# SOME CHARACTERIZATION THEOREMS FOR STARLIKE AND CONVEX FUNCTIONS INVOLVING A CERTAIN FRACTIONAL INTEGRAL OPERATOR

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The present paper is devoted to the investigation of the sufficient conditions that are satisfied by the generalized fractional integrals of certain analytic functions in the open unit disk in order to be starlike or convex. Further characterization theorems involving the Hadamard product (or convolution) are also considered.

## 1. INTRODUCTION

Let $\mathcal{A}_n$ $(n \in \mathcal{N} = \{1,2,3,\ldots\})$ be the class of functions of the form

$$(1.1) \qquad f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathcal{N})$$

which are analytic in the open unit disk $\mathcal{U} = \{z: |z| < 1\}$. Then a function $f(z)$ in $\mathcal{A}_n$ is said to be in the class $\mathcal{S}_n^*$ if and only if

$$(1.2) \qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in \mathcal{U}).$$

On the other hand, a function $f(z)$ in $\mathcal{A}_n$ is said to be in the class $\mathcal{K}_n$ if and only if

$$(1.3) \qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \qquad (z \in \mathcal{U}).$$

It is easily observed that

$$(1.4) \qquad f(z) \in \mathcal{K}_n \Leftrightarrow zf'(z) \in \mathcal{S}_n^* \qquad (\forall n \in \mathcal{N}),$$

and that $\mathcal{S}_1^*$ and $\mathcal{K}_1$ are the familiar classes of starlike and convex functions. Thus, using the corresponding results of Silverman [3], we immediately have the following lemmas which we shall require in our present investigation of the classes $\mathcal{S}_n^*$ and $\mathcal{K}_n$ $(\forall n \in \mathcal{N})$.

LEMMA 1. _If the function_ $f(z)$ _defined by_ (1.1) _satisfies_

(1.5) $$\sum_{k=n+1}^{\infty} k|a_k| \leqq 1 \qquad (n \in \mathcal{N}),$$

_then_ $f(z) \in \mathcal{S}_n^*$. _The equality in_ (1.5) _is attained by the function_

(1.6) $$g_1(z) = z + \frac{z^k}{k} \qquad (k \geqq n+1;\ n \in \mathcal{N};\ z \in \mathcal{U}).$$

LEMMA 2. _If the function_ $f(z)$ _defined by_ (1.1) _satisfies_

(1.7) $$\sum_{k=n+1}^{\infty} k^2|a_k| \leqq 1 \qquad (n \in \mathcal{N}),$$

_then_ $f(z) \in \mathcal{K}_n$. _The equality in_ (1.7) _is attained by the function_

(1.8) $$g_2(z) = z + \frac{z^k}{k^2} \qquad (k \geqq n+1;\ n \in \mathcal{N};\ z \in \mathcal{U}).$$

## 2. DEFINITIONS AND ELEMENTARY PROPERTIES OF THE FRACTIONAL INTEGRAL OPERATORS

Let $F(a,b;c;z)$ be the Gauss hypergeometric function defined, for $z \in \mathcal{U}$, by (cf., e.g., [4, p. 18])

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(1)_k} z^k, \tag{2.1}$$

where $(\lambda)_k$ denotes the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } k = 0, \\ \lambda(\lambda+1)\cdots(\lambda+k-1), & \forall k \in \mathcal{N}. \end{cases} \tag{2.2}$$

Making use of the Gauss hypergeometric function (2.1), Srivastava, Saigo and Owa [6] have introduced the fractional integral operators $I_{0,z}^{\alpha,\beta,\eta}$ and $J_{0,z}^{\alpha,\beta,\eta}$ defined below.

DEFINITION 1. For real $\alpha > 0$, $\beta$ and $\eta$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left[\alpha+\beta, -\eta; \alpha; 1 - \frac{\zeta}{z}\right] f(\zeta)d\zeta, \tag{2.3}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}), \qquad z \to 0,$$

where

$$\epsilon > \max\{0,\beta-\eta\} - 1,$$

and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

The operator $I_{0,z}^{\alpha,\beta,\eta}$ is a generalization of the fractional integral operator $I_{0,x}^{\alpha,\beta,\eta}$ introduced by Saigo [2], and studied subsequently by Srivastava and Saigo [5].

DEFINITION 2. Under the hypotheses of Definition 1, let

$$\alpha > 0,\ \min\{\alpha+\eta,-\beta+\eta,-\beta\} > -2, \quad \text{and} \quad n \geqq \frac{\beta(\alpha+\eta)}{\alpha} - 2 \qquad (n \in \mathcal{N}). \tag{2.4}$$

Then the fractional integral operator $J_{0,z}^{\alpha,\beta,\eta}$ is defined by

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z). \tag{2.5}$$

In order to derive our results, we shall also need the following lemma due to Srivastava, Saigo and Owa [6].

LEMMA 3. <u>Let</u> $\alpha > 0$, $\beta$ <u>and</u> $\eta$ <u>be real</u>, <u>and let</u> $\kappa > \beta - \eta - 1$. <u>Then</u>

$$I_{0,z}^{\alpha,\beta,\eta} z^{\kappa} = \frac{\Gamma(\kappa+1)\Gamma(\kappa-\beta+\eta+1)}{\Gamma(\kappa-\beta+1)\Gamma(\kappa+\alpha+\eta+1)} z^{\kappa-\beta}. \tag{2.6}$$

Now we prove

THEOREM 1. <u>Under the constraints</u> (2.4), <u>if the function</u> $f(z)$ <u>defined by</u> (1.1) <u>satisfies</u>

(2.7) $$\sum_{k=n+1}^{\infty} k|a_k| \leqq \frac{(2-\beta)_n(2+\alpha+\eta)_n}{(2-\beta+\eta)_n(1)_{n+1}} \qquad (n \in \mathcal{N}),$$

<u>then</u> $J_{0,z}^{\alpha,\beta,\eta} f(z)$ <u>belongs to the class</u> $\mathcal{S}_n^*$.

**Proof.** By virtue of Lemma 3 and Definition 2, we have

(2.8) $$J_{0,z}^{\alpha,\beta,\eta} f(z) = z + \sum_{k=n+1}^{\infty} \Phi(k) a_k z^k,$$

where, for convenience,

(2.9) $$\Phi(k) = \frac{(2-\beta+\eta)_{k-1}(1)_k}{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}} \qquad (k \geqq n+1; n \in \mathcal{N}).$$

Noting that $\Phi(k)$ is a non-increasing function of $k$, we have

(2.10) $$0 < \Phi(k) \leqq \Phi(n+1) = \frac{(2-\beta+\eta)_n(1)_{n+1}}{(2-\beta)_n(2+\alpha+\eta)_n} \qquad (n \in \mathcal{N}).$$

It follows from (2.7) and (2.10) that

(2.11) $$\sum_{k=n+1}^{\infty} k\Phi(k)|a_k| \leqq \Phi(n+1) \sum_{k=n+1}^{\infty} k|a_k| \leqq 1.$$

Hence, by Lemma 1, we conclude that $J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{S}_n^*$, which proves Theorem 1.

REMARK 1. As a function $f(z)$ satisfying (2.7), we can take the function

$$g_3(z) = z + \frac{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}}{k(2-\beta+\eta)_{k-1}\,(1)_k}\, z^k \qquad (k \geqq n+1; n\in\mathcal{N}; z\in\mathcal{U}). \tag{2.12}$$

Our next result (Theorem 2 below), characterizing the class $\mathcal{K}_n$, can be proven similarly.

THEOREM 2. <u>Under the constraints</u> (2.4), <u>if the function</u> $f(z)$ <u>defined by</u> (1.1) <u>satisfies</u>

$$\sum_{k=n+1}^{\infty} k^2 |a_k| \leqq \frac{(2-\beta)_n(2+\alpha+\eta)_n}{(2-\beta+\eta)_n(1)_{n+1}} \qquad (n \in \mathcal{N}), \tag{2.13}$$

<u>then</u> $J_{0,z}^{\alpha,\beta,\eta} f(z)$ <u>belongs to the class</u> $\mathcal{K}_n$.

REMARK 2. As a function $f(z)$ satisfying (2.13), we can take the function

$$g_4(z) = z + \frac{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}}{k^2(2-\beta+\eta)_{k-1}\,(1)_k}\, z^k \qquad (k \geqq n+1; n\in\mathcal{N}; z\in\mathcal{U}). \tag{2.14}$$

## 3. CHARACTERIZATION THEOREMS INVOLVING THE HADAMARD PRODUCT OR CONVOLUTION

Let the functions $f_j(z)$ $(j = 1,2)$ in $\mathcal{A}_n$ be given by

(3.1) $$f_j(z) = z + \sum_{k=n+1}^{\infty} a_{j,k} z^k \qquad (n \in \mathcal{N}).$$

We define the Hadamard product or convolution $(f_1 * f_2)(z)$ of the functions $f_1(z)$ and $f_2(z)$ by

(3.2) $$(f_1 * f_2)(z) = z + \sum_{k=n+1}^{\infty} a_{1,k} a_{2,k} z^k \qquad (n \in \mathcal{N}).$$

In order to prove our next characterization theorem, we recall here the following result due to Ruscheweyh and Sheil-Small [1].

LEMMA 4. *Let $\phi(z)$ and $g(z)$ be analytic in $\mathcal{U}$ and satisfy the condition*:

$$\phi(0) = g(0) = 0.$$

*Suppose also that*

(3.3) $$\phi(z) * \left\{\frac{1 + \rho\sigma z}{1 - \sigma z}\, g(z)\right\} \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

*for $\rho$ and $\sigma$ on the unit circle. Then, for a function $F(z)$ analytic in $\mathcal{U}$ and satisfying the inequality*:

$$\mathrm{Re}\{F(z)\} > 0 \qquad (z \in \mathcal{U}),$$

(3.4) $$\mathrm{Re}\left\{\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\right\} > 0 \qquad (z \in \mathcal{U}).$$

Applying Lemma 4, we shall prove

THEOREM 3. *In addition to the constraints* (2.4), *suppose that the function* f(z) *defined by* (1.1) *is in the class* $\mathcal{S}_n^*$ *and satisfies*

$$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} f(z)\right\} \neq 0 \qquad (z \in \mathcal{U} - \{0\}). \tag{3.5}$$

*for* $\rho$ *and* $\sigma$ *on the unit circle, where*

$$h(z) = z + \sum_{k=n+1}^{\infty} \frac{(2-\beta+\eta)_{k-1}(1)_k}{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}} z^k \qquad (n \in \mathcal{N}). \tag{3.6}$$

*Then* $J_{0,z}^{\alpha,\beta,\eta} f(z)$ *is also in the class* $\mathcal{S}_n^*$.

**Proof.** Notice that

$$J_{0,z}^{\alpha,\beta,\eta} f(z) = z + \sum_{k=n+1}^{\infty} \frac{(2-\beta+\eta)_{k-1}(1)_k}{(2-\beta)_{k-1}(2+\alpha+\eta)_{k-1}} a_k z^k = (h*f)(z), \tag{3.7}$$

which readily yields

$$\frac{z\left[J_{0,z}^{\alpha,\beta,\eta} f(z)\right]'}{J_{0,z}^{\alpha,\beta,\eta} f(z)} = \frac{z(h*f)'(z)}{(h*f)(z)} = \frac{(h*(zf'))(z)}{(h*f)(z)} . \tag{3.8}$$

Therefore, setting $\phi(z) = h(z)$, $g(z) = f(z)$, and $F(z) = zf'(z)/f(z)$ in Lemma 4, we find that

(3.9) $$\mathrm{Re}\left\{\frac{z\left[J_{0,z}^{\alpha,\beta,\eta} f(z)\right]'}{J_{0,z}^{\alpha,\beta,\eta} f(z)}\right\} > 0,$$

which implies that $J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{S}_n^*$.

Further, we have

THEOREM 4. Under the constraints (2.4), if the function $f(z)$ defined by (1.1) is in the class $\mathcal{K}_n$ and if

(3.10) $$h(z) * \left\{\frac{1+\rho\sigma z}{1-\sigma z} zf'(z)\right\} \neq 0 \qquad (z \in \mathcal{U} - \{0\})$$

for $\rho$ and $\sigma$ on the unit circle, where $h(z)$ is given by (3.6), then $J_{0,z}^{\alpha,\beta,\eta} f(z)$ is also in the class $\mathcal{K}_n$.

**Proof.** Using (1.4) and Theorem 3, we observe that

$$\begin{aligned} f(z) \in \mathcal{K}_n &\Leftrightarrow zf'(z) \in \mathcal{S}_n^* \Rightarrow J_{0,z}^{\alpha,\beta,\eta} zf'(z) \in \mathcal{S}_n^* \\ &\Leftrightarrow (h*zf')(z) \in \mathcal{S}_n^* \Leftrightarrow z(h*f)'(z) \in \mathcal{S}_n^* \\ &\Leftrightarrow (h*f)(z) \in \mathcal{K}_n \Leftrightarrow J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{K}_n, \end{aligned}$$

which completes the proof of Theorem 4.

The proof of our next result (Theorem 5 below) is much akin to that of Theorem 3; indeed, it is based upon

LEMMA 5 (Ruscheweyh and Sheil-Small [1]). Let $\phi(z)$ be convex and let $g(z)$ be starlike in $\mathcal{U}$. Then, for each function $F(z)$ analytic in $\mathcal{U}$ and satisfying the inequality:

$$\mathrm{Re}\{F(z)\} > 0 \qquad (z \in \mathcal{U}),$$

$$\mathrm{Re}\left\{\frac{(\phi * Fg)(z)}{(\phi * g)(z)}\right\} > 0 \qquad (z \in \mathcal{U}). \tag{3.11}$$

THEOREM 5. Under the constraints (2.4),

$$f(z) \in \mathcal{S}_n^* \quad \text{and} \quad h(z) \in \mathcal{K}_n \Rightarrow J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{S}_n^*,$$

where $h(z)$ is given by (3.6).

Finally, we have

THEOREM 6. Under the constraints (2.4),

$$f(z) \in \mathcal{K}_n \quad \text{and} \quad h(z) \in \mathcal{K}_n \Rightarrow J_{0,z}^{\alpha,\beta,\eta} f(z) \in \mathcal{K}_n,$$

where $h(z)$ is given by (3.6).

REMARK 3. The function $h(z)$ defined in (3.6) can be written in terms of the Clausenian hypergeometric series ${}_3F_2$ in the form (cf. [4, p. 19]):

$$h(z) = z + \frac{(2-\beta+\eta)_n (2)_n}{(2-\beta)_n (2+\alpha+\eta)_n} z^{n+1} \; {}_3F_2(1, 2-\beta+\eta+n, 2+n; 2-\beta+n, 2+\alpha+\eta+n; z), \tag{3.12}$$

which converges absolutely in $\mathcal{U}$. In fact, this ${}_3F_2$ series in (3.12) converges also for $z = 1$ when $\alpha > 1$, that is, when the order of integration is greater than one. However, it does not seem to be easy to determine the precise constraints on the parameters $\alpha$, $\beta$, and $\eta$ under which $h(z)$ would satisfy the hypotheses of Theorems 3 to 6.

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