SOME BILINEAR FORMULAS AND INTEGRAL EQUATIONS FOR CHEBYSHEV POLYNOMIALS

by

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Generating functions are used to construct bilinear expansion formulas for several kernel functions in terms of Chebyshev polynomials of the first and second kinds. These formulas lead to homogeneous integral equations that are satisfied by the polynomials.

1. Introduction

In a recent paper [1], it has been shown that the Chebyshev polynomials T_m $(m = 0,1,2,\cdots)$ are the characteristic functions of the homogeneous integral equation

$$\Phi(x) = \lambda \int_{-1}^{1} K(x,y)\ \Phi(y)dy,$$

corresponding to the characteristic values $\lambda_m := q^{-m}$ $(m = 0,1,2,\cdots)$. Here, in notation slightly different from that in [1],

$$K(x,y) = \frac{b}{\pi}\ \frac{a - xy}{x^2 + y^2 - 2axy + b^2}\ (1-y^2)^{-\frac{1}{2}},$$

$$a := \tfrac{1}{2}(q^{-1}+q), \qquad b := \tfrac{1}{2}(q^{-1}-q)$$

and $0 < q < 1$. This equation arose as a consequence of the bilinear formula; that is, the expansion of the kernel of a symmetric Fredholm integral equation in terms of its characteristic functions and values.

The purpose of the present note is to illustrate how this bilinear formula and integral equation, and others that involve the polynomials U_m as well, can be derived from generating functions. It would be surprising if these results were unknown. Yet a search through some of the literature has revealed few relations of this sort: of those we shall obtain, only two (namely 22.13.3 and 22.13.4 of [2]) were found. For this reason as well as for brevity, the subsequent treatment will be formal, although justification of the steps is straightforward.

2. Some Bilinear Formulas and Integral Equations

As our starting point, we take the relation

$$\frac{1-z^2}{1-2xz+z^2} = 1 + 2\sum_{n=1}^{\infty} T_n(x)\, z^n \qquad (-1 < x < 1, \quad |z| < 1), \tag{1}$$

although $(1-xz)/(1-2xz+z^2) = \sum_{n=0}^{\infty} T_n(x)\, z^n$ would serve just as well. On setting $z = qe^{i\theta}$, where $0 < q < 1$ and θ is real, and separating (1) into real and imaginary parts, one finds

$$\frac{b(a-x\cos\theta)}{x^2+\cos^2\theta-2ax\cos\theta+b^2} = 1 + 2\sum_{n=1}^{\infty} q^n\, T_n(x)\cos n\theta, \tag{2}$$

and

$$\frac{(ax-\cos\theta)\sin\theta}{x^2+\cos^2\theta-2ax\cos\theta+b^2} = 2\sum_{n=1}^{\infty} q^n\, T_n(x)\sin n\theta. \tag{3}$$

Assume now that $0 \le \theta \le \pi$ and set $y := \cos\theta$, with $(1-y^2)^{\frac{1}{2}} := \sin\theta$. Since $\cos n\theta = T_n(y)$ and $\sin n\theta = (1-y^2)^{\frac{1}{2}}\, U_{n-1}(y)$, (2) and (3) become

$$\frac{b(a-xy)}{x^2+y^2-2axy+b^2} = 1 + 2\sum_{n=1}^{\infty} q^n\, T_n(x)\, T_n(y), \tag{4}$$

and

$$\frac{ax - y}{x^2 + y^2 - 2axy + b^2} = 2 \sum_{n=1}^{\infty} q^n \, T_n(x) \, U_{n-1}(y). \tag{5}$$

Equation (4) is the bilinear formula found in [1], which led to the integral equation

$$T_m(x) = \frac{b}{\pi} q^{-m} \int_{-1}^{1} T_m(y) \frac{a - xy}{x^2 + y^2 - 2axy + b^2} \frac{dy}{(1-y^2)^{\frac{1}{2}}} \tag{6}$$

$$(-1 < x < 1)$$

for $m = 0,1,2,\cdots$.

There are, moreover, analogous consequences of (5). It will be seen that the right side is the expansion of the left side in terms of characteristic functions of two related symmetric kernels. Orthogonality of the polynomials of the first and second kinds gives

$$U_{m-1}(y) = \frac{1}{\pi} q^{-m} \int_{-1}^{1} T_m(x) \frac{ax - y}{x^2 + y^2 - 2axy + b^2} \frac{dx}{(1-x^2)^{\frac{1}{2}}} \tag{7}$$

$$(-1 < y < 1)$$

and

$$T_m(x) = \frac{1}{\pi} q^{-m} \int_{-1}^{1} U_{m-1}(y) \frac{ax - y}{x^2 + y^2 - 2axy + b^2} (1-y^2)^{\frac{1}{2}} \, dy \tag{8}$$

$$(-1 < x < 1),$$

for $m = 1,2,3,\cdots$.

Now the right sides of (7) and (8) must be independent of q. Thus we may take q close to 1 and let $q \to 1-$. Consider first (7), and express the integral in terms of a contour integral in the complex x-plane cut from -1 to 1, and with $\mathrm{Re}(1-x^2)^{\frac{1}{2}} > 0$. The integrand has poles at $ay \pm ib(1-y^2)^{\frac{1}{2}}$, which lie near the cut when q is close to 1. Initially the contour is assumed to enclose the two poles and the cut. Then, on letting $q \to 1-$ and collapsing the contour onto the cut, one finds

$$U_{m-1}(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{T_m(x)}{x-y} \frac{dx}{(1-x^2)^{\frac{1}{2}}}, \tag{9}$$

in which the integral is a Cauchy principal value at y. In a similar manner, (8) leads to

$$T_m(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{U_{m-1}(y)}{y-x} (1-y^2)^{\frac{1}{2}} \, dy. \tag{10}$$

These results agree with 22.13.3 and 22.13.4 in corrected printings of [2].

The insertion of $U_{m-1}(y)$ from (9) into (10) and inversion of orders of integration by means of the Hardy–Poincaré–Bertrand formula ([3], p. 171) produces an identity. Suppose again, however, that $q < 1$ in (7) and (8), and define a kernel function M by

$$M(x,y) := \frac{ax - y}{x^2 + y^2 - 2axy + b^2} \frac{(1-y^2)^{\frac{1}{4}}}{(1-x^2)^{\frac{1}{4}}} \qquad (-1 < x < 1, \ -1 < y < 1).$$

Let $V_m(x) := (1-x^2)^{-\frac{1}{4}} T_m(x)$ and $W_{m-1}(y) := (1-y^2)^{\frac{1}{4}} U_{m-1}(y)$. Then (7) and (8) give

$$V_m(x) = \left[\frac{1}{\pi} q^{-m}\right]^2 \int_{-1}^{1} V_m(y) \, K_R(x,y) dy, \tag{11}$$

where K_R is the (symmetric) right iterated kernel of M ([3], §3.16):

$$K_R(x,y) := \int_{-1}^{1} M(x,z) \, M(y,z) dz.$$

Similarly,

$$W_{m-1}(y) = \left[\frac{1}{\pi} q^{-m}\right]^2 \int_{-1}^{1} W_{m-1}(x)\ K_L(x,y)\,dx, \tag{12}$$

in which K_L is the left iterated kernel of M:

$$K_L(x,y) := \int_{-1}^{1} M(z,x)\ M(z,y)\,dz.$$

Simple or compact expressions for K_R and K_L have not been obtained, and we shall not proceed further in this direction. Note, however, that (11) differs from the equation for V_m determined by (6), and that various properties arising from relations of the form (7) and (8) are discussed in [3], §3.16. As observed earlier, (5) is, in effect, the expansion of $M(x,y)$ in terms of characteristic functions of K_R and K_L.

3. Concluding Remarks

It has been shown how a particular generating function leads to bilinear formulas and integral equations involving Chebyshev polynomials. We conclude with the observation that other generating functions can produce different results. For example, the real part of the expansion

$$-\tfrac{1}{2}\ \ell n(1-2xz+z^2) = \sum_{n=1}^{\infty} \frac{T_n(x)}{n} z^n, \tag{13}$$

with $z = qe^{i\theta}$, gives an integral equation for $T_m(x)$ with a logarithmic kernel. On differentiation with respect to x and use of the relation $T'_m(y) = m\ U_{m-1}(y)$, one again obtains (7). This also follows from the real part of the equation

$$\frac{z}{1 - 2xz + z^2} = \sum_{n=0}^{\infty} U_n(x)\ z^{n+1}, \tag{14}$$

found by differentiating (13) with respect to x. The imaginary part of (14), for $z = qe^{i\theta}$, yields

$$\frac{b}{2q(x^2+y^2-4axy+b^2)} = \sum_{n=0}^{\infty} q^n \; U_n(x) \; U_n(y). \tag{15}$$

On cancelling a factor z in (14), and taking real and imaginary parts, we find

$$\frac{b - y(x-qy)}{2q(x^2+y^2-2axy+b^2)} = \sum_{n=0}^{\infty} q^n \; U_n(x) \; T_n(y) \tag{16}$$

and

$$\frac{x - qy}{2q(x^2+y^2-2axy+b^2)} = \sum_{n=1}^{\infty} q^n \; U_n(x) \; U_{n-1}(y). \tag{17}$$

By combining (15), (16), and (17), the recurrence result (22.5.6 of [2])

$$T_n(y) = U_n(y) - y \; U_{n-1}(y) \qquad (n = 1,2,3,\cdots),$$

is easily obtained.

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