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Article

Characterizations of Continuous Fractional Bessel Wavelet Transforms

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Abstract: In this paper, we present a systematic study of the various characteristics and properties of some continuous and discrete fractional Bessel wavelet transforms. The method is based upon the theory of the fractional Hankel transform.

Keywords: Bessel function; continuous fractional Bessel wavelet transform; discrete fractional Bessel wavelet transform; fractional Hankel transform; fractional Hankel convolution

MSC: 26A33; 42C40; 46F12; 44A15; 44A20

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1. Introduction, Definitions and Preliminaries

Recently, the fractional Fourier transform of the real order α was introduced and studied by Luchko et al. [1]. This transform plays the same role for the fractional derivatives as the Fourier transform does for the ordinary derivatives. Moreover, in the case when $\alpha = 1$, the fractional Fourier transform reduces to the Fourier transform in the usual sense (see, for example, [2] Chapter 3). Several important properties of the fractional Fourier transform, including (for example) the inversion formula and the operational relations for the fractional derivatives, together with its applications in solving some partial differential equations of fractional order, were also given by Luchko et al. [1].

Motivated by these theoretical developments, Upadhyay and Khatterwani [3] considered the fractional Hankel transform and presented the relation between a two-dimensional fractional Fourier transform and the fractional Hankel transform in terms of radial functions. They also derived other operational properties of the Hankel transform and the fractional Hankel transform (see also [4] Chapter 8 and [5]).

The continuous and discrete Bessel wavelet transforms were investigated by Pathak and Dixit [6] by using Haimo's Hankel transform theory (see, for details, [7]). More recently, Srivastava et al. [8] studied a certain family of fractional wavelet transforms by applying the theory of the fractional Fourier transform. In the present sequel to the works of Pathak et al. (see [6,9]), Upadhyay and Khatterwani [3] and Srivastava et al. [8], our main objective is to develop the theory of the fractional wavelet transform by appealing to Haimo's Hankel transform theory. This theory is important in the sense that we can, thereby, study the fractional Bessel wavelet transform in a more efficient way.

We begin by giving some definitions and properties that are useful for our present work.

Let μ be a positive real number. Suppose also that

$$\sigma(x) = \frac{x^{2\mu+1}}{2^{\mu+\frac{1}{2}}\,\Gamma\left(\mu+\frac{3}{2}\right)} \tag{1}$$

and

$$j(x) = C_\mu x^{\frac{1}{2}-\mu}\, J_{\mu-\frac{1}{2}}(x), \tag{2}$$

where

$$C_\mu = 2^{\mu-\frac{1}{2}}\,\Gamma\left(\mu+\frac{1}{2}\right) \tag{3}$$

and $J_\nu(z)$ denotes the Bessel function of order ν (for details, see [10] Chapter 7 and [11]).

The space $L^p_\sigma(I)$, with $I = (0,\infty)$ and $1 \leqq p \leqq \infty$, is the space of those real measurable functions ϕ on $(0,\infty)$ for which

$$\|\phi\|_{p,\sigma} = \left(\int_0^\infty |\phi(x)|^p\, \mathrm{d}\sigma(x)\right)^{\frac{1}{p}} < \infty \qquad (1 \leqq p < \infty) \tag{4}$$

and

$$\|\phi\|_{\infty,\sigma} = \operatorname*{ess\,sup}_{0<x<\infty}\{|\phi(x)|\} < \infty. \tag{5}$$

We now recall the definition of the fractional Hankel transform.

Definition 1 (see [3,12])**.** *For each $\phi \in L^1_\sigma(I)$, the fractional Hankel transform of the function ϕ is defined by*

$$(h_{\mu,\alpha}\phi)(w) := \int_0^\infty j\left(w^{\frac{1}{\alpha}}x\right)\,\phi(x)\,\mathrm{d}\sigma(x) \tag{6}$$
$$(0 \leqq x < \infty;\ 0 < \alpha \leqq 1).$$

If $\phi \in L^1_\sigma(I)$ and $h_{\mu,\alpha}\phi \in L^1_\sigma(I)$, then the inversion formula of the fractional Hankel transform (6) is given for $0 < \alpha \leqq 1$ by

$$\phi(x) = \int_0^\infty j\left(w^{\frac{1}{\alpha}}x\right)(h_{\mu,\alpha}\phi)(w) \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right)\mathrm{d}\sigma(w) \qquad (0 < x < \infty). \tag{7}$$

The definitions of the Hankel and related integral transforms that we used in this article can be found in [13] (see also [14]).

Theorem 1 (see [13], p. 314, Theorem 1)**.** *For $f(x) \in L^p \ (1 < p \leqq 2)$, let*

$$g_a(t) := \int_0^a (xt)^{\frac{1}{2}}\, J_\mu(xt)\, f(x)\,\mathrm{d}x.$$

Also let $g(t)$ be the limit in the mean of $g_a(t)$, that is,

$$g(t) := \text{l.i.m.}\{g_a(t)\}.$$

If

$$f_a(x) := \int_0^a (xt)^{\frac{1}{2}}\, J_\mu(xt)\, g(t)\,\mathrm{d}t, \tag{8}$$

then

$$f_a(x) \in L^p \ (1 < p \leqq 2) \quad \textit{and} \quad f(x) := \text{l.i.m.}\{f_a(x)\}. \tag{9}$$

Remark 1. *Motivated by the results of Wing* [13], *in the present article, all of the results for the fractional Bessel wavelet transform hold true for* $p = 2$ *by exploiting the theory of the fractional Hankel transform. Thus, if* $\phi \in L^2_\sigma(I)$ *and* $\psi \in L^2_\sigma(I)$, *then the following Parseval-Goldstein formula holds true*:

$$\int_0^\infty \phi(x)\ \psi(x)\ \mathrm{d}\sigma(x) = \int_0^\infty (h_{\mu,\alpha}\phi)(w)(h_{\mu,\alpha}\psi)(w) \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) \mathrm{d}\sigma(w), \tag{10}$$

provided that each member of (10) *exists* (*see*, *for example*, [15]).

In order to define the continuous fractional Bessel wavelet transform, we shall need the definition of the fractional Hankel convolution, which is given below.

Definition 2 (see [12])**.** *Let* $\phi \in L^1_\sigma(I)$ *and* $\psi \in L^1_\sigma(I)$. *Then the fractional Hankel convolution is defined by*

$$(\phi\#\psi)(x) = \int_0^\infty \phi_\alpha(x,y)\ \psi(y)\ \mathrm{d}\sigma(y) \qquad (0 < \alpha \leqq 1), \tag{11}$$

where the fractional Hankel translation $\phi_\alpha(x,y)$ *is given by*

$$\phi_\alpha(x,y) = \int_0^\infty \phi(z)\ D_\alpha(x,y,z)\ \mathrm{d}\sigma(z) \qquad (0 < x,y < \infty) \tag{12}$$

and

$$D_\alpha(x,y,z) = \int_0^\infty j\left(w^{\frac{1}{\alpha}}x\right) j\left(w^{\frac{1}{\alpha}}y\right) j\left(w^{\frac{1}{\alpha}}z\right) \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) \mathrm{d}\sigma(w). \tag{13}$$

Using (6) and (7), we obtain

$$\int_0^\infty j\left(w^{\frac{1}{\alpha}}x\right) D_\alpha(x,y,z)\ \mathrm{d}\sigma(x) = j\left(w^{\frac{1}{\alpha}}y\right) j\left(w^{\frac{1}{\alpha}}z\right) \tag{14}$$

$$(0 < x,y < \infty;\ 0 \leqq w < \infty),$$

which, upon setting $w = 0$, yields

$$\int_0^\infty D_\alpha(x,y,z)\ \mathrm{d}\sigma(z) = 1. \tag{15}$$

Several properties of the fractional Hankel convolution given in Definition 2, which involves the fractional Hankel transform given in Definition 1, are being recorded below.

(i) If $\phi \in L^1_\sigma(I)$ and $\psi \in L^1_\sigma(I)$, then

$$\|\phi\#\psi]|_{1,\sigma} \leqq \|\phi\|_{1,\sigma} \cdot \|\psi\|_{1,\sigma} \qquad (0 < \alpha \leqq 1). \tag{16}$$

(ii) If $\phi \in L^1_\sigma(I)$ and $\psi \in L^p_\sigma(I)$, then

$$\|\phi\#\psi\|_{p,\sigma} \leqq \|\phi\|_{1,\sigma} \cdot \|\psi\|_{p,\sigma} \qquad (0 < \alpha \leqq 1). \tag{17}$$

(iii) If $\phi \in L^p_\sigma(I)$ and $\psi \in L^q_\sigma(I)$, then

$$\|\phi\#\psi\|_{r,\sigma} \leqq \|\phi\|_{p,\sigma} \cdot \|\psi\|_{q,\sigma} \qquad \left(0 < \alpha \leqq 1;\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1\right). \tag{18}$$

(iv) If $\phi \in L^1_\sigma(I)$ and $\psi \in L^1_\sigma(I)$, then

$$h_{\mu,\alpha}(\phi\#\psi) = (h_{\mu,\alpha}\phi)\,(h_{\mu,\alpha}\psi) \qquad (0 < \alpha \leqq 1). \tag{19}$$

Throughout this paper, the *dilation* is defined in the following way:

$$D_{a^{\frac{1}{\alpha}}}\phi_\alpha(x,y) = a^{-2\mu-\frac{1}{\alpha}}\,\phi_\alpha\left(\frac{x}{a^{\frac{1}{\alpha}}}, \frac{y}{a^{\frac{1}{\alpha}}}\right). \tag{20}$$

In this paper, we apply the aforementioned concepts and theories (which were developed in [6–9,13]) to introduce and investigate the fractional wavelet and the fractional Bessel wavelet transform. We also discuss the relationship between the fractional Bessel wavelet transform and the fractional Hankel transform. The Parseval-Goldstein formula for the fractional wavelet transform and the inversion formula for the fractional Bessel wavelet transform are also investigated.

It is widely recognized that various developments in wavelet theory and on the associated families of continuous, discrete and fractional wavelet transforms provide methods for solving several otherwise intractable problems in the mathematical, physical and engineering sciences.

Some of their modern applications are diverse, such as wave propagation, data compression, image processing, pattern recognition, computer graphics, the detection of aircraft and submarines, the resolution and synthesis of signals and improvements in CAT scans and other medical imaging technologies. Our present investigation is motivated essentially by today's remarkably greater demand for mathematical tools and techniques to provide both the theory and applications of wavelets and wavelet transforms to interested scientists and engineers.

Our plan in this paper is as follows. In the next section (Section 2), we introduce and present a detailed study of the continuous fractional Bessel wavelet transform by applying its relationship with the continuous fractional Hankel transform given by Definition 1. Section 3 deals with some applications of the fractional Bessel wavelet transform (see Definition 3) in a certain weighted Sobolev-type space by exploiting the theory of the fractional Hankel transform. Finally, in our concluding section (Section 4, we give several remarks and observations that are based upon the findings of our present investigation.

2. The Continuous Fractional Bessel Wavelet Transform

In this section, our main object is to study the continuous fractional Bessel wavelet transform and to develop its various properties by applying the theory of the fractional Hankel transformation.

Definition 3 (see [12])**.** *Let the function $\psi \in L^p_\sigma(I)$ be given for $1 \leqq p \leqq 2$. If $a > 0$, $b \geqq 0$ and $0 < \alpha \leqq 1$, the fractional Bessel wavelet $\psi_{b,a^{\frac{1}{\alpha}}}(x)$ is defined by*

$$\psi_{b,a^{\frac{1}{\alpha}}}(x) := D_{a^{\frac{1}{\alpha}}}\psi_\alpha(b,x) = a^{-2\mu-\frac{1}{\alpha}}\,\psi_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{x}{a^{\frac{1}{\alpha}}}\right) \tag{21}$$

$$= a^{-2\mu-\frac{1}{\alpha}}\int_0^\infty D_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{x}{a^{\frac{1}{\alpha}}}, z\right)\psi(z)\,\mathrm{d}\sigma(z), \tag{22}$$

in which the integral is convergent by virtue of the developments presented by Haimo [7] and Pathak et al. (see [6,9]).

Definition 4 (see [12]). *By taking the function $\psi \in L^2_\sigma(I)$ and the fractional wavelet $\psi_{b,a^{\frac{1}{\alpha}}}(x)$ given by Definition 3, the fractional Bessel wavelet transform $B_\psi f(b,a)$ is defined for $0 < \alpha \leqq 1$ by*

$$B_\psi f(b,a) := \langle f(t), \psi_{b,a^{\frac{1}{\alpha}}}(t) \rangle = \int_0^\infty f(t)\, \overline{\psi_{b,a^{\frac{1}{\alpha}}}(t)}\, \mathrm{d}\sigma(t) \tag{23}$$

$$= a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty \int_0^\infty f(t)\, D_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{t}{a^{\frac{1}{\alpha}}}, z\right) \cdot \overline{\psi(z)}\, \mathrm{d}\sigma(z)\, \mathrm{d}\sigma(t), \tag{24}$$

provided that the integral is convergent.

Theorem 2. *If $\psi \in L^2_\sigma(I)$ and $f \in L^2_\sigma(I)$, then the continuous fractional wavelet transform can be expressed as follows*:

$$B_\psi f(b,a) = \int_0^\infty j\left(w^{\frac{1}{\alpha}} b\right) (h_{\mu,\alpha} f)(w) \left(\overline{h_{\mu,\alpha}\psi}\right)(aw) \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) \mathrm{d}\sigma(w) \qquad (0 < \alpha \leqq 1). \tag{25}$$

Proof. Following the lines described in [6], if we use (24), we find that

$$B_\psi f(b,a) = a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty \int_0^\infty f(t)\, D_\alpha\left(\frac{b}{a^{\frac{1}{\alpha}}}, \frac{t}{a^{\frac{1}{\alpha}}}, z\right) \overline{\psi(z)}\, \mathrm{d}\sigma(z)\, \mathrm{d}\sigma(t).$$

Thus, upon substituting from (13), we obtain

$$B_\psi f(b,a) = a^{-2\mu-\frac{1}{\alpha}} \left[\int_0^\infty \int_0^\infty f(t) \left(\int_0^\infty j\left(\frac{b}{a^{\frac{1}{\alpha}}} \xi^{\frac{1}{\alpha}}\right) j\left(\frac{t}{a^{\frac{1}{\alpha}}}\, \xi^{\frac{1}{\alpha}}\right) \cdot j\left(\xi^{\frac{1}{\alpha}} z\right) \left(\frac{\xi^{\frac{1}{\alpha}-1}}{\alpha}\right) \mathrm{d}\sigma(\xi)\right) \overline{\psi(z)}\, \mathrm{d}\sigma(z)\, \mathrm{d}\sigma(t)\right],$$

which, in view of (6) and (7), yields

$$B_\psi f(b,a) = a^{-2\mu-\frac{1}{\alpha}} \int_0^\infty j\left(\frac{b}{a^{\frac{1}{\alpha}}}\, \xi^{\frac{1}{\alpha}}\right) \overline{(h_{\mu,\alpha}\psi)}(\xi)\, (h_{\mu,\alpha} f)\left(\frac{\xi}{a}\right) \cdot \left(\frac{\xi^{\frac{1}{\alpha}-1}}{\alpha}\right) \mathrm{d}\sigma(\xi). \tag{26}$$

Finally, upon setting $\xi = aw$, this last Equation (26) leads us to the result (25) as asserted by Theorem 2. □

Theorem 3. *For a function $\psi \in L^2_\sigma(I)$ and, for any signal $f \in L^2_\sigma(I)$, the following relation holds true*:

$$h_{\mu,\alpha}\left(B_\psi f(b,a)\right)(w) = (h_{\mu,\alpha} f(w)) \left(\overline{h_{\mu,\alpha}\psi}\right)(aw) \tag{27}$$

for $0 < \alpha \leqq 1$.

Proof. From (7) and (25), we have

$$B_\psi f(b,a) = h^{-1}_{\mu,\alpha}\left[(h_{\mu,\alpha} f(w)) \left(\overline{h_{\mu,\alpha}\psi}\right)(aw)\right](b). \tag{28}$$

Now, by using (6) in (28), we obtain (27). The proof of Theorem 3 is thus completed. □

Theorem 4. *Let $\psi \in L^2_\sigma(I)$. Then, for any $f, g \in L^2_\sigma(I)$, the following Parseval-Goldstein formula holds true for the fractional Bessel wavelet transform given by Definition 4:*

$$\int_0^\infty \int_0^\infty B_\psi f(b,a)\, \overline{B_\psi g(b,a)}\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) = C_{\psi,\alpha}\langle f, g\rangle, \tag{29}$$

where

$$C_{\psi,\alpha} := \int_0^\infty \left|(h_{\mu,\alpha}\psi)(aw)\right|^2 a^{-2\mu-1}\, d\sigma(a) < \infty. \tag{30}$$

Proof. From [6], p. 245, in conjunction with Theorem 3 and (29), we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty B_\psi f(b,a)\, \overline{B_\psi g(b,a)}\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) \\
&= \left[\int_0^\infty \left(\int_0^\infty B_\psi f(b,a)\overline{B_\psi g(b,a)} d\sigma(b)\right) a^{-2\mu-1}\, d\sigma(a)\right] \\
&= \left[\int_0^\infty \left(\int_0^\infty h^{-1}_{\mu,\alpha}\left[(h_{\mu,\alpha} f)(w)\, (\overline{h_{\mu,\alpha}\psi})(aw)\right](b)\right.\right. \\
&\quad \left.\left.\cdot\, h^{-1}_{\mu,\alpha}\left[(\overline{h_{\mu,\alpha} g})(w)\, (h_{\mu,\alpha}\psi)(aw)\right](b) d\sigma(b)\right) a^{-2\mu-1}\, d\sigma(a)\right].
\end{aligned} \tag{31}$$

Now, by using the Parseval-Goldstein formula (10) for the fractional Hankel transform in the $L^2_\sigma(I)$ sense, we find from (31) that

$$\begin{aligned}
&\int_0^\infty \int_0^\infty B_\psi f(b,a)\, \overline{B_\psi g(b,a)}\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) \\
&= \left[\int_0^\infty \left(\int_0^\infty (h_{\mu,\alpha} f)(w) (\overline{h_{\mu,\alpha}\psi})(aw) (\overline{h_{\mu,\alpha} g})(aw)\, (h_{\mu,\alpha}\psi)(aw)\right.\right. \\
&\quad \left.\left.\cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w)\right) a^{-2\mu-1}\, d\sigma(a)\right] \\
&= \int_0^\infty \left(\int_0^\infty \left|(h_{\mu,\alpha}\psi)(aw)\right|^2 a^{-2\mu-1}\, d\sigma(a)\right) (h_{\mu,\alpha} f)(w) (h_{\mu,\alpha} g)(w) \\
&\quad \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w) \\
&= C_{\psi,\alpha}\left\langle \frac{w^{\frac{1}{\alpha}-1}}{\alpha}\, (h_{\mu,\alpha} f)(w), (\overline{h_{\mu,\alpha} g})(w)\right\rangle,
\end{aligned}$$

which, by applying (10) in the $L^2_\sigma(I)$ sense, yields the Parseval-Goldstein Formula (29) as asserted by Theorem 4. □

Theorem 5. *Let $0 < \alpha \leqq 1$, $a > 0$ and $b > 0$. If $\psi \in L^2(I)$, then*

$$h_{\mu,b,\alpha}\left(\psi_{b,a^{\frac{1}{\alpha}}}(t)\right)(w) = j\left(w^{\frac{1}{\alpha}} t\right)(h_{\mu,\alpha}\psi)(aw). \tag{32}$$

Proof. Our demonstration of Theorem 5 is fairly straightforward. We choose to omit the details involved. □

Theorem 6. *Let $\psi \in L^2_\sigma(I)$. Then a signal $f \in L^2_\sigma(I)$ can be reconstructed by means of the following inversion formula*:

$$f(t) = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \int_0^\infty B_\psi f(b,a)\, \psi_{b,a^{\frac{1}{\alpha}}}(t)\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a), \tag{33}$$

where $C_{\psi,\alpha}$ is given by (30) *and* $0 < \alpha \leqq 1$.

Proof. We begin by observing that

$$\begin{aligned} &\frac{1}{C_{\psi,\alpha}} \int_0^\infty \int_0^\infty B_\psi f(b,a)\, \psi_{b,a^{\frac{1}{\alpha}}}(t)\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) \\ &\quad = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \left(\int_0^\infty B_\psi f(b,a)\, \psi_{b,a^{\frac{1}{\alpha}}}(t) d\sigma(b) \right) \\ &\qquad \cdot a^{-2\mu-1}\, d\sigma(a). \end{aligned} \tag{34}$$

Now, in view of the earlier work in [8], the Parseval-Goldstein Formula (10) for the fractional Hankel transform (10) and Theorem 5, we find that

$$\begin{aligned} &\frac{1}{C_{\psi,\alpha}} \int_0^\infty \int_0^\infty B_\psi f(b,a)\, \psi_{b,a^{\frac{1}{\alpha}}}(t)\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) \\ &\quad = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \left(\int_0^\infty h_{\mu,b,\alpha}\left[B_\psi f(b,a)\right](w)\, h_{\mu,b,\alpha}\left[\psi_{b,a^{\frac{1}{\alpha}}}(t)\right](w) \right. \\ &\qquad \left. \cdot \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w) \right) \frac{d\sigma(a)}{a^{2\mu+1}}, \end{aligned} \tag{35}$$

that is, that

$$\begin{aligned} &\frac{1}{C_{\psi,\alpha}} \int_0^\infty \int_0^\infty B_\psi f(b,a)\, \psi_{b,a^{\frac{1}{\alpha}}}(t)\, a^{-2\mu-1}\, d\sigma(b)\, d\sigma(a) \\ &\quad = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \left(\int_0^\infty (h_{\mu,\alpha} f)(w)\, \overline{(h_{\mu,\alpha}\psi)}(aw)\, j\left(w^{\frac{1}{\alpha}} t\right) \right. \\ &\qquad \left. \cdot (h_{\mu,\alpha}\psi)(aw) \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w) \right) a^{-2\mu-1}\, d\sigma(a) \\ &\quad = \frac{1}{C_{\psi,\alpha}} \int_0^\infty \left(\int_0^\infty \left|(h_{\mu,\alpha}\psi)(aw)\right|^2 a^{-2\mu-1}\, d\sigma(a) \right) \\ &\qquad \cdot j\left(w^{\frac{1}{\alpha}} t\right) (h_{\mu,\alpha} f)(w) \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w) \\ &\quad = \frac{C_{\psi,\alpha}}{C_{\psi,\alpha}} \int_0^\infty j\left(w^{\frac{1}{\alpha}} t\right) (h_{\mu,\alpha} f)(w) \left(\frac{w^{\frac{1}{\alpha}-1}}{\alpha}\right) d\sigma(w) \\ &\quad = f(t), \end{aligned} \tag{36}$$

which evidently completes the proof of Theorem 6. □

Theorem 7. *Let $\psi \in L^2_\sigma(I)$. Then the discrete fractional Bessel wavelet transform of a signal $f \in L^2_\sigma(I)$ is given by*

$$B_\psi f(m,n) = \int_0^\infty f(t)\, \overline{\psi_{\alpha,m,n}}(t)\, d\sigma(t) \qquad (0 < \alpha \leqq \infty), \tag{37}$$

where

$$\psi_{\alpha,m,n}(t) = a_0^{-m\left(2\mu+\frac{1}{\alpha}\right)} \psi\Big(nb_0, a_0^{-\frac{m}{\alpha}} t\Big). \tag{38}$$

Proof. We can easily obtain the result (37), which is asserted by Theorem 7, from [6] by using the equations (21) to (23). □

3. Application of the Fractional Bessel Wavelet Transform in a Weighted Sobolev Type Space

In this section, with the help of the developments in [16], we give applications of the fractional Bessel wavelet transform in weighted Sobolev-type space by exploiting the theory of the fractional Hankel transform.

Definition 5. *The convolution product for the fractional Bessel wavelet transform is formally defined by*

$$B_\psi(f \otimes g)(b,a) = \big(B_\psi f\big)(b,a)\big(B_\psi g\big)(b,a). \tag{39}$$

The relation between the convolution product for the fractional Bessel wavelet transform (39) and the fractional Hankel convolution (11) is now given below.

Lemma 1. *If $f, g, \psi \in L^1_\sigma(I)$, then*

$$\begin{aligned}&\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)(h_{\mu,\alpha}(f\otimes g))(w)\\ &\qquad\Big[\Big(\overline{h_{\mu,\alpha}\psi}\Big)(a\cdot)(h_{\mu,\alpha}f)(\cdot)\#\Big(\overline{h_{\mu,\alpha}\psi}\Big)(a\cdot)(h_{\mu,\alpha}g)(\cdot)\Big](w).\end{aligned} \tag{40}$$

Proof. In order to prove Lemma 1, we find from (27) that

$$h_{\mu,\alpha}\Big(B_\psi(f\otimes g)(b,a)\Big)(w) = \Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)\big(h_{\mu,\alpha}(f\otimes g)\big)(w),$$

which, by virtue of (39), leads us to

$$\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)\big(h_{\mu,\alpha}(f\otimes g)\big)(w) = h_{\mu,\alpha}\big[(B_\psi f)(b,a)(B_\psi g)(b,a)\big](w).$$

Now, from (28), we find that

$$\begin{aligned}&\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)(h_{\mu,\alpha}(f\otimes g))(w)\\ &\quad= h_{\mu,\alpha}\Big[h^{-1}_{\mu,\alpha}\Big(\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)(h_{\mu,\alpha}f)(w)\Big)(b)\\ &\qquad\cdot h^{-1}_{\mu,\alpha}\Big(\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)(h_{\mu,\alpha}g)(w)\Big)(b)\Big](w).\end{aligned}$$

Finally, by applying (19), we find

$$\begin{aligned}&\Big(\overline{h_{\mu,\alpha}\psi}\Big)(aw)\big(h_{\mu,\alpha}(f\otimes g)\big)(w)\\ &\quad= \Big[\Big(\overline{h_{\mu,\alpha}\psi}\Big)(a\cdot)(h_{\mu,\alpha}f)(\cdot)\#\Big(\overline{h_{\mu,\alpha}\psi}\Big)(a\cdot)\cdot(h_{\mu,\alpha}g)(\cdot)\Big](w),\end{aligned} \tag{41}$$

which evidently completes our demonstration of Lemma 1. □

Next, motivated by the developments in the earlier work [17], p. 142, Equation (1.5), we give the following definition of a weighted Sobolev space.

Definition 6. *Let $k(w)$ be an arbitrary weight function and suppose that $H'_\mu(I)$ is the dual of the Zemanian space $H_\mu(I)$ for $I = (0,\infty)$. Then a function $\phi \in H'_\mu(I)$ is said to belong to the* *weighted Sobolev space $G^p_{\mu,k}(I)$ for $\mu \in \mathbb{R}$ and $1 \leqq p < \infty$, if its fractional Hankel transform $h_{\mu,\alpha}\phi$ corresponding to a locally integrable function ϕ over $I = (0, \infty)$ satisfies the following norm:*

$$\|\phi\|_{p,\mu,\sigma,k} = \left(\int_0^\infty |k(w)(h_{\mu,\alpha}\phi)(w)|^p \, \mathrm{d}\sigma(w)\right)^{\frac{1}{p}} < \infty \tag{42}$$

$$(\mu \in \mathbb{R};\ 1 \leqq p < \infty).$$

In what follows, we first set

$$k(w) = \left(\overline{h_{\mu,\alpha}\psi}\right)(aw)$$

for fixed $a > 0$, and we then establish the following result.

Theorem 8. *Let $f \in G^1_{\mu,k}(I)$, $g \in G^p_{\mu,k}(I)$ and $1 \leqq p < \infty$. Then*

$$\|f \otimes g\|_{p,\mu,\sigma,k} \leqq \|f\|_{1,\mu,\sigma,k}\ \|g\|_{p,\mu,\sigma,k}. \tag{43}$$

Proof. In view of (39) and (42), we have

$$\begin{aligned}\|f \otimes g\|_{p,\mu,\sigma,k} &= \left(\int_0^\infty \left|k(w)\left(h_{\mu,\alpha}(f \otimes g)\right)(w)\right|^p \mathrm{d}\sigma(w)\right)^{\frac{1}{p}} \\ &= \left(\int_0^\infty \left|\left(\overline{h_{\mu,\alpha}\psi}\right)(aw)\left(h_{\mu,\alpha}(f \otimes g)\right)(w)\right|^p \mathrm{d}\sigma(w)\right)^{\frac{1}{p}}. \end{aligned} \tag{44}$$

Now, by using (40), Equation (44) becomes

$$\begin{aligned}\|f \otimes g\|_{p,\mu,\sigma,k} = \Bigg(& \int_0^\infty \Big|\Big[\left(\overline{h_{\mu,\alpha}\psi}\right)(a\cdot)(h_{\mu,\alpha}f)(\cdot) \\ & \#\left(\overline{h_{\mu,\alpha}\psi}\right)(a\cdot)(h_{\mu,\alpha}g)(\cdot)\Big](w)\Big|^p \mathrm{d}\sigma(w)\Bigg)^{\frac{1}{p}},\end{aligned}$$

and, in view of (17), we find that

$$\begin{aligned}\|f \otimes g\|_{p,\mu,\sigma,k} \leqq \|&\left(\overline{h_{\mu,\alpha}\psi}\right)(a\cdot)(h_{\mu,\alpha}f)(\cdot)\|_{1,\sigma}\| \\ &\cdot \left(\overline{h_{\mu,\alpha}\psi}\right)(a\cdot)(h_{\mu,\alpha}g)(.)\|_{p,\sigma}.\end{aligned}$$

Finally, by making use of (42), we obtain

$$\|f \otimes g\|_{p,\mu,\sigma,k} \leqq \|f\|_{1,\mu,\sigma,k}\ \|g\|_{p,\mu,\sigma,k},$$

which proves Theorem 8. □

Theorem 9. *Let $f \in G^p_{\mu,k}(I)$, $g \in G^q_{\mu,k}(I)$, $1 \leqq p, q < \infty$ and*

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$\|f \otimes g\|_{r,\mu,\sigma,k} \leqq \|f\|_{p,\mu,\sigma,k}\ \|g\|_{q,\mu,\sigma,k}. \tag{45}$$

Proof. The proof of Theorem 9 follows from (42) and (18). We choose to omit the details involved. □

Various other characteristics and properties of the fractional Bessel wavelet transform and the discrete fractional Bessel wavelet transform will be investigated in our next paper.

4. Concluding Remarks and Observations

In our present article, we have introduced the continuous fractional Bessel wavelet transform and, by following the concepts and the theoretical developments presented in [6–9], we have studied the Parseval-Goldstein formula and inversion formula for the continuous fractional Bessel wavelet transform by applying the theory of the fractional Hankel transform. We have also established the relationship between the fractional Hankel transform and the continuous fractional Bessel wavelet transform. The corresponding theory of the discrete fractional Bessel wavelet transform and its various other properties and characteristics can also be discussed by taking the above-mentioned theory.

More precisely, we have chosen to list our findings in this investigation as follows. The introductory section (Section 1) provides the relevant details about the preliminaries and the background material, as well as the motivation for our study, together with a potentially useful result (Theorem 1). In Section 2, we have defined and presented a systematic study of the continuous fractional Bessel wavelet transform given in Definition 3 by applying its relationship with the continuous fractional Hankel transform given by Definition 1. Our main results in Section 2 have been stated and proven as Theorems 2 to 7.

In Section 3, we have considered some applications of the fractional Bessel wavelet transform (see Definition 3) in a certain weighted Sobolev-type space by exploiting the theory of the fractional Hankel transform (see Definition 1). The main results in Section 3 have been presented as Lemma 1, Theorem 8 and Theorem 9.

The theory that we have developed in this article is potentially useful for a variety of applications of the fractional Bessel wavelet transform in signal processing, image processing, quantum mechanics and other areas of engineering and applied sciences. Some instances of applications have been presented in Section 3 (see also several recent developments involving continuous and discrete wavelet transforms in [5,18–27], each of which will presumably motivate further researches involving the continuous and discrete fractional Bessel wavelet transforms).

We conclude this article by further remarking that our motivation for choosing the fractional-order Bessel wavelet is that it is a potentially useful generalization of the widely-investigated Bessel wavelet, particularly, in the case when the order $\alpha = 1$, the fractional Bessel wavelet becomes the Bessel wavelet, which was investigated in [6]. Discussions such as those presented in this sequel were initiated by Srivastava et al. [8]. Some interesting applications of the fractional-order Bessel wavelet transform in the areas of time-invariant linear filters and integral equations involving the fractional wavelet in the kernel can be found in several recent works (see, for example, [12]; see also [28–30]).

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