POLE–FREE RATIONAL APPROXIMATION ON SUPERSETS

by

DAVID J. LEEMING AND GERALD D. TAYLOR

DM–489–IR November 1988

POLE–FREE RATIONAL APPROXIMATION ON SUPERSETS

by

DAVID J. LEEMING[†] AND GERALD D. TAYLOR[‡]

We consider characterization of the best uniform rational approximation subject to pole–free constraints.

1. INTRODUCTION

Given $K \subsetneq R^1$ compact, $W \subsetneq R^1$ such that $K \subsetneq W$, $f \in C(K)$ and set

$$R_n^m(W) = \Big\{p \in \pi_m,\ q \in \pi_n\colon (p,q) = 1,\ \ q(x) = q_0 + q_1 x + \cdots + q_n x^n > 0 \text{ on } W \text{ and } \max_{0 \le i \le n} |q_i| = 1\Big\}.$$

Define $\|h\|_K = \sup\{|h(x)| : x \in K\}$ and consider the problem: Find $r^* \in R_n^m(W)$ such that $\|f - r^*\|_K = \inf_{r \in R_n^m(W)} \|f - r\|_K$. This constrained problem has been studied earlier by Dunham [2] and Taylor [4] who considered questions of existence. Specifically, from [2] it follows that if W is compact then $r^* \in R_n^m(W)$ is a best approximation to f on K from $R_n^m(W)$ iff r^* is the best approximation to f on K from $R_n^m(K)$. That is, in this case the only time a given $f \in C(K)$ will have a best approximation on K from $R_n^m(W)$ is when it has a best approximation from $R_n^m(K)$ which is also in $R_n^m(W)$.

2. Characterization of Best Uniform Approximations on K from $R_n^m(W)$.

The following theorem characterizes those best approximations from $R_n^m(W)$ which are <u>not</u> the best approximations on $R_n^m(K)$.

Theorem 1: *Let* $f \in C(K)$ *and suppose* W *is bounded. Then* $r^* \in R_n^m(W)$ *is a best approximation to* f *on* K *from* $R_n^n(W)$ *implies either*

(i) r^* *is the best approximation to* f *on* K *from* $R_n^m(K)$, *or*

(ii) r^* *has at least one pole in* $\overline{W} \sim W$.

(*Note that if* W *is compact then* $\overline{W} \sim W = \emptyset$ *so that this result agrees with the existence result stated above.*)

Proof. Suppose $r^* \in R_n^m(W)$ is best from $R_n^m(W)$ and suppose that r^* has no poles in $\overline{W} \sim W$ (i.e. in $\partial\overline{W}$). Suppose there exists $\bar{r} \in R_n^m(K)$ with $\|f-\bar{r}\|_K < \|f-r^*\|_K$. Set

$$r_\epsilon(x) = \frac{(1-\epsilon)p^*(x) + \epsilon\bar{p}(x)}{(1-\epsilon)q^*(x) + \epsilon\bar{q}(x)} = \frac{p_\epsilon(x)}{q_\epsilon(x)}.$$

First we observe that since $(p^*,q^*) = 1$, $(p_\epsilon,q_\epsilon) = 1$ also for sufficiently small $\epsilon > 0$. Now, $\overline{W}$ is compact since W is bounded and therefore r^* is pole–free on $\overline{W}$. Hence there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$ we have that $(1-\epsilon)q^*(x) + \epsilon\bar{q}(x) > 0$ holds for $x \in W$ implying $r_\epsilon \in R_n^m(W)$. Thus, for $\epsilon > 0$ sufficiently small we have (see e.g. [1]) using standard continuity–compactness arguments that

$$\|f - r_\epsilon\|_K < \|f - r^*\|_K$$

and this contradiction proves the theorem. ∎

We now modify the standard definition of an extreme point for our particular setting.

DEFINITION: *The point* $x \in \overline{W}$ *is an* *extreme point* *for* $f - r^* \in R_n^m(W)$ *if either*

(i) $x \in K$ *and* $|f(x) - r^*(x)| = \|f - r^*\|_K$, *or*

(ii) $x \in \overline{W} \sim W$ *and* $q^*(x) = 0$.

We define $\sigma(x)$, *the sign of the extreme point* x *by*

$$\sigma(x) = \operatorname{sgn}[f(x) - r^*(x)] \quad \text{if } x \in K, \text{ or}$$

$$\sigma(x) = -\operatorname{sgn} p^*(x) \quad \text{if } x \in \overline{W} \sim W \text{ and } q^*(x) = 0.$$

We say $f - r^*$ *has an* *alternant of length* N *iff* $\exists\, x_1 < x_2 < \cdots < x_N$ *in* W, *with each* x_i *an extreme point for* $f - r^*$ *and* $\sigma(x_i) = -\sigma(x_{i+1})$, $i = 1,2,\cdots,N-1$.

Given this extended notion of extreme points, the standard characterization of a best approximation holds in this more general setting. Even more surprising is that the standard arguments actually establish this here also when looked at more carefully.

THEOREM 2: r^* *is the best approximation to* f *on* K *from* $R_n^m(W)$ *iff there exists an alternant of length* $n + m + 2 - d^*$ *for* $f - r^*$ *where* $d^* = \min(m - \partial p^*, n - \partial q^*)$.

Proof. ($\Leftarrow$) Let $x_1 < x_2 < \cdots < x_N$ be an alternant of $f - r^*$, $N = m + n + 2 - d^*$ and suppose there exists $\bar{r} \in R_n^m(W)$ such that $\|f - \bar{r}\|_K \le \|f - r^*\|_K$. As in the standard proof set $S(x) = p^*(x)\bar{q}(x) - \bar{p}(x)q^*(x) \in \Pi_{n+m-d^*}$. Since $\bar{q} > 0$ and $q^* > 0$ on W we have for $x_i \in K$ that

$$\sigma(x_i)S(x_i) = \sigma(x_i)[r^*(x_i) - \bar{r}(x_i)]q^*(x_i)\bar{q}(x_i)$$

$$= \Big[\sigma(x_i)[f(x_i) - \bar{r}(x_i)] - \sigma(x_i)]f(x_i) - r^*(x_i)]\Big] q^*x_i)\bar{q}(x_i)$$

$$= \left[\sigma(x_i)[f(x_i) - \bar{r}(x_i)] - \|f - r^*\|_K\right] q^*(x_i)\bar{q}(x_i) \le 0.$$

On the other hand if $x_i \in \overline{W}$ with $q^*(x_i) = 0$ then $\sigma(x_i)S(x_i) = \sigma(x_i)p^*(x_i)\bar{q}(x_i)$ $= -|p^*(x_i)|\bar{q}(x_i) \le 0$. Since this holds for $i = 1,\cdots,n+m-d^*+2$, it follows ([3], pg. 78) that $S(x) \equiv 0$ which is a contradiction. ■

($\Rightarrow$) For the proof of this the "hard" direction of the alternating theorem one proceeds exactly as in the standard theory ([1, pg. 55–56] or [3, pg. 78–80]). That is, assume $f - r^*$ has a maximal alternant $x_1 < x_2 < \cdots < x_k$ of length $k < N$ and select a set of disjoint intervals $\{(s_i,t_i)\}_{i=1}^{k}$ such that $x_i \in (s_i,t_i)$ $\forall i$, all extreme points of $f - r^*$ are contained in $\bigcup_{i=1}^{k} (s_i,t_i)$ and no alternations occur on any individual interval (s_i,t_i). Setting $z_i = \dfrac{s_{i+1} + t_i}{2}$, $i = 1,\cdots,k-1$, define $\psi(x) = \sigma \prod_{i=1}^{k-1} (x-z_i)$ where $\sigma = +1$ or -1 is chosen so that $\psi(x_1) = -\sigma(x_1)$. Since $(p^*,q^*) = 1$, there exist polynomials $\bar{p} \in \Pi_m$ and $\bar{q} \in \Pi_n$ such that $\psi(x) = p^*(x)\bar{q}(x) - \bar{p}(x)q^*(x)$. Set $r_\epsilon(x) = \dfrac{p^*(x) + \epsilon\bar{p}(x)}{q^*(x) + \epsilon\bar{q}(x)}$ for $\epsilon > 0$. Now, the standard argument using compactness of K, sign of $\psi(x)$ at extreme points in K and positivity of q on K shows that for all $\epsilon > 0$ sufficiently small we will have $\|f - r_\epsilon\|_K < \|f - r^*\|_K$ holding. To complete the argument we must show that $r_\epsilon \in R_n^m(W)$ holds (standard argument implies $r_\epsilon \in R_n^m(K)$) for all $\epsilon > 0$ sufficiently small. Now, we have that $q^*(x) \ge 0$ holds on $\overline{W}$. Suppose $x \in \overline{W}$ and $q^*(x) = 0$. Then x is an extreme point for $f - r^*$ and, hence, $\mathrm{sgn}(\psi(x)) = -\sigma(x) = \mathrm{sgn}(p^*(x))$ with $p^*(x) \ne 0$. Also, $\psi(x) = p^*(x)\bar{q}(x)$ showing that $\bar{q}(x) > 0$ must hold. Since there are only a finite number of points in $\overline{W}$ at which q^* can vanish, a simple continuity–compactness argument now shows that for all $\epsilon > 0$ sufficiently small $q^*(x) + \epsilon\bar{q}(x) > 0$ holds on W implying $r_\epsilon \in R_n^m(W)$. This gives the desired contradiction and the theorem is now proved. ■

Note, once again, that, if W is compact so that $\overline{W} \sim W = \emptyset$ this theorem is simply the standard alternation theorem which agrees with the existence result in this case noted at the beginning of this paper.

Acknowledgements

This research was supported in part by the National Science Foundation under grant no. ATM–88104541, by the Office of Naval Research under grant no. ONR–N00014–87–K–0535 and National Science and Engineering Research Council grant A–8061.

References

[1] Achieser, N.I., *Theory of Approximation*, Ungar, New York, 1956.

[2] Dunham, C.B., *Difficulties in Rational Chebyshev Approximation*, Constructive Theory of Functions, Sofia, 1984, 319–327.

[3] Rice, J.R., *The Approximation of Functions*, Vol. 1, Addison–Wesley, Reading, Mass., 1964.

[4] Taylor, G.D., *Existence of Pole–Free Rational Approximations on Supersets*, Approximation Theory V, Academic Press, 1986, 587–590.

†Department of Mathematics
University of Victoria
Victoria, B.C., Canada V8W 2Y2

‡Department of Mathematics
Colorado State University
Fort Collins, CO 80523