SOME COMBINATORIAL IDENTITIES ASSOCIATED WITH THE VANDERMONDE CONVOLUTION

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Abstract

The authors first present a unification (and generalization) of some combinatorial identities associated with the familiar Vandermonde convolution. A basic (or q-) extension of this general combinatorial identity is then obtained. An interesting open problem, relevant to the discussion in this paper, is also presented.

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1. INTRODUCTION

In terms of the familiar Gamma function, let

$$\binom{\lambda}{\mu} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)\,\Gamma(\mu+1)} \tag{1}$$

for arbitrary (real or complex) parameters λ and μ. The following combinatorial identities, associated with the Vandermonde convolution, are listed by Gould ([1, p. 22]; see also [2]):

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n} \tag{2}$$

and

$$\sum_{k=0}^{n} \binom{n}{k}\binom{x+y-n}{x-k} = \binom{x+y}{x}, \tag{3}$$

where the parameters x and y are unrestricted, in general.

The object of this note is first to give an interesting unification (and generalization) of the combinatorial identities (2) and (3). We then present a q-extension of this general combinatorial identity. An interesting open problem, relevant to our discussion in this paper, is also presented.

2. A UNIFICATION OF THE COMBINATORIAL IDENTITIES (2) AND (3)

Let λ, μ, ν, and ρ be essentially arbitrary (real or complex) parameters. Then

$$\sum_{k=0}^{\infty} \binom{\lambda}{k}\binom{\mu+\nu-n}{\rho-k} = \binom{\lambda+\mu+\nu-n}{\rho} \tag{4}$$

$$[\Re(\lambda+\mu+\nu-n+1) > 0 \quad (n = 0, 1, 2, \cdots)]$$

or, equivalently,

$$\sum_{k=0}^{\infty} \binom{\lambda}{k}\binom{\mu}{\rho-k} = \binom{\lambda+\mu}{\rho} \qquad [\Re(\lambda+\mu) > -1], \tag{5}$$

which would follow immediately from (4) upon replacing μ by $\mu-\nu+n$ $(n = 0, 1, 2, \cdots)$.

The combinatorial identity (4) corresponds to (2) when $\mu = \rho = n$ $(n = 0, 1, 2, \cdots)$. On the other hand, in its special case when $\rho = \mu$ and $\lambda = n$ $(n = 0, 1, 2, \cdots)$, (4) would correspond to the combinatorial identity (3).

In order to prove the general formula (4), let Ω denote the left-hand side of (4). Then, making use of the definition (1), it is easily seen that

$$\Omega = \frac{\Gamma(\mu+\nu-n+1)}{\Gamma(\rho+1)\,\Gamma(\mu+\nu-n-\rho+1)} \; {}_2F_1\left[\begin{matrix} -\lambda, \; -\rho; \\ \mu+\nu-\rho-n+1; \end{matrix} \; 1\right] \tag{6}$$

in terms of the (Gaussian) hypergeometric function (see [3] for details). Now apply the Gauss summation theorem [3, p. 28]:

$${}_2F_1\left[\begin{matrix} a, \; b; \\ c; \end{matrix} \; 1\right] = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad [\Re(c-a-b) > 0], \tag{7}$$

and (6) will lead us finally to the right-hand side of (4) under the parametric constraint stated already with (4).

3. FURTHER GENERALIZATIONS AND BASIC (OR q-) EXTENSIONS

For real or complex q $(|q| < 1)$, let

$$(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j) \tag{8}$$

for an arbitrary parameter λ. Define, as usual, the q-Gamma function by

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z} = \frac{(q;q)_{z-1}}{(1-q)^{z-1}}, \tag{9}$$

so that

$$\lim_{q \to 1} \Gamma_q(z) = \Gamma(z). \tag{10}$$

In terms of the q-binomial coefficient defined by

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_q = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\mu+1)\,\Gamma_q(\mu+1)}, \tag{11}$$

so that, by virtue of (1) and (10),

$$\lim_{q\to 1} \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_q = \binom{\lambda}{\mu}, \tag{12}$$

we may state the following basic (or q-) extension of the combinatorial identity (4) or (5):

$$\sum_{k=0}^{\infty} \begin{bmatrix} \lambda \\ k \end{bmatrix}_q \begin{bmatrix} \mu+\nu-n \\ \rho-k \end{bmatrix}_q q^{(\mu+\nu-\rho-n+k)k} = \begin{bmatrix} \lambda+\mu+\nu-n \\ \rho \end{bmatrix}_q \tag{13}$$

or, equivalently,

$$\sum_{k=0}^{\infty} \begin{bmatrix} \lambda \\ k \end{bmatrix}_q \begin{bmatrix} \mu \\ \rho-k \end{bmatrix}_q q^{(\mu-\rho+k)k} = \begin{bmatrix} \lambda+\mu \\ \rho \end{bmatrix}_q, \tag{14}$$

which is an immediate consequence of (13) when μ is replaced by $\mu-\nu+n$.

Our proof of the q-combinatorial identity (13) is much akin to that of its limiting case (4) when $q \to 1$. Indeed we make use of the definitions (8), (9), and (11), and then apply the Heine (or q-Gauss) summation theorem [3, p. 97, Equation (3.3.2.5)]. The details may be omitted.

Some special cases of the q-combinatorial identity (13) or (14) are worthy of mention. First of all, setting $\rho = n$ $(n = 0, 1, 2, \cdots)$ in (14) we get the q-Vandermonde convolution:

$$\sum_{k=0}^{n} \begin{bmatrix} \lambda \\ k \end{bmatrix}_q \begin{bmatrix} \mu \\ n-k \end{bmatrix}_q q^{(\mu-n+k)k} = \begin{bmatrix} \lambda+\mu \\ n \end{bmatrix}_q \tag{15}$$

or, upon reversal of the order of terms,

$$\sum_{k=0}^{n} \begin{bmatrix} \lambda \\ n-k \end{bmatrix}_q \begin{bmatrix} \mu \\ k \end{bmatrix}_q q^{(\mu-k)(n-k)} = \begin{bmatrix} \lambda+\mu \\ n \end{bmatrix}_q. \tag{16}$$

Formula (15) or (16) provides a q-extension of the combinatorial identity (2). A q-extension of the combinatorial identity (3) would follow from (13) when we let $\rho = \mu$ and $\lambda = n$ $(n = 0, 1, 2, \cdots)$, and we thus obtain

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \mu+\nu-n \\ \mu-k \end{bmatrix}_q q^{(\nu-n+k)k} = \begin{bmatrix} \mu+\nu \\ \mu \end{bmatrix}_q \tag{17}$$

or, on setting $\nu = \lambda + n$,

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} \lambda+\mu \\ \mu-k \end{bmatrix}_q q^{(\lambda+k)k} = \begin{bmatrix} \lambda+\mu+n \\ \mu \end{bmatrix}_q. \tag{18}$$

Now we turn to the combinatorial identity (5) which, upon setting

$$\rho = \mu - m \qquad (m = 0, 1, 2, \cdots),$$

yields

$$\sum_{k=0}^{\infty} \binom{\lambda}{k} \binom{\mu}{m+k} = \binom{\lambda+\mu}{\lambda+m} \qquad [\Re(\lambda+\mu) > -1]. \tag{19}$$

For $\lambda = n$ $(n = 0, 1, 2, \cdots)$, (19) assumes the form:

$$\sum_{k=0}^{n} \binom{n}{k} \binom{\mu}{m+k} = \binom{\mu+n}{m+n} \qquad (m, n = 0, 1, 2, \cdots). \tag{20}$$

In view of (19) and (20), it seems to be interesting to record here the general combinatorial identity:

$$\sum_{k=-\infty}^{\infty} \binom{\lambda}{\rho-k} \binom{\mu}{\sigma+k} = \binom{\lambda+\mu}{\rho+\sigma} \qquad [\Re(\lambda+\mu) > -1], \tag{21}$$

which obviously involves a *bilateral* series. For $\sigma = 0$, (21) immediately yields (5) with λ and μ interchanged.

The combinatorial identity (21) can be proven by using Dougall's generalization [3, p. 180, Equation (6.1.1)] of the Gauss summation theorem (7) in a manner analogous to our derivations which we have detailed already. Indeed it would be an interesting *open* problem to find an appropriate q-extension of the bilateral result (21).

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