

SPECTRUM-PRESERVING LINEAR MAPS ON THE
ALGEBRA OF REGULAR OPERATORS

By

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Dedicated to Professor H.H. Schaefer on his 60th birthday.

A bounded operator on a complex Banach lattice X is called regular if it is a linear combination of positive operators [10]. The algebra of all regular operators on X is denoted by $\mathcal{L}^r(X)$. For a regular operator T , the spectrum of T in the algebra $\mathcal{L}^r(X)$ is called the o-spectrum of T [11] and is denoted by $\sigma_o(T)$. The algebra of all bounded operators on X is denoted by $\mathcal{L}(X)$ and the usual spectrum of T , i.e. its spectrum in $\mathcal{L}(X)$, is denoted by $\sigma(T)$.

The purpose of this note is to make a few observations and raise some questions about spectrum-preserving (or o-spectrum-preserving) linear maps from $\mathcal{L}^r(X)$ to $\mathcal{L}^r(Y)$ for Banach lattices X and Y . (A linear map ϕ between algebras is called spectrum-preserving if $\sigma(\phi(a)) = \sigma(a)$ for every a). In particular we show that every (lattice and algebra) automorphism of $\mathcal{L}^r(X)$ is inner. Before stating our first result, we make two more definitions. A linear map ϕ between algebras is called a Jordan homomorphism if $\phi(ab+ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for every a, b , or equivalently if $\phi(a^2) = \phi(a)^2$ for every a . Furthermore ϕ is called an anti-isomorphism if it is bijective and $\phi(ab) = \phi(b)\phi(a)$ for every a and b .

In what follows X and Y denote complex Banach lattices.

THEOREM 1. Let ϕ be a linear map from $\mathcal{L}^r(X)$ onto $\mathcal{L}^r(Y)$. The following conditions are equivalent.

- (i) ϕ preserves the spectrum: $\sigma(\phi(T)) = \sigma(T)$ for all $T \in \mathcal{L}^r(X)$;
- (ii) ϕ preserves the o-spectrum: $\sigma_o(\phi(T)) = \sigma_o(T)$ for all $T \in \mathcal{L}^r(X)$;
- (iii) ϕ is a Jordan isomorphism;
- (iv) ϕ is either an algebra isomorphism or anti-isomorphism;
- (v) ϕ takes one of the following forms:
 - $\sigma(T) = ATA^{-1}$ for all $T \in \mathcal{L}^r(X)$,
 - or $\sigma(T) = BT^*B^{-1}$ for all $T \in \mathcal{L}^r(X)$,

where A (respectively B) is a bounded invertible operator from X (respectively X^*) onto Y.

We now consider the case when ϕ is also positive. (A mapping $\phi: \mathcal{L}^r(X) \rightarrow \mathcal{L}^r(Y)$ is called positive, or order preserving, if $\phi(T) \geq 0$ for every $T \geq 0$.) We say that an invertible operator $P: X \rightarrow Y$ is a lattice isomorphism if both P and P^{-1} are positive.

THEOREM 2. If ϕ satisfies the conditions of Theorem 1 and if, in addition, ϕ is order preserving, then the map A (respectively B) may be chosen to be a lattice isomorphism.

We note that $\mathcal{L}^r(X)$ is both an algebra and a vector lattice. We say that $\mathcal{L}^r(X)$ and $\mathcal{L}^r(Y)$ are isomorphic as lattice-algebras if there exists a map $\phi: \mathcal{L}^r(X) \rightarrow \mathcal{L}^r(Y)$ which is both an algebra isomorphism and a lattice isomorphism. The following corollaries are obvious consequences of Theorem 2.

COROLLARY 1. $\mathcal{L}^{\Gamma}(X)$ and $\mathcal{L}^{\Gamma}(Y)$ are isomorphic as lattice-algebras if and only if X and Y are isomorphic as Banach lattices.

COROLLARY 2. Every (lattice and algebra) automorphism of $\mathcal{L}^{\Gamma}(X)$ is inner.

Before discussing proofs, we wish to indicate the place these results occupy among similar results in the literature. Eidelheit [3] proved that, for a Banach space X , every algebra automorphism of $\mathcal{L}(X)$ is inner, and more generally every isomorphism between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ is spatially implemented (i.e. of the form $T \rightarrow ATA^{-1}$). The first result dealing with spectrum-preserving maps is due to Marcus and Moyls [8] where it was shown that a linear map ϕ on $M_n(\mathbb{C})$ which preserves eigenvalues and their multiplicity takes one of the forms: $\phi(T) = ATA^{-1}$ or $\phi(T) = BT^tB^{-1}$. This was generalized by Marcus and Purves [9]; a particular case of the results in [9] is that if ϕ is a unital invertibility-preserving linear map on $M_n(\mathbb{C})$, then ϕ takes one of the two forms indicated above. Here, a map ϕ is called unital if $\phi(1) = 1$; it is called invertibility-preserving if $\phi(a)$ is invertible for every invertible a . (We note that for a unital map ϕ between algebras, it is easy to see that ϕ is invertibility-preserving if and only if $\sigma\{\phi(a)\} \subset \sigma(a)$ for every a .)

In [7], Kaplansky raised the question whether every unital invertibility-preserving linear map between Banach algebras is a Jordan homomorphism. This was motivated by the result of Marcus and Purves as well as the following result of Kahane and Zelazko [6]: If ϕ is a unital invertibility-preserving linear map from a commutative Banach algebra into a semi-simple commutative Banach algebra, then ϕ is multiplicative. The answer to Kaplansky's question is negative [1, p. 28], but there are several positive results with additional assumptions on the Banach algebras or the map ϕ . We refer to [1, pp. 28-31] for generalizations

of the Marcus-Purves and the Kahane-Zelazko results and to [2] for maps between C^* -algebras.

In a recent article [4], it was shown that every spectrum-preserving linear map from $\mathcal{L}(X)$ onto $\mathcal{L}(Y)$, for Banach spaces X and Y , takes one of the forms indicated in Theorem 1, part (v). Minor modifications of the proofs in [4] lead to a proof of Theorem 1. We will give a brief outline of the proof of Theorem 1 with most of the details omitted.

OUTLINE OF PROOF OF THEOREM 1. The implications (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (i) and (iii) \Rightarrow (ii) are trivial. We also notice that the equivalence of conditions (iii) and (v) for maps ϕ from any ring into a prime ring (e.g. $\mathcal{L}(Y)$ or $\mathcal{L}^{\mathbb{R}}(Y)$) is due to Herstein [5, pp. 47-51], this equivalence is not needed for our purposes. We need only prove the implications (i) \Rightarrow (v) and (ii) \Rightarrow (v). We assume that ϕ satisfies (i) or (ii). The conclusion follows by examining the proof in [4] together with the following observations.

- (1) (The Fredholm alternative.) For every finite rank operator F , we have $\sigma(F) = \sigma_o(F) = \sigma_p(F)$ where $\sigma_p(F)$ is the point spectrum (eigenvalues) of F . If q is the minimal polynomial of F , it is easy to see that every one of the three spectra above coincides with the zero set of q .
- (2) Let $A \in \mathcal{L}^{\mathbb{R}}(X)$. If $\sigma_p(A+R) \subset \sigma_p(R)$ for every rank one operator R , then $A = 0$. (See the proof of Lemma 1 in [4].) In particular, each of the conditions " $\sigma(A+T) \subset \sigma(T)$ for every T " and " $\sigma_o(A+T) \subset \sigma(T)$ for every T " implies that $A = 0$.

(3) It follows from (2) that ϕ is injective and that $\phi(1) = 1$; see [4, Lemmas 2 and 3].

(4) If F has finite rank and if $\lambda \in \sigma_0(T+F)$ and $\lambda \notin \sigma_0(T)$, then $\lambda \in \sigma_p(T+F)$. This follows from the equation

$$\lambda - T - F = (\lambda - T) (1 - (\lambda - T)^{-1} F)$$

and the Fredholm alternative.

(5) For $T \in \mathcal{L}^r(X)$, $x \in X$, $f \in X^*$ and $\lambda \notin \sigma_0(T)$, we have $\lambda \in \sigma_0(T + x \otimes f)$ if and only if $\langle (\lambda - T)^{-1} x, f \rangle = 1$. (The duality between X and X^* is denoted by $\langle \cdot, \cdot \rangle$). This follows from (4) as in the proof of [4, Lemma 4].

(6) PROPOSITION. Let $A \in \mathcal{L}^r(X)$, $A \neq 0$. The following conditions are equivalent.

- (i) A has rank 1
- (ii) $\sigma(T+A) \cap \sigma(T+cA) \subset \sigma(T)$ for every $T \in \mathcal{L}^r(X)$ and every $c \neq 1$.
- (iii) $\sigma_0(T+A) \cap \sigma_0(T+cA) \subset \sigma_0(T)$ for every $T \in \mathcal{L}^r(X)$ and every $c \neq 1$.
- (iv) $\sigma_p(N+A) \cap \sigma_p(N+cA) \subset \{0\}$ for every finite rank nilpotent N and $c \neq 1$.

In [4, Theorem 1], the equivalence of (i) and (ii) for operators on a Banach space was established. The proof in [4] actually establishes the equivalence of (i) and (iv). The implication (i) \Rightarrow (iii) follows from (5) in the same way as the implication (i) \Rightarrow (ii) is proved in [4]. The implication (iii) \Rightarrow (iv) is obvious.

(7) From the preceding proposition we have that ϕ maps rank one operators to rank one operators. This implies (see [4]) that $\phi(x \otimes f) = Ax \otimes Cf$ or alternatively $\phi(x \otimes f) = Bf \otimes Dx$, where $A: X \rightarrow Y$ and $C: X^* \rightarrow Y^*$ (or $B: X^* \rightarrow Y$ and $D: X \rightarrow Y^*$). From (5) above we get that

$$\langle (1-\lambda T)^{-1}x, f \rangle = \langle (1-\lambda\phi(T))^{-1} Ax, Cf \rangle$$

or

$$\langle (1-\lambda T)^{-1}x, f \rangle = \langle (1-\lambda\phi(T))^{-1} Bf, Dx \rangle$$

for all λ in a deleted neighbourhood of 0. These equations and the closed graph theorem show that A and C (or B and D) are bounded. Taking the limit as $\lambda \rightarrow 0$, we get that $C = (A^{-1})^*$ (or that $D = (B^{-1})^*|X$). Taking the derivative at $\lambda = 0$, we get that $\phi(T) = ATA^{-1}$ or that $\phi(T) = BT^*B^{-1}$. \square

PROOF OF THEOREM 2. If ϕ is of the form $\phi(T) = ATA^{-1}$, and if x is a positive element in X and f a positive element in X^* , then

$$Ax \otimes (A^{-1})^*f = \phi(x \otimes f) \geq 0.$$

So, for every positive $y \in X$, we have

$$\langle A^{-1}y, f \rangle Ax = \phi(x \otimes f)y \geq 0.$$

Therefore, there exists a nonzero scalar c such that $cAx \geq 0$ for every $x \geq 0$ and that $c^{-1}\langle A^{-1}y, f \rangle \geq 0$ for every $y \geq 0$ and $f \geq 0$. Thus $cA \geq 0$ and $c^{-1}A^{-1} \geq 0$. We may replace A by cA which is a lattice isomorphism. The second case, $\phi(T) = BT^*B^{-1}$, is treated similarly. \square

OPEN QUESTIONS. From Theorem 1, we have that every algebra isomorphism from $\mathcal{L}^r(X)$ onto $\mathcal{L}^r(Y)$ is of the form $\phi(T) = ATA^{-1}$ for a bicontinuous linear map A from X onto Y . Therefore ATA^{-1} is regular for every regular T . We ask if this implies that A is regular. In particular we ask the question: Is every algebra automorphism of $\mathcal{L}^r(X)$ inner?

Our results have been motivated in part by Kaplansky's question, but that question remains unanswered for the algebras $\mathcal{L}^r(X)$ and $\mathcal{L}^r(Y)$ since we assumed that ϕ is spectrum-preserving rather than the weaker assumption of invertibility-preserving. So we ask the question: If ϕ is an invertibility preserving linear map from $\mathcal{L}^r(X)$ onto $\mathcal{L}^r(Y)$, must ϕ be of the form $\phi(T) = ATB$ for operators $A: X \rightarrow Y$ and $B: Y \rightarrow X$ or the form $\phi(T) = CT^*D$ for operators $C: X^* \rightarrow Y$ and $D: Y \rightarrow X^*$? The same question for $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ is also an open question.

We have assumed that ϕ is surjective. We don't know whether the equivalence of conditions (i)-(ii) of Theorem 1 is still valid without the surjectivity assumption or whether these conditions are equivalent to the condition that ϕ is a Jordan homomorphism. (No condition of the type (v) would follow as can be seen by considering the example: $X = \ell^2$, $Y = \ell^2 \oplus \ell^2$ and $\phi(T) = T \oplus T^*$.) A related question is whether (for spectrum-preserving linear maps) the surjectivity of ϕ is equivalent to the apparently weaker condition that the range of ϕ is irreducible in some sense.

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