ON THE PRODUCT OF UPPER IRREDUNDANCE NUMBERS OF A GRAPH AND ITS COMPLEMENT

By

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ABSTRACT

A set X of vertices of a graph is irredundant if the closed neighbourhood of each $x \in X$ is not contained in the union of closed neighbourhoods of the vertices of $X - \{x\}$. The upper irredundance number, $IR(G)$ is the largest number of vertices in any irredundant set of G. We prove that for any p-vertex graph G, $IR(G) \cdot IR(\overline{G}) \leq \left\lceil \frac{p(p+2)}{4} \right\rceil$ and exhibit all graphs which attain this bound.

1. Introduction

The vertex x in a subset X of vertices of a graph is called <u>redundant in X</u> if its closed neighbourhood is contained in the union of closed neighbourhoods of the vertices of $X - \{x\}$. The set X is called <u>irredundant</u> if it contains no vertex which is redundant in X. We will need the following equivalent definition in which $N_G(x)$ denotes the open neighbourhood of vertex x in the graph G.

The set X is irredundant in G if and only if for each $x \in X$

either (i) x is an isolated vertex of $G[x]$

or (ii) there exists a vertex x' such that

$$x' \in N_G(x) \cap (V-X) \text{ and } N_G(x') \cap X = \{x\}. \quad (1)$$

The concept of irredundance was introduced originally in [2]. It is closely related to domination and independence in graphs. An excellent bibliography of existing results is given in [3].

The upper irredundance number of G, denoted by $IR(G)$, is the maximum number of vertices in an irredundant set of G. Some results concerning this parameter were established in [1]. In this paper, we prove that for any p-vertex graph G, $IR(G) \cdot IR(\bar{G}) \le \left\lfloor \frac{p^2+2p}{4} \right\rfloor$ and exhibit all the graphs which attain this upper bound. This theorem is known as a Nordhaus and Gaddum type result due to their interest in the product of chromatic numbers of G and $\bar{G}$ (see [5]).

2. The Result

Theorem 1 (a) For any p-vertex graph G, $IR(G) \cdot IR(\bar{G}) \le \left\lfloor \frac{p^2+2p}{4} \right\rfloor$.

(b) G attains the bound in (a) if and only if G or its complement consists of

(i) A set X of $\left\lfloor \frac{p+1}{2} \right\rfloor$ independent vertices,

(ii) A set Y of $\left\lceil \frac{p+1}{2} \right\rceil$ vertices where $G[Y]$ is complete and $X \cap Y = \{x\}$

and (iii) any set S of edges joining vertices of $X - \{x\}$ to vertices of $Y - \{x\}$.

<u>Proof (a)</u>. Let X, Y be irredundant sets of G, $\bar{G}$ respectively where $|X| = m$ and $|Y| = n$. Further let $|X \cap Y| = s$ and $|V - (X \cup Y)| = t$. Then $m + n - s = p - t$, hence $mn = m(p-m+s-t)$. Using elementary calculus and the fact that m, n are integral, we have

$$mn \le \left\lfloor \left[\frac{p+(s-t)}{2}\right]^2 \right\rfloor . \tag{2}$$

It is easily verified that the right hand side of (2) is at most $\left\lfloor \frac{p^2+2p}{4} \right\rfloor$ provided that $s - t \le 1$ and hence the result is true in this case.

We now show that the situation $s - t \ge 2$ is impossible. Suppose not and let $X \cap Y = \{x_1, x_2, \ldots, x_s\}$. Our assumption implies $s \ge 2$. Either $G[X \cap Y]$ or $\bar{G}[X \cap Y]$ is isolate-free. Without losing generality suppose that $G[X \cap Y]$ has this property.

Since X is an irredundant set of G, it follows from (1) that for each $i = 1, \ldots, s$, there exists a vertex $f(x_i)$ such that

$$f(x_i) \in N_G(x_i) \cap (V-X)$$

$$\text{and} \qquad N_G\left[f(x_i)\right] \cap X = \{x_i\}. \tag{3}$$

Further, this definition of f implies that $f(x_1)$, $f(x_2)$, ..., $f(x_s)$ are distinct and since $s \ge t + 2$, one of these, say $f(x_s) \in Y - X$.

By (3) $f(x_s)$ is adjacent in $\bar{G}$ to each vertex of $X - Y$ and to each x_i, $i = 1, \ldots, s-1$. Therefore, $x_1, \ldots, x_{s-1}$ are not isolated vertices in $\bar{G}[Y]$. Since Y is an irredundant set of $\bar{G}$, it follows from (1) that for each $i = 1, \ldots, s-1$ there exists a vertex $g(x_i)$ such that

$$g(x_i) \in N_{\bar{G}}(x_i) \cap (V-Y)$$
$$\text{and} \quad N_{\bar{G}}\left[g(x_i)\right] \cap Y = \{x_i\}. \qquad (4)$$

This definition implies $g(x_1), \ldots, g(x_{s-1})$ are distinct. Further, no $g(x_i) \in X - Y$, for otherwise $N_{\bar{G}}\left[g(x_i)\right] \cap Y$ would contain both $f(x_s)$ and x_i contrary to (4). It follows that each $g(x_1), \ldots, g(x_{s-1})$ is in $V - (X \cup Y)$, i.e. $t \geq s - 1$, a contradiction.

<u>Proof (b)</u>. Let G attain the bound of the theorem. It is immediate from the proof above that $s - t = 1$. Moreover, the bound of (2) is attained only when (m,n) or $(n,m) = \left(\left\lfloor\frac{p+1}{2}\right\rfloor, \left\lceil\frac{p+1}{2}\right\rceil\right)$. It will now be shown in two steps that G has $s = 1$ and $t = 0$.

Firstly suppose $s \geq 3$ and without loss of generality let $G[X \cap Y]$ be isolate-free. By arguments identical to those used in the proof of (a), we assert the existence of $x_s \in X \cap Y$ such that $f(x_s) \in Y - X$ and since $t = s - 1$,

$$\{g(x_1), \ldots, g(x_{s-1})\} = V - (X \cup Y).$$

By definition of $g(x_i)$ and since $s \geq 3$, each vertex of $V - (X \cup Y)$ is adjacent in G to at least two vertices of $X \cap Y$. Therefore $f(x_1)$ cannot be in $V - (X \cup Y)$ and hence $f(x_1) \in Y - X$. The definition of $f(x_1)$ implies that $f(x_1)x_s$ is an edge of $\bar{G}$. Therefore x_s is not isolated in $\bar{G}$ and so there exists $g(x_s)$, distinct from $g(x_1), \ldots, g(x_{s-1})$ in $V - (X \cup Y)$, a contradiction.

Secondly suppose $s = 2$, $t = 1$ and without losing generality let x_1x_2 be an edge of G. Then as above, $f(x_2) \in Y - X$ and $V - (X \cup Y) = \{g(x_1)\}$. Therefore $f(x_1) \in Y - X$ and by the definition of $f(x_1)$, $x_2f(x_1)$ is an edge of $\bar{G}$. Since x_2 is not isolated in $\bar{G}$, there exists $g(x_2) \neq g(x_1)$ in $V - (X \cup Y)$, a contradiction showing that $s = 1$, $t = 0$.

We now prove that $G[X]$ has no edges. A similar argument shows that $\bar{G}[Y]$ has no edges i.e. $G[Y]$ is complete. Let $X \cap Y = \{x_1\}$ and xx' be an edge of $G[X]$, where $x \neq x_1$. Then $f(x) \in Y - X$ and the definition of $f(x)$ implies that $x_1 f(x)$ is an edge of $\bar{G}$. Hence x_1 is not isolated in $\bar{G}$, and there exists $g(x_1) \in V - (X \cup Y) = \emptyset$, a contradiction.

We have proved that if G attains the bound, then

(i) $V = X \cup Y$ where $|X \cap Y| = 1$ and (m,n) or $(n,m) = \left(\left\lfloor \frac{p+1}{2} \right\rfloor, \left\lceil \frac{p+1}{2} \right\rceil\right)$.

(ii) $G[X]$ and $\bar{G}[Y]$ have no edges.

Conversely any graph G satisfying (i) and (ii) has X, Y independent and hence irredundant in G, $\bar{G}$, respectively and hence $IR(G) \cdot IR(\bar{G}) \geq \left\lfloor \frac{p^2+2p}{4} \right\rfloor$. Therefore G attains the bound. This completes the proof.

3. Deductions

Corollary 1. For any graph G, $IR(G) + IR(\bar{G}) \leq p + 1$.

Proof. In the proof of theorem 1(a) we have $m + n - s = p - t$ and $s - t \leq 1$ hence the result. We note that K_p attains this bound.

Let $\gamma(G)$ and $\Gamma(G)$ ($i(G)$ and $\beta(G)$) denote the smallest and largest cardinalities of a minimal dominating (maximal independent) vertex subset of G. We abbreviate $\gamma(G)$ by γ, $\gamma(\bar{G})$ by $\bar{\gamma}$ etc. It is well known that for any graph G,

$$\gamma \leq i \leq \beta \leq \Gamma \leq IR. \qquad (5)$$

Jaegar and Payan [4] proved that $\gamma\bar{\gamma} \leq p$ for any p vertex graph.

Corollary 2. For any p-vertex graph G, the products $i\bar{i}$, $\gamma\bar{\beta}$, $\beta\bar{\beta}$, $\gamma\bar{\Gamma}$, $\Gamma\bar{\Gamma}$ (for example) are all bounded above by $\left\lceil \frac{p^2+2p}{4} \right\rceil$.

Proof. Immediate from the theorem and (5). The extremal graph of theorem 1(b) in which $S = \emptyset$, attains the upper bound of corollary 2 for all of the products except $i\bar{i}$.

It is easily verified that all the graphs described in theorem 1(b) have $i\bar{i} < IR \cdot \bar{IR}$. Therefore for any p-vertex graph, $i\bar{i} < \left\lceil \frac{p^2+2p}{4} \right\rceil$.

Let p be divisible by 4 and let $V(G)$ partition into sets of equal size X, Y where $G[X]$ is edge free, $G[Y]$ is complete and the bipartite graph induced by X, Y is regular of degree $p/4$. Then $i = \bar{i} = \frac{p}{4} + 1$ and hence $i\bar{i} = \frac{(p+4)^2}{16}$.

Corollary 3. The maximum value $h(p)$, of $i\bar{i}$ among p-vertex graphs G satisfies

$$\frac{(p+4)^2}{16} \leq h(p) < \left\lceil \frac{p^2+2p}{4} \right\rceil.$$

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