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A Criterion for Subfamilies of Multivalent Functions of Reciprocal Order with Respect to Symmetric Points

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Abstract: In the present research paper, our aim is to introduce a new subfamily of p-valent (multivalent) functions of reciprocal order. We investigate sufficiency criterion for such defined family.

Keywords: multivalent functions; starlike functions; close-to-convex functions

MSC: Primary 30C45, 30C10; Secondary 47B38

1. Introduction

Let us suppose that $\mathcal{A}_p$ represents the class of p-valent functions $f(z)$ that are holomorphic (analytic) in the region $\mathbb{E} = \{z : |z| < 1\}$ and has the following Taylor series representation:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}. \tag{1}$$

Two points p and p' are said to be symmetrical with respect to o if o is the midpoint of the line segment pp'.

If $f(z)$ and $g(z)$ are analytic in $\mathcal{E}$, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function, $w(z)$, which is analytic in $\mathcal{E}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathcal{E}$, then we have the following equivalence, see [1].

$$f(z) \prec g(z) \ \ (z \in \mathcal{E}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{E}) \subset g(\mathcal{E}).$$

Let $\mathcal{N}_\alpha$ denotes the class of starlike functions of reciprocal order α $(\alpha > 1)$ and is given below

$$\mathcal{N}_\alpha := \left\{ f(z) \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha, \ \ (z \in \mathbb{E}) \right\}. \tag{2}$$

This class was introduced by Uralegaddi et al. [2] amd further studied by the Owa et al. [3]. After that Nunokawa and his coauthors [4] proved that $f(z) \in \mathcal{N}_\alpha$, $0 < \alpha < \frac{1}{2}$, if and only if the following inequality holds

$$\left|\frac{2\alpha z f'(z)}{f(z)} - 1\right| < 1, \ \ (z \in \mathbb{E}).$$

Later on, Owa and Srivastava [5] in 2002 generalized this idea for the classes of multivalent convex and starlike functions of reciprocal order α $(\alpha > p)$, and further studied by Polatoglu et al. [6]. For more details of the related concepts, see the article of Dixit et al. [7], Uyanik et al. [8], and Arif et al. [9].

For $-1 \leq t < s \leq 1$ with $s \neq 0 \neq t$, $0 < \alpha < 1$, and $p \in \mathbb{N}$, we introduce a subclass of $\mathcal{A}_p$ consisting of all analytic p-valent functions of reciprocal order α, denoted by $\mathcal{N}_\alpha^p \mathcal{S}(s,t)$ and is defined as

$$\mathcal{N}_\alpha^p \mathcal{S}(s,t) = \left\{ f(z) \in \mathcal{A}_p : \operatorname{Re}\left(\frac{(s^p - t^p) z f'(z)}{f(sz) - f(tz)}\right) < \frac{p}{\alpha}, \ \ (z \in \mathbb{E}) \right\}, \tag{3}$$

or equivalently

$$\left|\frac{(s^p - t^p) z f'(z)}{f(sz) - f(tz)} - \frac{p}{2\alpha}\right| \leq \frac{p}{2\alpha}. \tag{4}$$

Many authors studied sufficiency conditions for various subclasses of analytic and multivalent functions, for details see [4,10–17].

We will need the following lemmas for our work.

Lemma 1 (Jack's lemma [18])**.** *Let Ψ be a non-constant holomorphic function in $\mathbb{E}$ and if the value of $|\Psi|$ is maximum on the circle $|z| = r < 1$ at $z_\circ$, then $z_\circ \Psi'(z_\circ) = k\,\Psi(z_\circ)$, where $k \geq 1$ is a real number.*

Lemma 2 (See [1])**.** *Let $\mathfrak{H} \subset \mathbb{C}$ and let $\Phi : \mathbb{C}^2 \times \mathbb{E}^* \to \mathbb{C}$ be a mapping satisfying $\Phi(ia, b, z) \notin \mathfrak{H}$ for $a, b \in \mathbb{R}$ such that $b \leq -\frac{1+a^2}{2}$. If $p(z) = 1 + c_1 z^1 + c_2 z^2 + \cdots$ is regular in $\mathbb{E}^*$ and $\Phi(p(z), zp'(z), z) \in \mathfrak{H}$ $\forall\, z \in \mathbb{E}^*$, then $\operatorname{Re}(p(z)) > 0$.*

Lemma 3 (See [15])**.** *Let $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{E}$ and η be analytic and starlike (with respect to the origin) univalent in $\mathbb{E}$ with $\eta(0) = 0$. If $zp'(z) \prec \eta(z)$, then*

$$p(z) \prec 1 + \int_0^z \frac{\eta(t)}{t} dt.$$

This result is the best possible.

2. Main Results

Theorem 1. *Let $f(z) \in \mathcal{A}_p$ and satisfies*

$$\sum_{n=1}^{\infty} \left(\alpha(p+n) + p\frac{(s^{p+n} - t^{p+n})}{(s^p - t^p)}\right) |a_{n+p}| \leq \frac{p}{2}(1 - |2\alpha - 1|). \tag{5}$$

Then $f(z) \in \mathcal{N}_\alpha^p \mathcal{S}(s,t)$.

Proof. Let us assume that the inequality (5) holds. It suffices to show that

$$\left|\frac{2\alpha (s^p - t^p) z f'(z)}{f(sz) - f(tz)} - p\right| \leq p. \tag{6}$$

Consider

$$\begin{aligned}
&\left|\frac{2\alpha\left(s^{p}-t^{p}\right)zf'(z)}{f(sz)-f(tz)}-p\right| \\
=&\left|\frac{p\left(2\alpha-1\right)\left(s^{p}-t^{p}\right)z^{p}+\sum_{n=1}^{\infty}(2\alpha\left(p+n\right)\left(s^{p}-t^{p}\right)-p\left(s^{n+p}-t^{n+p}\right))a_{n+p}z^{n+p}}{\left(s^{p}-t^{p}\right)z^{p}+\sum_{n=1}^{\infty}\left(s^{n+p}-t^{n+p}\right)a_{n+p}z^{n+p}}\right| \\
\leq&\frac{p\left|2\alpha-1\right|\left(s^{p}-t^{p}\right)+\sum_{n=1}^{\infty}(2\alpha\left(p+n\right)\left(s^{p}-t^{p}\right)+p\left(s^{n+p}-t^{n+p}\right))\left|a_{n+p}\right|}{\left(s^{p}-t^{p}\right)-\sum_{n=1}^{\infty}\left(s^{n+p}-t^{n+p}\right)\left|a_{n+p}\right|}
\end{aligned}$$

The last expression is bounded above by p if

$$\begin{aligned}
&p\left|2\alpha-1\right|\left(s^{p}-t^{p}\right)+\sum_{n=1}^{\infty}\left(2\alpha\left(p+n\right)\left(s^{p}-t^{p}\right)+p\left(s^{n+p}-t^{n+p}\right)\right)\left|a_{n+p}\right| \\
<\;& p\left\{\left(s^{p}-t^{p}\right)-\sum_{n=1}^{\infty}\left(s^{n+p}-t^{n+p}\right)\left|a_{n+p}\right|\right\}.
\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty}\left(\alpha\left(p+n\right)+p\frac{\left(s^{p+n}-t^{p+n}\right)}{\left(s^{p}-t^{p}\right)}\right)\left|a_{n+p}\right|\leq\frac{p}{2}\left(1-\left|2\alpha-1\right|\right).$$

This shows that $f(z)\in\mathcal{NS}_{p}\left(s,t,\alpha\right)$. This completes the proof. □

Theorem 2. *If $f(z)\in\mathcal{A}_{p}$ satisfies the condition*

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{z\left(f(sz)-f(tz)\right)'}{f(sz)-f(tz)}\right|<1-\alpha,\quad\left(\frac{1}{2}\leq\alpha<1\right), \tag{7}$$

then $f(z)\in\mathcal{N}_{\alpha}^{p}\mathcal{S}\left(s,t\right)$.

Proof. Let us set

$$q\left(z\right)=\frac{1-\frac{\alpha\left(s^{p}-t^{p}\right)zf'(z)}{p\left(f(sz)-f(tz)\right)}}{1-\alpha}-1. \tag{8}$$

Then clearly $q(z)$ is analytic in $\mathbb{E}$ with $q\left(0\right)=0$. Differentiating logarithmically, we have

$$1+\frac{zf''(z)}{f'(z)}-\frac{z\left(f(sz)-f(tz)\right)'}{f(sz)-f(tz)}=-\frac{\left(1-\alpha\right)zq'\left(z\right)}{\left(\alpha-\left(1-\alpha\right)q\left(z\right)\right)}.$$

So

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{z\left(f(sz)-f(tz)\right)'}{f(sz)-f(tz)}\right|=\left|-\frac{\left(1-\alpha\right)zq'\left(z\right)}{\left(\alpha-\left(1-\alpha\right)q\left(z\right)\right)}\right|.$$

From (7), we have

$$\left|\frac{\left(1-\alpha\right)zq'\left(z\right)}{\left(\alpha-\left(1-\alpha\right)q\left(z\right)\right)}\right|<1-\alpha.$$

Next, we claim that $\left|q\left(z\right)\right|<1$. Indeed, if not, then for some $z_{\circ}\in\mathbb{E}$, we have

$$\max_{|z|\leq|z_0|}\left|q\left(z\right)\right|=\left|q\left(z_0\right)\right|=1.$$

Applying Jack's lemma to $q(z)$ at the point z_0, we have

$$q(z_0) = e^{i\theta}, \ \frac{z_0 q'(z_0)}{q(z_0)} = k, \ k \geq 1.$$

Then

$$\begin{aligned}
\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z\left(f(sz_0) - f(tz_0)\right)'}{f(sz_0) - f(tz_0)}\right| &= \left|\frac{(1-\alpha) z_0 q'(z_0)}{(\alpha - (1-\alpha) q(z_0))}\right| \\
&= |1-\alpha| \left|\frac{z_0 q'(z_0)}{q(z_0)} \left(\frac{1}{(1-\alpha) - \alpha e^{-i\theta}}\right)\right| \\
&= |1-\alpha| \left|\frac{k}{\alpha e^{-i\theta} - (1-\alpha)}\right| \\
&\geq |1-\alpha| \left|\frac{1}{(1-\alpha) - \alpha e^{-i\theta}}\right|.
\end{aligned}$$

Therefore

$$\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z\left(f(sz_0) - f(tz_0)\right)'}{f(sz_0) - f(tz_0)}\right|^2 \geq \frac{(1-\alpha)^2}{(1-\alpha)^2 + \alpha^2 - 2\alpha(1-\alpha)\cos\theta}.$$

Now the right hand side has minimum value at $\cos\theta = -1$, therefore we have

$$\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z\left(f(sz_0) - f(tz_0)\right)'}{f(sz_0) - f(tz_0)}\right|^2 \geq (1-\alpha)^2.$$

But this contradicts (7). Hence we conclude that $|q(z)| < 1$ for all $z \in \mathbb{E}$, which shows that

$$\left|\frac{1 - \frac{\alpha(s^p - t^p) z f'(z)}{p(f(sz) - f(tz))}}{1-\alpha} - 1\right| < 1.$$

This implies that

$$\left|\frac{(s^p - t^p) z f'(z)}{p\left(f(sz) - f(tz)\right)} - 1\right| < \frac{1}{\alpha} - 1. \tag{9}$$

Now we have

$$\begin{aligned}
\left|\frac{(s^p - t^p) z f'(z)}{p\left(f(sz) - f(tz)\right)} - \frac{1}{2\alpha}\right| &\leq \left|\frac{(s^p - t^p) z f'(z)}{p\left(f(sz) - f(tz)\right)} - 1\right| + \left|1 - \frac{1}{2\alpha}\right| \\
&< \frac{1}{\alpha} - 1 + 1 - \frac{1}{2\alpha} \\
&= \frac{1}{2\alpha}.
\end{aligned}$$

This implies that $f(z) \in \mathcal{N}_\alpha^p \mathcal{S}(s,t)$. □

Theorem 3. *If $f(z) \in \mathcal{A}_p$ satisfies the condition*

$$\operatorname{Re}\left(-1 - \frac{z f''(z)}{f'(z)} + \frac{z\left(f(sz) - f(tz)\right)'}{f(sz) - f(tz)}\right) > \begin{cases} \frac{\alpha}{2(\alpha-1)}, & 0 \leq \alpha \leq \frac{1}{2} \\ \frac{\alpha-1}{2\alpha}, & \frac{1}{2} \leq \alpha < 1, \end{cases} \tag{10}$$

then $f(z) \in \mathcal{N}_\alpha^p \mathcal{S}(s,t)$ for $0 \leq \alpha < 1$.

Proof. Let

$$q(z) = \frac{\frac{p(f(sz)-f(tz))}{(s^p-t^p)zf'(z)} - \alpha}{1-\alpha}.$$

Then clearly $q(z)$ is analytic in $\mathbb{E}$. Applying logarithmic differentiation, we have

$$-1 - \frac{zf''(z)}{f'(z)} + \frac{z(f(sz)-f(tz))'}{f(sz)-f(tz)} = \frac{(1-\alpha)zq'(z)}{\alpha+(1-\alpha)q(z)} = \Psi(q(z), zq'(z), z),$$

where

$$\Psi(u, v; t) = \frac{(1-\alpha)v}{\alpha+(1-\alpha)u}.$$

Now for all $x, y \in \mathbb{R}$ satisfying the inequality $y \leq -\frac{1+x^2}{2}$, we have

$$\Psi(ix, y, z) = \frac{(1-\alpha)y}{\alpha+(1-\alpha)ix}.$$

Therefore

$$\begin{aligned} \operatorname{Re}(\Psi(ix, y, z)) &\leq -\frac{\alpha(1-\alpha)\left(1+x^2\right)}{2\left(\alpha^2+(1-\alpha)^2x^2\right)}, \\ &\leq \begin{cases} \frac{\alpha}{2(\alpha-1)}, & 0 \leq \alpha \leq \frac{1}{2}, \\ \frac{\alpha-1}{2\alpha}, & \frac{1}{2} \leq \alpha < 1. \end{cases} \end{aligned}$$

We set

$$\Lambda = \left\{\zeta : \operatorname{Re}(\zeta) > \begin{cases} \frac{\alpha}{2(\alpha-1)}, & 0 \leq \alpha \leq \frac{1}{2}, \\ \frac{\alpha-1}{2\alpha}, & \frac{1}{2} \leq \alpha < 1. \end{cases}\right\}$$

Then $\Psi(ix, y; z) \notin \Lambda$ for all real x, y such that $y \leq -\frac{1+x^2}{2}$. Moreover, in view of (10), we know that $\Psi(q(z), zq'(z), z) \in \Lambda$. So applying Lemma 2, we have

$$\operatorname{Re}(q(z)) > 0,$$

which shows that the desired assertion of Theorem 3 holds. □

Theorem 4. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \frac{f(sz)-f(tz)}{(s^p-t^p)zf'(z)}\left(1 - \beta\frac{zf''(z)}{f'(z)} + \beta\frac{z(f(sz)-f(tz))'}{f(sz)-f(tz)}\right) > \frac{2\alpha+\beta(3\alpha-1)}{2p}, \tag{11}$$

then $f(z) \in \mathcal{N}_\alpha^p\mathcal{S}(s,t)$ for $0 < \alpha < 1$ and $\beta \geq 0$.

Proof. Let

$$h(z) = \frac{\frac{p(f(sz)-f(tz))}{(s^p-t^p)zf'(z)} - \alpha}{1-\alpha}.$$

Where $h(z)$ is clearly analytic in $\mathbb{E}$ such that $h(0) = 1$. We can write

$$\frac{p(f(sz)-f(tz))}{(s^p-t^p)zf'(z)} = \alpha + (1-\alpha)h(z). \tag{12}$$

After some simple computation, we have

$$-\beta\frac{zf''(z)}{f'(z)}+\beta\frac{z\left(f(sz)-f(tz)\right)'}{f(sz)-f(tz)}=\beta\frac{\alpha+(1-\alpha)\left(h\left(z\right)+zh'\left(z\right)\right)}{\alpha+(1-\alpha)\,h\left(z\right)}$$

It follows from (12) that

$$\begin{aligned}&\frac{p\left(f(sz)-f(tz)\right)}{\left(s^{p}-t^{p}\right)zf'(z)}\left(1-\beta\frac{zf''(z)}{f'(z)}+\beta\frac{z\left(f(sz)-f(tz)\right)'}{f(sz)-f(tz)}\right)\\ =\;&\beta\left(1-\alpha\right)zh'\left(z\right)+\left(1-\alpha\right)\left(1+\beta\right)h\left(z\right)+\alpha\left(1+\beta\right)\\ =\;&\Psi\left(h\left(z\right),zh'\left(z\right),z\right)\end{aligned}$$

where

$$\Psi\left(u,v,t\right)=\beta\left(1-\alpha\right)v+\left(1-\alpha\right)\left(1+\beta\right)u+\alpha\left(1+\beta\right).$$

Now for some real numbers x and y satisfying $y\leq-\frac{1+x^{2}}{2}$, we have

$$\begin{aligned}\operatorname{Re}\left(\Psi\left(ix,y,z\right)\right)&\leq-\beta\left(1-\alpha\right)\frac{1+x^{2}}{2}+\alpha\left(1+\beta\right)\\&=\frac{1}{2}\left(2\alpha+\beta\left(3\alpha-1\right)\right).\end{aligned}$$

If we set

$$\Lambda=\left\{\zeta:\operatorname{Re}\left(\zeta\right)>\frac{1}{2}\left(2\alpha+\beta\left(3\alpha-1\right)\right)\right\},$$

then $\Psi\left(ix,y,z\right)\notin\Lambda$ Furthermore, by virtue of (11), we know that $\Psi\left(h\left(z\right),zh'\left(z\right),z\right)\in\Lambda$. Thus by using Lemma 2, we have

$$\operatorname{Re}\left(h\left(z\right)\right)>0,$$

which implies that the assertion of Theorem 4 holds true. □

Theorem 5. *If $f(z)\in\mathcal{A}_{p}$ satisfies the condition*

$$\left|\left(p-\frac{2\alpha\left(s^{p}-t^{p}\right)zf'(z)}{\left(f(sz)-f(tz)\right)}\right)'\right|\leq p\beta\left|z\right|^{\gamma}, \tag{13}$$

then $f(z)\in\mathcal{N}_{\alpha}^{p}\mathcal{S}\left(s,t\right)$ with $0<\alpha<1$, $0<\beta\leq\gamma+1$ and $\gamma\geq0$.

Proof. Let we define

$$F\left(z\right)=z\left(p-\frac{2\alpha\left(s^{p}-t^{p}\right)zf'(z)}{\left(f(sz)-f(tz)\right)}\right). \tag{14}$$

Then $F\left(z\right)$ is regular in $\mathbb{E}$ and $F\left(0\right)=0$. The condition (14) gives

$$\left|\left(p-\frac{2\alpha\left(s^{p}-t^{p}\right)zf'(z)}{\left(f(sz)-f(tz)\right)}\right)'\right|=\left|\left(\frac{F\left(z\right)}{z}\right)'\right|$$

It follows from (13) that

$$\left|\left(\frac{F\left(z\right)}{z}\right)'\right|\leq p\beta\left|z\right|^{\gamma}.$$

This implies that

$$\left|\left(\frac{F(z)}{z}\right)\right| = \left|\int_0^z \left(\frac{F(t)}{t}\right)' dt\right| \leq \int_0^z \left|\left(\frac{F(t)}{t}\right)'\right| dt \leq \frac{p\beta |z|^{\gamma+1}}{\gamma+1},$$

and therefore

$$\left|\left(\frac{F(z)}{z}\right)\right| < p,$$

which further gives

$$\left|\frac{(s^p - t^p) z f'(z)}{p(f(sz) - f(tz))} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha}.$$

Hence $f(z) \in \mathcal{N}_\alpha^p \mathcal{S}(s,t)$. □

Theorem 6. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\left|\frac{(s^p - t^p) z f'(z)}{f(sz) - f(tz)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(f(sz) - f(tz))'}{f(sz) - f(tz)}\right)\right| < p\left(\frac{1-\alpha}{\alpha}\right), \tag{15}$$

then $f(z) \in \mathcal{N}_\alpha^p \mathcal{S}(s,t)$, where $\frac{p}{p+1} < \alpha < 1$.

Proof. Let

$$q(z) = \frac{p(f(sz) - f(tz))}{(s^p - t^p) z f'(z)}. \tag{16}$$

Then $q(z)$ is clearly analytic in $\mathbb{E}$ such that $q(0) = 1$. After logarithmic differentiation and some simple computation, we have

$$z\left(\frac{1}{q(z)}\right)' q(z) = 1 + \frac{zf''(z)}{f'(z)} - \frac{z(f(sz) - f(tz))'}{f(sz) - f(tz)}. \tag{17}$$

From (16) and (17), we find that

$$z\left(\frac{1}{q(z)}\right)' = \frac{(s^p - t^p) z f'(z)}{p(f(sz) - f(tz))} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(f(sz) - f(tz))'}{f(sz) - f(tz)}\right).$$

Now by condition (15), we have

$$z\left(\frac{1}{q(z)}\right)' \prec p\left(\frac{1-\alpha}{\alpha}\right) z = \Theta(z),$$

where $\Theta(0) = 0$. Applying Lemma 3, we have

$$\frac{1}{q(z)} \prec 1 + \int_0^z \frac{\Theta(t)}{t} dt = \frac{\alpha + p(1-\alpha) z}{\alpha},$$

which implies that

$$q(z) \prec \frac{\alpha}{\alpha + p(1-\alpha) z} = H(z). \tag{18}$$

We can write

$$\begin{aligned} \operatorname{Re}\left(1+\frac{zH''(z)}{H'(z)}\right) &= \operatorname{Re}\left(\frac{\alpha-p(1-\alpha)z}{\alpha+p(1-\alpha)z}\right) \\ &\geq \frac{\alpha-p(1-\alpha)}{\alpha+p(1-\alpha)}. \end{aligned}$$

Now since $\frac{p}{1+p} < \alpha < 1$, therefore we have

$$\operatorname{Re}\left(1+\frac{zH''(z)}{H'(z)}\right) > 0.$$

This shows that H is convex univalent in $\mathbb{E}$ and $H(\mathbb{E})$ is symmetric about the real axis, therefore

$$\operatorname{Re}(H(z)) \geq H(1) \geq 0. \tag{19}$$

Combining (16), (18), and (19), we deduce that

$$\operatorname{Re}(q(z)) > \alpha,$$

which implies that $f(z) \in \mathcal{N}_{\alpha}^{p}\mathcal{S}(s,t)$. □

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