A NEW CONVOLUTION THEOREM FOR THE STIELTJES TRANSFORM AND ITS APPLICATION TO A CLASS OF SINGULAR INTEGRAL EQUATIONS

H.M. SRIVASTAVA AND VU KIM TUAN

DMS-648-IR

November 1993

A NEW CONVOLUTION THEOREM FOR THE STIELTJES TRANSFORM AND ITS APPLICATION TO A CLASS OF SINGULAR INTEGRAL EQUATIONS

by

H.M. SRIVASTAVA and VU KIM TUAN

Dedicated to the memory of Professor David Vernon Widder (1898-1990)

A new convolution theorem is proved for the Stieltjes transform and is then applied in solving a certain class of singular integral equations which are related rather closely to the Riemann-Hilbert boundary value problem. Some further extensions and consequences of the convolution theorem are also considered.

1991 *Mathematics Subject Classification.* Primary 44A15, 44A35; Secondary 45E05.

Key words and phrases. Convolution theorem, Stieltjes transform, singular integral equations, Riemann-Hilbert boundary value problem, Fourier transform, Laplace transform, Mellin transform, convolution integral equations, convolution transforms, Cauchy principal values, Hölder inequality, Hilbert transform, bounded operator, Riesz inequality, Lebesgue theorem, Stieltjes convolution, transcendental equation, classical inversion theorem.

1. INTRODUCTION

The convolution theorems for the Fourier, Laplace, and Mellin transforms are well-known (see, *e.g.*, [2] and [8]). Each of these results has a great potential for applications in solving convolution integral equations (*cf.* [7] and [8]), in the investigation of convolution transforms (*cf.* [4], [9], and [10]), and in the evaluation of definite integrals (*cf.* [5]). The object of the present paper is first to prove a new convolution theorem for the Stieltjes transform:

$$\mathcal{S}\{f(t):s\} = \int_0^\infty \frac{f(t)}{s+t}\,dt \qquad (s \in D), \tag{1}$$

which arises naturally from the iteration of the classical Laplace transform, D being an arbitrary region of the complex s-plane cut along the nonpositive real axis. We then consider an interesting deduction from this convolution theorem leading to a Titchmarsh type theorem. We also show how our convolution theorem can be applied in solving a certain class of singular integral equations which are usually investigated by reducing the problem to an equivalent Riemann-Hilbert boundary value problem.

2. THE CONVOLUTION THEOREM

Let the functions f and g be defined on the interval

$$\mathbb{R}_+ = (0,\infty) := \{x : 0 < x < \infty\}.$$

We define a function h on $\mathbb{R}_+$ by

$$h(t) = (f \otimes g)(t) := f(t)\int_0^\infty \frac{g(u)}{u-t}\,du + g(t)\int_0^\infty \frac{f(u)}{u-t}\,du, \tag{2}$$

where the integrals, when they exist, are understood as their *Cauchy principal values.*

Our main result is contained in the following

THEOREM. *Let*

$$f \in L_p(\mathbb{R}_+) \qquad \textit{and} \qquad g \in L_q(\mathbb{R}_+)$$

$$(1 < p < \infty; \quad 1 < q < \infty; \quad r^{-1} := p^{-1} + q^{-1} < 1).$$

Then the function h, defined by (2), belongs to $L_r(\mathbb{R}_+)$ and its Stieltjes transform is given by

$$\mathcal{S}\{h(t):s\} := \mathcal{S}\{(f \otimes g)(t):s\} = \mathcal{S}\{f(t):s\}\,\mathcal{S}\{g(t):s\}. \tag{3}$$

Proof. We begin by defining the functions f_1 and f^* by

$$\begin{cases} f_1(t) := \displaystyle\int_0^\infty \frac{f(u)}{u-t}\,dt \quad (f \in L_p(\mathbb{R}_+);\ t \in \mathbb{R}_+) \\ f^*(t) := \begin{cases} f(t) & (t > 0) \\ 0 & (t \leq 0). \end{cases} \end{cases} \tag{4}$$

Clearly, for $f \in L_p(\mathbb{R}_+)$, we have

$$f^* \in L_p(\mathbb{R}) \qquad (\mathbb{R} := \mathbb{R}_+ \cup \{0\}).$$

Furthermore, since the Hilbert transform

$$\tilde{f}(t) := \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{\infty} \right) \frac{f^*(u)}{t-u}\,du \qquad (t \in \mathbb{R}) \tag{5}$$

is a bounded operator in $L_p(\mathbb{R})$ (*cf.*, *e.g.*, [1, p. 315, Theorem 8.1.12]), we conclude that

$$\tilde{f}(t) \in L_p(\mathbb{R}).$$

Combining this observation with the fact that [*cf.* Equations (4) and (5)]

$$f_1(t) = -\tilde{f}(t) \qquad (t \in \mathbb{R}_+),$$

we have

$$f_1(t) \in L_p(\mathbb{R}_+),$$

which, in view of the Hölder inequality [6, p. 15], yields

$$g(t)\,f_1(t) \in L_r(\mathbb{R}_+)$$

under the hypothesis of the theorem.

In an analogous manner, we can show that

$$g_1(t) := \int_0^\infty \frac{g(u)}{u-t}\,du \in L_q(\mathbb{R}_+)$$

and

$$f(t)g_1(t) \in L_r(\mathbb{R}_+).$$

Therefore $h \in L_r(\mathbb{R}_+)$, which is precisely the first assertion of the theorem.

Next, since the Stieltjes transform (1) is a bounded operator in $L_r(\mathbb{R}_+)$ $(1 < r < \infty)$ (*cf.* [6, p. 225]), we can take the Stieltjes transform of the function h defined by (2), and we have

$$\begin{aligned} \mathcal{S}\{h(t) : s\} = & \int_0^\infty \frac{f(t)}{s+t} \left(\int_0^\infty \frac{g(u)}{u-t}\, du \right) dt \\ & + \int_0^\infty \frac{g(t)}{s+t} \left(\int_0^\infty \frac{f(u)}{u-t}\, du \right) dt. \end{aligned} \tag{6}$$

Putting

$$f_\varepsilon(t) := \left(\int_0^{t-\varepsilon} + \int_{t+\varepsilon}^\infty \right) \frac{f(u)}{u-t}\, dt \qquad (\varepsilon > 0),$$

and applying the Riesz inequality [1, p. 315, Theorem 8.1.12], we obtain

$$\|f_\varepsilon\|_{L_p(\mathbb{R}_+)} \le C\|f^*\|_{L_p(\mathbb{R})} = C\|f\|_{L_p(\mathbb{R}_+)}\,, \tag{7}$$

where f^* is defined by (4). We note also that

$$(s+t)^{-1} \in L_\rho(\mathbb{R}_+) \qquad (r^{-1} + \rho^{-1} = 1).$$

Hence, using the Hölder inequality for *three* functions (*cf.* [6, p. 154] and [8, p. 97]), we see that

$$\varphi_\varepsilon(t) := (s+t)^{-1}\, g(t)\, f_\varepsilon(t) \in L_1(\mathbb{R}_+)$$

and, moreover, that

$$\|(s+t)^{-1}\, g\, f_\varepsilon\|_1 \le \|(s+t)^{-1}\|_\rho\, \|f\|_p\, \|g\|_q\,. \tag{8}$$

Thus one can apply the Lebesgue theorem to obtain

$$\int_{\mathbb{R}_+} \left\{ \lim_{\varepsilon \to 0+} \varphi_\varepsilon(t) \right\} dt = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}_+} \varphi_\varepsilon(t)\, dt \tag{9}$$

or, equivalently,

$$\int_0^\infty \frac{g(t)}{s+t}\left(\int_0^\infty \frac{f(u)}{u-t}\,du\right)dt = \int_0^\infty f(u)\left(\int_0^\infty \frac{g(t)}{(s+t)(u-t)}\,dt\right)du\,, \tag{10}$$

and similarly for the first term on the right-hand side of (6). Consequently, the formula (6) becomes

$$\begin{aligned} \mathcal{S}\{h(t):s\} &= \int_0^\infty f(t)\left(\int_0^\infty \frac{g(u)}{(s+t)(u-t)}\,du\right)dt \\ &\quad + \int_0^\infty f(t)\left(\int_0^\infty \frac{g(u)}{(s+u)(t-u)}\,du\right)dt \\ &= \int_0^\infty f(t)\left(\int_0^\infty \frac{g(u)}{u-t}\left\{\frac{1}{s+t}-\frac{1}{s+u}\right\}du\right)dt. \end{aligned} \tag{11}$$

Simplifying this last double integral, we finally have

$$\begin{aligned} \mathcal{S}\{h(t):s\} &= \int_0^\infty \frac{f(t)}{s+t}\,dt \int_0^\infty \frac{g(u)}{s+u}\,du \\ &= \mathcal{S}\{f(t):s\}\,\mathcal{S}\{g(t):s\}. \end{aligned} \tag{12}$$

Since the Stieltjes transform, when it exists, is an analytic function in the complex plane cut along the nonpositive real axis (*cf.* [9, p. 328]), the formula (12) holds true everywhere in the complex s-plane *except* along the nonpositive real axis. This evidently completes the proof of the theorem.

REMARK. In view of the property (3), the function h defined by (2) may be called the *Stieltjes convolution* of the functions f and g.

Suppose now that the functions f and g satisfy the hypothesis of the convolution theorem (3) and that

$$f \otimes g = 0 \qquad \text{almost everywhere on } \mathbb{R}_+.$$

Then

$$\mathcal{S}\{f(t):s\}\,\mathcal{S}\{g(t):s\} = \mathcal{S}\{(f\otimes g)(t):s\} = 0$$

everywhere in the complex s-plane cut along the nonpositive real axis. Since

$$\mathcal{S}\{f(t):s\} \qquad \text{and} \qquad \mathcal{S}\{g(t):s\}$$

are analytic functions in this cut s-plane, it follows that *either*

$$\mathcal{S}\{f(t):s\} = 0$$

or

$$\mathcal{S}\{g(t):s\} = 0.$$

Making use of the uniqueness of the Stieltjes transform (*cf.* [9, p. 336]), we conclude that *either* $f = 0$ *or* $g = 0$ almost everywhere on $\mathbb{R}_+$. Thus we have proved the following result:

If the functions f and g satisfy the hypothesis of the theorem, and if $f \otimes g = 0$ almost everywhere on $\mathbb{R}_+$, then either $f = 0$ or $g = 0$ almost everywhere on $\mathbb{R}_+$.

This is a Titchmarsh type theorem.

3. AN APPLICATION OF THE CONVOLUTION THEOREM

We consider the following interesting class of *singular integral equations*:

$$f(t) + \lambda \int_0^\infty \frac{f(u)}{u-t}\,du = g(t) \qquad (\lambda \neq 0), \tag{13}$$

where $g(t)$ is prescribed and $f(t)$ is an unknown function to be determined. The solution of the integral equation (13) was investigated earlier by reducing the problem to an equivalent Riemann-Hilbert boundary value problem (see [3, Chapter 3, Section 21] for details). In this section we shall show how the convolution theorem (3) can be applied to solve the integral equation (13).

We begin by assuming α_0 to be a (unique) root of the transcendental equation:

$$\tan \pi\alpha = -\pi\lambda \qquad (0 < \Re(\alpha) < 1; \quad \lambda \neq 0). \tag{14}$$

Then, in view of the well-known integral (*cf.*, *e.g.*, [2, Vol. **II**, p. 249, Entry 15.2(28); p. 216, Entry 14.2(5)]):

$$\int_0^\infty \frac{u^{\alpha-1}}{u-\zeta}\,du = \begin{cases} -\pi\zeta^{\alpha-1}\cot\pi\alpha & (\Re(\zeta) > 0; \quad 0 < \Re(\alpha) < 1) \\ \pi(-\zeta)^{\alpha-1}\csc\pi\alpha & (\Re(\zeta) < 0; \quad 0 < \Re(\alpha) < 1), \end{cases} \tag{15}$$

the integral equation (13) can be written in the form:

$$f(t)\int_0^\infty \frac{u^{\alpha_0-1}}{u-t}\,du + t^{\alpha_0-1}\int_0^\infty \frac{f(u)}{u-t}\,du = -\pi t^{\alpha_0-1}\,g(t)\;\cot\,\pi\alpha_0$$

or, equivalently,

$$f(t)\otimes t^{\alpha_0-1} = -\pi\; t^{\alpha_0-1}\;g(t)\;\cot\,\pi\alpha_0\,, \tag{16}$$

where we have made use of the definition (2).

Applying the convolution theorem (3), this last relationship (16) yields

$$\mathcal{S}\{f(t):s\}\int_0^\infty \frac{u^{\alpha_0-1}}{u+s}\,du = -\pi\,\mathcal{S}\left\{t^{\alpha_0-1}\,g(t):s\right\}\;\cot\,\pi\alpha_0\,,$$

which, in view of the integral (15) again, becomes

$$\mathcal{S}\{f(t):s\} = -s^{1-\alpha_0}\;\mathcal{S}\left\{t^{\alpha_0-1}\;g(t):s\right\}\;\cos\,\pi\alpha_0\,. \tag{17}$$

Finally, by appealing to the classical inversion theorem for the Stieltjes transform (see, *e.g.*, [10, p. 126, Theorem 14.1]), we obtain

$$\begin{aligned} f(t) = \lim_{\varepsilon\to 0+}\frac{\cos\,\pi\alpha_0}{2\pi i}\Big[&(-t+i\varepsilon)^{1-\alpha_0}\,\mathcal{S}\left\{u^{\alpha_0-1}\,g(u):-t+i\varepsilon\right\} \\ &-(-t-i\varepsilon)^{1-\alpha_0}\,\mathcal{S}\left\{u^{\alpha_0-1}\,g(u):-t-i\varepsilon\right\}\Big], \end{aligned} \tag{18}$$

which provides the solution of the singular integral equation (13), α_0 being a (unique) root of the transcendental equation (14).

Acknowledgements

This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353, by the Vietnam National Basic Research Program in Natural Sciences, and by the Alexander von Humbold Foundation.

REFERENCES

[1] P.L. Butzer and R.J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser Verlag, Basel and Stuttgart, 1971.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Tables of Integral Transforms*, Vols. I and II, McGraw-Hill, New York, London, and Toronto, 1954.

[3] F.D. Gakhov, *Boundary Value Problems* (Edited translation prepared from the second revised and enlarged Russian edition, 1963), Pergamon Press, Oxford, New York, and London; Addison-Wesley Publishing Company, Reading, Massachusetts, and London, 1966; Reprinted by Dover Publications, New York, 1990.

[4] I.I. Hirschman and D.V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, New Jersey, 1955.

[5] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Brisbane, Chichester, and Toronto, 1983.

[6] G.O. Okikiolu, *Aspects of the Theory of Bounded Integral Operators in L^p-Spaces*, Academic Press, London and New York, 1971.

[7] H.M. Srivastava and R.G. Buschman, *Theory and Applications of Convolution Integral Equations*, Kluwer Academic Publishers, Dordrecht and Boston, 1992.

[8] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Second ed., Oxford University Press, Oxford, London, and New York, 1948.

[9] D.V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, New Jersey, 1946.

[10] D.V. Widder, *An Introduction to Transform Theory*, Academic Press, New York and London, 1971.

H.M. SRIVASTAVA
Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4
Canada

VU KIM TUAN
Institute of Mathematics
National Centre for Scientific Researches
Bo Ho, Hanoi 10000
Vietnam