{R}/(6^n\mathbb{Z})$.

Theorem (3.35)**.** *When parameterized by* $[1/3, 2] \times \varprojlim \mathbb{R}/(6^n\mathbb{Z})$*, left translation on* KC *can be written as a skew product and vertical translation is conjugate to an irrational rotation by* $\log 2/\log 6$.

Theorem (3.37)**.** KC *can be thought of as the closure of a line in the infinite-dimensional torus* $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}}$.

Theorem (4.11)**.** KC *is minimal with respect to the group action of* $\mathbb{Z}^2$ *by translation.*

Theorem (5.28)**.** *Let* $\eta = \eta(\log 2/\log 6)$. *Every legal* $n \times m$ *configuration in* KC *occurs in every* $B \times A$ *configuration in* KC *where*

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{\eta} < 6^{34.464m+25} n^{34.464} \qquad \text{and} \qquad B = 6^{5m+3} n^4$$

for sufficiently large $m+n$.

Further, for all m, n we have that a copy of every legal $n \times m$ configuration in KC occurs in every $B \times A$ configuration in KC where

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{14.3} \log 6 \qquad \textit{and} \qquad B = 6^{5m+3} n^4$$

These theorems link number-theoretic results to the dynamical system associated with KC and give a full characterization of KC in terms of more familiar dynamical systems.

1.2 Dynamical Systems

In its most basic sense, a *dynamical system* is a space of points coupled with a transformation that moves the points around according to some directed parameter (most often time). We will not deal with dynamical systems in this generality, but instead we will work with *discrete time* dynamical systems.

Definition 1.10 (Dynamical System)**.** *A* discrete time dynamical system *is a pair (T, X) where X is some set and $T : X \to X$ is a function.*

The essential property of a dynamical system is that the domain and range of T are the same, allowing us to iterate T and observe how points move about.

Definition 1.11 (Orbit)**.** *If (T, X) is a dynamical system, the* forward orbit *of $x \in X$ is the set $\mathcal{O}(x) = \{x, Tx, T^2x, \ldots\}$. The n-*orbit *of x is the set $\mathcal{O}^n(x) = \{x, Tx, T^2x, \ldots, T^{n-1}x\}$. If T is invertible, we define the* two-sided orbit *(sometimes just called the orbit) as $\mathcal{O}(x) = \{\ldots, T^{-1}x, x, Tx, T^2x, \ldots\}$.*

If there are multiple transformations on the same space X, we may specify which one we are taking the orbit under. For example, the T-orbit of a point x would be $\mathcal{O}_T(x) = \{\ldots, T^{-1}x, x, Tx, T^2x, \ldots\}$ (assuming T is invertible).

Definition 1.12 (Periodic Point)**.** *In a dynamical system (T, X), we call a point $x \in X$* periodic *if $T^ix = x$ for some $i > 0$.*

For ease of discussion, from now on, we will assume all dynamical systems are invertible (that is, we will only consider dynamical systems (T, X) where T is invertible).

Orbits give a notion of how a point moves over "time." Another part of the story are the *invariant* sets.

Definition 1.13 (Invariant Set)**.** *If (T, X) is an invertible, discrete time dynamical system, then a subset $A \subset X$ is said to be* invariant *if $TA = A$.*

The orbit of any point is always an invariant set, and invariant sets can be thought of as sets that contain the orbits of all their points. However, these notions are not much use unless we can also couple them with a notion of distance. This leads us to the first property we will insist upon in X, namely that it is a *metric space*.

1.2.1 Metric Spaces

Definition 1.14 (Metric)**.** *Given a set X, a* metric *on X is a function $d : X \times X \to \mathbb{R}$ so that for all $x, y, z \in X$ we have*

1. *$d(x, y) \geq 0$ with $d(x, x) = 0$;*
2. *$d(x, y) = 0$ implies $x = y$;*
3. *and $d(x, y) \leq d(x, z) + d(z, y)$ (the* triangle inequality*).*

A pair (X, d) where d is a metric on X is called a metric space*.*

In a metric space (X, d), we have a notion of convergence. Namely, a sequence (x_n) converges to a point x if $d(x_n, x) \to 0$. But, just having a metric space often is not good enough. What we really want is a *complete* metric space.

Definition 1.15 (Cauchy Sequence)**.** *A sequence (x_n) in a metric space (X, d) is called a* Cauchy sequence *if for all $\epsilon > 0$, there exists an N_ϵ so that $n, m > N_\epsilon$ implies $d(x_n, x_m) < \epsilon$.*

Definition 1.16 (Complete Metric Space)**.** *A metric space (X, d) is called* complete *if every Cauchy sequence in X converges in X.*

Given a metric space (X, d), a convergent sequence (x_n) in X is Cauchy. However, the converse may not be true for one reason: the point that (x_n) is "heading to" may not be in X at all. For example, consider the open interval $(0, 1)$ and the sequence $x_n = 1/n$. Clearly, $1/n$ is heading to 0 (under the usual notion of distance on the number line given by $|\cdot|$), but 0 is not in the set $(0, 1)$. Thus $((0, 1), |\cdot|)$ is not a complete metric space.

Definition 1.17 (Open & Closed). *In a metric space (X, d), let $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$ be the open ball of radius epsilon about x. We then define a set $A \subset X$ to be* open *if A is the union of (possibly infinitely many) open balls. The complement of the set A is $A^c = \{x \in X : x \notin A\}$, and a* closed *set is defined to be the complement of an open set.*

For subsets of a complete metric space, there are alternate definitions of closed and open sets.

Definition 1.18 (Closed). *Let (X, d) be a complete metric space. A set $A \subset X$ is* closed *if (A, d) is a complete metric space.*

Proposition 1.19. *An arbitrary intersection of closed sets is closed.*

Proof. Suppose that $A_\lambda \subset X$ are closed subsets of the metric space X indexed by $\lambda \in \Lambda$. Let $A = \bigcap_{\lambda \in \Lambda} A_\lambda$. Fix a Cauchy sequence $\mathbf{x} = (x_0, x_1, \ldots)$ where $x_i \in A$. Since $\mathbf{x}$ is Cauchy in every A_λ, $x_i \to x \in A_\lambda$.

Since $x \in A_\lambda$ for every $\lambda \in \Lambda$, $x \in A$ and so A is complete. Thus A is also a closed subset of X. □

Definition 1.20 (Open). *Let (X, d) be a complete metric space. A set $A \subset X$ is* open *if $(X \backslash A, d)$ is a compete metric space. That is, an open set is the complement of a closed set.*

This is not the typical definition of a closed set and it only allows us to consider subsets of complete metric spaces, but it captures the moral essence of what it means to be closed in a metric space. More generally, closed sets are defined in terms of open sets.

Definition 1.21 (Topology). *A* topology *τ on a metric space (X, d) is the collection of all open subsets of X.*

If $A \subset X$, the relative topology *on A is the collection $\tau' = \{A \cap B : B \in \tau\}$.*

Topologies can be defined much more generally than presented here, but we will not need such generality. Topologies are intimately connected to the metric they came from and one can define convergence strictly in terms of the topology on a metric space without ever using the metric (for a detailed introduction to the theory of topology and metric spaces, see [14]). However, there are some counterintuitive differences between metrics and topologies. For instance, given two different metrics

$d, \hat{d}$ on X, they both may generate the same topology. This is not too hard to believe, but what is strange is that while they both generate the same topology, they may have different Cauchy sequences! In general,

$$\text{metric spaces} \subsetneq \text{uniform spaces} \subsetneq \text{topological spaces.}$$

An example that we will use later is $\mathbb{Q}^c$, where both the standard metric and the Baire metric give the same topology, but $\mathbb{Q}^c$ with the Baire metric is complete.

1.2.2 Minimality

Now that we have the notion of a metric space and closed sets, we can start defining some interesting properties. For simplicity, we will now always assume any dynamical system we talk about is also a metric space and if not specified otherwise, the metric will be called d. The first property we will define is *recurrence*.

Definition 1.22 (Recurrent)**.** *A dynamical system* (T, X) *is* recurrent *if every open set* $A \subset X$ *has the property that* $x \in A$ *implies there exists some* $i > 0$ *so* $T^i x \in A$.

Recurrence is nice because if a dynamical system is recurrent, it ensures that all the pieces of the dynamical system are actually interesting. If a system were not recurrent, in some sense there would be a strict subspace of the system that absorbed all points and then continues mixing them around.

Of course, in a recurrent dynamical system, there can still be parts of the system that have nothing to do with each other (for example, we could take two disjoint recurrent dynamical systems and glue them together). This is a motivation for the definition of a *minimal* system.

Definition 1.23 (Minimal)**.** *A non-empty dynamical system* (T, X) *is* minimal *if for any non-empty closed subset* $A \subset X$, $TA = A$ *implies that* $A = X$.

Given a dynamical system (T, X), a minimal subsystem can be thought of as a "smallest" closed dynamical system contained in (T, X)—there are no pieces that can be broken off. However, the definition given above is not always the easiest to work with or to use to prove that a certain dynamical system is minimal. Further, minimal systems provide a link between the transformation on a space and the underlying topology. For this relationship to be meaningful, we need $T : X \to X$ to be *continuous*.

Definition 1.24. *A function $f : X \to X$ on a metric space is* continuous *if $f^{-1}(A) = \{x \in X : f(x) \in A\}$ is an open set whenever $A \subset X$ is an open set.*

From now on, we will assume that T is continuous. This allows us to produce several equivalent definitions of a minimal dynamical system.

Definition 1.25 (Closure)**.** *Given a set $A \subset X$, its* closure*, denoted $\bar{A}$, is the intersection of all closed sets containing A.*

Note that by Proposition 1.19, the closure of a set is indeed closed.

Definition 1.26 (Dense)**.** *A subset $A \subset X$ is* dense in X *if $\bar{A} = X$.*

Definition 1.27 (Minimality characterization II)**.** *A non-empty dynamical system (T, X) where T is a continuous function is* minimal *if $\overline{\mathcal{O}x} = X$ for every $x \in X$. That is, the orbit of every point is dense.*

The equivalence of these two definitions of minimality is straightforward. If there were an orbit in X that were not dense, its closure $A \subsetneq X$ would not be equal to the entire set and consequently (by continuity of T), A would be a closed, proper invariant subset. Alternatively, if there exists a closed subset $A \subsetneq X$ so that $TA = A$, then by invariance, $\overline{\mathcal{O}x} \subset A \subsetneq X$ and so the orbit of some points would not be dense.

We will soon see yet another characterization of minimality applicable in the symbolic case.

1.3 Symbolic Dynamics

Although general dynamical systems on a metric space provide enough structure to prove many interesting theorems, there are advantages to moving to a space where orbits consist of sequences of symbols. Most dynamical systems can be translated to a space of symbols that still preserves the important dynamical properties. This brings us into the realm of *symbolic dynamics*.

Definition 1.28. *Given a set X, a* finite partition *of X is a set $P = \{P_0, P_1, \ldots, P_n\}$ so that $X = \bigcup P_i$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$.*

We motivate symbolic dynamics as follows. Suppose that (T, X) is a dynamical system and that T is invertible. Let $P = \{P_0, P_1, \ldots, P_n\}$ be a finite partition of X.

Then, given a point $x \in X$, we can write down a sequence $x' = (x'_i)_{i=-\infty}^{\infty}$ corresponding to x where

$$x'_i = j \qquad \text{if} \quad T^i x \in P_j.$$

In this way, we have recoded $x \in X$ to $x' \in \{0, \ldots, n\}^{\mathbb{Z}}$ and Tx corresponds to Sx' where

$$S(\ldots, x_0, x_1, \ldots) = (\ldots, x_1, x_2, \ldots)$$

shifts all symbols to the left. We now have two dynamical systems, (T, X) and $(S, \{0, \ldots, n\}^{\mathbb{Z}})$, and if the partition P was chosen in the right way, the dynamics of both systems will closely mirror each other.

Definition 1.29 (Symbolic Dynamical System)**.** *A* one-dimensional symbolic dynamical system *is a pair (S, X) where $X \subset \mathfrak{L}^{\mathbb{Z}}$ is an S-invariant set, $\mathfrak{L}$ is some finite set, and $S : X \to X$ is defined by $(x_i)_{i=-\infty}^{\infty} \mapsto (x_{i+1})_{i=-\infty}^{\infty}$.*

We call $\mathfrak{L}$ the symbols, letters, *or* digits *of (S, X) and we call S the* shift *map.*

Being able to refer to points in a dynamical system as sequences of symbols allows one to explicitly construct examples with strange properties as well as give simpler definitions than in the case of general dynamical systems.

Notation 1.30. *If $x = (\ldots, x_0, x_1, \ldots) \in \mathfrak{L}^{\mathbb{Z}}$, then $(x)_i^j = (x_i, x_{i+1}, \ldots, x_j)$ is the* subword *of x from position i to j. $(x)_i$ is short for $(x)_i^i$. We write $w \subset x$ if $w = (x)_i^j$ for some i, j (possibly infinite). In this case, $|w| = j - i + 1$ is the length of the word w. Further, if $A \subset \mathbb{Z}$, then $x|_A$ is the restriction of x to the indices in A.*

Given a symbolic dynamical system (S, X), we call $\mathcal{L}_n(X) = \{w : |w| = n$ and $w \subset x$ for some $x \in X\}$ the n-language *of X and call $\mathcal{L}(X) = \bigcup \mathcal{L}_n(X)$ the* language *of X.*

Definition 1.31 (Standard Metric on Sequences)**.** *Given a set of sequences $X = \mathfrak{L}^{\mathbb{Z}}$, we define the* standard metric on sequences *to be d where*

$$d(x, y) = \inf\{2^{-n} : (x)_{-n}^{n} = (y)_{-n}^{n}\}.$$

The standard metric on sequences says that two points are close together if they agree for a great many symbols about the origin. It turns out that the shift is continuous with respect to this metric. Further, $\mathfrak{L}^{\mathbb{Z}}$ endowed with this metric is complete and compact.

Definition 1.32 (Minimality characterization III)**.** *If* (S, X) *is a non-empty symbolic dynamical system,* (S, X) *is* minimal *if there exists an* M_n *such that for every* x *and every subword* $w \subset x$ *with* $|w| = M_n$*, we have that every word in* $\mathcal{L}_n(X)$ *is contained in* w*.*

Proof. We will show that the third characterization of minimality is equivalent to the others.

Suppose for every n that $\mathcal{O}x$ contains every word in $\mathcal{L}_n(X)$. Then by the definition of the standard metric, $\mathcal{O}x$ is dense in X. Thus, it is clear that our third characterization of minimality implies the orbit of every point is dense.

Of course, if the orbit of every point is dense, then every point must contain every word in $\mathcal{L}_n(X)$. The subtlety is that there must be an upper bound on how long it takes to see every word in $\mathcal{L}_n(X)$. Suppose x is a point such that there is no upper bound on the waiting time for $w \in \mathcal{L}_n(X)$. Since X is a closed subset of a compact metric space and therefore compact, there must be an accumulation point of $\mathcal{O}x$ in which w does not occur. Thus, not only does every point in a minimal symbolic dynamical system contain every word of $\mathcal{L}_n(X)$, but there is an upper bound on how long a segment must be to contain every word of $\mathcal{L}_n(X)$.

□

Definition 1.33 (Subshift)**.** *If* (S, X) *is a symbolic dynamical system, a* subshift *is a dynamical system* (S, A) *where* $A \subset X$ *is a closed, invariant set.*

Definition 1.34 (Full Shift)**.** *If* (S, X) *is a symbolic dynamical system, we call* (S, X) *a* full shift *if* $X = \mathfrak{L}^{\mathbb{Z}}$ *for some finite set* $\mathfrak{L}$*.*

Definition 1.35 (Subshift of Finite Type)**.** *If* (S, X) *is a full shift, we call* (S, A) *a* subshift of finite type *(*SFT*) if* (S, A) *is a subshift and there exists some finite set of forbidden words* $\mathcal{F}$ *so that*

$$\mathcal{L}(A) = \{w \in \mathcal{L}(X) : f \not\subset w \text{ for all } f \in \mathcal{F}\}.$$

For a full introduction to symbolic dynamics and subshifts of finite type, see [10]. We will only discuss a few of the relevant highlights here.

Definition 1.36 (Nearest Neighbour SFT)**.** *A* nearest-neighbour *subshift of finite type is a subshift of finite type whose forbidden words are all of length two.*

Definition 1.37. *A* block presentation *of a subshift of finite type* (S, X) *is a subshift of finite type* (S', Y) *such that there exists a continuous bijection* $\Phi : X \to Y$ *satisfying* $\Phi \circ S = S' \circ \Phi$.

Proposition 1.38. *Given an subshift of finite type* (S, X)*, there exists a block presentation of* (S, X) *as a nearest-neighbour subshift of finite type* (S', Y)*.*

Proof. Let ℓ be the length of the longest forbidden word in (S, X). Since (S, X) is a subshift of finite type, $\ell < \infty$. Let $W = \mathcal{L}_\ell(X)$ be the set of all words in X of length ℓ. We may then consider the dynamical system $(S', W^{\mathbb{Z}})$, where S' is the usual shift on $W^{\mathbb{Z}}$. For clarity, define $Y = W^{\mathbb{Z}}$. (S', Y) is a block presentation of (S, X) via the function $\Phi : X \to Y$ sending

$$\ldots, x_0, x_1, \ldots \mapsto \ldots, (x_0, x_1, \ldots, x_\ell), (x_1, x_2, \ldots, x_{\ell+1}), \ldots.$$

Φ is continuous and so Y is closed, making (S', Y) a subshift. Further, if $w, w' \in W$, then ww' is a valid word in (S', Y) if and only if $w = (x_0, \ldots, x_\ell)$ and $w' = (x_1, \ldots, x_{\ell+1})$ for some $x_0 \cdots x_{\ell+1}$ a valid subword of (S, X). Thus, a subshift of finite type can always be block-presented as a nearest-neighbour subshift of finite type. □

Nearest neighbour subshifts of finite type are easier to work with and without loss of generality, we may always assume to be working with one. The only consequence of doing so is potentially increasing the size of our alphabet.

Proposition 1.39. *If* (S, X) *is a subshift of finite type and* X *is non-empty, then* X *contains a periodic point.*

Proof. By Proposition 1.38, we may assume that (S, X) is a nearest-neighbour subshift of finite type. Suppose (S, X) is non-empty, and pick a point $x \in X$. Since $x = \cdots x_0 x_1 \cdots$ is an infinite sequence of symbols, there must be some pair of consecutive symbols $x_i x_{i+1}$ that occurs twice in x. Let i, j be positions of the start of such an occurrence. We then have that the word x' formed by repeating the symbols $(x)_{i+1}^{j}$ must be in X and by construction x' is periodic. □

Proposition 1.40. *If* (S, X) *is a subshift of finite type and* (S, X) *is minimal, then* $X = \mathcal{O}x$ *for some periodic point* x*.*

Proof. Since (S, X) is a subshift of finite type, Proposition 1.39 gives us that (S, X) must contain a periodic point. Since for any periodic point x, $\mathcal{O}x$ is a closed, invariant set (since $\mathcal{O}x$ is finite and invariant), by the definition of minimality, $\mathcal{O}x = X$. □

Proposition 1.40 will stand in stark contrast to the analogous statement about two-dimensional subshifts of finite type explored in the next section.

1.3.1 $\mathbb{Z}^2$ Symbolic Dynamics

One-dimensional symbolic dynamical systems are well studied and we have a fairly complete theory of many subsystems (for example, subshifts of finite type). However, when we introduce another commuting transformation on our symbolic space, all bets are off.

Definition 1.41. *A* $\mathbb{Z}^2$-symbolic dynamical system *is a triplet* (T, S, X) *such that* $X \subset \mathfrak{L}^{\mathbb{Z}^2}$ *is a closed, invariant set and* $T, S : X \to X$ *are commuting maps given by*

$$(Tx)|_{i,j} = x|_{i+1,j} \qquad \text{and} \qquad (Sx)|_{i,j} = x|_{i,j+1}.$$

Here $x|_{i,j}$ *is the symbol of* x *at position* $(i, j) \in \mathbb{Z}^2$.

The language of a $\mathbb{Z}^2$-symbolic dynamical system is defined analogously to a one-dimensional symbolic dynamical system except that instead of subwords consisting of contiguous lists of symbols from points in X, now subwords consists of rectangular configurations of symbols occurring in points in X.

Notation 1.42. *Given* $x \in \mathfrak{L}^{\mathbb{Z}^2}$ *and* $A \subset \mathbb{Z}^2$ *by* $x|_A$ *we mean the configuration of symbols of* x *at the indices in* A.

By convention, when we write $x|_A$, we only care about the relative position of symbols at coordinates in A. That is $x|_A = x|_B$ is a valid comparison if B is some $\mathbb{Z}^2$-translate of A. For example, if $A = \{0, 1\} \times \{0, 1\}$ and $B = \{3, 4\} \times \{7, 8\}$, a statement like $x|_A = x|_B$ would make sense. However, if $A = \{0, 1\} \times \{0, 1\}$ and $B = \{3, 4\} \times \{7, 10\}$, the statement $x|_A = x|_B$ would always be false since A and B cannot be translated to coincide.

Definition 1.43 (Language)**.** *Given a subset* $X \subset \mathfrak{L}^{\mathbb{Z}^2}$ *and* $A = \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$*, the* $m \times n$ language *of* X *is*

$$\mathcal{L}_{m\times n}(X) = \{w : w = T^i S^j x|_A \textit{ for some } x \in X \textit{ and } (i, j) \in \mathbb{Z}^2\}.$$

The language *of* X *is* $\mathcal{L}(X) = \bigcup \mathcal{L}_{m\times n}$.

Definition 1.44 ($\mathbb{Z}^2$-Subshift of Finite Type)**.** *Given a $\mathbb{Z}^2$-symbolic dynamical system (T, S, X), X is a* $\mathbb{Z}^2$-subshift of finite type *($\mathbb{Z}^2$* SFT*) if there exists a finite subset $\mathcal{F} \subset \mathcal{L}(\mathfrak{L}^{\mathbb{Z}^2})$ such that*

$$\mathcal{L}(X) = \{w \in \mathcal{L}(\mathfrak{L}^{\mathbb{Z}^2}) : f \not\subset w \text{ for all } f \in \mathcal{F}\}.$$

Similar to one-dimensional subshifts of finite type, a *nearest-neighbour* $\mathbb{Z}^2$ subshift of finite type is one where the forbidden words are 1×2 or 2×1 rectangles, and any $\mathbb{Z}^2$ SFT can be recoded to a nearest-neighbour $\mathbb{Z}^2$ SFT.

Notation 1.45. *Given a point $x \in \mathfrak{L}^{\mathbb{Z}^2}$, we denote by $(x)_i$ the ith row of x. That is, $(x)_i = x|_{\mathbb{Z}\times\{i\}}$.*

Definition 1.46 (Standard Metric)**.** *Let d be the standard metric on sequences. We define $d_{\mathbb{Z}^2} : \mathfrak{L}^{\mathbb{Z}^2} \times \mathfrak{L}^{\mathbb{Z}^2} \to \mathbb{R}$ to be the* standard metric on $\mathbb{Z}^2$ configurations *where*

$$d_{\mathbb{Z}^2}(x, y) = \sup_{i\in\mathbb{Z}}\{2^{-i} d((x)_i, (y)_i)\}.$$

Using the metric $d_{\mathbb{Z}^2}$ endows $\mathfrak{L}^{\mathbb{Z}^2}$ with the product topology. Where it is unambiguous, we will write d instead of $d_{\mathbb{Z}^2}$.

Definition 1.47 (Periodic in $\mathbb{Z}^2$)**.** *Given a $\mathbb{Z}^2$-symbolic dynamical system (T, S, X), we say $x \in X$ is* weakly periodic *if $T^i S^j x = x$ for some $(i, j) \neq (0, 0)$. We say x is* strongly periodic *if $T^i x = x$ and $S^j x = x$ for some $i, j > 0$.*

We call X aperiodic *if X contains no weakly periodic points.*

Having multiple directions in which periodicity may exist can be a hassle. Fortunately, if we restrict ourselves to $\mathbb{Z}^2$ SFTs, weakly periodic implies strongly periodic.

Proposition 1.48. *If (T, S, X) is a $\mathbb{Z}^2$ SFT, then the existence of a weakly periodic point implies the existence of a strongly periodic point.*

Proof. Suppose (T, S, X) is a $\mathbb{Z}^2$ SFT that contains a weakly periodic point x. Without loss of generality, we may assume X is a nearest-neighbour $\mathbb{Z}^2$ SFT and that $T^i x = x$ for some $i > 0$. Now consider $x' = x|_{\{0,\dots,i-1\}\times\mathbb{Z}}$. Since only a finite number of words may appear in the rows of x', we know that there must be a consecutive pair of rows in x' that occurs twice. Let k, l be the indices of two such rows. We may now form a strongly periodic point in X by repeating $x|_{\{0,\dots,i-1\}\times\{k,\dots,l-1\}}$. $\square$

The characterizations of minimality directly carry over from the one-dimensional case, however we have contrasting propositions to Proposition 1.39 and Proposition 1.40.

Proposition 1.49 (Berger [3]). *There exists a non-empty $\mathbb{Z}^2$ subshift of finite type that contains no (weakly or strongly) periodic points.*

Proposition 1.50 (Raphael Robinson). *There exists a non-empty, minimal $\mathbb{Z}^2$ subshift of finite type that contains no (weakly or strongly) periodic points.*

As cited by Makowsky in [13], Proposition 1.50 is attributed to Raphael Robinson who explained in private communications a way of making a robust version of the Robinson tilings. This result has been accepted as a folk theorem, but this dissertation does not depend on this result.

For a readable exposition of how a $\mathbb{Z}^2$ SFT with no periodic points can exist, see [9] where an example of Robinson is explained in detail.

1.3.2 Wang Tilings

In general, a tiling is a covering of $\mathbb{R}^2$ by infinitely many translates of a finite number of bounded polygonal regions that overlap only on their boundaries. A *Wang tiling* is a restriction of this idea where $\mathbb{R}^2$ is covered edge-to-edge by translations of identically sized squares whose edges are colored. Two squares are allowed to lie adjacent to each other if the colors (or labels) of their shared edge match.

We can avoid the technicalities of general tiling systems by noticing that Wang tilings can all be viewed as subshifts of finite type.

Proposition 1.51 (Wang Tiling). *Every* Wang tiling *is a nearest-neighbour $\mathbb{Z}^2$-subshift of finite type.*

To see Proposition 1.51, let $\mathfrak{L}$ be a set of Wang tiles. The set of all valid Wang tilings may now be interpreted as subset of $\mathfrak{L}^{\mathbb{Z}^2}$ and the rules of the Wang tiling system translate directly to nearest-neighbour adjacency restrictions. Combined now with the action of translation by one unit, a Wang tiling is a nearest-neighbour SFT.

When viewed as a subshift of finite type, the symbols of a Wang tiling are called *tiles*.

Chapter 2

Sturmian Sequences

Sturmian sequences are a particular type of minimal dynamical system with very interesting combinatoric and geometric properties. They arise naturally in several contexts including the Kari-Culik tilings and have many equivalent characterizations. In this chapter, we develop tools for working with Sturmian sequences.

2.1 Equivalent Classifications

Definition 2.1 (Recurrent). *A bi-infinite sequence s is said to be* recurrent *if for all subwords w, w appears infinitely often.*

Definition 2.2 (Complexity). *For a sequence s, the* complexity *of s is the function $\sigma_s : \mathbb{N} \to \mathbb{N}$ where $\sigma_s(n)$ is the number of distinct subwords of s of length n.*

The Morse-Hedlund theorem states that if s is a sequence that satisfies $\sigma_s(n) \leq n$ for some n, then s is periodic.

Definition 2.3 (Balanced). *Let s be a sequence of integers. For any subword w, let Σw be the sum of the digits of w.*

We call s a balanced *sequence if there exists a sequence (a_n) such that*

$$a_n \leq \Sigma w \leq a_n + 1$$

whenever $|w| = n$.

Definition 2.4 (Rotation Sequence). *For parameters $\alpha, t \in \mathbb{R}$, a* rotation sequence *with* angle α *and* phase t *is either the bi-infinite sequence* $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha,t)$ *given by* r *where*

$$(r)_i = \lfloor (i+1)\alpha + t \rfloor - \lfloor i\alpha + t \rfloor ,$$

or the bi-infinite sequence $\mathbf{R}_{\lceil\cdot\rceil}(\alpha,t)$ *given by* r' *where*

$$(r')_i = \lceil (i+1)\alpha + t \rceil - \lceil i\alpha + t \rceil .$$

We will now list several equivalent definitions of Sturmian sequence.

Definition 2.5 (Sturmian Sequence I)**.** *A* Sturmian sequence *is a bi-infinite recurrent sequence* s *such that* $\sigma_s(n) \leq n+1$.

Definition 2.6 (Sturmian Sequence II)**.** *A* Sturmian sequence *is a bi-infinite recurrent sequence* s *such that* s *is balanced.*

Definition 2.7 (Sturmian Sequence III)**.** *A* Sturmian sequence *is a bi-infinite rotation sequence.*

Theorem 2.8. *All given definitions of Sturmian sequences are equivalent.*

Notation 2.9. *The set of all Sturmian sequences is denoted* $\mathcal{S}$.

For a proof of Theorem 2.8, see [7, 12]. Differing from some authors, we allow Sturmian sequences to be periodic, and we will rely most heavily on Definition 2.7.

Definition 2.10. *For a bi-infinite sequence* s*, define*

$$\overline{\alpha}(s) = \limsup_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(s)_i \qquad \underline{\alpha}(s) = \liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(s)_i$$

and

$$\alpha(s) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}(s)_i$$

is the average of the digits in s*, if the limit exists.*

Notice the α in Definition 2.10 can be applied to any sequence whose digits have an average and not just Sturmian sequences.

Proposition 2.11. *If* s *is a Sturmian sequence, then* $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha,t)$ *or* $s = \mathbf{R}_{\lceil\cdot\rceil}(\alpha,t)$ *where* α *is the average of the digits of* s.

Proof. Since all three definitions of Sturmian sequence are equivalent, we know that $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t')$ or $s = \mathbf{R}_{\lceil\cdot\rceil}(\alpha'', t'')$ for some α' or α''. For simplicity, assume $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t')$. Since the average of the digits of $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t')$ necessarily equal α', the proposition is proved. $\square$

The down side to rotation sequences is that their parameterization in terms of angles and phases and a choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$ is not in one-to-one correspondence with Sturmian sequences. In particular, while the angle of a rotation sequence is uniquely determined as the average of its digits, the fact that $\lfloor a \rfloor + 1 = \lceil a \rceil$ if and only if $a \notin \mathbb{Z}$, gives

$$\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) = \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t)$$

if $i\alpha + t \notin \mathbb{Z}$ for all i and so the choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$ is not uniquely determined. Further, if $\alpha \in \mathbb{Q}$, there is an interval of phases that produce the same rotation sequence.

We will address the non-uniqueness of $\mathbf{R}_{\lfloor\cdot\rfloor}$ vs $\mathbf{R}_{\lceil\cdot\rceil}$ by introducing infinitesimals into the phases.

Definition 2.12. *Let ϵ be defined as an infinitesimal such that $0 < n\epsilon < r$ for any positive real number r and any $n \in \mathbb{N} \cup \{\infty\}$. Define the set $R = \mathbb{R} + \epsilon\bar{\mathbb{Z}}$, where $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$. Further, define $\mathbb{R}^{\pm} = \mathbb{R} \pm \epsilon$.*

Notation 2.13. *If $a \in R$, we use $\mathrm{Re}(a)$ to denote the real component of a and $\mathrm{Inf}(a)$ to denote the coefficient of the infinitesimal component of a.*

We will extend $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ in the natural way to functions on R.

Proposition 2.14. *The sequence s is a Sturmian sequence if and only if $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ for some $\alpha \in \mathbb{R}$ and $t \in R$.*

Proof. Let s be a Sturmian sequence. If $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ for some $\alpha, t \in \mathbb{R}$, we are done. If not, we must have $s = \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t)$ for some parameters $\alpha, t \in \mathbb{R}$. However,

$$\mathbf{R}_{\lceil\cdot\rceil}(\alpha, t) = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t - \epsilon)$$

for all real parameters α, t.

Similarly, if $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t + n\epsilon)$ for $n \in \bar{\mathbb{Z}}$ and $t \in \mathbb{R}$, then either $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ or $s = \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t)$. $\square$

It may seem strange that we take the phase to be in R when $\mathbb{R}^{\pm}$ would suffice, however taking the phase to be in R will be essential when we start talking about generalized Sturmian sequences.

Definition 2.15. *For a Sturmian sequence s, define $t(s) = \mathrm{Re}(t')$ where $t' = \inf\{t \in R : s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ for some $\alpha \in \mathbb{R}^{\pm}\}$.*

We extend the t in Definition 2.15 to apply to any rotation sequence with potentially infinitesimal parameters. Later we will see that rotation sequences requiring infinitesimal parameters can arise as limits of Sturmian sequences.

Again, in order to have α and t take real values, we must resort to a choice between $\mathbf{R}_{\lfloor\cdot\rfloor}$ and $\mathbf{R}_{\lceil\cdot\rceil}$ when representing our Sturmian sequence as a rotation sequence. Further, while α is well behaved (i.e., continuous), t is not.

Proposition 2.16. *The map $s \mapsto \alpha(s)$, when restricted to $\mathcal{S}$, is continuous in $\mathcal{S}$ endowed with the product topology.*

Proof. Since $s \in \mathcal{S}$ implies that s is a balanced sequence, we see that the average of the first n digits of s determines $\alpha(s)$ to an error bounded by $1/n$. Thus α is continuous. $\square$

Proposition 2.17. *The map $s \mapsto t(s)$, when restricted to $\mathcal{S}$, is not continuous in $\mathcal{S}$ endowed with the product topology.*

Proof. Consider the sequence

$$y_i = \mathbf{R}_{\lfloor\cdot\rfloor}(\sqrt{2}/i, 1/2).$$

We have that $t(y_i) = 1/2$ for all i, but $y_i \to y = \mathbf{R}_{\lfloor\cdot\rfloor}(0, 0)$ and so $t(y_i) \nrightarrow t(y)$. $\square$

2.2 Irrational Rotations and Continued Fractions

From Definition 2.7, we see that Sturmian sequences and rotations are closely related via rotation sequences. Rotations, in turn, can be analyzed using continued fractions, and so we will explore some of the theory of continued fractions.

Definition 2.18 (Rotation)**.** *A* rotation *by the* rotation angle *α is a function $R_\alpha : [0, 1) \to [0, 1)$ defined by*

$$R_\alpha(x) = x + \alpha \bmod 1.$$

A rotation is called an irrational rotation *if $\alpha \notin \mathbb{Q}$.*

For any α, $(R_\alpha, [0,1))$ is an example of a uniquely-ergodic, minimal dynamical system, and we can now see that Sturmian sequences with angle α are just the symbolic recoding of the system $(R_\alpha, [0,1))$ with the partition $\mathcal{P} = \{[0,\alpha), [\alpha, 1)\}$.

Definition 2.19 (Continued Fraction)**.** *Given a number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, the* continued fraction representation of α *is the sequence* $[a_0; a_1, a_2, a_3, \ldots]$ *such that*

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

We define the nth convergent of α *to be*

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}.$$

We always assume $\frac{p_n}{q_n}$ is in lowest terms.

We can extend the definition of continued fraction to include $\alpha \in \mathbb{Q}$, but we lose uniqueness of the continued fraction representation. For example,

$$\frac{1}{3} = 0 + \frac{1}{3} \qquad \text{and} \qquad \frac{1}{3} = 0 + \cfrac{1}{2 + \cfrac{1}{1}},$$

giving both $[0;3]$ and $[0;2,1]$ as valid representations of $\frac{1}{3}$. We can fix this issue by defining the continued fraction representation of a rational to be the shortest possible representation. For simplicity, we may write $\alpha = [a_0; a_1, a_2, \ldots]$ to mean that α is the real number with continued fraction representation $[a_0; a_1, a_2, \ldots]$.

Proposition 2.20. *If α has a continued fraction representation of* $[a_0; a_1, a_2, a_3, \ldots]$, *then $\frac{p_n}{q_n}$, the nth convergent of α, is given by the recursive formula*

$$p_n = a_n p_{n-1} + p_{n-2} \qquad \textit{and} \qquad q_n = a_n q_{n-1} + q_{n-2}$$

with $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, and $q_{-1} = 0$. Further, p_n, q_n are relatively prime.

A proof of Proposition 2.20 can be found in [19]. I think it is remarkable that there is such a simple recursion for the convergents.

Proposition 2.21. *If α has convergents $\frac{p_n}{q_n}$ then for any n, either*

$$\frac{p_n}{q_n} \leq \alpha \leq \frac{p_{n+1}}{q_{n+1}} \qquad \textit{or} \qquad \frac{p_{n+1}}{q_{n+1}} \leq \alpha \leq \frac{p_n}{q_n}.$$

Proposition 2.21 is well known and has a proof in [19]. Since successive convergents for α always approximate α better, Proposition 2.21, along with the fact that $|\alpha - \frac{p_i}{q_i}| \leq |\alpha - \frac{p_j}{q_j}|$ for $j < i$ implies that the even and the odd sequence of convergents are both monotone sequences with one increasing and the other decreasing.

Definition 2.22 (Best Approximation)**.** *Given a number $\alpha \in \mathbb{R}$, the* best rational approximation to α with denominator bounded by q *is the fraction p/q' with $q' \leq q$ such that*

$$|\alpha - \tfrac{p}{q'}| \leq |\alpha - \tfrac{a}{b}|$$

for all $\frac{a}{b} \in \mathbb{Q}$ with $b \leq q$. If there are multiple approximations satisfying this property, then the best approximation is taken to be p/q' where $|p| + |q'|$ is minimal.

Proposition 2.23. *If $\frac{p_n}{q_n}$ is the nth convergent of $\alpha \in \mathbb{R}$, then $\frac{p_n}{q_n}$ is the best rational approximation to α with denominator bounded by q_n.*

A proof of this proposition can be found in any standard number theory textbook, for example [19]. It is worth noting that the converse to Proposition 2.23 is not necessarily true. That is, for a number $\alpha \in \mathbb{R}$, the continued fraction convergents of α may not completely enumerate the best rational approximations of α (the denominators of the convergents may grow very fast, but the next-best rational approximation may have a denominator that does not grow as quickly).

We will call the best rational approximation to a number α a *type 1* rational approximation and we will call the convergents of α *type 2* rational approximation (noting that an approximation can be both type 1 and type 2).

Proposition 2.24. *If $\frac{p_n}{q_n}$ is the nth convergent of α, then*

$$\frac{1}{q_n(q_{n+1}+q_n)} < |\alpha - \tfrac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}.$$

For a proof, consult [8].

Definition 2.25. *For a number $\alpha \in \mathbb{R}$, let*

$$\|\alpha\|_n = \min_{k\in\mathbb{Z}}\{|\alpha - nk|\}.$$

$\|\cdot\|$ denotes the special case $\|\cdot\|_1$.

Proposition 2.26. *For $\alpha \in \mathbb{R}$,*

$$\{q \in \mathbb{N} : \|q\alpha\| < \|a\alpha\| \text{ for all } a < q\} = \{q_n : \tfrac{p_n}{q_n} \text{ is a convergent of } \alpha\}.$$

A formal proof of Proposition 2.26 can be seen in [8]. We will not give a proof, but we will explore the geometry of rotations in relation to continued fractions.

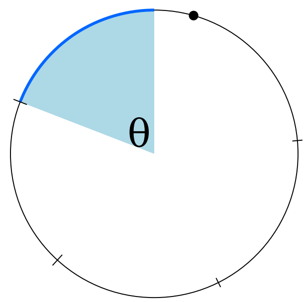
Fix θ, and notice that we can view $\{\|a\theta\| : a \in \mathbb{Z}\}$ as the set of distances of $a\theta$ mod 1 from 0 on the unit circle. For illustration, we will fix $\theta = [0; 5, 4, 3, 5, \ldots] \approx 0.1911357$. The convergents of θ are

$$\frac{p_0}{q_0} = \frac{0}{1} \qquad \frac{p_3}{q_3} = \frac{13}{68}$$

$$\frac{p_1}{q_1} = \frac{1}{5} \qquad \frac{p_4}{q_4} = \frac{69}{261}$$

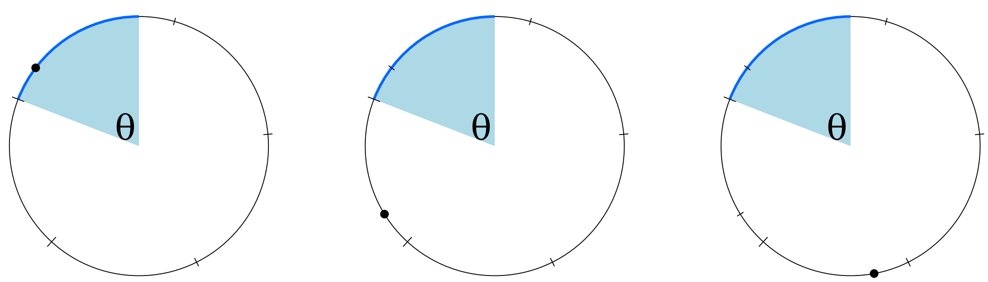
$$\frac{p_2}{q_2} = \frac{4}{21}.$$

Notice that after 5 iterates, the rotation R_θ defined by $R_\theta(x) = x + \theta$ mod 1 obtains a new closest return to 0. That is, if $\mathcal{O}_n = \{\theta, 2\theta, \ldots, n\theta\}$ is the n-orbit under R_θ of 0 (excluding 0 itself), then $\mathcal{O}_n$ achieves a new closest return to 0 when $n = 5$. $\mathcal{O}_5$ is illustrated in the figure below, with 5θ mod 1 marked with a dot and rotations are performed clockwise.

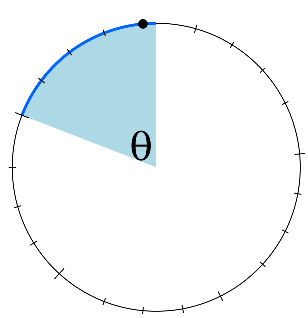


At this point, $\mathcal{O}_5$ partitions $[0,1)$ into 5 intervals of width θ and one interval of width $\|5\theta\|$. It turns out, for a number $\alpha = [a_0; a_1, a_2, \ldots]$, $\lfloor \|q_n\alpha\|/\|q_{n+1}\alpha\| \rfloor = a_{n+2}$ exactly recovers the continued fraction coefficients. We see this in the figure illustrated by the fact that 4 intervals of width $\|5\theta\|$ fit in a single interval of width $\|\theta\|$.

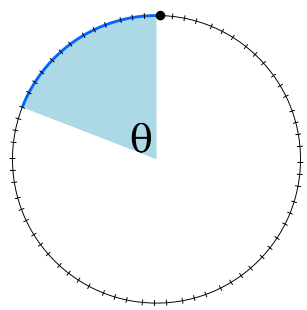
Using the fact that $\lfloor \|q_n\alpha\|/\|q_{n+1}\alpha\| \rfloor = a_{n+2}$ for a number α, we see that each of the 5 intervals of width $\|\theta\|$ can fit 4 intervals of width $\|5\theta\|$. Observing $\mathcal{O}_6$, $\mathcal{O}_7$, and $\mathcal{O}_8$, we can see that the intervals of width $\|\theta\|$ each have smaller intervals of width $\|5\theta\|$ "munched away" from the left side.



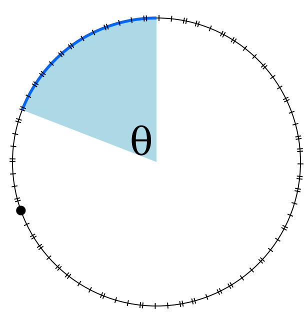
After this process is repeated $a_{n+1}q_n + q_{n-1} = q_{n+1}$ times, we see a new closest return time to 0. That is, $\mathcal{O}_{21}$ contains a new closest return time to 0.



$\mathcal{O}_{21}$ partitions $[0,1)$ into intervals of width $\|21\theta\|$, $\|5\theta\|$, and a few intervals of width $\|5\theta\| + \|21\theta\|$. Now the process repeats again, with each larger interval getting chunks of size $\|21\theta\|$ "munched away" until a new closest return to 0 is obtained in $\mathcal{O}_{68}$.



The process again repeats until the next closest return time in $\mathcal{O}_{261}$. For illustration, below is $\mathcal{O}_{102}$.



Given the link between rotations and Sturmian sequences, we see that for some angle α, the backwards n orbit, $\mathcal{O}_{-n}$ partitions the set of phases in a way identical to partitioning the set of phases by the first n symbols of a Sturmian sequence with angle α.

For a fixed α, it is clear the partition by the first n symbols of a Sturmian sequence gives us a partition of $[0,1)$ by intervals exactly corresponding to "cutting" $[0,1)$ by $\mathcal{O}_{-n}$. In [18], a precise description of these intervals is given, which in general is known as the Stienhaus 3-length conjecture.

Proposition 2.27 (Stienhaus 3-length Conjecture, Slater [18])**.** *Fix* $\alpha \in [0,1)$ *and let* $[0; a_1, a_2, \ldots]$ *be its continued fraction representation and let* $\frac{p_n}{q_n}$ *be the nth convergent* *of α. Fix $n = cq_k + q_{k-1} + \ell$ with $1 \le c \le a_{k+1}$ and $0 \le \ell < q_k$ for some k. Then, the partition $\mathcal{P}$ of $[0, 1)$ generated by $\{\alpha, 2\alpha, \ldots, n\alpha\} \bmod 1$ has intervals of exactly three lengths*

$$l_{short} = \|q_k\alpha\| \qquad l_{med} = \|q_{k-1}\alpha\| - c\|q_k\alpha\| \qquad l_{long} = \|q_{k-1}\alpha\| - (c-1)\|q_k\alpha\|.$$

Further, the number of intervals of length l_{short}, l_{med}, and l_{long} is $n - q_k + 1, \ell + 1$, and $q_k - \ell - 1$.

Proposition 2.27 gives some precision to the previous pictures of $[0, 1)$ being divided up by the orbit of $\theta = [0; 5, 4, 3, 5, \ldots]$.

Lemma 2.28. *Let $\frac{p_n}{q_n}$ be the convergents of α. If $cq_k + q_{k-1} < q_{k+1}$ for $c \in \mathbb{N}$, then*

$$\|(cq_k + q_{k-1})\alpha\| = \|q_{k-1}\alpha\| - c\|q_k\alpha\|.$$

Proof. Consider the quantities $a = q_k\alpha \bmod 1$ and $b = q_{k-1}\alpha \bmod 1$ interpreted as lying in the interval $[-1/2, 1/2)$. By Proposition 2.21, we have that exactly one of a or b is negative. Further, the restriction on c ensures $c|a| \le |b|$ and so $|ca + b| = ||b| - c|a|| = |b| - c|a|$, which proves the claim. $\square$

Proposition 2.29. *Fix $\alpha \in \mathbb{Q}^c$ and let $\mathcal{O}_n = \{\alpha, 2\alpha, \ldots, n\alpha\} \bmod 1$, $a = \min \mathcal{O}_n$, and $b = \max \mathcal{O}_n$. Then one of $\|a\|$ or $\|b\|$ is l_{short} and the other l_{med} as specified in Proposition 2.27.*

Proof. Assume $\alpha \in \mathbb{Q}^c$ and let $n = cq_k + q_{k-1} + \ell = m + \ell$ with c and ℓ as in Proposition 2.27. By Lemma 2.28, $\|m\alpha\| = \|q_{k-1}\alpha\| - c\|q_k\alpha\| = l_{\text{med}}$. Further, $\|q_k\alpha\| = l_{\text{short}}$.

Now, by Proposition 2.21, when we interpret $x = (cq_k\alpha + q_{k-1})\alpha \bmod 1$ and $y = q_k\alpha \bmod 1$ as points in $[-1/2, 1/2)$, one is negative and one is positive. Thus if $\|a\| = \|q_k\alpha\| = l_{\text{short}}$ then $\|b\| \le \|m\alpha\| = l_{\text{med}}$ and visa versa. Finally, noting that $\|a\| = \|b\| = l_{\text{short}}$ implies $\alpha \in \mathbb{Q}$ and that an interval of the partition generated by $\mathcal{O}_n$ shorter than l_{med} must be length l_{short} completes the proof. $\square$

Proposition 2.29 can be extended to work on rationals with the obvious exception that we might have $\|a\| = \|b\| = l_{\text{short}}$.

2.3 Properties of Sturmian Sequences

Recurrence was a required hypothesis for many of the equivalent definitions for Sturmian sequences. However, Sturmian sequences satisfy the much stronger property of minimality.

Proposition 2.30. *Let s be a Sturmian sequence with angle α, and let $O = \overline{\mathcal{O}s}$ be its orbit closure. Then $O = \{\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t) : t \in R\}$ and O is minimal.*

Proof. Fix a Sturmian sequence s with $\alpha(s) = \alpha$ and let $O' = \{\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t) : t \in R\}$. Since s is balanced and recurrent, any accumulation point of $\mathcal{O}s$ must also be balanced and recurrent. Thus, any $y \in O$ is a Sturmian sequence. Since α is continuous and $\alpha(s) = \alpha(T^i s)$, we have that $\alpha(y) = \alpha(s)$ for any $y \in O$, and so by Proposition 2.14, $y \in O'$.

This shows that $O \subset O'$. Let $W(n)$ be the maximum waiting time for a length-n subword of s. As we will see in Theorem 2.33, $W(n) < \infty$. From the definition of convergence, we have that every point in O has a waiting time bounded by $W(n)$, so since the set of subwords of $y \in O$ is identical to the set of subwords of points in $\mathcal{O}s$, O is minimal. However, the set of subwords of $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ is identical for all $t \in R$, and $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ is a Sturmian sequence for all $t \in R$, and so $O = O'$. □

Corollary 2.31. *Every Sturmian sequence s is the limit of rotation sequences of the form $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ where $\alpha, t \in \mathbb{R}$.*

Corollary 2.31 follows from a proof very similar to that of Proposition 2.14.

Notation 2.32 ($\mathbb{T}$)**.** *The torus $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbb{T}$ and is assumed to have the quotient topology unless otherwise specified.*

Where convenient, we may think of the phases of a Sturmian sequence as lying in $\mathbb{T}$ instead of $\mathbb{R}$.

We can also very precisely bound the waiting times for any word as well as the best periodic approximation to a Sturmian word.

Theorem 2.33. *Fix α. Let $s = \mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, 0)$, and fix a subword $w \subset s$. Let λ_w be the frequency of w in s and let W_w be the maximum waiting time between successive (possibly overlapping) occurrences of w. Then,*

$$\frac{1}{\lambda_w} \leq W_w \leq \frac{1}{\lambda_w} + |w|.$$

Proof. The lower bound follows immediately: if $W_w < \frac{1}{\lambda_w}$, then the frequency of w must exceed λ_w, a contradiction.

Fix α and w with $|w| = n$, and let $\mathcal{P}_n$ be the partition of the phase space $[0, 1)$ such that t, t' lie in the same partition element if the first n symbols of $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ and $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t')$ agree. That is $(\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_0^{n-1} = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t'))_0^{n-1}$.

Let $[a_0; a_1, \dots]$ be the continued fraction expansion of α and let $\frac{p_k}{q_k}$ be the convergent of α such that $n = cq_k + q_{k-1} + \ell$ with c, ℓ as in Proposition 2.27. By Proposition 2.27, $\mathcal{P}_n$ consists of intervals of exactly three lengths: l_{short}, l_{med}, and l_{long}. These lengths are given by

$$l_{\text{short}} = \|q_k\alpha\| \qquad l_{\text{med}} = \|q_{k-1}\alpha\| - c|I_{\text{short}}| \qquad l_{\text{long}} = \|q_{k-1}\alpha\| - (c-1)|I_{\text{short}}|.$$

Let $\mathcal{O}_i = \{0, \alpha, 2\alpha, \dots, (i-1)\alpha\} \bmod 1$ be the i-orbit of 0 under rotation by α. Let $I_w \in \mathcal{P}_n$ be the partition element such that $(\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_0^{n-1} = w$ for any $t \in I_w$. Since $|I_w| = \lambda_w$, the upper bound will be proved if we can show that $\mathcal{O}_{1/|I_w|+|w|} + t$ for any $t \in I_w$ intersects I_w in at least two places. Equivalently, we may show that $\mathcal{O}_{1/|I_w|+|w|}$ intersects any interval containing 0 of width $|I_w|$ in at least two places (one of those places being 0).

We will consider cases based on $|I_w|$. Suppose $|I_w| = l_{\text{med}}$ or l_{long}. In either of these cases, $1/|I_w| \geq q_k$, which follows quickly from the bound $|I_w| \leq \|q_{k-1}\alpha\| < \frac{1}{q_k}$.

Let $m = q_k + |w| \leq 1/|I_w| + |w|$. Let $a = \min(\mathcal{O}_{|w|}\backslash\{0\})$ and $b = \max(\mathcal{O}_{|w|}\backslash\{0\})$. By Proposition 2.29, $\|a\|$ and $\|b\|$ are l_{short} and l_{med}. For simplicity, assume $\|b\| = l_{\text{short}}$ and $\|a\| = l_{\text{med}}$. Under this assumption, $q_k\alpha \bmod 1 = 1 - \|q_k\alpha\|$. Thus, $a - \|q_k\alpha\| \in \mathcal{O}_{|w|+q_k} = \mathcal{O}_m$. Since the distance between $a - \|q_k\alpha\|$ and b is $l_{\text{med}} \leq |I_w|$, any interval of width $|I_w|$ containing 0 must intersect a non-zero point of $\mathcal{O}_m \subset \mathcal{O}_{1/|I_w|+|w|}$.

Finally, consider the case where $|I_w| = l_{\text{short}}$. In this case, $1/|I_w| \geq q_{k+1}$. Let $m = q_{k+1} + |w|$. Let $a = \min(\mathcal{O}_{|w|}\backslash\{0\})$ and $b = \max(\mathcal{O}_{|w|}\backslash\{0\})$. By Proposition 2.29, $\|a\|$ and $\|b\|$ are l_{short} and l_{med}. For simplicity, assume $\|b\| = l_{\text{short}}$ and $\|a\| = l_{\text{med}}$. Under this assumption and by Proposition 2.21 we have that $q_{k+1}\alpha \bmod 1 = \|q_{k+1}\alpha\|$. Thus, we have the following relation,

$$\mathcal{O}_m = \mathcal{O}_{q_{k+1}+|w|} = \mathcal{O}_{q_{k+1}} \cup (\mathcal{O}_{|w|} + \|q_{k+1}\alpha\|),$$

showing that both $\|q_{k+1}\alpha\|$ and $b + \|q_{k+1}\alpha\|$ are in $\mathcal{O}_m$. Since the distance between $\|q_{k+1}\alpha\|$ and $b + \|q_{k+1}\alpha\|$ is $b = l_{\text{short}} = |I_w|$, the claim is proved. $\square$

Theorem 2.33 may be interpreted as a type of higher block balanced property.

Definition 2.34. *If w is a subword of some Sturmian sequence, define*

$$q(w) = \inf_{p,q \in \mathbb{N}} \{q : w \subset \mathbf{R}_{\lfloor\cdot\rfloor}(p/q, 0)\}.$$

Notice that $q(w)$ is the shortest period of a periodic Sturmian sequence that contains the word w.

Proposition 2.35. *If w is a subword of a Sturmian sequence,*

$$q(w) \leq |w|.$$

In other words, w is contained in a periodic Sturmian sequence with period $\leq |w|$.

Proof. Fix n. Let $\mathcal{P} = \{P_0, P_1, \ldots\}$ be the partition of $[0,1]^2$ generated by the relation

$$(\alpha, t) \sim (\alpha', t') \qquad \text{if} \qquad (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_0^{n-1} = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t'))_0^{n-1}.$$

That is, $(\alpha, t) \sim (\alpha', t')$ if the sequences $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t)$ and $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t')$ start with the same n-word.

If we can show that every P_i contains a point $(p/q, t)$ where $q \leq n$, then we will have shown that for any Sturmian word w with $|w| = n$, we have $q(w) \leq |w|$.

Fix α and consider now the partition $\mathcal{X}_\alpha = \{X_0, X_1, \ldots\}$ where $t \sim t'$ if $(\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_0^{n-1} = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t'))_0^{n-1}$. Notice that this is a partition into half-open intervals whose endpoints are the point $\{0, -\alpha, \ldots, -n\alpha \bmod 1\}$.

Notice now that $\mathcal{X}_\alpha$ is precisely the fiber of $\mathcal{P}$ along the line $\{\alpha\} \times [0,1]$, and so the boundaries of the partition $\mathcal{P}$ are the set

$$\{-i\alpha \bmod 1 : 0 \leq i \leq n \text{ and } \alpha \in [0,1]\}.$$

This set is identical to the set of graphs of lines of the form $L_a(\alpha) = -a\alpha \bmod 1$ for $0 \leq a \leq n$ or equivalently, $L_{a,b}(\alpha) = -a\alpha + b$ for $0 \leq a, b \leq n$ restricted to $[0,1]^2$.

Computing, we see the intersection between the lines $L_{a,b}$ and $L_{a',b'}$ occurs at

$$\alpha = \frac{b' - b}{a' - a} \qquad \text{and} \qquad L_{a,b}(\alpha) = \frac{a'b - ab'}{a' - a}.$$

Since $|a' - a| \leq n$, we see that the corners of every P_i have rational coordinates with

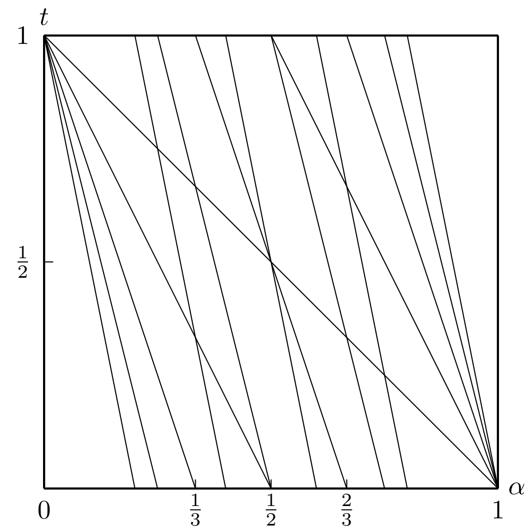


Figure 2.1: The partition $\mathcal{P}$ when $n = 5$.

denominator $\leq n$. To complete the proof, notice that except for the two extreme cases (the two triangles with vertical edges), every edge of every polygon in $\mathcal{P}$ has negative slope. Therefore, every polygon must contain a point (e, t) where e is the α-component of some corner and $t \in [0, 1]$. For the remaining two partition elements, it is easy to check that one contains $(0, 0)$ and the other contains $(\frac{n-1}{n}, 1)$. $\square$

2.4 Generalized Sturmians

Although the set of all Sturmian sequences that share a common angle is closed, the same cannot be said if we take a union over all angles.

Definition 2.36. *The set of* generalized *Sturmian sequences, $\bar{\mathcal{S}}$, is the closure of $\mathcal{S}$ under d, the usual metric on sequences.*

Proposition 2.37. *$\bar{\mathcal{S}}$ is strictly bigger than $\mathcal{S}$.*

Proof. Consider the sequence of Sturmian sequences

$$s_n = \mathbf{R}_{\lfloor \cdot \rfloor}(1/n, -1/(2n)),$$

and observe that $s_n \to s = (\ldots, 0, 0, 1, 0, 0, \ldots)$, the sequence of all zeros with a single 1 at position 0. Since s is not recurrent, s is not Sturmian, however $s \in \bar{\mathcal{S}}$. $\square$

From Proposition 2.37, it is clear that we cannot write every element in $\bar{\mathcal{S}}$ as a rotation sequence with real parameters, however, allowing infinitesimals in the angle as well as the phase will allow us to represent $\bar{\mathcal{S}}$ as rotation sequences.

Proposition 2.38 (Infinitesimal Representation)**.**

$$\bar{\mathcal{S}} = \{\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) : \alpha \in \mathbb{R}^{\pm} \textit{ and } t \in R\}.$$

Proof. Let $I = \{\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) : \alpha \in \mathbb{R}^{\pm}, t \in R\}$. We will first show that $I \subset \bar{\mathcal{S}}$.

Fix $y \in I$ with parameters $\alpha_y = \alpha + \epsilon$ and $t_y = t + n\epsilon$ or $\alpha_y = \alpha - \epsilon$ and $t_y = t + n\epsilon$. To be concise, we will write $\alpha_y = \alpha \pm \epsilon$ and consistently use $+$ or $-$ in the following equations.

Consider the sequence $z_i \in \mathcal{S}$ where

$$\begin{aligned}(z_i)_k &= \left\lfloor (k+1)(\alpha \pm \frac{1}{i}) + t + \frac{n}{i} \right\rfloor - \left\lfloor k(\alpha \pm \frac{1}{i}) + t + \frac{n}{i} \right\rfloor \\ &= \left\lfloor (k+1)\alpha + t + \frac{n \pm (k+1)}{i} \right\rfloor - \left\lfloor k\alpha + t + \frac{n \pm k}{i} \right\rfloor .\end{aligned}$$

Comparatively,

$$(y)_k = \lfloor (k+1)\alpha + t + (n \pm (k+1))\epsilon \rfloor - \lfloor k\alpha + t + (n \pm k)\epsilon \rfloor .$$

Since $\lfloor\cdot\rfloor$ is continuous off the integers, it is clear that if $(k\alpha + t), ((k+1)\alpha + t) \notin \mathbb{Z}$ then $(z_i)_k \to (y)_k$. Further, $1/i$ becomes arbitrarily small as $i \to \infty$, and so $(z_i)_k \to (y)_k$ for all k.

We will now show $\bar{\mathcal{S}} \subset I$. Fix $y \in \bar{\mathcal{S}}$. By Corollary 2.31, we may find $y_i \to y$ with $y_i = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha_{y_i}, t_{y_i}) \in \mathcal{S}$ (note α_{y_i} and t_{y_i} are real). We then have that $\alpha_{y_i} \to \alpha_y$ where α_y is the average of the digits of y, and by passing to a subsequence if necessary, we may assume $t_{y_i} \to t_y \in [0, 1]$.

Define

$$d_k(i) = k\alpha_{y_i} + t_i$$

and note that by passing to a subsequence, we may assume for every k that $\lfloor d_k(i) \rfloor$ is eventually constant in i (This is clear since $\alpha_{y_i} \to \alpha_y$ and $t_{y_i} \to t_y$ imply that $\{\lfloor d_k(i) \rfloor : i \in \mathbb{Z}\}$ has at least one accumulation point). Let $d_k = \lim_{i\to\infty} d_k(i) = k\alpha_y + t_y$.

We now see that if $d_{k+1}, d_k \notin \mathbb{Z}$, $\lfloor d_{k+1}(i) \rfloor \to \lfloor d_{k+1} \rfloor$ and $\lfloor d_k(i) \rfloor \to \lfloor d_k \rfloor$, and so the kth coordinate of y agrees with the kth coordinate of $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha_y, t_y)$.

If $d_k \in \mathbb{Z}$, we see that because $\lfloor d_k(i) \rfloor$ is eventually constant, $d_k(i) \to d_k$ must converge one-sidedly (We will notate one-sided convergence from above as $d_k(i) \searrow d_k$ and from below as $d_k(i) \nearrow d_k$).

Let $K = \{k : d_k \in \mathbb{Z}\}$. If $d_k(i) \searrow d_k$ for all $k \in K$ or $d_k(i) \nearrow d_k$ for all $k \in K$, then $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha_y, t_y + \epsilon)$ or $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha_y, t_y - \epsilon)$ respectively.

If not, $|K| \geq 2$. Let q be the minimum gap between numbers in K and note that $d_k, d_{k+q} \in \mathbb{Z}$ implies $\alpha_y = \frac{p}{q} \in \mathbb{Q}$. From this we may deduce that $K = \{k_0 + nq : n \in \mathbb{Z}\}$ for some k_0.

Fix $k_0 \in K$ so that $d_{k_0}(i) \searrow d_{k_0}$ and either $d_{k_0+q}(i) \nearrow d_{k_0+q}$ or $d_{k_0-q}(i) \nearrow d_{k_0-q}$. Assume $d_{k_0+q}(i) \nearrow d_{k_0+q}$ (since the other case follows similarly).

This means $k_0\alpha_{y_i} + t_{y_i}$ converges from above, but $(k_0+q)\alpha_{y_i} + t_{y_i} = (k_0\alpha_{y_i} + t_{y_i}) + q\alpha_{y_i}$ converges from below. From this we conclude that $\alpha_{y_i} \nearrow \alpha_y$ and that if $n \geq 0$,

$$d_k(i) \nearrow d_k$$

for all $k = k_0 + nq > k_0$ and

$$d_k(i) \searrow d_k$$

for all $k = k_0 - nq \leq k_0$.

Thus, upon inspection, we see $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha_y - \epsilon, t_y + k_0\epsilon)$. □

Using infinitesimal representation, we can now write the generalized Sturmian sequence $(\ldots, 0, 0, 1, 0, 0, \ldots)$ as $\mathbf{R}_{\lfloor\cdot\rfloor}(\epsilon, 0)$.

Although Proposition 2.38 gives a parameterization of generalized Sturmian sequences, it suffers from non-uniqueness just as the parameterization of Sturmian sequences by rotation sequences does. Further, the topology induced on the parameter space $\mathbb{R}^{\pm} \times R$ by d, the standard metric on sequences, is quite unwieldy. However, if we restrict ourselves to generalized Sturmians whose angle is irrational, we have near-uniqueness in our representation as a rotation sequence.

Notation 2.39.

$$\bar{\mathcal{S}}_{\mathbb{Q}} = \{s \in \bar{\mathcal{S}} : \alpha(s) \in \mathbb{Q}\} \qquad \textit{and} \qquad \bar{\mathcal{S}}_{\mathbb{Q}^c} = \{s \in \bar{\mathcal{S}} : \alpha(s) \notin \mathbb{Q}\}.$$

Since $s \in \bar{\mathcal{S}}$ is still a balanced sequence, $\alpha(s)$ is defined, and so Notation 2.39 is well defined.

Proposition 2.40. *For $y \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$, the following properties hold:*

1. $y \in \mathcal{S}$*;*

2. *if t_y is a phase for y, $\mathrm{Re}(t_y)$ is uniquely determined and is equal to $t(y)$;*

3. $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha(y), t(y) + \epsilon)$ *or* $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha(y), t(y) - \epsilon)$ *and this representation is unique if $n\alpha(y) + t(y) \in \mathbb{Z}$ for some n, and $y = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha(y), t(y) + \epsilon) = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha(y), t(y) - \epsilon)$ if $n\alpha(y) + t(y) \notin \mathbb{Z}$ for any n.*

Proof. First note that if $r \in \mathcal{S}$ with $\alpha(r) \notin \mathbb{Q}$, then $t(r)$ is uniquely determined and so property 2 holds as well as property 3. Suppose $s \in \bar{S}$ and $\mathrm{Re}(\alpha(s)) \notin \mathbb{Q}$. By the prior observation, it will be sufficient to show that $s \in \mathcal{S}$.

By Proposition 2.38, we may write

$$s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha \pm \epsilon, t + n\epsilon).$$

If $m\alpha + t \notin \mathbb{Z}$ for all m, then $\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha \pm \epsilon, t + n\epsilon) = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha \pm \epsilon, t)$, and so $s \in \mathcal{S}$. Otherwise, $m\alpha + t \in \mathbb{Z}$ for precisely one m. In this case, if $\pm m + n$ is positive, $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) \in \mathcal{S}$ and if $\pm m + n$ is negative, $s = \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t) \in \mathcal{S}$. □

Proposition 2.40 shows that $\bar{\mathcal{S}}_{\mathbb{Q}^c} \subset \mathcal{S}$, and so by restricting our attention to $\bar{\mathcal{S}}_{\mathbb{Q}^c}$, we can avoid many technical issues in our analysis of generalized Sturmians (this is one reason many authors define Sturmian sequences to be aperiodic).

Definition 2.41. *Let $q_n : \bar{\mathcal{S}} \to \mathbb{N}$ be defined such that*

$$q_n(s) = \min\{q \in \mathbb{N} : (s)_{-n}^{n} \textit{ is a subword of } \mathbf{R}_{\lfloor\cdot\rfloor}(p/q \pm \epsilon, \epsilon\mathbb{Z}) \textit{ for some } p \in \mathbb{N}\}.$$

Note that q_n is defined so that $s \in \bar{\mathcal{S}}_{\mathbb{Q}}$ implies $\lim_{n\to\infty} q_n(s) < \infty$ (because s is periodic) and $s \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$ implies $\lim_{n\to\infty} q_n(s) = \infty$ (because s is necessarily aperiodic). Further, by an application of Proposition 2.35 we see

$$q_n(s) \leq 2n + 1$$

for any $s \in \bar{\mathcal{S}}$.

Definition 2.42. *Define a metric $\hat{d}$ on $\bar{\mathcal{S}}$ in the following way.*

If $y = z$ then $\hat{d}(y,z) = 0$. Otherwise, let $n = \sup\{i : (y)_{-i}^{i} = (z)_{-i}^{i}\}$ and define

$$\hat{d}(y,z) = \frac{1}{q_n(y)} = \frac{1}{q_n(z)}.$$

Proposition 2.43. *$\hat{d}$ is a metric.*

Proof. Reflexivity, non-degeneracy, and symmetry are clear. The triangle inequality is also deduced quite quickly. Fix $x, y, z \in \bar{\mathcal{S}}$. Suppose that $d(y,z) = 2^{-a}$, $d(y,x) = 2^{-b}$, and $d(x,z) = 2^{-c}$ and note that either $b \leq a$ or $c \leq a$. Suppose $b \leq a$. Since for $i \leq n$, $q_i(y) = q_i(z)$ and $q_i(y), q_i(z)$ are monotone in i we have

$$\hat{d}(y,z) = \frac{1}{q_a(y)} \leq \frac{1}{q_b(y)} \leq \frac{1}{q_b(y)} + \frac{1}{q_c(z)} = \hat{d}(y,x) + \hat{d}(x,z).$$

If $c \leq a$, then

$$\hat{d}(y,z) = \frac{1}{q_a(y)} \leq \frac{1}{q_c(z)} \leq \frac{1}{q_b(y)} + \frac{1}{q_c(z)} = \hat{d}(y,x) + \hat{d}(x,z).$$

□

In fact, $\hat{d}$ is an *ultra metric* which means it satisfies a stronger version of the triangle inequality: $\hat{d}(x,y) \leq \max\{\hat{d}(x,z), \hat{d}(z,y)\}$.

Note that since $q_n(y) \leq n$, we have that $d(y,z) \leq \hat{d}(y,z)$ for all $y, z \in \bar{\mathcal{S}}$, and that $\hat{d}$ is constructed so that sequences whose angles are heading towards a rational number diverge. This leads to the following proposition.

Proposition 2.44. *$\bar{\mathcal{S}}_{\mathbb{Q}^c}$ is a complete metric space with respect to $\hat{d}$ and the topology induced by $\hat{d}$ is the relative topology induced by d.*

Proof. We will first show that the topologies induced by $\hat{d}$ and d are the same. Since $\hat{d}(y,z) \geq d(y,z)$, the topology induced by $\hat{d}$ is at least as fine as that induced by d, so we need only to show that a sequence that converges in d converges in $\hat{d}$.

Let $y_i, y \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$ with $y_i \xrightarrow{d} y$. Let $2n_i$ be be the largest window about the origin where y_i and y agree. By convergence in d, $n_i \to \infty$. Since $y \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$, $q_n(y) \to \infty$ as $n \to \infty$. Because of this, $q_{n_i}(y_i) \to \infty$ as $i \to \infty$ and so $\hat{d}(y_i, y) \to 0$.

Next, we will show that $\bar{\mathcal{S}}_{\mathbb{Q}^c}$ is complete with respect to $\hat{d}$. First, consider a Cauchy sequence y_i. Being Cauchy in $\hat{d}$ implies that you are Cauchy in d. We may therefore conclude that $y_i \to y \in \bar{\mathcal{S}}$. What remains to be shown is that $y \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$.

Suppose $y \in \bar{\mathcal{S}}_{\mathbb{Q}}$. This means that $q_n(y)$ is bounded for all n. Thus $\hat{d}(y_i, y) \nrightarrow 0$. □

Proposition 2.45. *Both α and t are continuous on $\bar{\mathcal{S}}_{\mathbb{Q}^c}$ with respect to $\hat{d}$.*

Proof. By the balanced property of sequences in $\bar{\mathcal{S}}_{\mathbb{Q}^c}$, α is determined up to an error of $1/n$ by the first n digits of a sequence, and is therefore continuous.

We will now prove the continuity of t using the properties of q_n. Fix α and observe that for a Sturmian sequence s with $\alpha(s) = \alpha$, if $q_n(s) = q$, then $(s)_{-n}^{n}$ determines the phase of s up to an interval of width $1/q$. We now have that for an arbitrary Sturmian sequence s, $(s)_{-n}^{n}$ determines the phase of s up to an interval of width $2/q_n(s) + 1/(2n+1)$, where $1/(2n+1)$ comes from our bound on $\alpha(s)$. Proposition 2.35 gives that $q_n(s) \leq 2n+1$ and so in fact $(s)_{-n}^{n}$ determines the phase of s up to an interval of width $3/q_n(s)$.

Fix $s \in \bar{\mathcal{S}}_{\mathbb{Q}^c}$ and $\delta > 0$ and choose n so that $q_n(s)/3 > 1/\delta$. Now, $(s)_{-n}^{n}$ determines the phase of s up to an interval of width δ. Call this interval I_δ. We now have that for any y with $d(s, y) \leq 2^{-(n+1)}$ (i.e., any y with $(y)_{-n}^{n} = (s)_{-n}^{n}$), $t(y) \in I_\delta$, and so t is continuous.

□

2.5 2-d Sturmian Configurations

We will now explore configurations on $\mathbb{Z}^2$ where every row is Sturmian.

Notation 2.46.

$$\Omega = \{y : \textit{every row of } y \textit{ is in } \bar{\mathcal{S}}\}$$

and

$$\Omega_{\mathbb{Q}^c} = \{y : \textit{every row of } y \textit{ is in } \bar{\mathcal{S}}_{\mathbb{Q}^c}\}.$$

We may abuse notation by saying $x \in \Omega$ if the rows of x are generalized Sturmian sequences for *some* two letter alphabet. That is, the rows of x do not have to be Sturmian sequences on the alphabet $\{0, 1\}$. This will allow us to more cleanly talk about the Kari-Culik tilings where Sturmian sequences on the alphabet $\{1, 2\}$ arise.

Extend $\hat{d}$ to a metric on Ω by

$$\hat{d}_\Omega(y, z) = \sup_{i \in \mathbb{Z}} 2^{-|i|} \hat{d}((y)_i, (z)_i),$$

which gives the product topology on Ω with respect to the topology induced by $\hat{d}$ on the fibers. When unambiguous, we will write $\hat{d}$ instead of $\hat{d}_\Omega$. Further, extend α and t to Ω by

$$\alpha(y) = (\ldots, \alpha((y)_0), \alpha((y)_1), \ldots)$$

and

$$t(y) = (\ldots, t((y)_0), t((y)_1), \ldots)$$

as well as $\overline{\alpha}$ and $\underline{\alpha}$ in the analogous way (in the future, we will be applying $\overline{\alpha}$ and $\underline{\alpha}$ to non-Sturmian sequences). We will call $\alpha(y)$ the *vector of angles* of y and $t(y)$ the *vector of phases* of y.

Note the following relations

$$\alpha \circ T = \alpha \qquad \text{and} \qquad \alpha \circ S = S' \circ \alpha$$

$$t \circ T = t + \alpha \bmod \vec{1} \qquad \text{and} \qquad t \circ S = S' \circ t$$

where T, S are the horizontal and vertical shifts on $\Omega_{\mathbb{Q}^c}$ and S' is the shift on vectors indexed by $\mathbb{Z}$.

Proposition 2.47. *$\Omega_{\mathbb{Q}^c}$ is complete with respect to $\hat{d}$.*

Proof. This follows directly from the definition of $\hat{d}$. □

Proposition 2.48. *Both α and t are continuous on $\Omega_{\mathbb{Q}^c}$ with respect to $\hat{d}$.*

Proof. The Cartesian product of a finite number of continuous functions is always continuous in the product topology. Further, a function in the product topology is continuous if and only if all projections onto a finite number of coordinates are continuous. It follows that countable Cartesian products of continuous functions are continuous in the product topology.

The proof is complete by observing that $\alpha, t : \Omega_{\mathbb{Q}^c} \to \mathbb{R}^{\mathbb{Z}}$ are countable Cartesian products of continuous functions (with respect to $\hat{d}$). □

Definition 2.49. *For $y \in \Omega$, define $P(y) \subset \mathbb{T}^{\mathbb{Z}}$ by*

$$P(y) = \overline{\{n\alpha(y) + t(y) \bmod \vec{1} \;\; \textit{for } n \in \mathbb{Z}\}},$$

where the closure is with respect to the product topology.

Definition 2.50. *Let $\Omega_{\mathbb{Q}^c}^{\text{rat}} \subset \Omega_{\mathbb{Q}^c}$ be the set of points whose rows have rationally related angles. That is*

$$\Omega_{\mathbb{Q}^c}^{\text{rat}} = \left\{ y \in \Omega_{\mathbb{Q}^c} : \frac{\alpha((y)_i)}{\alpha((y)_j)} \in \mathbb{Q} \textit{ for all } i, j \in \mathbb{Z} \right\}.$$

A priori, the set $P(y)$ may tell us very little about y, however, when we restrict to $y \in \Omega_{\mathbb{Q}^c}^{\text{rat}}$ (those points in $\Omega_{\mathbb{Q}^c}$ whose rows have rationally related angles), $P(y)$ will allow for an easy comparison between points. We see that

$$P \circ T = P \qquad \text{and} \qquad P \circ S = S' \circ P$$

where S' is the shift operator on $\mathbb{T}^{\mathbb{Z}}$.

The set $\Omega_{\mathbb{Q}^c}^{\text{rat}}$ will play an important role as a naturally occurring object in the analysis of Kari-Culik tilings.

Definition 2.51. *For vectors $\alpha, t \in \mathbb{R}^{\mathbb{Z}}$, define the line with direction α through the point t as*

$$L(\alpha, t) = \{\ell\alpha + t \mod \vec{1} : \ell \in \mathbb{R}\}.$$

Proposition 2.52. *If $y \in \Omega_{\mathbb{Q}^c}^{\text{rat}}$, then $P(y)$ is the closure of the graph of a line $\mod \vec{1}$. Specifically,*

$$P(y) = \overline{L(\alpha(y), t(y))}.$$

Further, $P(y)$ is one dimensional in the sense that $\text{proj}_{-n}^{n} P(y) = \text{proj}_{-n}^{n} L(\alpha(y), t(y))$ is a one-dimensional line, where proj_{-n}^{n} is projection onto the $-n$ to n coordinates.

Proof. Notice that $L_n = \text{proj}_{-n}^{n} L(\alpha(y), \vec{0})$ is a one-dimensional subgroup of $\mathbb{T}^{2n+1}$ under addition. Since the coordinates of $\alpha(y)$ are rationally related, L_n is closed.

Further, notice that $G_n = \text{proj}_{-n}^{n}\{i\alpha(y) \mod \vec{1} : i \in \mathbb{Z}\} \subset L_n$ is a subgroup of L_n under addition. Since every coordinate of $\alpha(y)$ is irrational, G_n is dense in L_n, showing $L_n = \overline{G_n}$.

Let $L = L(\alpha(y), \vec{0})$ and $G = \{i\alpha(y) \mod \vec{1} : i \in \mathbb{Z}\}$. From the definition of the product topology, we now conclude

$$\bar{L} = \bar{G}.$$

Lastly, since $L(\alpha(y), t(y)) = L + t(y)$, we see

$$\overline{L(\alpha(y), t(y))} = P(y).$$

$\square$

Proposition 2.52 gives a relationship between orbits of points in $\Omega^{\mathrm{rat}}_{\mathbb{Q}^c}$ and lines, and we will exploit this later on.

Chapter 3

The Kari-Culik Tilings

Recall that a Wang tiling is a nearest-neighbour $\mathbb{Z}^2$ subshift of finite type whose rules are typically given by labeling the edges of square tiles and insisting that two tiles may lie adjacent only if their shared edge labels match. Currently, the smallest known set of Wang tiles that tile the plane only aperiodically is the Kari-Culik tile set. Discovered in 1995, it has only 13 tiles. We call the set of Kari-Culik tiles $\mathfrak{K}$ and note that rotations and reflections of tiles in $\mathfrak{K}$ are allowed. The Kari-Culik tiles are listed in Figure 3.1.

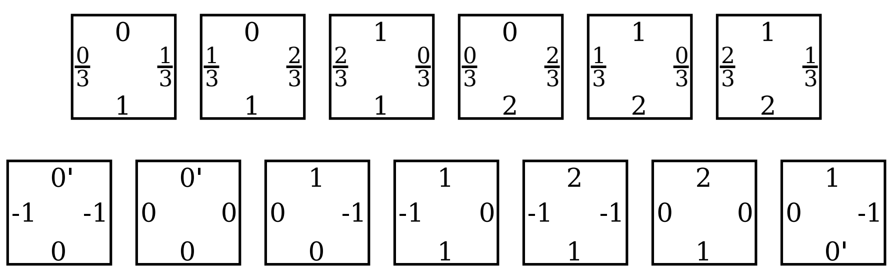


Figure 3.1: List of the 13 Kari-Culik tiles.

A detailed exposition and proof of the Kari-Culik tiling's aperiodic properties can be found in [6]. For completeness, we will give a brief recap of the proof given by Eigen et. al. in [6], which somewhat differs from the original proof given by Kari and Culik.

3.1 Aperiodicity of the Kari-Culik tilings

The explanation of aperiodicity we will give relies on some number-theoretic properties of the Kari-Culik tiles. Namely, every tile satisfies the soon-to-be-introduced *multiplier property* and the tiles are constructed so that if one averages the bottom labels of the rows of a Kari-Culik tiling (treating $0'$ as 0), the sequence of averages behave like an orbit under an easy-to-understand homeomorphism.

Definition 3.1. $\Phi : \mathfrak{K}^{\mathbb{Z}^2} \to \{0,1,2\}^{\mathbb{Z}^2}$ *is projection onto the bottom labels of tiles in* $\mathfrak{K}$ *followed by mapping the symbol* $0'$ *to* 0.

Notice that for any Kari-Culik tiling, the rows fall into two distinct categories: those where every tile has left-right edge labels in $\{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\}$ and those where every tile has left-right edge labels in $\{0, -1\}$. We will call these rows, as well as the tiles in each row, *type* $\frac{1}{3}$ and *type* 2 respectively. The convention in this paper will be to refer to the labels of a tile in $\mathfrak{K}$ in clockwise order starting with the bottom label. That is, the labels a, b, c, d of a tile will correspond to the figure:

$$\boxed{\begin{array}{ccc} & c & \\ b & & d \\ & a & \end{array}}$$

Part of the cleverness of the Kari-Culik tilings is that every tile satisfies the following.

Definition 3.2 (Multiplier Property)**.** *A Kari-Culik tile with bottom, left, top, and right labels of* a, b, c, d *satisfies the relationship*

$$\lambda a + b = c + d \tag{3.1}$$

where $\lambda \in \{\frac{1}{3}, 2\}$ *corresponds to the type of the tile. We also refer to* λ *as the* multiplier *of the tile.*

Proposition 3.3. *Fix a Kari-Culik configuration* x *and let* $r_0 = \Phi((x)_0)$ *and* $r_1 = \Phi((x)_1)$*. Then, if the average of* r_0 *exists, it satisfies the relation*

$$\lambda\alpha(r_0) = \alpha(r_1)$$

where $\lambda \in \{\frac{1}{3}, 2\}$ *is the type of* $(x)_0$*.*

Proof. This is a direct result of the telescoping nature of the multiplier property when rewritten as $\lambda a - c = d - b$. Notice that in any row, every tile is the same type and therefore has the same multiplier. Let a_i be the bottom labels and c_i be the top labels of $(x)_0$. Summing along a central segment of length $2n+1$, we have

$$\lambda \sum_{i=-n}^{n} a_i - \sum_{i=-n}^{n} c_i = d - b \tag{3.2}$$

where b, d are the left and right labels of the central segment. Since

$$\alpha(r_0) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} a_i \qquad \text{and} \qquad \alpha(r_1) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} c_i$$

and b, d are bounded, dividing both sides of Equation (3.2) by $2n+1$ and taking a limit produces the desired relationship. □

Proposition 3.4. *For a Kari-Culik tiling x, $\overline{\alpha}((\Phi(x))_i) \in [1/3, 2]$ and $\underline{\alpha}((\Phi(x))_i) \in [1/3, 2]$ for all i.*

Proof. Fix i and let $\alpha = \overline{\alpha}((\Phi(x))_i)$. Inspecting the tile set, we see that the largest symbol on the bottom of any tile is 2 and so $\alpha \leq 2$. Now, let $\alpha = \underline{\alpha}((\Phi(x))_i)$. To see that $\alpha \geq 1/3$, we will consider rows by type. For a row of type $\frac{1}{3}$, the smallest symbol appearing on the bottom is 1, and so $\alpha \geq 1 \geq 1/3$.

For a row of type 2, notice that the bottom labels may contain 0 or $0'$, but not both (if a row of type 2 had both 0 and $0'$ on the bottom, the row below it would need to have tiles of both type $\frac{1}{3}$ and type 2). If the bottom labels only contain 0, then the row below $(x)_i$ must be of type $\frac{1}{3}$. Inspecting the type $\frac{1}{3}$ tiles, we see that no more than two consecutive 0 symbols may occur as top labels and so $(x)_i$ cannot have more than two 0 symbols in a row as bottom labels giving $\alpha \geq 1/3$. Finally, by inspecting the tile set, we notice that as bottom labels all occurrences of $0'$ must be isolated. Thus, if the row below $(x)_i$ is of type 2, $\alpha \geq 1/2 \geq 1/3$. □

We will now introduce a map that is intimately related to the Kari-Culik tilings.

Definition 3.5. *For $x \in [1/3, 2]$, define*

$$\lambda_x = \begin{cases} 2 & \text{if } x \in [1/3, 1) \\ 1/3 & \text{if } x \in [1, 2] \end{cases} \qquad \textit{and} \qquad f(x) = \lambda_x x = \begin{cases} 2x & \text{if } x \in [1/3, 1) \\ x/3 & \text{if } x \in [1, 2] \end{cases}.$$

Definition 3.6 (Conjugate)**.** *We say two maps $g : X \to X$ and $h : Y \to Y$ are* conjugate *if there exists a continuous bijection $\phi : X \to Y$ so that $g = \phi^{-1} \circ h \circ \phi$. In this case, ϕ is called a* conjugacy*.*

When two maps are conjugate, we can study the easier-to-understand map and then use the conjugacy to carry desired properties over to the more difficult map.

Proposition 3.7 (Liousse [11])**.** *The map f is conjugate to an irrational rotation by* $\log 2/\log 6$*.*

Proof. An explicit conjugacy $\phi : [1/3, 2] \to [0, 1]$ is given by $\phi(x) = \frac{\log x+\log 3}{\log 6}$. □

Proposition 3.8 (Durand, Gamard, and Grandjean [5])**.** *For a Kari-Culik tiling x, $\alpha((\Phi(x))_i) = \overline{\alpha}((\Phi(x))_i) = \underline{\alpha}((\Phi(x))_i)$ for all i.*

Proof. Fix a Kari-Culik tiling x and notice that a single row of x must be made entirely of type $\frac{1}{3}$ tiles or entirely of type 2 tiles. Further notice that if $\alpha((\Phi(x))_i)$ exists for some i, then by the multiplier property, it exists for all i.

Suppose $\overline{\alpha} = \overline{\alpha}((\Phi(x))_0) \neq \underline{\alpha}((\Phi(x))_0) = \underline{\alpha}$. By compactness, we may find accumulation points $\overline{x}, \underline{x}$ of $\mathcal{O}_T(x)$ such that $\alpha((\Phi(\overline{x}))_0) = \overline{\alpha}$ and $\alpha((\Phi(\underline{x}))_0) = \underline{\alpha}$. That is, the averages exists, which implies the vectors $\alpha(\Phi(\overline{x}))$ and $\alpha(\Phi(\underline{x}))$ exist.

Observe that for a number $\gamma \in [1/3, 2]$ with $\gamma \neq 1$, if γ satisfies $\lambda\gamma \in [1/3, 2]$ for some $\lambda \in \{\frac{1}{3}, 2\}$, then λ is uniquely determined. Since $\alpha(\Phi(\overline{x})), \alpha(\Phi(\underline{x})) \in [1/3, 2]^{\mathbb{Z}}$, if neither vector contains 1 as a component, then they satisfy the equations

$$(\alpha(\Phi(\overline{x})))_{i+1} = f((\alpha(\Phi(\overline{x})))_i) \qquad (\alpha(\Phi(\underline{x})))_{i+1} = f((\alpha(\Phi(\underline{x})))_i),$$

where f is given by Definition 3.5.

Since f is conjugate to an irrational rotation (and this conjugacy is continuous), given any two points a, b, there exists some i so that $f^i(a) > 1 > f^i(b)$ or $f^i(a) < 1 < f^i(b)$. Now, using the observation that if $\alpha(\Phi(\overline{x}))_i < 1$, the type of $(\overline{x})_i$ is 2 and if $\alpha(\Phi(\overline{x}))_i > 1$, the type of is $\frac{1}{3}$, we deduce that for some i, $(\overline{x})_i$ and $(\underline{x})_i$ are of different types. However, the type of $(\overline{x})_i$ = type of $(\underline{x})_i$ = type of $(x)_i$ for all i (by virtue of taking accumulation points of the horizontal orbit only), which is a contradiction. We conclude $\overline{\alpha} = \underline{\alpha}$.

Finally, to handle the case where 1 is a component of $\alpha(\Phi(\overline{x}))$ or $\alpha(\Phi(\underline{x}))$, notice that the branch of f taken exactly corresponds to the multiplier of a row. Thus, $(\alpha(\Phi(\overline{x})))_i$ and $(\alpha(\Phi(\underline{x})))_i$ must always take the same branches of f, which allows us to complete f to a well-defined function $f_x : [1/3, 2] \to [1/3, 2]$. Using f_x now leads to the same contradiction. □

Corollary 3.9. *For a Kari-Culik tiling x, $\alpha((\Phi(x))_i) \in [1/3, 2]$ for all i.*

Proof. This is a direct result of Proposition 3.4 and Proposition 3.8. □

Corollary 3.10. *Fix a Kari-Culik configuration x and let $r_i = \Phi((x)_i)$. Then,*

$$\alpha(r_{i+1}) = f(\alpha(r_i))$$

provided $\alpha(r_i) \neq 1$.

Proof. Since $\alpha(r_{i+1}) = \lambda\alpha(r_i)$ for some $\lambda \in \{\frac{1}{3}, 2\}$, the constraint that both $\alpha(r_{i+1}), \alpha(r_i) \in [1/3, 2]$ uniquely determines λ when $\alpha(r_i) \neq 1$. □

Even if $\alpha(r_i) = 1$, there are still only two options for $\alpha(r_{i+1})$ and as shown in Proposition 3.7, orbits under f are aperiodic, ensuring that a choice can only be made at most once.

Theorem 3.11 (Kari, Culik, Eigen, et al)**.** *The Kari-Culik tile set admits no periodic points.*

Proof. Recall that if a $\mathbb{Z}^2$ SFT admits a periodic point, it admits a doubly periodic point. That is, there exists some rectangular configuration that can be repeated to fill the plane.

Suppose such a rectangle exists and fix it, labeling the top labels a_i, the left b_i, the bottom c_i, and the right d_i and let A, B, C, D be the respective sums of the side labels. Further, let λ_i be the multiplier of the ith row.

$$A = \sum a_i$$

	a_{i-1}	a_i	a_{i+1}	
b_i				d_i
	c_{i-1}	c_i	c_{i+1}	

$$C = \sum c_i$$

When rewritten as $\lambda c - a = d - b$, notice that the multiplier property telescopes when summing across a row. Let $\vec{b}$ be the vector with components b_i and let $\vec{d}$ be the vector with components d_i. Summing across columns, the multiplier property again telescopes, which leaves us with the equation

$$\Lambda C - A = \vec{\gamma} \cdot (\vec{d} - \vec{b})$$

where $\Lambda = \prod \lambda_i$, $\vec{\gamma}$ is a vector whose entries are sums and products of λ_i and $\cdot$ is the dot product. Since we assumed that this rectangular configuration could be used to tile the plane, we have $a_i = c_i$ and $b_i = d_i$, which implies $A = C$ and $\vec{d} = \vec{b}$. Expanding the equation with these substitutions yields

$$\Lambda C = C.$$

Since $C \neq 0$ for blocks larger than 2×2, we conclude that $\Lambda = \frac{2^n}{3^m} = 1$, which is impossible if $n, m > 0$. Finally, by inspection we verify that there are no periodic points with period 2, and so there cannot exist a periodic point. $\square$

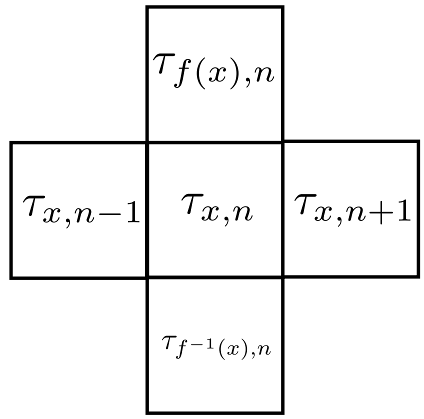
Theorem 3.12 (Kari, Culik, Eigen, et al)**.** *The Kari-Culik tile set admits uncountably many tilings of the plane.*

Proof. For a detailed version of the proof to Theorem 3.12, see [6]. We will give an outline of the proof and later present a slight generalization.

Fix $x \in [1/3, 2]$. We will show how to produce a valid tiling from x. Define the Kari-Culik tile $\tau_{x,n}$ to be the tile with edge labels given by the following.

$$\begin{array}{ccc} & \lfloor nf(x) \rfloor - \lfloor (n-1)f(x) \rfloor & \\ \lambda_x \lfloor (n-1)x \rfloor - \lfloor (n-1)f(x) \rfloor & \boxed{\tau_{x,n}} & \lambda_x \lfloor nx \rfloor - \lfloor nf(x) \rfloor \\ & \lfloor nx \rfloor - \lfloor (n-1)x \rfloor & \end{array}$$

Replace a bottom label of 0 with $0'$ if $f^{-1}(x) \in [1/3, 1/2]$ and a top label 0 with $0'$ if $x \in [1/3, 1/2]$. It can be checked that $\tau_{x,n}$ is always in $\mathfrak{K}$. Further, the following is a valid configuration.



Since this is defined for all $n \in \mathbb{Z}$ and every $x \in [1/3, 2]$ has a bi-infinite orbit under f, a tiling corresponding to every x is ensured. Further, given x and x' with $x \neq x'$, the zeroth row of the constructions coming from x and x' will differ in some place, ensuring distinct $x, x' \in [1/3, 2]$ construct distinct tilings. $\square$

3.2 Sturmian Kari-Culik Configurations

Until recently, it was unknown whether fundamentally different Kari-Culik tilings than those arising from the construction used in the proof of Theorem 3.12 existed. However, Durand, Gamard, and Grandjean recently showed that the Kari-Culik tilings have positive topological entropy.

Definition 3.13 (Entropy)**.** *Given a subshift $X \subset \mathfrak{A}^{\mathbb{Z}^2}$, the* topological entropy *of X is*

$$\mathcal{H}_{\text{top}}(X) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_{n\times n}(X)|}{n^2}.$$

Topological entropy measures the exponential growth rate in the number of configurations verses a configuration's diameter, and subshifts with positive entropy are considered to be "big."

Theorem 3.14 (Durand et al [5])**.** *The set of all Kari-Culik tilings has positive topological entropy.*

They do this by showing there exists substitutive pairs, that is pairs of 2×2 configurations with identical edge labels, occurring with positive density in any Kari-Culik tiling. This shows that the number of globally admissible $m \times n$ Kari-Culik configurations grows exponentially, yielding positive entropy.

Proposition 3.15. *The set Ω (the set of all $\mathbb{Z}^2$ configurations whose rows are Sturmian) has zero topological entropy.*

Proof. The number of Sturmian sequences on two symbols of length n is of order n^3 [1]. Thus, the number of $n \times n$ configurations occurring in Ω (restricted to two symbols) is $(n^3)^n$. Since $\frac{n^{3n}}{2^{n^2}} \to 0$ as $n \to \infty$, Ω has zero topological entropy. $\square$

Corollary 3.16. *There exists a Kari-Culik tiling that does not arise from the construction given in the proof of Theorem 3.12.*

Proof. Notice that a Kari-Culik tiling y, arising from the construction given in the proof of 3.12, satisfies $\Phi(y) \in \Omega$. By Proposition 3.23, Φ is finite-to-one, and so $\Phi^{-1}(\Omega)$ has zero entropy. Since the set of all Kari-Culik tilings has positive entropy, there must be a Kari-Culik tiling x so that $\Phi(x) \notin \Omega$. $\square$

Durand et. al. likely knew Corollary 3.16, but did not explicitly state so in [5]. Despite the existence of tilings not in $\Phi^{-1}(\Omega)$, we will restrict our study to this set. That is, we will study the set of Kari-Culik tilings whose rows have bottom labels that form generalized Sturmian sequences.

Definition 3.17. *Let*

$$KC = \{KC \text{ tilings } y : \Phi(y) \text{ has rows in } \bar{\mathcal{S}}\}$$

and

$$KC_{\mathbb{Q}^c} = \{KC \text{ tilings } y : \Phi(y) \text{ has rows in } \bar{\mathcal{S}}_{\mathbb{Q}^c}\}.$$

Using observations about the multiplier of tiles and the averages of sequences of bottom labels, we can refine our classification of rows of the Kari-Culik tilings.

Definition 3.18. *Let $x \in KC$ and $r_i = (x)_i$ be the ith row of x. We define the* general type *of r_i based on the tiles in r_{i-1}, r_i, r_{i+1} in the following way.*

Type $\frac{1}{3}$*: r_i is of general type $\frac{1}{3}$ if r_i consists of type $\frac{1}{3}$ tiles.*

Type 2.1*: r_i is of general type 2.1 if r_i consists of type 2 tiles and r_{i+1}, r_{i-1} both consist of type $\frac{1}{3}$ tiles.*

Type 2.2t*: r_i is of general type 2.2t if r_i consists of type 2 tiles and r_{i+1} consists of type $\frac{1}{3}$ tiles while r_{i-1} consists of type 2 tiles.*

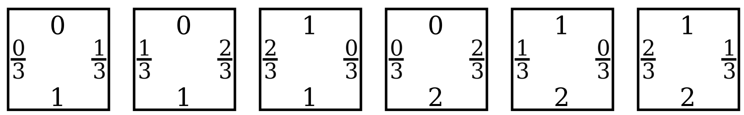
Type 2.2b*: r_i is of general type 2.2b if r_i consists of type 2 tiles and r_{i-1} consists of type $\frac{1}{3}$ tiles while r_{i+1} consists of type 2 tiles.*

We consider a pair of rows whose top row is of general type 2.2t *and whose bottom row is of general type* 2.2b *as* type 2.2*.*

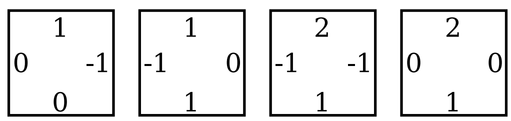
Since general type $\frac{1}{3}$ exactly corresponds to type $\frac{1}{3}$ and we have no previous definition for type 2.1, 2.2t, 2.2b, or 2.2, without ambiguity we may from now on refer to the general type of a row as simply the type of that row. Further, every row is exactly one of these types. That is, we never have three consecutive instances of a type 2 row.

Proposition 3.19. *Let $x \in KC$ and $r_i = (x)_i$ be the ith row of x. The general type of r_i is unique and the tiles that may appear in r_i are contained in exactly one of the following (non-disjoint) sets based on general type.*

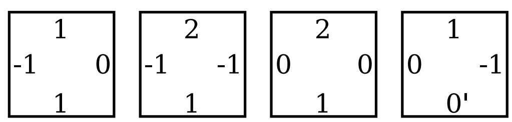
Type $\frac{1}{3}$*:*



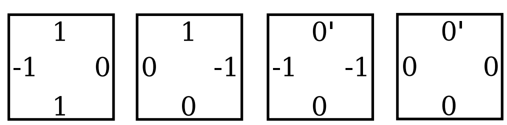
Type 2.1*:*



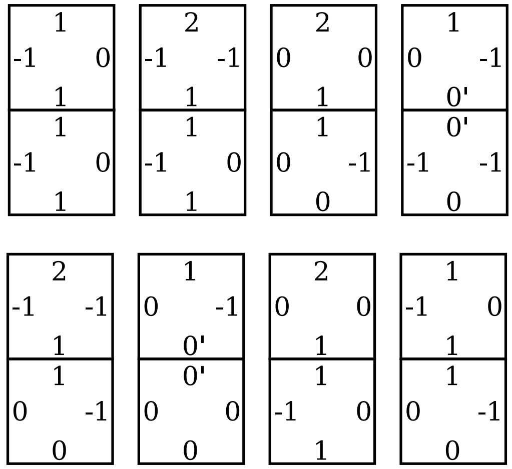
Type 2.2t*:*



Type 2.2b*:*



A pair of rows whose top tiles are type 2.2t *and bottom tiles are type* 2.2b *taken together and considered as type* 2.2 *consists of the stacked tiles:*



Proof. Fix a Kari-Culik tiling x. Let $r_i = (x)_i$ be the ith row of x and let λ_i be its multiplier. Let α be the average of the bottom labels of r_i. We will show something slightly stronger than is stated in the proposition, namely that except for $\alpha \in \{1/2, 2/3, 1\}$, α uniquely determines the type r_i.

If $\alpha \in (1, 2]$, then $\lambda = \frac{1}{3}$, and so r_i must consist of tiles of type $\frac{1}{3}$, making r_i of general type $\frac{1}{3}$.

If $\alpha \in (1/2, 2/3)$, r_{i+1}, r_{i-1} must be of type $\frac{1}{3}$, and so $0'$ cannot occur as a label, making r_i of general type 2.1 and leaving the only available tiles those listed as type 2.1.

If $\alpha \in [1/3, 1/2)$, the rows r_i and r_{i+1} are both of type 2 and r_{i-1} and r_{i+2} are of type $\frac{1}{3}$. Thus, r_i must be of general type 2.2b and must consist of the tiles listed as type 2.2b.

Finally, if $\alpha \in (2/3, 1)$, r_{i-1} is of type 2 and r_{i+1} and r_{i-2} are both of type $\frac{1}{3}$. Thus, r_i must be of general type 2.2t and consists of the tiles listed as type 2.2t.

The tiles listed as type 2.2 consist of the ways to stack type 2 tiles to be compatible on tops and bottoms with type $\frac{1}{3}$ tiles and so correspond exactly to the cases where 2.2t and 2.2b tiles arise in consecutive rows.

In the remaining cases of $\alpha \in \{1/2, 2/3, 1\}$, the type of r_i is not strictly determined by α, but nonetheless the tiles in r_i fall into one of the four categories and the classification is unique. $\square$

The tiles listed as type 2.1 have non-trivial intersection with the tiles listed as type 2.2t and type 2.2b tiles, however the condition that no two rows of type 2.1

occur consecutively ensures that the categorization is unique. We call the pairs of tiles listed as type 2.2 *stacked tiles*. When we think of a row of a Kari-Culik tiling as being type 2.2, we may think of its multiplier as being 4 (since it is composed of two consecutive rows with multiplier 2).

We will now formalize the construction of an infinite Kari-Culik tiling given in the proof of Theorem 3.12.

Definition 3.20 (BC Property). *A pair of vectors $(\vec{\alpha}, \vec{t}) \in [1/3, 2]^{\mathbb{Z}} \times [0, 1]^{\mathbb{Z}}$ satisfies the* BC property *(Basic Construction property) if*

$$\lambda_i = \frac{\alpha_{i+1}}{\alpha_i} \in \{\tfrac{1}{3}, 2\}$$

and

$$\begin{aligned} 2t_i &= t_{i+1} \bmod 1 \quad \text{if } \lambda_i = 2 \\ t_i &= 3t_{i+1} \bmod 1 \quad \text{if } \lambda_i = \tfrac{1}{3} \end{aligned}$$

for all i.

Given a pair of vectors $(\vec{\alpha}, \vec{t})$ satisfying the BC property, we can construct a point $y \in KC$ via the following procedure. The tile at position m, n in y has bottom, left, top, and right edges given by

$$\begin{aligned} \text{bottom} &= \lfloor (n+1)\alpha_m + t_m \rfloor - \lfloor n\alpha_m + t_m \rfloor \\ \text{left} &= \lambda \lfloor n\alpha_m + t_m \rfloor - \lfloor \lambda n\alpha_m + t_m \rfloor \\ \text{top} &= \lfloor (n+1)\alpha_{m+1} + t_{m+1} \rfloor - \lfloor n\alpha_{m+1} + t_{m+1} \rfloor \\ \text{right} &= \lambda \lfloor (n+1)\alpha_m + t_m \rfloor - \lfloor \lambda(n+1)\alpha_m + t_m \rfloor \end{aligned}$$

where $\lambda = \alpha_m/\alpha_{m+1}$. Further, if either the bottom or the top label is computed to be 0, then 0 is replaced with $0'$ if $\alpha_{m-1} \in [1/3, 1/2]$ (respectively $\alpha_m \in [1/3, 1/2]$). We can also do the same construction using $\lceil \cdot \rceil$ instead of $\lfloor \cdot \rfloor$. We call a tiling constructed in this way a *Basic Construction* with parameters $(\vec{\alpha}, \vec{t})$.

Proposition 3.21 (Robinson [16]). *If $(\vec{\alpha}, \vec{t})$ satisfies the BC property, then the resulting Basic Construction using either $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$ is an element of KC.*

Proof. First observe that if y is the result of a Basic Construction, then $\Phi(y)$ consists of rows that are rotation sequences and therefore Sturmian. Further, by definition, the top labels of each row of y are guaranteed to be compatible with the bottom labels of the next row, and the right labels of each column of y are guaranteed to be compatible with the left labels of the next column.

The remainder of the proof involves checking for all ranges of α_m, t_m that the resulting bottom, left, top, and right labels correspond to an actual tile in $\mathfrak{K}$. The details of this are straightforward, and after substituting $t_{m+1} = 2t_m \bmod 1$ or $t_m = 3t_{m+1} \bmod 1$ depending on the ratio α_m/α_{m+1}, it requires only examining what cases result from the choice of α_m,t_m or α_m, t_{m+1}. □

Proposition 3.22. *If $y \in KC_{\mathbb{Q}^c}$ and $(\vec{\alpha}, \vec{t}) = (\alpha(\Phi(y)), t(\Phi(y)))$ are the angle and phase vectors of y, then y is the result of a Basic Construction arising from $(\vec{\alpha}, \vec{t})$ using either $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$.*

Proof. First note that since $y \in KC_{\mathbb{Q}^c}$, the rows of $\Phi(y)$ are Sturmian sequences and therefore rotation sequences (since rows in $\bar{\mathcal{S}} \backslash \mathcal{S}$ are excluded).

We will first show that $(\vec{\alpha}, \vec{t})$ satisfies the BC property. Fix $k \in \mathbb{Z}$. Since Corollary 3.10 already shows that $\vec{\alpha}$ is determined by f and α_k (that is $\alpha_{k+i} = f^i(\alpha_k)$), we only need to show that either $t_{k+1} = 2t_k \bmod 1$ or $t_k = 3t_{k+1} \bmod 1$ in accordance with α_k. For simplicity, call $\alpha = \alpha_k$, $t = t_k$, and $t' = t_{k+1}$. Let $\lambda = \alpha_k/\alpha_{k+1}$ be the type of the kth row of y and let a_i, b_i, c_i, d_i be the bottom, left, top, and right labels of the ith tile in $(y)_k$. We divide the proof into two similar cases depending on λ.

Case $\lambda = 1/3$: We will assume the Sturmian sequences $(\Phi(y))_k$ and $(\Phi(y))_{k+1}$ may both be represented using $\mathbf{R}_{\lfloor \cdot \rfloor}$, but note that for every combination $(\mathbf{R}_{\lfloor \cdot \rfloor}, \mathbf{R}_{\lfloor \cdot \rfloor})$, $(\mathbf{R}_{\lfloor \cdot \rfloor}, \mathbf{R}_{\lceil \cdot \rceil})$, $(\mathbf{R}_{\lceil \cdot \rceil}, \mathbf{R}_{\lfloor \cdot \rfloor})$, and $(\mathbf{R}_{\lceil \cdot \rceil}, \mathbf{R}_{\lceil \cdot \rceil})$ of ways to represent $(\Phi(y))_k$ and $(\Phi(y))_{k+1}$, upon replacing $\lfloor \cdot \rfloor$ with $\lceil \cdot \rceil$ where appropriate, the same argument still works. By Corollary 3.10, $\alpha_{k+1} = \lambda \alpha_k = \lambda \alpha$, and so we have the following relationship for the bottom and top labels:

$$a_i = \lfloor (i+1)\alpha + t \rfloor - \lfloor i\alpha + t \rfloor \qquad \text{and} \qquad c_i = \left\lfloor \frac{(i+1)\alpha + 3t'}{3} \right\rfloor - \left\lfloor \frac{i\alpha + 3t'}{3} \right\rfloor.$$

Exploiting the telescoping nature of the Multiplier Property (shown in Equation (3.2)) and summing from $i = n$ to $m - 1$, we get

$$(b_n - d_m) + \frac{1}{3}\left(\lfloor \alpha m + t \rfloor - \lfloor \alpha n + t \rfloor\right) = \left\lfloor \frac{\alpha m + 3t'}{3} \right\rfloor - \left\lfloor \frac{\alpha n + 3t'}{3} \right\rfloor. \tag{3.3}$$

Since $\alpha \notin \mathbb{Q}$, we can pick n, m so that

$$\frac{\alpha m + 3t'}{3} = k_m + \varepsilon_m \qquad \text{and} \qquad \frac{\alpha n + 3t'}{3} = k_n - \varepsilon_n$$

where $k_m, k_n \in \mathbb{Z}$ and $\varepsilon_m, \varepsilon_n$ are arbitrarily small positive numbers. Upon this choice, the right side of Equation (3.3) simplifies to $k_m - k_n + 1$. By rearranging and substituting into Equation (3.3), we get

$$\lfloor 3(k_m + \varepsilon_m) + (t - 3t')\rfloor - \lfloor 3(k_n - \varepsilon_n) + (t - 3t')\rfloor = 3(k_m - k_n) + 3 - 3(b_n - d_m),$$

but since $b_n - d_m$ is bounded above by $2/3$, we conclude

$$\lfloor 3(k_m + \varepsilon_m) + (t - 3t')\rfloor - \lfloor 3(k_n - \varepsilon_n) + (t - 3t')\rfloor \geq 3(k_m - k_n) + 1. \tag{3.4}$$

If $(t - 3t' \bmod 1) = \gamma \neq 0$, choosing $\varepsilon_n, \varepsilon_m \ll \gamma$ gives a contradiction (with the left hand side of Equation (3.4) yielding $3(k_m - k_n)$). Thus, $t = 3t' \bmod 1$.

Case $\lambda = 2$: Since this case is nearly identical to the $\lambda = 1/3$ case, we will omit the details, noting only that in this case the relationship between bottom labels and top labels is reversed. That is, in this case fix $\alpha = \alpha_{k+1}$ and rewrite $\alpha_k = \alpha/2$.

We have now shown that $(\vec{\alpha}, \vec{t})$ satisfies the BC property. To complete the proof and show that y arises as a Basic Construction, we need to show that every Sturmian sequence in $\Phi(y)$ can be written with exclusively $\mathbf{R}_{\lfloor \cdot \rfloor}$ or exclusively $\mathbf{R}_{\lceil \cdot \rceil}$.

Notice that since every component of $\vec{\alpha}$ is rationally related to α_0, we have either $\vec{\alpha} \in \mathbb{Q}^{\mathbb{Z}}$ or $\vec{\alpha} \in (\mathbb{Q}^c)^{\mathbb{Z}}$. By assumption however, $\vec{\alpha} \in (\mathbb{Q}^c)^{\mathbb{Z}}$ and so $n\vec{\alpha} + \vec{t} \in \mathbb{Q}^{\mathbb{Z}}$ for at most one n. Further, since the components of $\vec{t}$ are rationally related, for each n we have either $n\vec{\alpha} + \vec{t} \in \mathbb{Q}^{\mathbb{Z}}$ or $n\vec{\alpha} + \vec{t} \in (\mathbb{Q}^c)^{\mathbb{Z}}$. Define

$$r_{i,n} = n\alpha_i + t_i.$$

Suppose that the ith row of $\Phi(y)$ requires $\mathbf{R}_{\lfloor \cdot \rfloor}$ or $\mathbf{R}_{\lceil \cdot \rceil}$ to be written as a rotation sequence. This implies that for some n we have $r_{i,n} \in \mathbb{Z}$. Fix this n. By our observation that $m\vec{\alpha} + \vec{t} \in \mathbb{Q}^{\mathbb{Z}}$ for at most one m, we may conclude that $r_{j,n'} \notin \mathbb{Z}$ for any $n' \neq n$ and $j \in \mathbb{Z}$.

Let

$$B = \{i : (\Phi(y))_i \text{ requires } \mathbf{R}_{\lfloor \cdot \rfloor} \text{ or } \mathbf{R}_{\lceil \cdot \rceil}\},$$

and notice again that by the uniqueness of n (with n still being fixed as above),

$B = \{j : r_{j,n} \in \mathbb{Z}\}$. We will now show that B consists of a contiguous sequence of integers.

Exploiting the fact that $(\vec{\alpha}, \vec{t})$ satisfies the BC property, we may conclude $r_{j+1,n} = 2r_{j,n}$ or $r_{j+1,n} = \frac{1}{3}(r_{j,n} + i)$ where $i \in \{0, 1, 2\}$. This implies that if $|r_{j,n}|_3 > 0$ then $|r_{j+1,n}|_3 > 0$ where $|\cdot|_3$ is the 3-valuation. That is, for $q \in \mathbb{Q}$, if

$$q = \prod_{p \text{ prime}} p^{n_p}$$

is the prime decomposition of q with $n_p \in \mathbb{Z}$, then $|q|_3 = -n_3$.

If $b \in B$ and $b + 1 \notin B$, this means $r_{b,n} \in \mathbb{Z}$ but $r_{b+1,n} \notin \mathbb{Z}$ and so $|r_{b+1,n}|_3 > 0$ (since multiplying by 2 keeps us in $\mathbb{Z}$, the only way to leave $\mathbb{Z}$ is to divide by 3). However, $|r_{b+1,n}|_3 > 0$ implies that $r_{b+i,n} \notin \mathbb{Z}$ for all $i > 0$. We conclude that B cannot contain any gaps.

Since B consists of a contiguous set of integers, it will complete the proof if we show that there do not exist two adjacent rows in $\Phi(y)$ where one requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the other requires $\mathbf{R}_{\lceil\cdot\rceil}$. We will conclude the proof by showing that the rules of the Kari-Culik tiling forbid such an occurrence.

Suppose $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) \neq \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t) = s'$ and $\alpha \notin \mathbb{Q}$, and notice s and s' differ only by a transposition of two adjacent coordinates. For simplicity, assume s and s' differ at coordinates 1 and 2 and that $\alpha(s) \in [0, 1]$ so that s and s' consist of the symbols 0 and 1. We then have

$$s = \cdots s_0 s_1 s_2 s_3 \cdots \qquad \text{and} \qquad s' = \cdots s_0 s_2 s_1 s_3 \cdots,$$

and in particular $s_1 \neq s_2$. Since s and s' are both valid Sturmian sequences, we may conclude that $s_0 = s_3$, since if $s_0 \neq s_3$ either s or s' would contain both a $1, 1$ and a $0, 0$. Further, since s requires $\lfloor\cdot\rfloor$, we know $s_1 > s_2$.

In general, we will call a length four word $w_{\alpha,t} = w_0 w_1 w_2 w_3$ or $w_{\alpha,t} = w_0 w_2 w_1 w_3$ a *straddle word* of a Sturmian sequence if

$$w_0 w_1 w_2 w_3 = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_i^{i+3} \qquad \text{and} \qquad w_0 w_2 w_1 w_3 = (\mathbf{R}_{\lceil\cdot\rceil}(\alpha, t))_i^{i+3}$$

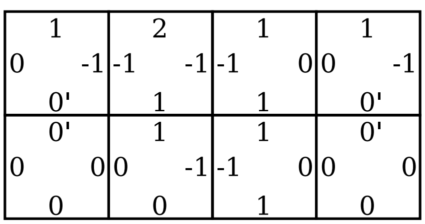
for some i (or vice versa). The previous argument shows that if $w_{\alpha,t}$ is a straddle word, then $w_0 = w_3$ and $w_1 \neq w_2$. It also shows that if a Sturmian sequence requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$, it necessarily contains a straddle word.

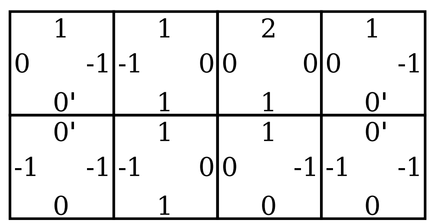
Now, consider a row r of y where the sequence of bottom labels requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the top labels require $\mathbf{R}_{\lceil\cdot\rceil}$ (or vice versa) and let w^t and w^b be the straddle words for the labels on the top of r and the bottom of r respectively. Since the top sequence requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the bottom sequence requires $\mathbf{R}_{\lceil\cdot\rceil}$, we conclude that $w_1^t > w_2^t$ and $w_1^b < w_2^b$ (or vice versa if the roles of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are reversed). We will call a pair of straddle words like these, whose middle two symbols satisfy opposite inequalities, *misaligned.*

We will complete the proof by observing that misaligned straddle words cannot occur in y.

By enumerating all pairs of length-four words (w^t, w^b) that arise as tops and corresponding bottoms of rows of type 2.1, we see that out of the 64 possibilities, only four have the property that $w_1^t \neq w_2^t$ and $w_1^b \neq w_2^b$ (which is necessary to be a straddle word). Out of those four, none are misaligned. Similarly, in a row of type $\frac{1}{3}$, of the 96 possibilities, 24 differ in their middle symbols and out of those, none are misaligned straddle words.

Now consider pair of rows of type 2.2. Out of the 128 possible sequences of length 4, there are exactly two ways to obtain misaligned straddle words, namely:





This gives misaligned straddle pairs of $(w^t, w^b) = (1211, 0010)$ and $(w^t, w^b) = (1121, 0100)$. Since we are considering a type 2.2 row, the Sturmian angle for the sequence of bottoms must be in $[1/3, 1/2]$. Thus, there cannot be three 0s in a row. We therefore conclude the symbol before the word $w^b = 0010$ must be a 1 and the symbol after the word $w^b = 0100$ must be a 1. Since we are considering 0010 and 0100 as straddle subwords of some pair of Sturmian sequences and these Sturmian sequences must agree everywhere except for a single transposition of symbols, we conclude $w^b = 0010$ and $w^b = 0100$ must be subwords of 100101 and 101001. By a similar argument, the top straddle words must be subwords of 211212 and 212112. Thus, the stacked tile to the left or right of the designated blocks must have a bottom label of 1 and a top label of 2. Inspecting the two stacked tiles with this property reveals that neither of them are compatible with the potential misaligned straddle words shown, and thus misaligned straddle words cannot appear in y. $\square$

Theorem 3.23. $\Phi|_{KC_{\mathbb{Q}^c}}$ *is one-to-one and* Φ *is at most sixteen-to-one.*

Proof. Fix $x \in KC$ and consider a row r of x. Let r^t be the Sturmian sequence formed by the top labels of r and let r^b be the Sturmian sequence formed by the bottom labels of r. Further, let $\alpha^t = \alpha(r^t)$ and $\alpha^b = \alpha(r^b)$, and if $w^t = (r^t)_i^j$ is a subword of r^t, then $w^b = (r^b)_i^j$ is the corresponding subword of r^b.

Notice that by the multiplier property (Equation (3.1)), r is completely determined by (r^t, r^b) and a single left label of one of the tiles in r.

Define Φ_2 by $\Phi_2(r) = (r^t, r^b)$ and extend Φ_2 to work on subwords of r. By our previous observation, if there is a subword $r' \subset r$ such that $\Phi_2^{-1}(\Phi_2(r'))$ contains a single element, then r is completely determined by (r^t, r^b). We will show that if $x \in KC_{\mathbb{Q}^c}$, this is always the case. Moving forward, we analyze r separately depending on its type.

Case r is of type $\frac{1}{3}$: In this case, we know $\alpha^t \in [1/3, 2/3]$ and $\alpha^b \in [1, 2]$.

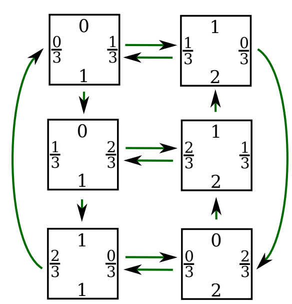


Figure 3.2: Transition graph for type $\frac{1}{3}$ tiles.

Figure 3.2 shows the transition graph moving left to right in a type $\frac{1}{3}$ row. Notice that there is only one way for 11 or 00 to appear as top labels in a type $\frac{1}{3}$ row. In particular, if $w^t = 11$ then $w^b = 22$ and if $w^t = 00$ then $w^b = 11$ and $|\Phi_2^{-1}(11, 22)| = |\Phi_2^{-1}(00, 11)| = 1$. Thus, if r^t contains the word 11 or 00, r is uniquely determined.

If r^t contains neither 11 nor 00, then $\alpha^t = 1/2$ and $r^t = \cdots 101010 \cdots$. Analysing further, if $w^t = 01$ then $w^b \in \{12, 21, 11, 22\}$. We note that $|\Phi_2^{-1}(01, 11)| = |\Phi_2^{-1}(01, 22)| = |\Phi_2^{-1}(01, 21)| = 1$ and $|\Phi_2^{-1}(01, 12)| = 2$. Thus, Φ on a row of type $\frac{1}{3}$ is only non-unique if $\alpha^t = 1/2$ which further implies $\alpha^b = 3/2$.

Case r is of type 2.1: In this case, we know $\alpha^t \in [1, 2]$ and $\alpha^b \in [1/2, 1]$.

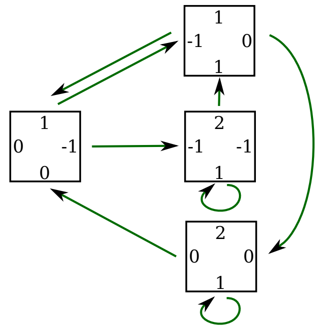


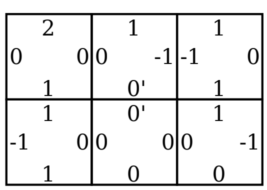
Figure 3.3: Transition graph for a type 2.1 row.

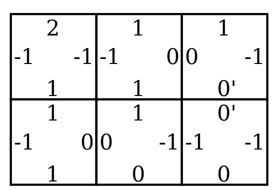
Figure 3.3 shows the transition graph moving left to right in a type 2.1 row. Notice that there is only one way a type 2.1 row can contain 0 as a bottom label. This means r is uniquely determined unless $\alpha^b = 1$ (since $\alpha^b < 1$ implies a zero occurs in r^b) and consequently $r^b = \cdots 111 \cdots$. If $w^b = 1$ then $w^t \in \{1,2\}$ with $|\Phi_2^{-1}(1,1)| = 1$ and $|\Phi_2^{-1}(2,1)| = 2$.

Case r is of type 2.2: In this case, we know $\alpha^t \in [4/3, 2]$ and $\alpha^b \in [1/3, 1/2]$ (since we interpret our multiplier as 4 in this case).

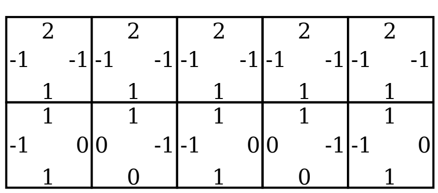
Because $\alpha^b \in [1/3, 1/2]$, we know that r^b cannot contain 11 (since if it did, $\alpha(r^b) > 1/2$). Therefore, it must contain 100 or 10101 as a subword. Further, if $\alpha^b \in (1/3, 1/2)$, r^b must contain 10100 as a subword. Let $w^b \in \{100, 10101, 10100\}$. Below is a list of all pairs (w^t, w^b) such that $|\Phi_2^{-1}(w^t, w^b)| > 1$.

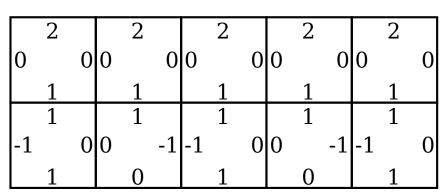
$(w^t, w^b) = (211, 100)$, which can be obtained in exactly two ways, namely



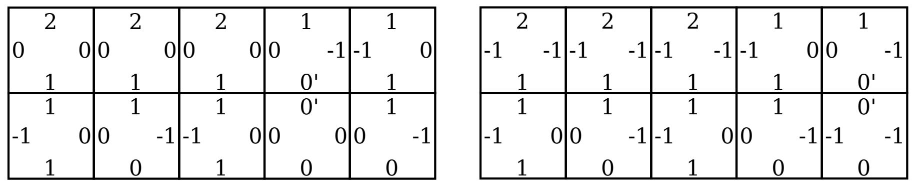


$(w^t, w^b) = (22222, 10101)$, which can be obtained in exactly two ways, namely





and $(w^t, w^b) = (22211, 10100)$, which can be obtained in exactly two ways, namely



Considering the pair $(w^t, w^b) = (22211, 10100)$, we see that 22211 is not a generalized Sturmian sequence (since it contains both 11 and 22 as subwords), and so this situation never occurs. This means if Φ_2 is not invertible, $\alpha^b = 1/3$, which corresponds to the first case, or $\alpha^b = 1/2$ which corresponds to the second case.

As a result of our case-by-case analysis, we see that if $\alpha^b \notin \{1/3, 1/2, 1, 3/2\}$, then r is uniquely determined and so when restricted to $KC_{\mathbb{Q}^c}$, Φ is one-to-one. Further, since for $\alpha^b \in \{1/3, 1/2, 1, 3/2\}$ we have Φ_2 is at most two-to-one, we conclude in general that Φ is at most sixteen-to-one (since f prevents α^b from taking any particular value in $\{1/3, 1/2, 1, 3/2\}$ more than once). $\square$

Computer code for enumerating the valid configurations used in the proof of Proposition 3.23 is included in Appendix A. In light of Theorem 3.23, we will treat points in KC and points in $\Phi(KC)$ interchangeably, differentiating only when needed.

Proposition 3.24. *For $x \in KC$, we have*

$$\alpha(Sx) = f(\alpha(x)),$$

where f is applied component-wise and if $x \in KC_{\mathbb{Q}^c}$,

$$t(Tx) = t(x) + \alpha(x).$$

Proof. This immediately follows from Corollary 3.10 and the definition of a rotation sequence. $\square$

3.2.1 Parameterization of KC

We will now produce a parameterization of points in $KC_{\mathbb{Q}^c}$ in a similar fashion to the way a Sturmian sequence may be parameterized by an angle, phase, and choice of floor or ceiling function.

Definition 3.25 (Inverse Limit)**.** *Given a sequence of groups G_n for $n \in \mathbb{N}$ with surjective homomorphisms $h_n : G_n \to G_{n-1}$, the* inverse limit *of the groups G_n is*

$$L = \{x \in \prod_{i=1}^{\infty} G_i : h_i(x_i) = x_{i-1} \text{ for all } i\}$$

and is denoted $L = \varprojlim G_n$. The group operation on L is defined by applying the group operation on each G_n component-wise and L is endowed with the product topology.

We will be consider the special case of the inverse limit of the groups $\mathbb{Z}/(q^n\mathbb{Z})$ where the homomorphism h_n acts by $h_n(x) = x \bmod q^{n-1}$.

Definition 3.26. *Suppose $\mathcal{I} = \varprojlim \mathbb{R}/(q^n\mathbb{Z})$ is the inverse limit of the sequence of groups $\mathbb{R}/(q^n\mathbb{Z})$ for some $q \in \mathbb{N}$. For any $d \in \mathbb{N}$ such that $d|q$, define*

$$M_d : \mathcal{I} \to \mathcal{I}$$

by $M_d(i_0, i_1, i_2, \ldots) = (di_0, di_1, di_2, \ldots)$ and

$$M_{1/d} : \mathcal{I} \to \mathcal{I}$$

by $M_{1/d}(i_0, i_1, i_2, \ldots) = (\frac{q}{d})(i_1/q, i_2/q, i_3/q, \ldots)$.

Proposition 3.27. *Let $\mathcal{I} = \varprojlim \mathbb{R}/(q^n\mathbb{Z})$ and suppose $d|q$. Then M_d and $M_{1/d}$ are bijective group homomorphisms on $\mathcal{I}$.*

Proof. Notice that for a point $x \in \mathcal{I}$, component-wise multiplication, M_a, by any integer a is always group homomorphism. Further, $M_{1/q}$ as defined is a group homomorphism, and $M_{1/d} = M_a \circ M_{1/q}$ whenever $q = da$.

To show M_d is bijective, it is sufficient to show that $M_d(x) = 0$ has only the trivial solution. Suppose $x = (x_0, x_1, \ldots) \in \mathcal{I}$ satisfies $M_d(x) = 0$. The consistency condition on the coordinates of x ensures

$$dx_i = 0 \bmod q^i$$

which implies

$$x_i = 0 \bmod \left(\frac{q}{d}\right) q^{i-1}$$

which further implies

$$x_i = 0 \bmod q^{i-1},$$

but $x_{i-1} = x_i = 0 \bmod q^{i-1}$. Thus $x = 0$.

To complete the proof, observe that $M_d \circ M_{1/d} = id$ is the identity and that M_d is bijective. Therefore, $M_{1/d}$ must also be bijective. □

Definition 3.28. *Let*

$$\mathcal{T} = \varprojlim \mathbb{R}/(6^n\mathbb{Z})$$

be the inverse limit of the groups $\mathbb{R}/(6^n\mathbb{Z})$ as $n \to \infty$.

Corollary 3.29 (of Proposition 3.27)**.** *$M_2, M_3, M_{1/2}, M_{1/3} : \mathcal{T} \to \mathcal{T}$ are bijective homomorphisms.*

The proof of Corollary 3.29 follows from Proposition 3.27. From now on, for $t \in \mathcal{T}$ we may write at or t/a instead of $M_a t$ and $M_{1/a} t$. Further, for $r \in \mathbb{R}$, we may define scalar addition $A_r : \mathcal{T} \to \mathcal{T}$ by

$$A_r(t_0, t_1, \ldots) = (t_0 + r \bmod 1, t_1 + r \bmod 6, t_2 + r \bmod 6^2, \ldots)$$

and we may write $t + r$ instead of $A_r(t)$.

Proposition 3.30. *Let $\mathcal{I} \subset \varprojlim \mathbb{R}/(3^n\mathbb{Z}) \times \varprojlim \mathbb{R}/(2^n\mathbb{Z})$ be defined by*

$$\mathcal{I} = \{((a_0, a_1, \ldots), (b_0, b_1, \ldots)) : a_0 = b_0\}.$$

Then we have $\mathcal{I}$ is isomorphic to $\mathcal{T}$ as a group.

Proof. First note that $\mathbb{Z}/(2^n\mathbb{Z}) \times \mathbb{Z}/(3^n\mathbb{Z}) \cong \mathbb{Z}/(6^n\mathbb{Z})$ via the Chinese remainder theorem.

Next, observe that $\varprojlim \mathbb{R}/(b^n\mathbb{Z}) \cong [0, 1) \times \varprojlim \mathbb{Z}/(b^n\mathbb{Z})$. The right hand side is a group via the map

$$(x, y) + (x', y') = (x + x' \bmod 1, (y \oplus_b y') \vec{\oplus} \lfloor x + x' \rfloor)$$

where $\oplus_b$ is addition in $\varprojlim \mathbb{Z}/(b^n\mathbb{Z})$ and $\vec{\oplus} : (\varprojlim \mathbb{Z}/(b^n\mathbb{Z})) \times \mathbb{Z} \to \varprojlim \mathbb{Z}/(b^n\mathbb{Z})$ is given by $(r_0, r_1, \ldots) \vec{\oplus} l = (r_0 + l, r_1 + l, \ldots)$.

Since the coordinates of $a = (a_0, a_1, \ldots) \in \varprojlim \mathbb{R}/(b^n\mathbb{Z})$ must satisfy $a_i = a_{i+1} \bmod b^i$, we see that $a_{i+1} = a_i + j_{i+1} b^i$ for some $0 \leq j_{i+1} < b$. Thus the sequence $(0, j_1, j_2 b, j_3 b^2, \ldots) \in \varprojlim \mathbb{Z}/(b^n\mathbb{Z})$ and the point $a_0 \in [0, 1)$ is all we need to reconstruct a.

Endow $[0,1) \times \varprojlim \mathbb{Z}/(2^n\mathbb{Z}) \times \varprojlim \mathbb{Z}/(3^n\mathbb{Z})$ with a group operation similarly to before. It is now clear, since the first coordinates of each component of $\mathcal{I}$ must agree, that

$$\mathcal{I} \cong [0,1) \times \varprojlim \mathbb{Z}/(2^n\mathbb{Z}) \times \varprojlim \mathbb{Z}/(3^n\mathbb{Z}) \cong [0,1) \times \varprojlim \mathbb{Z}/(6^n\mathbb{Z}) \cong \mathcal{T}.$$

$\square$

Definition 3.31. *Define* $\mathrm{proj}_j : \mathcal{T} \to \mathbb{R}/(6^j\mathbb{Z})$ *to be projection onto the* j*th coordinate of* $\mathcal{T}$*.*

Notation 3.32. *Extend* f *from Definition 3.5 to a function* $\hat{f} : [1/3,2] \times \mathcal{T}$ *by*

$$\hat{f}(\alpha,t) = \begin{cases} (2\alpha, 2t) & \text{if } \alpha \in [1/3,1) \\ (\alpha/3, t/3) & \text{if } \alpha \in [1,2] \end{cases}.$$

Notice that $\hat{f} : [1/3,2] \times \mathcal{T} \to [1/3,2] \times \mathcal{T}$ is a bijection. Morally, we will show that $[1/3,2] \times \mathcal{T}$ is a parameterization of KC. However, since trouble arises for generalized Sturmian sequences with rational angles and Sturmian sequences whose phase vector contains zero, we focus our attention to the sets $([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ and $KC_{\mathbb{Q}^c}$.

Lemma 3.33. *Let* $\mathcal{O}_1$ *be the orbit of* 1 *under* f*, and let*

$$X = \{(\vec{\alpha},\vec{t}) : (\vec{\alpha},\vec{t}) \text{ satisfies the BC property and } 1 \notin \vec{\alpha}\}.$$

There exists a bijection $W : ([1/3,2]\backslash\mathcal{O}_1) \times \mathcal{T} \to X$ *that respects the dynamical relationships. That is, for* $(\alpha,t) \in ([1/3,2]\backslash\mathcal{O}_1) \times \mathcal{T}$ *such that* $W(\alpha,t) = (\vec{\alpha},\vec{t})$*, we have*

$$W(\alpha, t+\alpha) = (\vec{\alpha}, \vec{t} + \vec{\alpha} \bmod 1) \qquad \text{and} \qquad W \circ \hat{f}(\alpha,t) = (\sigma(\vec{\alpha}), \sigma(\vec{t})).$$

Proof. Defining W is straightforward. Fix $(\alpha,t) \in ([1/3,2]\backslash\mathcal{O}_1) \times \mathcal{T}$ and define $W(\alpha,t) = (\vec{\alpha},\vec{t})$ where $(\alpha_i,t_i) = (\mathrm{id} \times \mathrm{proj}_0) \circ \hat{f}^i(\alpha,t)$. The definition of $\hat{f}$ acting on $([1/3,2]\backslash\mathcal{O}_1) \times \mathcal{T}$ ensures that $(\vec{\alpha},\vec{t})$ satisfies the BC property and respects the desired dynamical relationships.

The map W is clearly one-to-one in the first coordinate. Further, if $t,t' \in \mathcal{T}$ with $t \neq t'$, we have that $\mathrm{proj}_j t \neq \mathrm{proj}_j t'$ for some j. From this we deduce that

$\mathrm{proj}_0 \circ \hat{f}^{j'}(t) \neq \mathrm{proj}_0 \circ \hat{f}^{j'}(t')$ for some $j' \geq j$, and so W is one-to-one. After the construction of W^{-1} it will be evident that it is onto.

Constructing W^{-1} is slightly more difficult. Fix $(\vec{\alpha}, \vec{t}) \in X$. Let $\Lambda_i = \alpha_i/\alpha_0$. Let $|\cdot|_2$ and $|\cdot|_3$ be the 2-valuation and 3-valuation respectively.

Let $j(i) = \min\{n \geq 0 : |\Lambda_n|_3 = i\}$ and define the subsequences $(t_i^{(3)})_{i=0}^{\infty}$ and $(\Lambda_i^{(3)})_{i=0}^{\infty}$ of $(t_i)_{i=0}^{\infty}$ and $(\Lambda_i)_{i=0}^{\infty}$ by

$$t_i^{(3)} = t_{j(i)} \qquad \text{and} \qquad \Lambda_i^{(3)} = \Lambda_{j(i)}.$$

Similarly, let $j'(i) = \min\{n \geq 0 : |\Lambda_{-n}|_2 = i\}$ and define the subsequences $(t_i^{(2)})_{i=0}^{\infty}$ and $(\Lambda_i^{(2)})_{i=0}^{\infty}$ by $t_i^{(2)} = t_{-j'(i)}$ and $\Lambda_i^{(2)} = \Lambda_{-j'(i)}$. We've constructed $t^{(2)}$ and $t^{(3)}$ to be the values of $\vec{t}$ corresponding to where we "divide α by 2" or "divide α by 3" respectively.

To construct $W^{-1}(\vec{\alpha}, \vec{t})$, first consider the point $z^{(3)} = (z_0^{(3)}, z_1^{(3)}, \ldots) \in \varprojlim \mathbb{R}/(3^n\mathbb{Z})$ defined inductively in the following way. Let $z_0^{(3)} = t_0^{(3)} = t_0^{(2)}$. Fix j and suppose for all $i < j$ we have

$$\Lambda_i^{(3)} z_i^{(3)} = t_i^{(3)} \bmod 1.$$

Let p be such that $\Lambda_j^{(3)} = \frac{2^p}{3}\Lambda_{j-1}^{(3)}$. We now have that $3t_j^{(3)} = 2^p t_{j-1}^{(3)} \bmod 1$ and so along with our induction hypothesis there exist unique $r', r'' \in \{0,1,2\}$ so

$$\begin{aligned} 3t_j^{(3)} &= 2^p t_{j-1}^{(3)} + r' \bmod 3 \\ \Lambda_{j-1}^{(3)} z_{j-1}^{(3)} - t_{j-1}^{(3)} &= r'' \bmod 3. \end{aligned}$$

Define $z_j^{(3)} = z_{j-1}^{(3)} + r3^{j-1}$ where $r \in \{0,1,2\}$ is the unique solution to $2^p r'' - r' + r2^p\Lambda_{j-1}^{(3)}3^{j-1} = 0 \bmod 3$. Note that since $2^p\Lambda_{j-1}^{(3)}3^{j-1} \in \mathbb{Z}$ and contains no multiples of 3, such an r always exists. By construction, $z_j^{(3)}$ satisfies $z_{j-1}^{(3)} = z_j^{(3)} \bmod 3^{j-1}$. We will now verify that $\Lambda_j^{(3)} z_j^{(3)} - t_j^{(3)} = 0 \bmod 1$.

Substituting, we see

$$\Lambda_j^{(3)} z_j^{(3)} - t_j^{(3)} = \tfrac{2^p}{3}\Lambda_{j-1}^{(3)} z_j^{(3)} - t_j^{(3)} = \tfrac{2^p}{3}\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - t_j^{(3)}.$$

Finally, multiplying by 3, we have

$$2^p\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - 3t_j^{(3)} = 2^p\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - 2^p t_{j-1}^{(3)} - r' \bmod 3$$

$$= 2^p\left(\Lambda_{j-1}^{(3)} z_{j-1}^{(3)} - t_{j-1}^{(3)} + r\Lambda_{j-1}^{(3)} 3^{j-1}\right) - r' = 2^p r'' - r' + r2^p \Lambda_{j-1}^{(3)} 3^{j-1} = 0 \bmod 3$$

as desired.

We have shown that $z^{(3)}$ exists and is unique. In an analogous way, construct $z^{(2)} \in \varprojlim \mathbb{R}/(2^n\mathbb{Z})$. Finally, since $z_0^{(3)} = z_0^{(2)}$, by the Chinese remainder theorem we may produce $z = (z_0, z_1, \ldots) \in \mathcal{T}$ such that $z_i^{(3)} = z_i \bmod 3^i$ and $z_i^{(2)} = z_i \bmod 2^i$.

It is worth noting now that by construction, z_i is the unique simultaneous solution to

$$\begin{aligned}
\Lambda_i^{(3)} z_i &= t_i^{(3)} \bmod 1 \\
\Lambda_i^{(2)} z_i &= t_i^{(2)} \bmod 1
\end{aligned}$$

where $z_i \in \mathbb{R}/(6^i\mathbb{Z})$ and $z_{i-1} = z_i \bmod 6^{i-1}$.

Having established existence and uniqueness of z, we may define $W^{-1}(\vec{\alpha}, \vec{t}) = (\alpha_0, z)$. The fact that W^{-1} is an inverse is now immediate by construction: if $|\Lambda_j|_3 = k \geq 0$, then $\Lambda_j \mathrm{proj}_k(z) = t_j^{(3)} \bmod 1$, and if $|\Lambda_j|_2 = k \geq 0$, then $\Lambda_j \mathrm{proj}_k(z) = t_{-j}^{(2)} \bmod 1$. $\square$

We restricted the domain of W to $([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T}$ instead of $[1/3, 2] \times \mathcal{T}$ because the function f could be defined so that $f(1) = 1/3$ or $f(1) = 2$, and both ways would be consistent with the rules of the Kari-Culik tilings. As such, this presents a small obstruction to W being a bijection on $[1/3, 2] \times \mathcal{T}$.

To allow for a clearer statement of Proposition 3.34, we will extend notation so that $\hat{f}(\alpha, t, R) = (\alpha', t', R)$ where $(\alpha', t') = \hat{f}(\alpha, t)$.

Proposition 3.34. *There exists maps $K : [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\} \to KC$ and $K' : KC_{\mathbb{Q}^c} \to [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ so that $K \circ K' = \mathrm{id}$ and for any $y \in KC_{\mathbb{Q}^c}$, K' satisfies the following relationships:*

$$\begin{aligned}
K'(y) &= (\alpha, t, R) \\
K'(Ty) &= (\alpha, t + \alpha, R) \\
K'(Sy) &= \hat{f}(\alpha, t, R)
\end{aligned}$$

for some $(\alpha, t, R) \in [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$.

Proof. This proposition is a corollary of Lemma 3.33.

Let $X = \{(\vec{\alpha}, \vec{t}) : (\vec{\alpha}, \vec{t})$ satisfies the BC property$\}$. Observe that by Proposition 3.22, we have explicit maps $A : X \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\} \to KC$ and $A' : KC_{\mathbb{Q}^c} \to X \times$

$\{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ so that $A \circ A' = \text{id}$. Ignoring the choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$, we also have if $A'(x) = (\vec{\alpha}, \vec{t})$, then $A'(Tx) = (\vec{\alpha}, \vec{t} + \vec{\alpha} \bmod \vec{1})$ and $A'(Sx) = (\sigma(\vec{\alpha}), \sigma(\vec{t}))$, where σ is the shift on bi-infinite vectors. Thus, the proof is complete by noting that Lemma 3.33 gives us an invertible map $W : [1/3, 2] \times \mathcal{T} \to X$ that respects the dynamical relationships. □

Where convenient, we will think of $K : [1/3, 2] \times \mathcal{T} \to KC$ and $K' : KC_{\mathbb{Q}^c} \to [1/3, 2] \times \mathcal{T}$ without worrying about the choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$. Considering the relationships outlined in the proof of Proposition 3.34, we may now deduce the following theorem.

Theorem 3.35. *$KC_{\mathbb{Q}^c}$ is a skew-product. That is, if K' is the map defined in Proposition 3.34 and $y \in KC_{\mathbb{Q}^c}$, we have the following relationships:*

$$\begin{aligned} K'(y) &= (\alpha, t) \\ K'(Ty) &= (\alpha, t + \alpha) \\ K'(Sy) &= \hat{f}(\alpha, t). \end{aligned}$$

We have established that $KC_{\mathbb{Q}^c}$ can be parameterized by $[1/3, 2] \times \mathcal{T}$, and that using this parameterization, $KC_{\mathbb{Q}^c}$ can be viewed as a skew product. Using this fact, we will be able to easily prove the minimality of the actions T, S on KC. However, we will first explore another way to characterize elements in $KC_{\mathbb{Q}^c}$ that satisfy the BC property.

Definition 3.36 (Respecting f)**.** *A vector $\vec{\alpha} \in [1/3, 2]^{\mathbb{Z}}$* respects f *if*

$$\alpha_{i+1} = f(\alpha_i).$$

Further, we say a point $y \in \Omega$ (the set of all points with Sturmian rows) respects f if $\alpha(y)$ respects f.

The BC property has an alternate characterization in terms of the function P defined in the previous chapter.

Theorem 3.37. *(Geometric Characterization of $KC_{\mathbb{Q}^c}$) Suppose $y \in KC_{\mathbb{Q}^c}$ is the result of a basic construction arising from $(\vec{\alpha}, \vec{t})$. Then, $(\vec{\alpha}, \vec{t})$ has the BC property if and only if*

(i) $\vec{\alpha}$ respects f, and

(ii) $\vec{0} \in P(y)$.

Proof. For simplicity, call the two conditions of the BC property BC_1 and BC_2.

$((i) \iff BC_1)$

Notice that $\vec{\alpha}$ respects f if and only if $\alpha_{i+1}/\alpha_i \in \{\frac{1}{3}, 2\}$.

$(BC_1, BC_2 \implies (i), (ii))$

Suppose $(\vec{\alpha}, \vec{t})$ satisfies the first and second conditions of the BC property. Let X be the set of points satisfying the BC property equipped with the product topology, and let $W : [1/3, 2] \times \mathcal{T} \to X$ be the map from Lemma 3.33. Note that W is continuous when $\mathcal{T}$ is equipped with the product topology. Define $(\alpha, t) = W^{-1}(\vec{\alpha}, \vec{t})$ and note that $(\alpha, t) \in ([1/3, 2]\backslash\mathbb{Q}) \times \mathcal{T}$.

Fix n. Since $\alpha \in \mathbb{Q}^c$, we know that the set $\overline{\{i\alpha + t\}}$ contains a point $t_n \in \mathcal{T}$ whose first n coordinates are simultaneously zero. Similarly, by continuity of W, $\overline{\{W(\alpha, t + i\alpha)\}} = \overline{\{(\vec{\alpha}, \vec{t} + i\vec{\alpha})\}}$ contains a point $(\vec{\alpha}, \vec{t_n})$ such that the first n coordinates of $\vec{t_n}$ are simultaneously zero. Since $\vec{t_n} \in P(y)$, $P(y)$ contains a point where the first n coordinates are simultaneously zero for arbitrary n. It now follows from the definition of the product topology that $\vec{0} \in P(y)$.

$((i), (ii) \implies BC_1, BC_2)$

Conversely, assume $\vec{0} \in P(y)$ and fix n. By Proposition 2.52, the projection of $P(y)$ onto a finite number of coordinates is the graph of a line, and since $\vec{0} \in P(y)$, that line contains $\vec{0}$. Because $\vec{t} \in P(y)$, there is some $\gamma_n \in \mathbb{R}$ so that $(\vec{t})_{-n}^{n} = (\gamma_n \vec{\alpha})_{-n}^{n} \mod 1$. Since $\vec{\alpha}$ respects f, we immediately have that $\lambda\vec{\alpha} \mod \vec{1}$ satisfies BC_2 for any $\lambda \in \mathbb{R}$, and so $(\vec{t})_{-n}^{n}$ satisfies BC_2.

Because n was arbitrary, we have that in fact $\vec{t}$ satisfies BC_2, which completes the proof. $\square$

Chapter 4

Minimality of KC

We have defined what it means for a dynamical system with only one transformation to be minimal, namely that any non-empty set X such that $TX = X$ must be the whole space. If we have two transformations, the definition of minimality is quite similar.

Definition 4.1 (Minimality with Respect to Multiple Transformations)**.** *If (S, T, Ω, τ) is an invertible topological dynamical system with $S, T : \Omega \to \Omega$ and $S \circ T = T \circ S$, then (S, T, Ω, τ) is* minimal *if any non-empty closed subset $X \subset \Omega$ with the property that $S^a T^b X = X$ for all $a, b \in \mathbb{Z}$ must be Ω.*

If the underlying topological space is clear, we simply refer to the action of (S, T) as minimal.

In light of Theorem 3.35, we will first consider the minimality of the $\mathbb{Z}^2$ action of $(\hat{T}, \hat{S})$ on $[1/3, 2] \times \mathcal{T}$ where $\hat{T}$ and $\hat{S}$ are defined by

$$\hat{T}(\alpha, t) = (\alpha, t + \alpha) \qquad \text{and} \qquad \hat{S}(\alpha, t) = \hat{f}(\alpha, t).$$

However, trouble arises when considering (α, t) with $\alpha \in \mathbb{Q}$. To deal with this, we will introduce a different metric.

Definition 4.2. *Define the metric $\hat{d}_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\hat{d}_{\mathbb{R}}(x, y) = 0$ if $x = y$. Otherwise,*

$$\hat{d}_{\mathbb{R}}(x, y) = \hat{d}\Big(\mathbf{R}_{\lfloor \cdot \rceil}(x, 0), \mathbf{R}_{\lfloor \cdot \rceil}(y, 0)\Big)$$

where $\hat{d}$ is as in Definition 2.42.

Proposition 4.3. *$\mathbb{Q}^c$ with the subspace topology is completely metrizable using $\hat{d}_{\mathbb{R}}$.*

The proof of Proposition 4.3 follows from the fact that $\mathcal{S}_{\mathbb{Q}^c}$ is a complete metric space with respect to $\hat{d}$ and $\mathcal{S}_{\mathbb{Q}^c}$ is precisely the image of $\mathbb{R}\backslash\mathbb{Q}$ under $\mathbf{R}_{\lfloor\cdot\rfloor}$. (It may be helpful to recall that $\mathcal{S}_{\mathbb{Q}^c} = \bar{\mathcal{S}}_{\mathbb{Q}^c}$).

A simple application of Proposition 4.3 shows that the space $([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T}$ is completely metrizable (it can be endowed with a metric that makes it a complete metric space).

Proposition 4.4. *The $\mathbb{Z}^2$ action of $(\hat{T},\hat{S})$ on $([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T}$ is minimal.*

Proof. We first claim that since $\alpha \notin \mathbb{Q}$, the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense in $\mathcal{T}$. For any k, it is clear that the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense modulo 6^k. That is, the second coordinate of $\mathrm{id} \times \mathrm{proj}_k(\mathcal{O}_{\hat{T}}(\alpha,t))$ is dense in in $\mathrm{proj}_k(\mathcal{T})$. We therefore have the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense modulo 6^i for any $i \leq k$. Denseness now follows from the definition of the product topology on $\mathcal{T}$.

Using this observation, it is clear that for any $(\alpha,t'),(\alpha,t) \in ([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T}$, we have $(\alpha,t) \in \overline{\mathcal{O}_{\hat{T}}(\alpha,t')}$.

To complete the proof, fix $(\alpha',t'),(\alpha,t) \in ([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T}$. Since the orbit of any point under $f : [1/3,2] \to [1/3,2]$ is dense, we may find a point $(\alpha,t'') \in \overline{\mathcal{O}_{\hat{S}}(\alpha',t')}$. Using our previous observation, $(\alpha,t) \in \overline{\mathcal{O}_{\hat{T}}(\alpha,t'')}$. Since $\hat{T}$ and $\hat{S}$ are continuous, $(\alpha,t) \in \overline{\mathcal{O}(\alpha',t')}$, and so the orbit of every point is dense. □

Definition 4.5 (G_δ)**.** *A set is called a G_δ* set *if it is a countable intersection of open sets.*

Definition 4.6 (Residual Set)**.** *A set is called a* residual set *if it is a dense G_δ.*

If the function $K : ([1/3,2]\backslash\mathbb{Q}) \times \mathcal{T} \to KC_{\mathbb{Q}^c}$ were continuous, this would give us a quick proof of the minimality of $KC_{\mathbb{Q}^c}$. Unfortunately this is not the case, but the set of points where K is continuous is a dense G_δ.

Proposition 4.7. *The set of points of continuity of $K : [1/3,2] \times \mathcal{T} \to KC$ is a dense residual set G and $K(G)$ dense in KC.*

Proof. Let $A_{m\times n} = \{0,\ldots,m-1\} \times \{0,\ldots,n-1\}$. Suppose $E \subset [1/3,2] \times \mathcal{T}$ is an open set on which K is not continuous. Then, for some $m,n \in \mathbb{N}$, E must contain a point (α,t) so that $K(\alpha,t,\mathbf{R}_{\lfloor\cdot\rfloor})|_{A_{m\times n}} \neq K(\alpha,t,\mathbf{R}_{\lceil\cdot\rceil})|_{A_{m\times n}}$. Any point (α,t) that satisfies this is a point of discontinuity, and any point of discontinuity satisfies this condition for some m,n.

Define

$$B_{m\times n} = \{(\alpha, t) : K(\alpha, t, \mathbf{R}_{\lfloor\cdot\rfloor})|_{A_{m\times n}} \neq K(\alpha, t, \mathbf{R}_{\lceil\cdot\rceil})|_{A_{m\times n}}\},$$

and notice that since the only points of discontinuity of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are $\mathbb{Z}$, $B_{m\times n}$ is a closed, nowhere dense set implying that $B^c_{m\times n}$ is a dense open set. Let

$$G = \bigcap_{m,n\in\mathbb{N}} B^c_{m\times n}.$$

We now have that by construction, the set of continuity points of K is the dense residual set G. Further, since $K(B^c_{m\times n})|_{A_{m\times n}}$ contains every $m \times n$ configuration that appears in KC, $K(G)$ is dense in KC. □

Corollary 4.8. *If $O \subset KC$ is a non-empty open set, then $K^{-1}(O)$ contains a non-empty open set.*

Proof. Let G be a dense set of points of continuity of K whose image is dense. Fix an open set $O \subset KC$. Since $K(G)$ is dense, $K(G) \cap O \neq \emptyset$ and so O is a neighbourhood of the image of a point of continuity of K. Thus $K^{-1}(O)$ is a neighbourhood and so contains an open set. □

Lemma 4.9. *Suppose (X, σ) and $(Y, \hat{\sigma})$ are dynamical systems and that $g : X \to Y$ is a surjective map that satisfies $g \circ \sigma = \hat{\sigma} \circ g$. If the set of points of continuity of g is a dense set G and $g(G)$ is dense in Y, then (X, σ) minimal implies $(Y, \hat{\sigma})$ minimal.*

Proof. We will first show that if $D \subset X$ is dense, then $g(D)$ is dense. Fix $y \in Y$ and some neighbourhood N_y of y. Let G be a dense set of points of continuity for g such that $g(G)$ is dense. Then, $N_y \cap g(G) \neq \emptyset$ and so N_y is a neighbourhood of the image of a point of continuity. Thus $g^{-1}(N_y)$ is a neighbourhood of some point. Since D is dense, $D \cap g^{-1}(N_y) \neq \emptyset$, and so $g(D)$ intersects every neighbourhood and must be dense.

To complete the proof, fix $y \in Y$, and by surjectivity of g find $x \in X$ so $g(x) = y$. Suppose X is minimal. We have that $\mathcal{O}x$ is dense, and so $g(\mathcal{O}x) = \mathcal{O}g(x) = \mathcal{O}y$ is dense. □

Proposition 4.10. $\Phi(KC) = \overline{\Phi(KC_{\mathbb{Q}^c})}$.

Proof. Fix $y \in \Phi(KC)$, $A = \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$, and the cylinder set $C = \{x \in \Phi(KC) : x|_A = y|_A\}$. C is open and so by Corollary 4.8, $K^{-1}(C)$ contains an open set. Thus, there is some $(\alpha, t) \in K^{-1}(C)$ where $\alpha \notin \mathbb{Q}$. □

Theorem 4.11. *KC is minimal with respect to the group action of $\mathbb{Z}^2$ by translation.*

Proof. We will first show that every orbit of every point in $KC_{\mathbb{Q}^c}$ is dense in KC', the subset of KC where Φ is one-to-one, and then that the orbit closure of any point in KC intersects $KC_{\mathbb{Q}^c}$. Finally we will show that KC' is dense in KC. Here we must take special care to differentiate between KC and $\Phi(KC)$.

By Proposition 3.34, we have that

$$K\Big(([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}\times\{\mathbf{R}_{\lfloor\cdot\rfloor},\mathbf{R}_{\lceil\cdot\rceil}\}\Big)=KC_{\mathbb{Q}^c}.$$

Now, by Proposition 4.7, K satisfies the conditions of Lemma 4.9. Thus, since Proposition 4.4 states that $([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}$ is minimal with respect to $(\hat{T},\hat{S})$, Lemma 4.9 gives us that the orbit of every point in $KC_{\mathbb{Q}^c}$ is dense in $KC_{\mathbb{Q}^c}$.

Let

$$KC'=\{x\in KC:\alpha(x)\text{ does not contain }1/3,1/2,1,\text{ or }3/2\}.$$

and recall that the proof of Theorem 3.23 shows that Φ is one-to-one exactly on KC'. Thus, since $\Phi(KC_{\mathbb{Q}^c})$ is dense in $\Phi(KC)$, by continuity of Φ, we may conclude that the orbit of any point in $KC_{\mathbb{Q}^c}$ is dense in KC'.

Next, we will show that for any $y\in KC$, we have $\overline{\mathcal{O}y}\cap KC_{\mathbb{Q}^c}\neq\emptyset$. Fix $y\in KC$ and let $\alpha(y)=(\ldots,\alpha_0,\alpha_1,\ldots)$. Choose f be the function

$$f(x)=\begin{cases}2x & \text{if } x\in[1/3,1)\\ x/3 & \text{if } x\in[1,2]\end{cases}$$

or

$$f(x)=\begin{cases}2x & \text{if } x\in[1/3,1]\\ x/3 & \text{if } x\in(1,2]\end{cases}$$

such that $f(\alpha_i)=\alpha_{i+1}$. Since the orbit of every point under f is dense, we may find $y'\in\overline{\mathcal{O}y}$ so that $\alpha(y')$ contains only irrationals. However, since $\alpha(y')$ contains only irrationals, $y'\in KC_{\mathbb{Q}^c}$ and so $\mathcal{O}(y')$ is dense in $KC_{\mathbb{Q}^c}$ showing that $\mathcal{O}y$ is dense in $KC_{\mathbb{Q}^c}$.

To complete the proof, we will now show KC' is dense in KC. We will do this by considering cases. Suppose $y\in KC\backslash KC'$. This means for some j, $\alpha((y)_j)\in\{1/3,1/2,1,3/2\}$. For simplicity, assume this occurs at $j=0$ and let $\alpha=\alpha((y)_0)$.

Case $\alpha=3/2$: If $\alpha=3/2$, the sequence of bottom labels must be $\cdots 121212\cdots$. Looking at the transition graph in Figure 3.2, a sequence to bottom labels of $\cdots 121212\cdots$

can be realized in two ways. Call the configuration using the tiles in the bottom of the diagram configuration A and the configuration using the tiles in the top of the diagram configuration B, and notice that if $\alpha = 3/2 + \delta$ for $0 < \delta$ small, then the row will contain arbitrarily long runs of tiles in configuration A. Similarly, if $\alpha = 3/2 - \delta$, the row will contain arbitrarily long rows of tiles in configuration B.

Case $\alpha = 1$. Looking at the transition graph in Figure 3.3, we see that if $\alpha = 1$, a bottom sequence of $\cdots 111 \cdots$ can be obtain in two different ways. Call these configurations configuration A and configuration B. Notice that we can force the bottom labels of the row to contain arbitrarily long sequences of 1's separated by 0's by picking $\alpha = 1 - \delta$ for some small $0 < \delta$. Further, the only way this can happen is by alternating arbitrarily long occurrences of configuration A with arbitrarily long sequences of configuration B.

Similarly for cases $\alpha = 1/2$ and $\alpha = 1/3$, a small perturbation of α will produce arbitrarily long occurrences of each type of configuration. Since KC' contains all perturbations of angles of points in KC, KC' is dense in KC. $\square$

Chapter 5

Explicit Return Time Bounds

We will now give explicit bounds on the on the size of the smallest rectangular configuration in KC that contains every $m \times n$ sub-configuration. The strategy will be to analyze the parameter space $[1/3, 2] \times \mathcal{T}$ to find intervals of parameters that have short $\hat{T}$-return times and then bound the $\hat{S}$-return times to such intervals. These return time bounds will then carry forward to (KC, T, S).

Definition 5.1. *Let $\mathcal{P}_{m,n}$ be the partition of $[1/3, 2] \times \mathcal{T}$ given by $m \times n$ configurations in KC. Specifically, $(\alpha, t) \sim (\alpha', t')$ if for $A = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ we have*

$$K(\alpha, t, \mathbf{R}_{\lfloor \cdot \rfloor})|_A = K(\alpha', t', \mathbf{R}_{\lfloor \cdot \rfloor})|_A.$$

Definition 5.2. *For a partition $\mathcal{P}$ of $[1/3, 2] \times X$, let $\pi_\alpha(\mathcal{P})$ be the restriction of $\mathcal{P}$ to the fiber $\{\alpha\} \times X$.*

We will familiarize ourselves with the structure of $\mathcal{P}_{m,n}$. Let us consider $\mathcal{P}_{1,n}$. Putting the inverse-limit space $\mathcal{T}$ aside for a moment, let $\mathcal{P}_n$ be the partition of $[1/3, 2] \times [0, 1)$ such that $(\alpha, t) \sim (\alpha', t')$ if $(\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t))_0^{n-1} = (\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha', t'))_0^{n-1}$.

After considering pre-images under rotation by an angle α, we see that, as in the proof of Proposition 2.35, $\pi_\alpha(\mathcal{P}_n)$ is precisely the partition generated by intervals whose endpoints are consecutive elements of $C_\alpha = \{0, -\alpha, -2\alpha, \ldots, -n\alpha \bmod 1\}$. We view C_α as the places $[0, 1)$ needs to be "cut" to produce $\pi_\alpha(\mathcal{P}_n)$. Now, varying α, we see that $\mathcal{P}_n$ is produced by cutting $[1/3, 2] \times [0, 1)$ by the set of lines $L = \{(x, y) \in [1/3, 2] \times [0, 1) : y = -ix \bmod 1 \text{ for some } i \leq n\}$.

Recall for the next definition that $\mathrm{proj}_j : \mathcal{T} \to \mathbb{R}/(6^j\mathbb{Z})$ is projection onto the jth coordinate of $\mathcal{T}$.

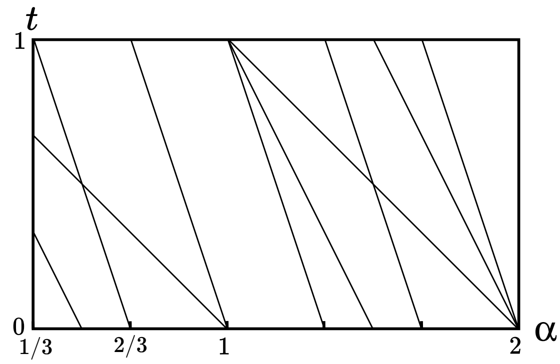


Figure 5.1: The partition $\mathcal{P}_3$.

Definition 5.3. *For $j \in \mathbb{N}$, define the σ-algebra $\mathscr{B}_j = (\mathrm{id} \times \mathrm{proj}_j)^{-1}(\mathscr{B})$ on $[1/3, 2] \times \mathcal{T}$ where $\mathscr{B}$ is the Borel σ-algebra defined on $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$.*

Informally, a partition $\mathcal{P}$ being $\mathscr{B}_j$-measurable means that for a point (α, t), α and $t \bmod 6^j$ are all you need to determine in which element of $\mathcal{P}$ it lies. Rephrased, $\mathcal{P}$ gives no extra information after the jth coordinate of $\mathcal{T}$. Consequently, $\mathcal{P}_{m,n}$ is $\mathscr{B}_j$-measurable for all $j \geq m$. Further, we may interchangeably talk about a $\mathscr{B}_j$-measurable partition of $[1/3, 2] \times \mathcal{T}$ and a Borel-measurable partition of $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$. Where a distinction is needed, we will say that a partition of $[1/3, 2] \times \mathcal{T}$ coming from $\mathcal{P}$, a partition of $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$, is the *$\mathscr{B}_j$-measurable extension* of $\mathcal{P}$ or just the *measurable extension* of $\mathcal{P}$.

Let $A = [1/3, 2] \times [0, 6^i)$ which we will identify with $[1/3, 2] \times \mathbb{R}/(6^i\mathbb{Z})$. Given a finite collection, L, of lines in A, we form a partition of A up to a Lebesgue measure-zero set by taking the connected components of L^c. We call this the *geometric partition generated by L*.

Let's consider how $\mathcal{P}_{m,n}$ and our description of $\mathcal{P}_n$ arising from lines relate.

Definition 5.4. *Let $L^j_{a,\gamma} = \{(x, y) \in [1/3, 2] \times \mathbb{R}/(j\mathbb{Z}) : y = -ix + \gamma \bmod j$ for some $0 \leq i \leq a\}$ be the set of lines with slopes in $\{0, -1, -2, \dots, -a\}$ and offset γ.*

We can now view $\mathcal{P}_{1,n}$ as being the $\mathscr{B}_0$-measurable extension of the partition on $[1/3, 2] \times [0, 1)$ generated by $L^1_{n,0}$. Further, the boundary points of $\mathrm{id} \times \mathrm{proj}_j(\mathcal{P}_{1,n})$ are precisely the set $\bigcup_{i<6^j} L^{6^j}_{n,i}$.

Consider $\hat{f}^{-1}(\mathcal{P}_{1,n})$. Since $\hat{f}^{-1} : [1/3, 2] \times \mathcal{T} \to [1/3, 2] \times \mathcal{T}$ either multiplies by 3 or divides by 2 (and does so in each coordinate if we view $\hat{f}$ as acting on $(\mathbb{R}^2)^{\mathbb{N}}$), we see

$$\hat{f}|^{-1}_{[1/3,2/3]\times\mathbb{R}/(6^j\mathbb{Z})}\left(L^{6^j}_{n,i}\right) =$$
$$= \left\{(3x, 3y \bmod 6^j) : (x,y) \in L^{6^j}_{n,i} \cap ([1/3,2/3) \times \mathbb{R}/(6^j\mathbb{Z}))\right\}$$
$$= L^{6^j}_{n,3i} \cap ([1,2] \times \mathbb{R}/(6^j\mathbb{Z})) \subset L^{6^j}_{n,3i}$$

and

$$\hat{f}|^{-1}_{[2/3,2]\times\mathbb{R}/(6^{j+1}\mathbb{Z})}\left(L^{6^{j+1}}_{n,i}\right) =$$
$$= \left\{(\tfrac{x}{2}, \tfrac{y}{2} \bmod 6^j) : (x,y) \in L^{6^{j+1}}_{n,i} \cap ([2/3,2] \times \mathbb{R}/(6^{j+1}\mathbb{Z}))\right\}$$
$$= L^{6^j}_{n,\frac{i}{2}} \cap ([1/3,1] \times \mathbb{R}/(6^{j+1}\mathbb{Z})) \subset L^{6^j}_{n,\frac{i}{2}}.$$

Illustrated in Figure 5.2 is a truncation of $\mathrm{proj}_3(\hat{f}^{-i}\mathcal{P}_{1,1})$.

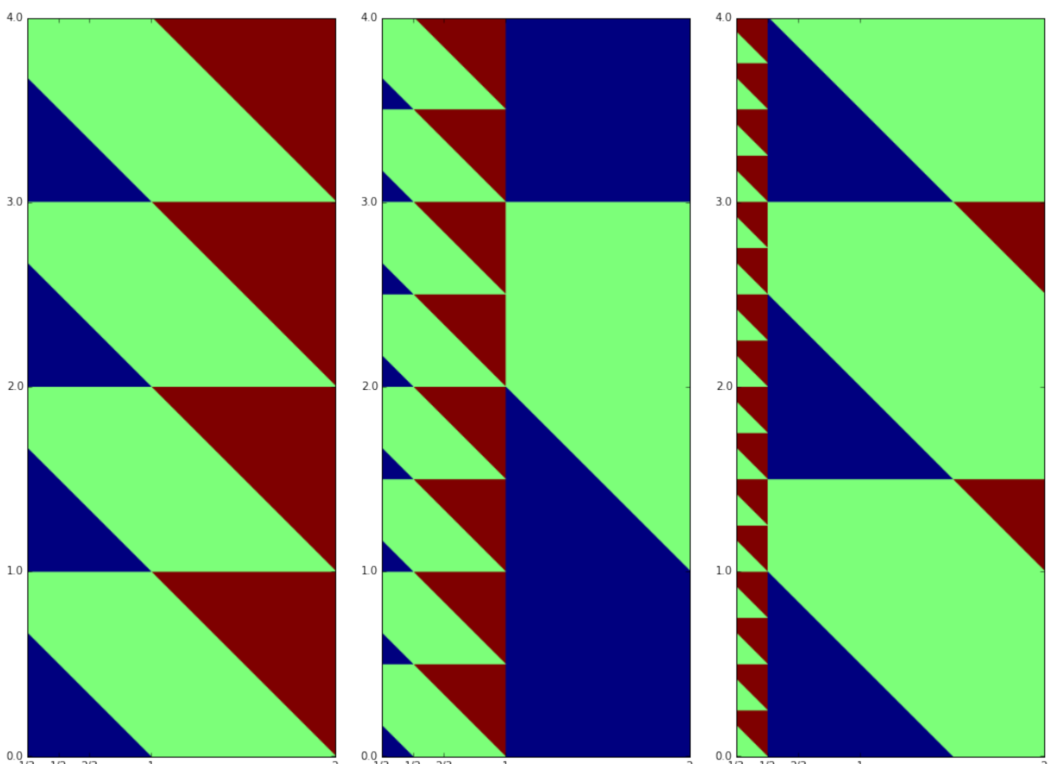


Figure 5.2: From left to right, the projection of $\mathcal{P}_{1,1}$, $\hat{f}^{-1}\mathcal{P}_{1,1}$, $\hat{f}^{-2}\mathcal{P}_{1,1}$ onto the third coordinate, truncated to lie in $[1/3,2] \times [0,4)$, and colored by whether the symbol at the zero position is 0, 1, or 2.

Definition 5.5. *Define* $\mathrm{rnd}_\alpha : \mathbb{R} \to \mathbb{Z}$ *by* $\mathrm{rnd}_\alpha(x) = n \in \mathbb{Z}$ *whenever* $x \in (n - \frac{\log 3}{\log 6} -$

$\frac{\log \alpha}{\log 6}, n + \frac{\log 2}{\log 6} - \frac{\log \alpha}{\log 6})$.

Note that $(n - \frac{\log 3}{\log 6} - \frac{\log \alpha}{\log 6}, n + \frac{\log 2}{\log 6} - \frac{\log \alpha}{\log 6})$ is an interval of length 1, so rnd_α is defined everywhere but the countable set of endpoints.

Lemma 5.6. *Suppose $\alpha \notin \mathbb{Q}$. Then, $\hat{f}^{-j}(\alpha, t) = (\frac{3^a}{2^b}\alpha, \frac{3^a}{2^b}t)$ and $f^{-j}(\alpha) = \frac{3^a}{2^b}\alpha$ where*

$$a = j - b \approx \frac{\log 2}{\log 6} j \qquad \text{and} \qquad b = \mathrm{rnd}_\alpha\left(\frac{\log 3}{\log 6} j\right) \approx \frac{\log 3}{\log 6} j.$$

Proof. The proof of Lemma 5.6 follows directly from solving the system $a + b = j$ and $\frac{3^a}{2^b}\alpha \in [1/3, 2]$ with the restriction that $a, b \in \mathbb{Z}$. We will include the derivation here.

Since $\alpha \notin \mathbb{Q}$, $\frac{3^a}{2^b}\alpha \notin \{1/3, 2\}$, and so we may solve the simpler inclusion $\frac{3^a}{2^b}\alpha \in (1/3, 2)$. Using the fact that $a = j - b$, we may solve for an integer b so that

$$\frac{3^{j-b}}{2^b}\alpha \in (1/3, 2).$$

Taking logs and rearranging slightly, we see that we must have

$$j \log 3 - b \log 6 = (j - b) \log 3 - b \log 2 \in (-\log 3 - \log \alpha, \log 2 - \log \alpha).$$

Dividing by $\log 6$ shows that we must have $j \frac{\log 3}{\log 6} - b \in (-\frac{\log 3}{\log 6} - \frac{\log \alpha}{\log 6}, \frac{\log 2}{\log 6} - \frac{\log \alpha}{\log 6})$, which is precisely satisfied by $\mathrm{rnd}_\alpha(j \frac{\log 3}{\log 6})$.

□

Proposition 5.7. *Let B be the boundary of* $\mathrm{id} \times \mathrm{proj}_m(\mathcal{P}_{m,n})$ *and let and $b = \mathrm{rnd}_{f^{-m}(\alpha)}(m \log 3 / \log 6)$. Then $\pi_\alpha(B)$ is*

$$\pi_\alpha\left(\bigcup_{k < 2^b 6^m} L^{6^m}_{n, \frac{k}{2^b}}\right).$$

Proof. Because the boundary of $\mathcal{P}_{m,n}$ is the boundary of $\bigvee_{i=0}^m f^{-i}(\mathcal{P}_{1,n})$, we see that $\pi_\alpha(L^{6^m}_{n,\frac{k}{2^b}})$ arises from applying $\hat{f}^{-m}$ to $\pi_{f^{-m}(\alpha)}(L^{6^{m+b}}_{n,k})$. This holds for every k, which completes the proof. □

Iterating $\hat{f}^{-1}$ and observing how it moves the boundaries of partition elements motivates us to define the following refinement of $\mathcal{P}_{m,n}$.

Definition 5.8. *Let $\mathcal{X}_{m,n}$ be the $\mathscr{B}_m$-measurable extension of the partition generated* *by* L *where*

$$L = \bigcup_{k<2^m 6^m} L^{6^m}_{n,\frac{k}{2^m}}.$$

Proposition 5.9. $\mathcal{X}_{m,n}$ *is a refinement of* $\mathcal{P}_{m,n}$.

Proof. For a fixed α, let b_α be the b from Proposition 5.7, and notice that $b_\alpha \leq m$. It now immediately follows that the set L defining $\mathcal{X}_{m,n}$ is a superset of the set of boundaries of $\mathrm{id} \times \mathrm{proj}_m(\mathcal{P}_{m,n})$, which completes the proof. $\square$

Definition 5.10. *For a partition* $\mathcal{P}$ *of* $\mathbb{R}$ *into intervals, let* $\kappa\mathcal{P}$ *be the* coarseness *of* $\mathcal{P}$. *That is,*

$$\kappa\mathcal{P} = \inf_{I \in \mathcal{P}} \mathit{length}(I).$$

If $\mathcal{P}$ *is a* $\mathscr{B}_j$*-measurable partition of* $\mathcal{T}$, *then* $\kappa\mathcal{P} = \kappa\mathrm{proj}_j(\mathcal{P})$.

Given $(\alpha, t) \in [1/3, 2] \times \mathcal{T}$, we would like to bound j such that $\mathcal{O}^j_{\hat{T}}(\alpha, t) = \{(\alpha, t), \hat{T}(\alpha, t), \ldots, \hat{T}^{j-1}(\alpha, t)\}$, the j-orbit of (α, t) under $\hat{T}$, intersects every partition element of $\pi_\alpha(\mathcal{P}_{m,n})$. We can address this in the following way.

Definition 5.11. *Let* $D^i_\ell(\alpha)$ *be the smallest* n *such that* $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha, t))$ *is* ℓ*-dense in* $\mathbb{R}/(6^i\mathbb{Z})$ *for any* t.

Note that the density of $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha, t))$ is equal to the density of $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha, t'))$, and so when computing $D^i_\ell(\alpha)$ we only need to consider a single t.

For a fixed α, we consider points in KC whose 0th row has rotation number α. Consider the $n \times m$ configuration that arises based at $(0, 0)$ corresponding to (α, t) for some t. We now see that for any $t' \in \mathcal{T}$, the maximum $\hat{T}$-waiting time to see an occurrence of this $n \times m$ configuration in the point corresponding to (α, t') is bounded by

$$D^m_{\kappa\pi_\alpha(\mathcal{P}_{m,n})}(\alpha) \leq D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha).$$

Proposition 5.12. $\kappa\pi_\alpha(\mathcal{P}_{m,n}) \geq \kappa\pi_\alpha(\mathcal{X}_{m,n}) \geq \min\{|\alpha - \frac{p}{2^m q}| : q \leq n\}$.

Proof. Since $\mathcal{X}_{m,n}$ is a refinement of $\mathcal{P}_{m,n}$, $\kappa\pi_\alpha(\mathcal{P}_{m,n}) \geq \kappa\pi_\alpha(\mathcal{X}_{m,n})$ follows trivially.

Let $\hat{\mathcal{X}}_{m,n}$ be the geometric partition generated by the lines $L = \bigcup_{k<2^m} L^1_{n,\frac{k}{2^m}}$. Recalling our description of $\mathcal{X}_{m,n}$ in terms of lines, we see that L corresponds exactly to the image under $\mathrm{id} \times \mathrm{proj}_0$ of the boundaries of the partition elements in $\mathcal{X}_{m,n}$. This shows that

$$\kappa\pi_\alpha(\hat{\mathcal{X}}_{m,n}) = \kappa\pi_\alpha(\mathcal{X}_{m,n}).$$

Thus, we will focus our attention on $\hat{\mathcal{X}}_{m,n}$.

Upon inspecting L, we see partition elements in $\pi_\alpha(\hat{\mathcal{X}}_{m,n})$ have endpoints in the set $E = \{\frac{k}{2^m}, -\alpha + \frac{k}{2^m}, \ldots, -n\alpha + \frac{k}{2^m} \bmod 1 : k < 2^m\}$.

Fix α and observe $\kappa\pi_\alpha(\hat{\mathcal{X}}_{m,n}) = d$ is now given by the minimum distance between two points in E, which is

$$d = |-i\alpha + \tfrac{k}{2^m} - (-j\alpha + \tfrac{k'}{2^m}) + p| = |a\alpha + \tfrac{b}{2^m} + p|,$$

for appropriate $p, a, b \in \mathbb{Z}$. Since $\frac{d}{a} = |\alpha + \frac{b+p2^m}{a2^m}|$ for some $p \in \mathbb{Z}$, $a \leq q$, and $-k < b < k$, the results follow immediately. $\square$

Having obtained a lower bound ℓ for $\kappa\pi_\alpha(\mathcal{P}_{m,n})$, we will now bound above the time it takes an orbit to become ℓ-dense.

Definition 5.13. *Define*

$$\mathcal{G}^{a,b} = \{\alpha : |\alpha - \tfrac{p}{q}| > \tfrac{1}{b} \text{ for } q \leq a \text{ and } p, q \in \mathbb{N}\}.$$

Note that $\mathcal{G}^{a,b}$ could be empty, but a simple estimate shows that $\mathcal{G}^{a,b}$ is non-empty if $b > a^2$.

Proposition 5.14. *$\alpha \in \mathcal{G}^{ka,b}$ implies $\{0, \alpha, 2\alpha, \ldots, (kb-1)\alpha \bmod k\}$ is $\frac{1}{a}$-dense in $\mathbb{R}/(k\mathbb{Z})$.*

Proof. For $x \in \mathbb{R}$, let $\|x\|_k$ represent the distance of x from $k\mathbb{Z}$. Suppose $\mathcal{G}^{ka,b} \neq \emptyset$, fix $\alpha \in \mathcal{G}^{ka,b}$, and let $q \in \mathbb{N}$ be the smallest number such that

$$\|q\alpha\|_k \leq \tfrac{1}{a}.$$

Let $p \in \mathbb{Z}$ be such that $\|q\alpha\|_k = |q\alpha - kp|$. By the pigeonhole principle, $q \leq ka$. By assumption, we have $|\alpha - \frac{kp}{q}| > \frac{1}{b}$ and so

$$\|q\alpha\|_k = |q\alpha - kp| > \tfrac{q}{b}.$$

It would suffice to show that the points

$$\begin{aligned} X &= \{0, \|q\alpha\|_k, 2\|q\alpha\|_k, \ldots, (\lceil b/q \rceil k - 1)\|q\alpha\|_k\} \\ &= \{0, q\alpha, 2q\alpha, \ldots, (\lceil b/q \rceil k - 1)q\alpha \bmod k\} \end{aligned}$$

are $\frac{1}{a}$-dense since they form a subset of $\{0, \alpha, 2\alpha, \ldots, (kb-1)\alpha \bmod k\}$. Since consecutive points in X are separated by a distance of less that $\frac{1}{a}$, we need only show that the last point satisfies $(\lceil b/q \rceil\, k - 1)\|q\alpha\|_k \geq k - 1/a$. But this is implied by the fact that $\frac{b}{q}\|q\alpha\|_k > 1$, which completes the proof. □

Proposition 5.15. *$\alpha \in \mathcal{G}^{ka,b}$ implies $\{0, \alpha, 2\alpha, \ldots, ka\alpha \bmod k\}$ is $\frac{1}{b}$-sparse in $\mathbb{R}/(k\mathbb{Z})$. That is, no two points are within $1/b$ of each other.*

Proof. Suppose $\mathcal{G}^{ka,b} \neq \emptyset$. Fix $\alpha \in \mathcal{G}^{ka,b}$ and note that to prove $\frac{1}{b}$-sparsity of $\{0, \alpha, 2\alpha, \ldots, ka\alpha \bmod k\}$ we only need to show $\|r\alpha\|_k > \frac{1}{b}$ for all $0 < r \leq ka$.

Choose p, q to minimize $|q\alpha - kp|$ subject to $0 < q \leq ka$. We then have that $\|r\alpha\|_k$ is minimized by

$$\|q\alpha\|_k = |q\alpha - kp| \geq |\alpha - \tfrac{kp}{q}| > \tfrac{1}{b},$$

with the last inequality following by assumption. □

The previous propositions show a symmetry in $\mathcal{G}^{a,b}$. Namely, if $\alpha \in \mathcal{G}^{a,b}$, then the b-orbit of rotation by α is $1/a$-dense and the $(a+1)$-orbit of rotation by α is $1/b$-sparse.

Corollary 5.16. *If $\alpha \in \mathcal{G}^{2^m n,b}$ then $\kappa\pi_\alpha(\mathcal{X}_{m,n}) > \frac{1}{b}$.*

Proof. Fix $\alpha \in \mathcal{G}^{2^m n,b}$. By Proposition 5.12, $\kappa\pi_\alpha(\mathcal{X}_{m,n}) \geq \min\{|\alpha - \frac{p}{2^m q}| : q \leq n \text{ and } p, q \in \mathbb{N}\}$. By the assumption that $\alpha \in \mathcal{G}^{2^m n,b}$, we have $|\alpha - \frac{p}{2^m q}| > \frac{1}{b}$. □

Proposition 5.17. *If $\alpha \in \mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$ then*

$$D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha) \leq 6^m c.$$

Proof. Fix $\alpha \in \mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$. Since $\alpha \in \mathcal{G}^{2^m n,b}$, Corollary 5.16 implies $\kappa\pi_\alpha(\mathcal{X}_{m,n}) > \frac{1}{b}$ and so $\kappa\pi_\alpha(\mathrm{proj}_m(\mathcal{X}_{m,n})) > \frac{1}{b}$. By Proposition 5.14 applied to $\mathcal{G}^{6^m b,c}$, we have that $E = \{0, \alpha, \ldots, (6^m c - 1)\alpha \bmod 6^m\}$ is $\frac{1}{b}$-dense in $[0, 6^m]$, and so E intersects every partition element of $\pi_\alpha(\mathcal{X}_{m,n})$, which completes the proof. □

We have identified α's that give us good return times, but $\mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$ could be empty. Next we will find constraints on b, c to avoid this and guarantee us some useful properties.

Definition 5.18. *Given a set X and a collection of sets $\mathcal{C}$, we say $(X, \mathcal{C})$ has the* intersection property *if for all $I \in \mathcal{C}$, $X \cap I \neq \emptyset$. If $X \subset \mathbb{R}$, we say X is δ-*fat *relative to $\mathcal{C}$ if for all $I \in \mathcal{C}$, $X \cap I$ contains an interval of width δ.*

Definition 5.19. *Let $\mathcal{F}_n$ be the partition of $\mathbb{R}$ whose elements are of the form $[a, b)$ where a, b are consecutive points in $\{\frac{p}{q} : q \leq n\}$. That is $\mathcal{F}_n$ is the partition of $\mathbb{R}$ into half-open intervals whose endpoints are consecutive Farey fractions with denominator bounded by n.*

Proposition 5.20. *Let $p : [1/3, 2] \times \mathcal{T} \to [1/3, 2]$ be projection onto the first coordinate. Let $X \subset \mathbb{R}$. If $(X, \mathcal{F}_{2^m n})$ has the intersection property, then for any element $E \in \mathcal{X}_{m,n}$, $X \cap p(E) \neq \emptyset$.*

Proof. Let $\hat{\mathcal{X}}_{m,n} = \mathrm{id} \times \mathrm{proj}_0(\mathcal{X}_{m,n})$ and note it is sufficient to show that if $(X, \mathcal{F}_{2^m n})$ has the intersection property, then for any element $E \in \hat{\mathcal{X}}_{m,n}$, we have $X \cap p(E) \neq \emptyset$.

Recalling the description of $\hat{\mathcal{X}}_{m,n}$ in terms of lines, we see that $\hat{\mathcal{X}}_{m,n}$ consists of polygonal regions whose corners have coordinates of the form $\frac{p}{2^m q}$ for some $q \leq n$. Since every element of $\hat{\mathcal{X}}_{m,n}$ contains an open set, we see that for all $P \in \hat{\mathcal{X}}_{m,n}$, there exists $I \in \mathcal{F}_{2^m n}$ so that $I \subset p(P)$ (possibly ignoring some points along the boundary of P), which completes the proof. $\square$

Proposition 5.21. *If $b \geq 4a^2$, $c^2 \geq 4b$, and $d \geq 4c^2$ then $\mathcal{G}^{a,b} \cap \mathcal{G}^{c,d}$ is $\frac{2}{d}$-fat relative to $\mathcal{F}_a$.*

Proof. By definition $\mathcal{G}^{x,y}$ is constructed by removing balls of radius $1/y$ centered at points $\frac{p}{q}$ with $q \leq x$. If $q, q' \leq x$, then $|\frac{p}{q} - \frac{p'}{q'}| > \frac{1}{x^2}$. Thus, if $y > 4x^2$, not only will $\mathcal{G}^{x,y}$ intersect every element of $\mathcal{F}_x$, but it will do so with diameter at least

$$\frac{1}{x^2} - \frac{2}{y} = \frac{1}{2x^2}.$$

Suppose a, b, c, d satisfy $b \geq 4a^2$, $c^2 \geq 4b$, and $d \geq 4c^2$. Every gap in $\mathcal{G}^{c,d}$ is of size $\frac{2}{d} < \frac{1}{2c^2}$ and every interval in $\mathcal{G}^{c,d}$ has size at least $\frac{1}{2c^2}$. Thus, the intersection of $\mathcal{G}^{c,d}$ with an interval of width $\frac{1}{2b}$ must contain an interval of width at least

$$\min\left\{\frac{1}{2c^2}, \frac{1}{2b} - \frac{2}{2c^2}\right\} \geq \min\left\{\frac{1}{2c^2}, \frac{2}{c^2} - \frac{1}{c^2}\right\} = \frac{1}{2c^2} \geq \frac{2}{d}.$$

Noticing that the smallest interval in $\mathcal{G}^{a,b}$ is of size at least $\frac{2}{b} > \frac{1}{2b}$ completes the proof. $\square$

We can now identify a set of α's that have good waiting times.

Definition 5.22. *Let $\mathcal{W}_{n\times m} = \mathcal{G}^{a,b} \cap \mathcal{G}^{c,d}$ where $a = 2^m n$, $b = 2^{2m+2}n^2$, $c = 6^m 2^{2m+2} n^2$, and $d = 6^{4m+3}n^4$.*

Notice that the parameters a, b, c, d in $\mathcal{W}_{n\times m}$ were carefully chosen to satisfy the conditions of Proposition 5.21 and Proposition 5.17.

Theorem 5.23. *Let c be an $n \times m$ configuration in KC and $A = \{0, \dots, m-1\} \times \{0, \dots, n-1\}$. Then there exists an interval $I_c \subset \mathcal{W}_{n\times m}$ of width $2/(6^{4m+3}n^4)$ so that for every $\alpha \in I_c$ and every $t \in \mathcal{T}$, there exists a $j < 6^{5m+3}n^4$ so that*

$$K \circ \hat{T}^j(\alpha, t)|_A = c.$$

Proof. Given the framework we have established, the proof is straightforward.

Proposition 5.21 tells us that $\mathcal{W}_{n\times m}$ is $2/(6^{4m+3}n^4)$-fat relative to $\mathcal{F}_{2^m n}$, and so by Proposition 5.20, we have that there exists an interval $I_c \subset \mathcal{W}_{n\times m}$ of width $2/(6^{4m+3}n^4)$ so that for every $\alpha \in I_c$ there exists $t \in \mathcal{T}$ so $K(\alpha, t)|_A = c$.

Fix I_c and $\alpha \in I_c$. By Proposition 5.17,

$$D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha) \le 6^m 6^{4m+3} n^4 = 6^{5m+3} n^4,$$

and so we will see c in less than $6^{5m+3}n^4$ applications of $\hat{T}$, which completes the proof. □

Theorem 5.23 gives the bulk of the proof of Theorem 5.28. If we have an $n \times m$ configuration c in mind, we know there is an open interval I_c of angle parameters where we will see c in a horizontal orbit of no more than $6^{5m+3}n^4$ steps. Since orbits under $\hat{S}$ are dense in the first coordinate, we know that if we bound how long it takes for an $\hat{S}$-orbit (equivalently an f-orbit) to become $|I_c|$-dense, we have a bound on the minimum size of a rectangle that contains the configuration c.

5.1 Asymptotic Density of Orbits Under f

Definition 5.24 (Irrationality Measure)**.** *For a number $\alpha \in \mathbb{R}$, the* irrationality measure *of α is*

$$\eta(\alpha) = \inf\left\{\gamma : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\gamma} \text{ for only finitely many } p, q \in \mathbb{Z}\right\}.$$

Proposition 5.25 (Rhin [15])**.** *For $u_0, u_1, u_2 \in \mathbb{Z}$ and $H = \max\{|u_1|, |u_2|\}$, we have* *that if H is sufficiently large,*

$$|u_0 + u_1 \log 2 + u_2 \log 3| \geq H^{-7.616},$$

and if $H \geq 2$, we have the universal bound

$$|u_0 + u_1 \log 2 + u_2 \log 3| \geq H^{-13.3}.$$

Corollary 5.26. *$\eta(\log 2/\log 6) \leq 8.616$ and $\left|\frac{\log 2}{\log 6} - \frac{p}{q}\right| \geq \frac{1/\log 6}{q^{14.3}}$ if $q \geq 2$.*

Proof. Let $x = \left|\frac{\log 2}{\log 6} - \frac{p}{q}\right|$. By algebraic manipulation, we deduce

$$xq \log 6 = |(q-p)\log 2 - p \log 3|.$$

And so by Proposition 5.25 and the fact that $\max\{|q-p|, |p|\} \leq q$, we have that asymptotically, $xq \log 6 \geq q^{-7.616}$, which produces a bound of $x \geq \frac{1/\log 6}{q^{8.616}}$. Alternatively, we may use the bound $xq \log 6 \geq q^{-13.3}$, which holds for all $q \geq 2$. $\square$

Proposition 5.27. *Fix $\delta > 0$ and let $k_\ell \geq (\frac{3}{\ell \log 6})^{8.616+\delta}$. Then, for sufficiently small ℓ, the k_ℓ-orbit of any $x \in [1/3, 2]$ under f is ℓ-dense. That is*

$$\{x, f(x), f^2(x), \ldots f^{k_\ell - 1}(x)\}$$

is ℓ-dense for any $x \in [1/3, 2]$. Further, if $k_\ell \geq (\frac{3}{\ell \log 6})^{14.3} \log 6$ and $1/\ell \geq 2$, then the k_ℓ orbit of any point $x \in [1/3, 2]$ under f is ℓ-dense.

Proof. Let ϕ be the conjugacy from Proposition 3.7 between f and rotation by $\frac{\log 2}{\log 6}$. We have that $|\phi'|$ attains a maximum value of $\frac{3}{\log 6}$. Thus, to ensure an orbit segment under f is ℓ-dense, we must have that the image of an orbit segment under $\phi \circ f \circ \phi^{-1} = R_{\frac{\log 2}{\log 6}}$ is $\frac{\ell \log 6}{3}$-dense. Let $\eta = \eta(\log 2/\log 6)$ be the irrationality measure of $\log 2/\log 6$. Fix $\delta > 0$. We then have, by the definition of the irrationality measure, $\log 2/\log 6 \in \mathcal{G}^{k, k^{\eta+\delta}}$ for all sufficiently large k. Applying Proposition 5.14 and using Corollary 5.26 to bound η now completes the proof of the first claim.

For the second claim, note that Corollary 5.26 implies that $\frac{\log 2}{\log 6} \in \mathcal{G}^{k, k^{14.3} \log 6}$ for any $k \geq 2$. The proof then follows similarly. $\square$

Theorem 5.28. *Let $\eta = \eta(\log 2/\log 6)$. Every legal $n \times m$ configuration in KC* *occurs in every $B \times A$ configuration in KC where*

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{\eta} < 6^{34.464m+25} n^{34.464} \qquad \text{and} \qquad B = 6^{5m+3} n^4$$

for sufficiently large $m + n$.

Further, for all m, n we have that a copy of every legal $n \times m$ configuration in KC occurs in every $B \times A$ configuration in KC where

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{14.3} \log 6 \qquad \text{and} \qquad B = 6^{5m+3} n^4$$

Proof. Let $C = \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ and let c be a legal $n \times m$ configuration. Fix $I_c \subset \mathcal{W}_{n\times m}$ as in Theorem 5.23. We now have that for any $(\alpha, t) \in I_c \times \mathcal{T}$, $K \circ \hat{T}^j(\alpha, t)|_C = c$ for some $j < 6^{5m+3} n^4$.

Since I_c is of length at least $2/(6^{4m+3} n^4)$, by Proposition 5.27 with $\ell = 2/(6^{4m+3} n^4)$, we see that for any $(\alpha, t) \in [1/3, 2] \times \mathcal{T}$, we have $\hat{S}^j(\alpha, t) \in I_c \times \mathcal{T}$ for some $j < (3 \cdot 6^{4m+3} n^4/(2 \log 6))^{\eta}$.

We now have a bound on how many applications of $\hat{T}$ and $\hat{S}$ it takes to land in a particular element of $\mathcal{P}_{n,m}$, which gives bounds on A and B.

Alternatively, using the second part of proposition 5.27 with $\ell = 2/(6^{4m+3} n^4)$ we get a bound for all $m \geq 2$. □

5.1.1 Alternative Bound on the Return of $n \times 1$ Words

Fix a particular length n Sturmian word w, and suppose w occurs in $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, 0)$. By Theorem 2.33, we know the waiting time for w is bounded by $n + 1/|I|$, where

$$I = \{t : (\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t))_0^{n-1} = w\}$$

is the interval of phases t such that w occurs as the starting word of $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$. Thus, bounding $|I|$ would bound the waiting time for w. Of course, I is just an element of $\pi_\alpha(P_n)$, and as we have seen in Proposition 5.15, if $\alpha \in \mathcal{G}^{a,b}$, then $|I| > 1/b$.

Proposition 5.29. *If $\alpha \in \mathcal{G}^{n,m}$, the maximum waiting time for a length-n subword of $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, 0)$ is $n + m$.*

Proof. This is a direct application of Theorem 2.33 and 5.15. □

Proposition 5.30 (Refinement of Theorem 5.28)**.** *Fix $\delta > 0$ and let w be a $n \times 1$ configuration in KC. Then, for large enough n, w occurs in every $A_n \times B_n$ configuration in KC where $A_n = 2^{24} n^{17.232+\delta}$ and $B_n = 4n^2 + n$.*

Proof. By arguments similar to those leading up to the proof of Theorem 5.28, for any length-n word w, there is an interval $I_w \subset \mathcal{G}^{n,4n^2}$ so that for $\alpha \in I_w$ and any t, $\mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ contains w in every block of size at least $n + 4n^2$ and $|I_w| \geq 1/(4n^2)$. We also have a bound of $(12n^2/\log 6)^{8.616+\delta} < 2^{24} n^{17.232+\delta}$ for how long it takes an f-orbit to become $1/(4n^2)$ dense. Combining these facts completes the proof. $\square$

Appendices

Appendix A

Code for Enumerating Kari-Culik Configurations

Below is Python code that enumerates all misaligned straddle words that can occur in the Kari-Culik tileset. The main function is `enumerate_misaligned_straddle_words()` which when given a set of tiles will first create all possible words of length four with compatible left-right edges, then filters those words to make sure the top and bottom labels form valid Sturmian subwords (by testing that they are balanced), and then it filters to and only returns the set of words whose tops and bottoms are misaligned straddle words.

The output from the code is

```
There are 0 pairs of misaligned straddle words of type 1/3
There are 0 pairs of misaligned straddle words of type 2.1
There are 2 pairs of misaligned straddle words of type 2.2
```

```
from __future__ import division, print_function
from math import *
from collections import defaultdict
from fractions import Fraction
F = Fraction

class Tile(dict):
    """ A class to represent a Wang tile """
    def __init__(self, top='', left='', bottom='', right='', stacked_middle=None)
    :
        dict.__init__(self)
        self['top'] = top
        self['left'] = left
        self['bottom'] = bottom
```

```
        self['right'] = right
        self['stacked_middle'] = stacked_middle
        self.type = Fraction(1,3) if self['left'] in ['0/3','1/3','2/3'] else Fraction(2,1)
    def __hash__(self):
        return self.__repr__().__hash__()
    def __repr__(self):
        return self.__str__()
    def __str__(self):
        return "[{} {} {} {}]".format(self['left'],self['top'],self['bottom'],self['right'])

# initialize the basic tilesets

TYPE_13 = [Tile('0','0/3','1','1/3'),Tile('0','1/3','1','2/3'),Tile('1','2/3','1','0/3'),
           Tile('0','0/3','2','2/3'),Tile('1','1/3','2','0/3'),Tile('1','2/3','2','1/3')]
TYPE_2  = [Tile("0'","-1","0","-1"),Tile("0'","0","0","0"),Tile("1","0","0","-1"),
           Tile("1","-1","1","0"),Tile("2","-1","1","-1"),Tile("2","0","1","0"),
           Tile("1","0","0'","-1")]
BLANK_TILE = Tile()
KC_TILES = TYPE_13 + TYPE_2

def create_stacked_tileset(input_tiles):
    """creates a tile set formed by stacking tiles from input_tiles on top of eachother in
    every possible way"""

    ret = set()
    for top_t in input_tiles:
        for bottom_t in input_tiles:
            if top_t['bottom'] == bottom_t['top']:
                ret.add(Tile(top=top_t['top'], bottom=bottom_t['bottom'], left=top_t['left']+" "+bottom_t['left'], right=top_t['right']+" "+bottom_t['right'], stacked_middle=top_t['bottom']))
    return ret

# initialize the type 2.1 and 2.2 tiles

# if two consecutive rows of tiles of type 2 appear, we can group those tiles into one stacked tileset, TYPE_22
TYPE_22 = list(filter(lambda x: x['top'] != "0'" and x['bottom'] != "0'", create_stacked_tileset(TYPE_2)))
# if TYPE_2 tiles are sandwitched between two rows of type 1/3, they must be these tiles
TYPE_21 = list(filter(lambda x: x['top'] != "0'" and x['bottom'] != "0'", TYPE_2))


#
#### tools for enumerating tiles ####
```

```python
#
def create_adjacency_graph(tiles):
    """returns a dictionary giving tiles that follow to the right of a given tile
    """
    ret = {}
    for t in tiles:
        ret[t] = set()
        for t2 in tiles:
            if t['right'] == t2['left']:
                ret[t].add(t2)
    return ret

def enumerate_tiles(graph, length=4, ret=None):
    """ returns a list of all walks on the graph of length 'length' """
    # no more tiles to add
    if length <= 0:
        return ret
    # base case, we're just starting out
    if ret == None:
        ret = [[t] for t in graph.keys()]
        return enumerate_tiles(graph, length - 1, ret)
    # we have initial lists, we're going to append to
    new_ret = []
    for row in ret:
        prev_tile = row[-1]
        for poss_tile in graph[prev_tile]:
            new_ret.append(row + [poss_tile])
    return enumerate_tiles(graph, length - 1, new_ret)

def enumerate_misaligned_straddle_words(tiles):
    """ given a tile set, enumerate all words of length 4 that can be constructed
    and return a list of ones that contain misaligned straddle words """
    g = create_adjacency_graph(tiles)
    all_words = enumerate_tiles(g, length=4)
    misaligned = []
    for word in all_words:
        tops = proj(word, 'top')
        bottoms = proj(word, 'bottom')

        if is_straddle_word(tops) and is_straddle_word(bottoms) and (
    straddle_alignment(tops) != straddle_alignment(bottoms)):
            misaligned.append(word)
    return misaligned

def balanced(l,n=2):
    """ returns whether sums of n consecutive labels differ by at most one """
    sums = set()
    for i in range(len(l)-n+1):
        s = sum(int(w) for w in l[i:i+n])
        sums.add(s)
    return max(sums) - min(sums) <= 1

def is_sturmian(l):
```

```
    """ returns whether the sequence l is Sturmian"""
    for n in range(1,len(l)):
        if balanced(l, n) == False:
            return False
    return True

def is_straddle_word(l):
    """ returns whether or not l is a straddle word.
    I.e., it is length 4, Sturmian, its first and last
    symbols agree and its middle symbols disagree. """
    if len(l) != 4:
        raise Error("A straddle word must be length 4. \"{}\" isn't".format(l))
    if is_sturmian(l) and (l[0] == l[3]) and (l[1] != l[2]):
        return True
    return False

def straddle_alignment(l):
    """ returns the alignment of a straddle word, 'left' or 'right'
    depending on which symbol is bigger, the left, or the right of the middle """
    if l[1] > l[2]:
        return "left"
    return "right"

def proj(l, determined='top'):
    """ Project onto the determined label """
    return [x[determined] for x in l]

# detect if we're being executed as opposed to imported as a module
if __name__ == "__main__":
    misaligned = enumerate_misaligned_straddle_words(TYPE_13)
    print("There are {} pairs of misaligned straddle words of type 1/3".format(
    len(misaligned)))
    misaligned = enumerate_misaligned_straddle_words(TYPE_21)
    print("There are {} pairs of misaligned straddle words of type 2.1".format(
    len(misaligned)))
    misaligned = enumerate_misaligned_straddle_words(TYPE_22)
    print("There are {} pairs of misaligned straddle words of type 2.2".format(
    len(misaligned)))
```

code/enumerate_misaligned.py

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