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Graphs of minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and large enough maximum degree embed every tree with d vertices

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Abstract

For $d \in \mathbb{N}$, we show that there exists a function $f(d)$ such that every graph G with $\Delta(G) \geq f(d)$ and $\delta(G) \geq \lfloor \frac{d}{2} \rfloor$ contains every tree on d vertices as a subgraph.

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1. Introduction

Determining conditions on the degree sequence of a host graph which ensure that it contains a graph F as a subgraph is a vibrant area of extremal graph theory.

For a fixed graph F, if we simply bound the average degree then we are determining the function ex_F defined by setting $ex_F(n)$ to be the maximum number of edges in a graph on n vertices containing no subgraph isomorphic to F. Clearly, if $r = \chi(F)$, the chromatic number of F, then no $r-1$ colourable graph contains a member of F, so $ex_F(n)$ is at least the maximum number of edges in such a graph. Turán's theorem [1] states that if F is a clique then $ex_F(n)$ is exactly this number, which is the number of edges in a complete $r-1$ partite graph with parts of as equal size as possible (the so-called *Turán*-graph). The Erdős-Stone-Simonovits theorem [2, 3] states that $ex_F(n)$ is within a $o(1)$ factor of this number for any F. Determining ex_F when F is a bipartite graph is more difficult (see [4, 5, 6, 7] and the survey of Füredi and Simonovits [8]).

We can ask a related question where we replace average degree by minimum degree. Clearly, any graph of minimum degree exceeding $ex_F(n)/n$ also has average degree exceeding $ex_F(n)/n$ and hence contains F as a subgraph. On the other hand, every graph of average degree at least d contains a subgraph of minimum degree at least $d/2$ which can be

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obtained by repeatedly deleting vertices of smaller degree until none are left. Thus, the answer to this related question for a given graph F is at least $ex_F(n)/2n$, i.e. the answers to the two questions can differ by at most a multiplicative factor of two. We note that since the complete $r-1$ partite graphs described above are close to regular, the Erdős-Stone theorem implies that for non-bipartite graphs the answers differ by at most a factor of $1 + o(1)$.

We can also allow the graph F to vary with the host graph and consider a family of graphs rather than just one graph. The same argument shows that the answer to the two related questions will still differ by a multiplicative factor of at most two. Sometimes they do indeed differ. For example, Bondy [9] showed that any graph with minimum degree greater than $\frac{n}{2}$ is pancyclic, i.e. it contains a copy of every cycle of length between 3 and n. In contrast, the average degree required in order to guarantee a graph is pancyclic exceeds $n-3$, as the disjoint union of a clique on $n-1$ vertices and a single vertex shows.

We focus on when every member of the family $\mathcal{T}_d$ of all trees with d vertices for some fixed d is guaranteed as a subgraph of a graph G. If the minimum degree $\delta(G)$ is at least $d-1$ then we can greedily embed any such tree in $\mathcal{T}_d$ by traversing it in post-order from the root, embedding each node in turn in an unused neighbour of the vertex in which its parent was embedded. Furthermore, for even d, the complete graph on d vertices with a perfect matching removed is a graph of minimum degree $d-2$ which does not embed every tree with d vertices, namely the star.

Thus, every graph G with $\delta(G) > d-2$ contains every tree with d vertices as a subgraph. The Erdős-Sós conjecture (see [10]) states that we can replace minimum degree by average degree in this statement. The example of the last paragraph shows that this bound, if correct, is tight. This conjecture has received considerable attention. Ajtai, Komlós, Simonovits and Szemerédi announced they had proved it for sufficiently large d in the 1990s. Despite this there have been a number of partial results. The conjecture is trivially true for stars. It can also be proven for trees containing a node t adjacent to $d/2$ leaves, by rooting the tree at t and embedding t in a vertex of degree at least $d-1$, then greedily embedding the tree, embedding the leaves adjacent to t lastly. Further, the conjecture has been confirmed for paths [11], trees containing no path on 5 edges [12] and trees with at most one vertex of degree exceeding two (called *spiders*) [13]. It has also been proved for bounded degree trees and large dense host graphs [14]. See [15] for a plethora of other results relating to the Erdős-Sós conjecture.

In this paper we show that if we also impose a bound on the maximum degree of the graph, then we can reduce the bound on the minimum degree by a factor of 2.

Theorem 1. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph G with $\Delta(G) \geq f(d)$ and $\delta(G) > \lfloor (d-2)/2 \rfloor$ contains every tree on d vertices as a subgraph.*

We note that for any function f, the statement obtained from Theorem 1 by replacing $\delta(G) > \lfloor (d-2)/2 \rfloor$ by $\delta(G) > x$ is false for any $x < \lfloor (d-2)/2 \rfloor$. Two examples which show this are (i) $K_{f(d),x}$ and the path on d vertices, and (ii) for even d, the graph obtained by adding a universal vertex - a vertex that sees every other vertex - to the disjoint union of $f(d)$ copies of K_x and a tree obtained by adding an edge between two trees of size $d/2$. We also note that the magnitude of $f(d)$ is much larger than 2^d, which is very likely far from optimal.

We present the proof of Theorem 1 in the next section and conclude with discussion of some related results and open problems.

2. The Proof of Theorem 1

Clearly, Theorem 1 holds for $d \leq 3$ with $f(d) = 2$. Hence we can assume $d \geq 4$.

For each d, we set $g(\lfloor d/2 \rfloor, d) = d^2$ and $h(\lfloor d/2 \rfloor, d) = 2$. Then, for each x from $\lfloor d/2 \rfloor - 1$ down to 1, and i from 1 to x we define $g(x,d)$, $h(x,d)$, and $k^i(x,d)$ recursively. We set

$$k^1(x,d) := g(x+1,d)\binom{d}{x+1}$$

and then for $i = 2$ up to x we set

$$k^i(x,d) := g(x+1,d)\binom{d + \sum_{j=1}^{i-1} k^j(x,d)}{x+1}.$$

We define

$$h(x,d) := 2d + \sum_{i=1}^{d-1} k^i(x,d).$$

We set $g(x,d) := 2g(x+1,d) \cdot h(x,d) \cdot d^2$ and set $f(d) := g(1,d)$ for all d. We note that $g(x,d) \geq g(\lfloor d/2 \rfloor, d) = d^2$ for all x between 1 and $\lfloor d/2 \rfloor$.

Assume for a contradiction that there exists $d \geq 4$ such that Theorem 1 is false and let G be a counterexample. Let s be the maximum $x \leq \lfloor d/2 \rfloor$ for which $K_{x,g(x,d)} \subseteq G$. Let T be a tree on d vertices which is not a subgraph of G.

Fix a copy of $K_{s,g(s,d)}$ in G and let A and B' be the sides of this copy of $K_{s,g(s,d)}$ with $|A| = s$. Note that if $s = \lfloor d/2 \rfloor$ then we can embed the smaller colour class of the unique 2-colouring of T in A and the other colour class in B'. This contradiction implies $s < \lfloor d/2 \rfloor$.

We construct a subset B of B' as follows. We repeatedly add the vertex v of lowest degree in $B' - B$ to B. If v has degree less than d we remove from B': (i) all vertices other than v in $N(v)$ and (ii) all vertices adjacent to a neighbour of v which is not in A. We note that we remove no vertex of B.

By our choice of s, no neighbour of v which is outside A has more than $g(s+1,d) - 1$ neighbours in $B - v$. Hence

$$|B| \geq \frac{|B'|}{dg(s+1,d)} > h(s,d).$$

We note that no vertex outside A sees two vertices of degree at most d in B, and no vertex of B sees even one such vertex.

We now need the following claim. Note that for two subgraphs T' and $\hat{T}$ of a tree T we set $T' + \hat{T} := T[V(T') \cup V(\hat{T})]$.

Claim 2. *Suppose we have embedded a subgraph $\hat{T}$ of T into G and T' is a subtree of T, none of whose vertices have been embedded into G yet. If*

(i) precisely one vertex t of T' has at least one neighbour in $\hat{T}$,
(ii) every neighbour of t in $\hat{T}$ is embedded in A, and
(iii) $|V(T')| \leq \lfloor d/2 \rfloor - s + 1$

then we can extend our embedding of $\hat{T}$ to an embedding of $T' + \hat{T}$.[1]

Proof. Let X_0 be the set of vertices of G into which we have embedded $\hat{T}$. We can assume $|X_0| > s$ as otherwise, by (*iii*) and the fact that $\delta(G) \geq \lfloor d/2 \rfloor$, we can greedily embed T'. Let X_1 be the set of vertices in $V \setminus X_0$ which have more than s neighbours in X_0. By our choice of s, we have $|X_1| < g(s+1,d)\binom{|X_0|}{s+1}$. Now, for $i \in [2, d-1]$, recursively define X_i to be the set of vertices in $V - \cup_{j<i} X_j$ which have more than s neighbours in $\cup_{j<i} X_j$. Similarly as before, by our choice of s, X_i has less than $g(s+1,d)\binom{|\cup_{j<i} X_j|}{s+1}$ vertices. Finally, let $X_d := V - \cup_{i=0}^{d-1} X_i$. Since $\delta(G) \geq \lfloor \frac{d}{2} \rfloor$, for each

[1] By 'extend' we mean that the vertices and edges of $\hat{T}$ are embedded in the same way in both the initial embedding of $\hat{T}$ and the embedding of $T' + \hat{T}$.

$i \in [2, d]$ every vertex in X_i has at least $\lfloor \frac{d}{2} \rfloor - s$ neighbours in $V - \cup_{j<i-1} X_j$. Then, by definition of $h(s, d)$, we have

$$\begin{aligned} h(s,d) - d &> |X_0| + |X_1| + |X_2| + \dots \\ &= |\cup_{i=0}^{d-1} X_i| \end{aligned}$$

where we have used that $|X_0| \leq d$ and $|X_i| < g(s+1, d)\binom{|\cup_{j<i} X_j|}{s+1}$. Now, since $h(s, d) - d \leq |B|$ we have $|B \cap X_d| \geq d$. It follows that we can embed T' in $\cup_{i=2}^{d} X_i$ in a breadth first fashion with t embedded at a vertex v_d in $B \cap X_d$. Indeed, the vertex in $B \cap X_d$ we embed t into is in B and so is adjacent to every vertex in A. Further, this vertex has at least $\lfloor d/2 \rfloor - s \geq |V(T')| - 1$ neighbours in $V - \cup_{j<d-1} X_j$, hence we may embed $N_T(t)$ in $X_d \cup X_{d-1}$ We may then continue similarly, embedding the i^{th} neighbourhood of t into $X_d \cup X_{d-1} \cup \dots \cup X_{d-i}$. □

We also need the following.

Fact 3. *Let T be a tree on t vertices. Then there exist $z \in V(T)$ such that every component of $T - z$ has $t/2$ or fewer vertices.*

Fix such a vertex z. We now break into two cases.

Case 1. *$T - z$ contains no component with more than $\lfloor d/2 \rfloor - s + 1$ vertices.*

We can embed z into a vertex $a \in A$ and then repeatedly apply Claim 2 to embed each component of $T - z$ one by one.

Case 2. *$T - z$ contains a component with more than $\lfloor d/2 \rfloor - s + 1$ vertices.*

Let U be the largest component of $T - z$. Root T at z and let u be a vertex of U such that the subtree T_u rooted at u has more than $\lfloor d/2 \rfloor - s + 1$ vertices but the subtree rooted at each child of u has at most $\lfloor d/2 \rfloor - s + 1$ vertices. Embed u into a vertex $a \in A$. We note that the following property holds; we state it in a general way so that we may argue later that the property still holds after we have embedded more vertices of T.

(∗) Every embedded vertex is embedded into A. The components of the unembedded part of $T - T_u$ are trees, each of which has only a single vertex with neighbours in the embedded part of T. The number of such components is at least the number of vertices embedded into A.

We now further embed some of $T - T_u$, maintaining (∗) as follows.

For each component S in the unembedded part of $T - T_u$, we let t_S be the node of S with neighbours in $T - S$. While there is an unembedded component S such that some component of $S - t_S$ is not a singleton, we choose such a component (of $S - t_S$) of maximum size. We then embed the unique vertex of this component adjacent to t_S into A. Importantly, (∗) still holds after we do this. We repeat this process until either we have embedded s vertices into A, or until every remaining such component S is a star centred at t_S.

If we embed s vertices into A, we have that $|V(T_u)| > \lfloor d/2 \rfloor - s + 1$, there is a set Z of $s - 1$ vertices of $T - T_u$ embedded in $A - a$, and there are at least s components of $T - T_u - Z$ (by (∗)). Therefore, each component of $T - T_u - Z$ has at most $\lfloor d/2 \rfloor - s + 1$ vertices. So we can embed them by applying Claim 2[2]. To complete our embedding we can apply Claim 2 to embed the components of $T_u - u$ since each such component has at most $\lfloor d/2 \rfloor - s + 1$ vertices.

Hence we can assume we are in the second situation, in which case at some point before we embed s vertices into A, every component S of the unembedded part of $T - T_u$ is a star centered at t_S. If $|S| \leq \lfloor d/2 \rfloor - s + 1$ for each such star S, then we can iteratively apply Claim 2 to embed these stars and the components of $T_u - u$. Thus we can assume there exists a component star S' with $|S'| > \lfloor d/2 \rfloor - s + 1$ in T. Set $u' := t_{S'}$ and $T_{u'} := S'$, and repeat the procedure above replacing u with u' and T_u with $T_{u'}$. Again, we can assume that the first situation does not occur and that we are

[2] Again by (∗), particularly that each embedded component has a single vertex with neighbours in the rest of T and all these neighbours are in A.

in the second situation where we embed fewer than s vertices into A. Observe that every unembedded component S of $T - T_u{}'$ is thus a star, and that once again we succeed unless one of these stars has more than $\lfloor \frac{d}{2} \rfloor - s + 1$ vertices.

So, we can assume that there are two nodes t_1 and t_2 of T each incident to more than $\lfloor \frac{d}{2} \rfloor - s$ leaves which are on opposite sides of the unique bipartition of T. We label them so that t_1 is in a largest side of the bipartition of T. Letting T' be the tree obtained from T by deleting all the leaves incident to t_1 and t_2, we see that our choice ensures that T' has at most s vertices on the side of its unique bipartition containing t_2. We embed T' into $G[A \cup B]$ so that the side of the bipartition containing t_2 is embedded into A. We now embed the leaves incident to t_1, which we can do by our choice of B and that t_1 is in a largest side of the bipartition of T. We then embed the leaves incident to t_2 in B. This completes the proof of Theorem 1.

3. Conclusion

If G is a counterexample to the Erdős-Sós conjecture for a given d which contains a vertex v of degree at most $\frac{d-1}{2}$ then $G - v$ is also a counterexample. Thus the Erdős-Sós Conjecture is equivalent to the following a-priori stronger statement:

Every graph of average degree at least $d - 1$ and minimum degree at least $\lfloor d/2 \rfloor$ contains every tree with d vertices as a subgraph.

It was this version of the Erdős-Sós Conjecture which motivated interest in Theorem 1, as the strengthening of Theorem 1 obtained by replacing $f(d)$ by $d - 1$, implies the Erdős-Sós Conjecture. However, this common strengthening of Theorem 1 and the Erdős-Sós Conjecture is false as was shown by Havet, Reed, Stein and Wood in [16].

To see this, consider for $k > 2$ and $d = 3k + 1$ a tree T containing a node t such that $T - t$ has three components, each with k vertices, and a graph G obtained from the disjoint union of two cliques of size $2k - 1$ by adding a vertex v adjacent to all the vertices in both these cliques. It is easy to see that in any embedding of T into G, two of the three components of $T - t$ must be embedded into the same component of $G - v$, which is impossible. Hence T cannot be embedded into G. But G has maximum degree $4k - 2 \geq d - 1$, and minimum degree $2k - 1 \geq \lfloor \frac{d}{2} \rfloor$.

This example shows that if we only know $\Delta(G) \geq d - 1$, then we certainly need $\delta(G) \geq \frac{2d-2}{3}$ in order to conclude that G contains every tree on d vertices as a subgraph. In [16], it is conjectured that this is also sufficient, that is, $\Delta(G) \geq d - 1$ and $\delta(G) \geq \frac{2d-2}{3}$ implies that every tree on d vertices is a subgraph of G. Furthermore, in [16] a weakening of Theorem 1 obtained by replacing $\lfloor \frac{d}{2} \rfloor$ with $\lfloor \frac{2d-2}{3} \rfloor$ is proved.

Let us also recall from earlier (just after the statement of Theorem 1) that if $\delta(G) < \lfloor \frac{d}{2} \rfloor$ then G does not necessarily contain all trees on d vertices. For $\delta(G) = \lfloor \frac{d}{2} \rfloor$, it is conjectured by Besomi, Pavez-Signé and Stein [17] that $\Delta(G) \geq 2d - 2$ is needed in order to guarantee every tree on d vertices is a subgraph. They illustrate this asymptotically using the following example: Let $\varepsilon > 0$ and $d \in \mathbb{N}$, and $G_{\varepsilon,d}$ consist of two copies of the complete bipartite graph with parts of size $(1-\varepsilon)d$ and $(1-\varepsilon)d/2$, and one vertex that is adjacent to every vertex in the parts of size $(1-\varepsilon)d$.[3] It is easy to see that, provided that $d = d(\varepsilon)$ is sufficiently large, $G_{\varepsilon,d}$ does not contain the tree T_d consisting of $\sqrt{d} - 1$ stars of size $\sqrt{d}$ and 1 star of size $\sqrt{d} - 1$, all of whose centers are adjacent to a central vertex. Besomi, Pavez-Signé and Stein [17] prove an approximate version of this conjecture for trees of bounded degree and dense host graphs. Unfortunately, this conjecture is false, as witnessed by the following example. Suppose that d is odd and let $k = d + 1$. Let G be a graph on $n > 2k + 2$ vertices which has a universal vertex whose deletion yields a $\frac{d-3}{2}$-regular graph J where every two vertices have a common neighbour. Let T be the tree constructed from taking two stars of size $\frac{d-1}{2}$ and connecting their roots to another vertex v. Observe that T does not embed into G. Indeed, we cannot place the centres of both stars wholly into J, as every pair of vertices in J have a common neighbour and J is $\frac{d-3}{2}$-regular. If we place one of these centres into the universal vertex then the other star and v form a star of size $\frac{d+1}{2}$, which we cannot embed into J since J is $\frac{d-3}{2}$-regular. Such graphs J exist for sufficiently large d and even n in a strong sense if $\frac{(d-3)^2}{4n} > \omega(1)\log n$ and $\frac{d-3}{2} < n - \frac{c \log n}{n}$ for some $c > 2/3$ (see [18, Theorem 2.1]). Observe that such graphs J have at most $\frac{(d-3)^2}{4} + 1$

[3] Their example is for *d edge* trees, so we adapt it slightly for *d vertex* trees.

vertices and, moreover, if $\frac{d-3}{2} \notin \{1, 2, 3, 7, 57\}$[4] then such graphs have at most $\frac{(d-3)^2}{4} - 1$ vertices. As such, we believe that either the maximum degree condition in the above conjecture of Besomi, Pavez-Signé and Stein should be raised to $\lfloor d/2 \rfloor^2$ (+1) or the minimum degree should be raised to $\lfloor d/2 \rfloor + 1$.

Later, the same authors, in [20], conjectured that for any $\alpha \in [0, 1/3)$, we can take $\delta(G) = (1+\alpha)d/2$ and $\Delta(G) = 2(1-\alpha)d$ (to embed any tree on *d edges*), and proved that this is asymptotically best possible for infinitely many values of α using a variant of their construction above (see [20] for more information). Note that the $\alpha = 1/3$ case is left out of this conjecture as the authors believe that $\Delta(G)$ at least d should suffice, as conjectured by Havet, Reed, Stein and Wood [16]. They similarly prove an approximate version of their conjecture for trees of bounded degree and dense host graphs.

In a slightly different vein, for sufficiently large d, Reed and Stein [21, 22] proved that every graph on d vertices that has a universal vertex and minimum degree at least $\lfloor \frac{2d-2}{3} \rfloor$ contains every tree on d vertices as a subgraph. Observe that this confirms the conjecture of Havet, Reed, Stein and Wood [16] mentioned earlier in the case when $\Delta(G) = |G| - 1$ (for large $|G|$). Observe that we cannot reduce the minimum degree condition here to $\lfloor d/2 \rfloor$ or lower here. Indeed, consider that the tree whose root has 3 children, each of which is the centre of a star with $(d-1)/3$ leaves cannot be embedded into a copy of $K_{\lfloor d/2 \rfloor - 1, \lceil d/2 \rceil}$ with a universal vertex attached.

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[4] Note that the inclusion of 57 here is speculative since a so-called 'Moore graph' for 57 has not been either discovered or proven to not exist yet. See [19] for more details.