

A THEOREM ON GENERATING FUNCTIONS
AND ITS APPLICATIONS

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Abstract. Motivated by several recently demonstrated applications of a theorem on generating functions in (for example) the derivation of a number of interesting expansions for the generalized hypergeometric ${}_rF_s$ function in series of Bessel functions, we present here a generalization of the theorem and briefly indicate how the general result and its several consequences can also be applied in various directions.

We begin by recalling the following result on generating functions, which was applied recently by H.M. Srivastava and R.M. Shreshtha [7] with a view to deriving various interesting expansions of the generalized hypergeometric ${}_rF_s$ function in series of the Bessel functions $I_\nu(z)$ and $J_\nu(z)$, and of their such products as $I_\lambda(z)I_\mu(z)$ and $I_\nu(z)J_\nu(z)$ (cf., e.g., [9]):

THEOREM 1 (Srivastava and Panda [6, p. 472, Theorem 2]). *Corresponding to the given sequences $\{\Lambda_n\}_{n=0}^\infty$ and $\{\Omega_n\}_{n=0}^\infty$, let*

$$P_n^{(\lambda)}(x; m) = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (\lambda+n)_{mk}}{k!} \Lambda_k x^k \quad (1)$$

and

$$\theta_n(t) = \sum_{r=0}^{\infty} \frac{\Omega_{n+r}}{(\lambda+2n+1)_r} \frac{t^r}{r!} \quad (|t| < T_0), \quad (2)$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$, m is an arbitrary positive integer, and the complex parameter λ is neither zero nor a negative integer. Suppose also that $G(z)$ is defined by

$$G(z) = \sum_{n=0}^{\infty} \Lambda_n \Omega_{mn} \frac{z^n}{n!} \quad (|z| < S_0). \quad (3)$$

Then

$$\sum_{n=0}^{\infty} \frac{(-t)^n}{n!(\lambda+n)_n} P_n^{(\lambda)}(x; m) \theta_n(t) = G(xt^m), \quad (4)$$

provided that $\Lambda_0 \Omega_0 \neq 0$, $|xt^m| < S_0$, $|t| < T_0$, and the series on the left has a meaning.

The object of the present paper is to prove a generalization of Theorem 1 and to show how our main result (Theorem 2 below) and its various consequences would apply not only in the aforementioned context, but also in the derivation of new classes of generating functions for such familiar orthogonal polynomials as Jacobi, Laguerre, Hermite, and Bessel polynomials, and also for numerous interesting generalizations of some of these polynomials studied in the

literature.

THEOREM 2. *Given two suitably bounded sequences $\{\Lambda_n\}_{n=0}^{\infty}$ and $\{\Omega_n\}_{n=0}^{\infty}$ of complex numbers, define*

$$\begin{aligned} \theta_n^{(\lambda, \mu)}(x, t; m) = & \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (\lambda+n)_{mk}}{(\mu-n+1)_{mk} (\lambda+\mu+2mk+1)_{n-mk}} \Lambda_k \frac{x^k}{k!} \\ & \cdot \sum_{r=0}^{\infty} \frac{(\mu)_r (\lambda+\mu+n+r+mk)_{n-mk}}{(\lambda+2n+1)_r} \Omega_{n+r} \frac{t^r}{r!} \end{aligned} \quad (5)$$

$$(|x| < \infty; |t| < T_0; 0 < T_0 < \infty),$$

where m is an arbitrary positive integer, and $\lambda \neq 0, -1, -2, \dots$ and μ are suitable (real or complex) parameters. Suppose also that $G(z)$ is defined, for $0 < S_0 < \infty$, by Equation (3) above.

Then

$$\sum_{n=0}^{\infty} \frac{(-\mu)_n}{(\lambda+n)_n} \theta_n^{(\lambda, \mu)}(x, t; m) \frac{t^n}{n!} = G(xt^m), \quad (6)$$

provided that $\Lambda_0 \Omega_0 \neq 0$, $|xt^m| < S_0$, $|t| < T_0$, and the series on the left-hand side of (6) has a meaning.

PROOF. For the sake of convenience, let $\omega(x, t)$ denote the left-hand side of the generating function (6). Then, we find from the definition (5) that

$$\begin{aligned} \omega(x, t) = & \sum_{n, r=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-\mu)_{n-mk} (\mu)_r (\lambda)_{n+mk} (\lambda+1)_{2n} (\lambda+\mu)_{2n+r}}{(n-mk)! (\lambda)_{2n} (\lambda+1)_{2n+r} (\lambda+\mu)_{n+r+mk}} \\ & \cdot \frac{(\lambda+\mu+1)_{2mk}}{(\lambda+\mu+1)_{n+mk}} \Lambda_k \Omega_{n+r} \frac{x^k}{k!} \frac{t^{n+r}}{r!}. \end{aligned} \quad (7)$$

Since

$$(-n)_{mk} = (-1)^{mk} n! / (n-mk)! \quad (0 \leq k \leq [n/m]; m = 1, 2, 3, \dots), \quad (8)$$

upon interchanging the order of summation and setting

$$r = N - n \quad (0 \leq n \leq N; N = 0, 1, 2, \dots),$$

(7) yields

$$\omega(x,t) = \sum_{N,k=0}^{\infty} \frac{(\mu)_N}{(\lambda+2mk+1)_N} \Lambda_k \Omega_{N+mk} \frac{x^k}{k!} \frac{t^{N+mk}}{N!} \cdot {}_5F_4 \left[\begin{matrix} \lambda+2mk, \frac{1}{2}\lambda+mk+1, & -\mu, \lambda+\mu+N+2mk, & -N; \\ & & & 1 \end{matrix} \right], \quad (9)$$

$$\left[\begin{matrix} \frac{1}{2}\lambda+mk, \lambda+\mu+2mk+1, & 1-\mu-N, \lambda+N+2mk+1; \end{matrix} \right]$$

where ${}_rF_s$ denotes, as usual, a generalized hypergeometric series with r numerator and s denominator parameters (*cf.*, *e.g.*, [4]).

The hypergeometric ${}_5F_4$ series occurring in (9) is well-poised. By applying a terminating version of a known summation theorem [4, p. 244, Equation (III.13)], its sum can easily be seen to be the Kronecker delta $\delta_{N,0}$. Thus (9) reduces immediately to

$$\omega(x,t) = \sum_{k=0}^{\infty} \Lambda_k \Omega_{mk} \frac{(xt^m)^k}{k!} \quad (m = 1, 2, 3, \dots), \quad (10)$$

which proves the assertion (6) of Theorem 2 under the hypothesis that the various interchanges of the order of summation are permissible by absolute convergence of the series involved.

REMARK 1. Replacing t by t/μ , and x by $x\mu^m$, and letting $\mu \rightarrow \infty$, Theorem 2 would yield Theorem 1.

REMARK 2. A limiting case of Theorem 2 when t is replaced by $\lambda t/\mu$, and x by $x(\mu/\lambda)^m$, and $\lambda, \mu \rightarrow \infty$, leads us to a result on generating functions due to Srivastava and Panda [6, p. 468, Theorem 1].

REMARK 3. If, in Theorem 2, we replace t by λt , and x by $x\lambda^{-m}$, and let $\lambda \rightarrow \infty$, we shall obtain the generating function:

$$\sum_{n=0}^{\infty} \frac{(-\mu)_n}{n!} \Phi_n^{(\mu)}(x,t;m) t^n = G(xt^m), \quad (11)$$

where $G(z)$ is given, as before, by (3), and

$$\Phi_n^{(\mu)}(x,t;m) = \lim_{\lambda \rightarrow \infty} \Theta_n^{(\lambda,\mu)}(x\lambda^{-m}, \lambda t; m)$$

$$= \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{(\mu-n+1)_{mk}} \Lambda_k \frac{x^k}{k!} \sum_{r=0}^{\infty} (\mu)_r \Omega_{n+r} \frac{t^r}{r!}. \quad (12)$$

It may be observed in passing that, since $\Phi_n^{(\mu)}(x,t;m)$ given by (12) is actually a product of two series (just as in the case of Theorem 1), the assertion (11) can be applied to derive bilateral and bilinear generating functions as well.

For suitable special values of the coefficients Λ_n and Ω_n ($n = 0,1,2,\dots$), Theorem 2 and its various consequences indicated above can be applied with a view to obtaining further expansion formulas in series of Bessel functions (and of their aforementioned products) and numerous (linear, bilinear, and bilateral) generating functions for the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite (*cf.* [5] and [8]), the Bessel polynomials of Krall and Frink [3], the many classes of generalized hypergeometric polynomials studied in the literature ([1], [2], and [6, Section 3]), and so on. The details of such interesting derivations from the results considered here are already provided in the earlier works [6] and [7].

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