ASYMPTOTIC PROPERTIES OF BAYES RISK FOR ONE-SIDED TESTS

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ABSTRACT

Consider a given sequence $\{T_n\}$ of estimators for a real valued parameter θ. This paper studies asymptotic properties of restricted Bayes tests of the form: reject H_o: $\theta \le \theta_o$ in favor of the alternative $\theta > \theta_o$ if $T_n > c_n$, where the critical point c_n is determined to minimize among all tests of this form the expected probability of error with respect to the prior distribution. Such tests may or may not be fully Bayes tests, so are called T_n-Bayes. Under fairly broad conditions it is shown that

$$c_n = \theta_o + a_n b(\theta_o)\tilde{\mu} + o(a_n)$$

and the T_n-Bayes risk

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$$B_n(c_n) = a_n b(\theta_o)\rho(\theta_o)\int_{-\infty}^{\infty} |x-\tilde{\mu}|dF(x) + o(a_n),$$

where a_n is the order of the standard error of T_n, ρ is the prior density, and $\tilde{\mu}$ is the median of F, the limit distribution of $(T_n-\theta)/a_n b(\theta)$. Several examples are given.

1. INTRODUCTION

This paper studies the asymptotic behavior of certain Bayes testing procedures. Limiting results in general (Bayesian or not) are useful in comparing tests whose risks can be quite complicated for fixed n. For example, comparison of the Bayes risk with the risk of a competing, perhaps simpler test is possible. This is similar to looking at asymptotic relative efficiencies.

Most of the previous studies of the asymptotic behavior of Bayes tests concerned sampling from some member of an exponential family (Efron, 1967; Johnson and Truax, 1974, 1978; Johnson, 1976). In these papers the shape of the Bayes region was obtained, as well as the rate at which the Bayes risk converged to zero and the dependence of these on the prior distribution. For example, in one-sided testing problems for typical prior distributions, the risk is $O(1/\sqrt{n})$. Johnson (1980) pointed out that in nonexponential settings the order of convergence for the Bayes risk could be different from $O(1/\sqrt{n})$.

In this paper we give a simple unified approach to finding the order of the Bayes risk in a wide variety of situations. In order to keep things as simple as possible we will assume we have a real valued parameter θ and we are only interested in testing one-sided hypotheses H_o: $\theta \leq \theta_o$. A typical Bayes test (at least in the case where there is a sufficient statistic having a monotone likelihood ratio) rejects if $T_n > c_n$, where T_n is a real valued statistic and $\{c_n\}$ is a sequence of constants determined by the prior distribution to minimize the expected risk. For ease in exposition consider tests of this form even if they are not truly Bayes. We call them T_n-Bayes. Their risks provide upper bounds for the risks of Bayes tests and are useful for comparison with competing non-Bayes procedures. Under fairly broad conditions we argue that if the standard error of T_n is $O(a_n)$, then the T_n-Bayes risk is of the same order. Several examples will be given.

2. NOTATION AND MATHEMATICAL SETTING

Consider the problem of testing

$$H_o: \theta \leq \theta_o \quad \text{versus} \quad H_a: \theta > \theta_o$$

in the following Bayesian model.

(i) The sequence $\{T_n\}$ is a sequence of estimators for a real valued parameter θ. (Typically, T_n will be some function of a random sample of size n taken from one member of a family of distributions indexed by θ.)

(ii) The parameter space Ω is an interval which contains θ_o in its interior.

(iii) The prior density $\rho(\cdot)$ is positive and continuous at θ_o, has finite mean, and has support contained in Ω.

(iv) Zero-one loss functions are used.

We will restrict our attention to tests having the form that reject H_o whenever $T_n > $ constant. With zero-one loss functions, the risk at θ is just the probability of making the wrong decision. Therefore, the Bayes risk corresponding to the critical region $\{T_n > c\}$ is

$$B_n(c) = \int_{-\infty}^{\theta_o} P_\theta(T_n>c)\rho(\theta)d\theta + \int_{\theta_o}^{\infty} P_\theta(T_n\leq c)\rho(\theta)d\theta .$$

Let c_n be a critical point satisfying

$$B_n(c_n) = \inf_c B_n(c).$$

Usually c_n will be unique. (In particular, c_n is unique if T_n is a sufficient statistic having a monotone likelihood ratio; see Karlin and Rubin (1956), page 281). The test with rejection region $\{T_n > c_n\}$ will be called a T_n-Bayes test.

3. REGULARITY CONDITIONS

We will study asymptotic properties of T_n-Bayes tests under the following regularity conditions, which are satisfied by a wide variety of examples.

REGULARITY CONDITION 1. *Under* θ *the test statistic* T_n *satisfies*

$$T_n = \theta + a_n b(\theta) Z_n,$$

where a_n *decreases to* 0 *as* $n \to \infty$, $b(\cdot)$ *is positive valued, continuous and integrable with respect to the prior distribution, and* Z_n *converges in distribution to a distribution which does not depend on* θ.

REGULARITY CONDITION 2. *Under* θ *the cumulative distribution function of* Z_n *satisfies*

$$F_n(x;\theta) = F(x) + a_n Q_n(x;\theta),$$

where the limit distribution F *has finite mean, and* $\{Q_n\}$ *satisfies*

Condition Q: $Q_n\left[\frac{\theta_0-\theta+o_n(1)}{a_n b(\theta)};\ \theta\right]$ *is bounded uniformly in* n and θ, *and for each* $\theta \neq \theta_o$ *converges to* 0 *as* $n \to \infty$.

Actually, many examples satisfy the stronger

Condition Q:* $\{Q_n\}$ *is a uniformly bounded sequence of functions, and for each* θ

$$Q_n(x_n;\theta) \to 0 \quad as \quad n \to \infty$$

along all sequences $\{x_n\}$ *such that* $x_n \to \pm\infty$.

4. RESULTS

LEMMA 1. *If regularity condition 1 is satisfied, then the sequence of critical points* $\{c_n\}$ *converges to* θ_o *as* $n \to \infty$.

Proof: Suppose, on the contrary, that $\liminf_{n\to\infty} c_n < \theta_o$. Then we can find $\epsilon > 0$ and a subsequence $\left\{c_{n_k}\right\}$ such that $c_{n_k} \le \theta_o - \epsilon$ for all k. Now we observe that the Bayes risk corresponding to the critical point θ_o goes to 0 as $n \to \infty$, because

$$B_n(\theta_o) = \int_{-\infty}^{\theta_o} P_\theta(T_n > \theta_o)\rho(\theta)d\theta + \int_{\theta_o}^{\infty} P_\theta(T_n \le \theta_o)\rho(\theta)d\theta$$

$$= \int_{-\infty}^{\theta_o} P_\theta\left[Z_n > \frac{\theta_o - \theta}{a_n b(\theta)}\right]\rho(\theta)d\theta + \int_{\theta_o}^{\infty} P_\theta\left[Z_n \le \frac{\theta_o - \theta}{a_n b(\theta)}\right]\rho(\theta)d\theta,$$

and the right-hand side goes to zero as $n \to \infty$ by dominated convergence. Hence, $B_n(c_n) \to 0$ as $n \to \infty$. However, by Fatou's lemma we see that

$$\liminf_{k\to\infty} B_{n_k}(c_{n_k}) \ge \liminf_{k\to\infty} \int_{\theta_o-\epsilon}^{\theta_o} P_\theta\left[T_{n_k} > c_{n_k}\right]\rho(\theta)d\theta$$

$$\ge \int_{\theta_o-\epsilon}^{\theta_o} \liminf_{k\to\infty} P_\theta\left[Z_{n_k} > \frac{c_{n_k} - \theta}{a_{n_k} b(\theta)}\right]\rho(\theta)d\theta = \int_{\theta_o-\epsilon}^{\theta_o} \rho(\theta)d\theta > 0.$$

From this contradiction we conclude that $\liminf_{n\to\infty} c_n \ge \theta_o$.

Similarly, the supposition that $\limsup_{n\to\infty} c_n > \theta_o$ leads to a contradiction. ∎

THEOREM 1. *If regularity conditions* 1 *and* 2 *are satisfied, then the Bayes risk corresponding to the rejection region*

$$\{T_n > \theta_o + a_n b(\theta_o)c + o(a_n)\}$$

has the asymptotic behavior

$$B_n\left[\theta_o + a_n b(\theta_o)c + o(a_n)\right] = a_n b(\theta_o)\rho(\theta_o)\int_{-\infty}^{\infty} |x-c|\, dF(x) + o(a_n).$$

Proof: First we will establish the asymptotic behavior of the type 1 risk

$$\int_{-\infty}^{\theta_o} P_\theta\left[T_n > \theta_o + a_n b(\theta_o)c + o(a_n)\right]\rho(\theta)\,d\theta.$$

From regularity conditions 1 and 2, and dominated convergence, we obtain

$$\int_{-\infty}^{\theta_o} P_\theta\left[T_n > \theta_o + a_n b(\theta_o)c + o(a_n)\right]\rho(\theta)\,d\theta = \int_{-\infty}^{\theta_o} P_\theta\left[Z_n > \frac{\theta_o - \theta}{a_n b(\theta)} + \frac{b(\theta_o)c + o_n(1)}{b(\theta)}\right]\rho(\theta)\,d\theta$$

$$= \int_{-\infty}^{\theta_o}\left\{1 - F\left[\frac{\theta_o - \theta}{a_n b(\theta)} + \frac{b(\theta_o)c + o_n(1)}{b(\theta)}\right] - a_n\, Q_n\left[\frac{\theta_o - \theta + a_n b(\theta_o)c + o(a_n)}{a_n b(\theta)};\ \theta\right]\right\}\rho(\theta)\,d\theta$$

$$= \int_{-\infty}^{\theta_o}\left\{1 - F\left[\frac{\theta_o - \theta}{a_n b(\theta)} + c + \frac{(b(\theta_o) - b(\theta))c + o_n(1)}{b(\theta)}\right]\right\}\rho(\theta)\,d\theta + o(a_n).$$

Letting $0 < \epsilon < 1$ be given, we find K sufficiently large and $\delta > 0$ sufficiently small that

$$(1-\epsilon)b(\theta_o) < b(\theta) < (1+\epsilon)b(\theta_o),\ (1-\epsilon)\rho(\theta_o) < \rho(\theta) < (1+\epsilon)\rho(\theta_o),$$
$$\text{and}\quad |\{(b(\theta_o)-b(\theta))c + o_n(1)\}/b(\theta)| < \epsilon,$$

whenever $n \geq K$ and $\theta_o - \delta < \theta < \theta_o$. Then, for $n \geq K$

$$\int_{\theta_o-\delta}^{\theta_o} \left\{1-F\left[\frac{\theta_o - \theta}{a_n b(\theta)} + c + \frac{(b(\theta_o)-b(\theta))c+o_n(1)}{b(\theta)}\right]\right\}\rho(\theta)d\theta$$

$$\leq (1+\epsilon)\rho(\theta_o) \int_{\theta_o-\delta}^{\theta_o} \left\{1-F\left[\frac{\theta_o - \theta}{a_n b(\theta_o)(1+\epsilon)} + c - \epsilon\right]\right\}d\theta$$

$$= a_n(1+\epsilon)^2\, b(\theta_o)\rho(\theta_o) \int_{c-\epsilon}^{c-\epsilon+\delta/a_n b(\theta_o)(1+\epsilon)} (1-F(x))dx,$$

and

$$\int_{\theta_o-\delta}^{\theta_o} \left\{1-F\left[\frac{\theta_o - \theta}{a_n b(\theta)} + c + \frac{(b(\theta_o)-b(\theta))c+o_n(1)}{b(\theta)}\right]\right\}\rho(\theta)d\theta$$

$$\geq (1-\epsilon)\rho(\theta_o) \int_{\theta_o-\delta}^{\theta_o} \left\{1-F\left[\frac{\theta_o - \theta}{a_n b(\theta_o)(1-\epsilon)} + c + \epsilon\right]\right\}d\theta$$

$$= a_n(1-\epsilon)^2\, b(\theta_o)\rho(\theta_o) \int_{c+\epsilon}^{c+\epsilon+\delta/a_n b(\theta_o)(1-\epsilon)} (1-F(x))dx.$$

Also, since $b(\cdot)$ is integrable with respect to $\rho(\cdot)$ and the distribution F has finite mean, we have

$$\int_{-\infty}^{\theta_o-\delta}\left\{1-F\left[\frac{\theta_o-\theta}{a_n b(\theta)}+\frac{b(\theta_o)c+o_n(1)}{b(\theta)}\right]\right\}\rho(\theta)d\theta \le \int_{-\infty}^{\theta_o-\delta}\left\{1-F\left[\frac{\delta+a_n b(\theta_o)c+o(a_n)}{a_n b(\theta)}\right]\right\}\rho(\theta)d\theta$$

$$= \frac{a_n}{\delta+a_n b(\theta_o)c+o(a_n)}\int_{-\infty}^{\theta_o-\delta}\frac{\delta+a_n b(\theta_o)c+o(a_n)}{a_n b(\theta)}\left\{1-F\left[\frac{\delta+a_n b(\theta_o)c+o(a_n)}{a_n b(\theta)}\right]\right\}b(\theta)\rho(\theta)d\theta$$

$= o(a_n)$, by dominated convergence.

Therefore,

$$\liminf_{n\to\infty}\frac{1}{a_n}\int_{-\infty}^{\theta_o} P_\theta(T_n > \theta_o + a_n b(\theta_o)c + o(a_n))\rho(\theta)d\theta$$

is bounded below by

$$(1-\epsilon)^2\, b(\theta_o)\,\rho(\theta_o)\int_{c+\epsilon}^{\infty}(1-F(x))dx,$$

and

$$\limsup_{n\to\infty}\frac{1}{a_n}\int_{-\infty}^{\theta_o} P_\theta(T_n > \theta_o + a_n b(\theta_o)c + o(a_n))\rho(\theta)d\theta$$

is bounded above by

$$(1+\epsilon)^2\, b(\theta_o)\,\rho(\theta_o)\int_{c-\epsilon}^{\infty}(1-F(x))dx.$$

Since ϵ can be arbitrarily small, we obtain

$$\int_{-\infty}^{\theta_o} P_\theta(T_n > \theta_o + a_n b(\theta_o)c + o(a_n))\rho(\theta)d\theta = a_n b(\theta_o)\rho(\theta_o)\int_{c}^{\infty}(1-F(x))dx + o(a_n).$$

Similarly, the type 2 risk satisfies

$$\int_{\theta_o}^{\infty} P_\theta(T_n \le \theta_o + a_n b(\theta_o)c + o(a_n))\rho(\theta)d\theta = a_n\, \rho(\theta_o)\, b(\theta_o) \int_{-\infty}^{c} F(x)dx + o(a_n).$$

The desired result now follows immediately. ∎

From Theorem 1 we see that among tests with critical regions of the form

$$\{T_n > \theta_o + a_n b(\theta_o)c + o(a_n)\}$$

the optimum asymptotic behavior is attained when c is a median of F. We will see in Theorem 2 that such tests are asymptotically T_n-Bayes (i.e. the ratio of their risk to that of the T_n-Bayes test converges to 1 as $n \to \infty$). First we will use Theorem 1 to prove the following lemma concerning the rate of convergence of the T_n-Bayes sequence of critical points $\{c_n\}$.

LEMMA 2. *If regularity conditions* 1 *and* 2 *are satisfied, then*

$$c_n = \theta_o + O(a_n) \quad \text{as} \quad n \to \infty.$$

Proof: Suppose, on the contrary, that $\liminf_{n\to\infty} (c_n - \theta_o)/a_n = -\infty$. Then we can find a subsequence $\{n_k\}$ such that $\left[c_{n_k} - \theta_o\right]/a_{n_k} \to -\infty$ as $k \to \infty$. Since $\rho(\cdot)$ is continuous and positive at θ_o, $c_{n_k} < \theta_o$ for all sufficiently large k, and $c_{n_k} \to \theta_o$, we obtain

$$\liminf_{k\to\infty} B_{n_k}(c_{n_k})/a_{n_k} \geq \liminf_{k\to\infty} \frac{1}{a_{n_k}} \int_{c_{n_k}}^{\theta_o} P_\theta\left[T_{n_k} > c_{n_k}\right]\rho(\theta)d\theta$$

$$\geq \liminf_{k\to\infty} \frac{\rho(\theta_o)}{2a_{n_k}} \int_{c_{n_k}}^{\theta_o} P_\theta\left[Z_{n_k} > \frac{c_{n_k} - \theta}{b(\theta)a_{n_k}}\right]d\theta$$

$$= \liminf_{k\to\infty} \frac{\rho(\theta_o)}{2a_{n_k}} \int_{c_{n_k}}^{\theta_o} \left\{1 - F\left[\frac{c_{n_k} - \theta}{b(\theta)a_{n_k}}\right] - a_{n_k} Q_{n_k}\left[\frac{(\theta_o - \theta) + (c_{n_k} - \theta_o)}{a_{n_k} b(\theta)};\ \theta\right]\right\}d\theta$$

$$\geq \liminf_{k\to\infty} \frac{\rho(\theta_o)}{2a_{n_k}} \left\{\int_{c_{n_k}}^{\theta_o} \left[1 - F\left[\frac{c_{n_k} - \theta}{M_k\ a_{n_k}}\right]\right]d\theta + o(a_{n_k})\right\},$$

where $M_k = \sup\left\{b(\theta):\ c_{n_k} \leq \theta \leq \theta_o\right\}$.

$$= \liminf_{k\to\infty}\left\{\frac{M_k\ \rho(\theta_o)}{2} \int_{\frac{c_{n_k} - \theta_o}{M_k\ a_{n_k}}}^{0} (1-F(x))dx + o_k(1)\right\} = +\infty.$$

However, this is a contradiction because

$$B_n(c_n) \leq B_n(\theta_o) \quad \text{and} \quad B_n(\theta_o) = O(a_n) \quad \text{by Theorem 1.}$$

Similarly, the assumption that $\limsup_{n\to\infty} (c_n - \theta_o)/a_n = +\infty$ leads to a contradiction. ∎

Theorem 1 and Lemma 2 combine to yield the following theorem concerning the asymptotic behaviors of the T_n-Bayes critical point c_n and the T_n-Bayes risk.

THEOREM 2. *If regularity conditions* 1 *and* 2 *are satisfied and* F *has a unique median* $\tilde{\mu}$, *then*

$$c_n = \theta_o + a_n b(\theta_o)\tilde{\mu} + o(a_n),$$

and

$$B_n(c_n) = a_n b(\theta_o)\rho(\theta_o) \int_{-\infty}^{\infty} |x-\tilde{\mu}| dF(x) + o(a_n).$$

REMARK. The formula for $B_n(c_n)$ remains valid even if the median of F is not unique. In this case both $\liminf_{n\to\infty} (c_n-\theta_o)/a_n b(\theta_o)$ and $\limsup_{n\to\infty} (c_n-\theta_o)/a_n b(\theta_o)$ are medians of F.

Proof: From Lemma 2 we know that $L = \limsup_{n\to\infty} (c_n-\theta_o)/a_n$ is finite. Let $\{n_k\}$ be a subsequence such that

$$c_{n_k} = \theta_o + a_{n_k} L + o(a_{n_k}).$$

By Theorem 1, the risk of the T_n-Bayes test must satisfy

$$B_{n_k}(c_{n_k}) = a_{n_k} b(\theta_o) \rho(\theta_o) \int_{-\infty}^{\infty} |x-L/b(\theta_o)| dF(x) + o(a_{n_k}).$$

Since $\int_{-\infty}^{\infty} |x-c| dF(x)$ is minimized if and only if $c = \tilde{\mu}$, it follows that $L = \tilde{\mu}\, b(\theta_o)$.

Similarly,

$$\liminf_{n\to\infty} (c_n-\theta_o)/a_n = \tilde{\mu}\, b(\theta_o). \quad ■$$

5. EXAMPLES

Regularity conditions 1 and 2 are satisfied in a wide variety of situations. Several examples are given in Table 1. In each of these examples $T_n = h(X_1,X_2,\ldots,X_n)$, where $X_1, X_2, \ldots, X_n$ are random samples from the given population distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the corresponding order statistics.

The rates for the exponential family examples (normal$(\theta,1)$, normal$(0,\theta)$, exponential(θ), gamma, Bernoulli, Poisson) could be obtained from earlier results (Johnson and Truax, 1974; Johnson, 1976); also, the exponential distribution with translation parameter θ was covered previously (Johnson, 1980). We believe the remaining non-exponential family examples (uniform, Pareto, Laplace, Cauchy, Weibull) are new.

TABLE 1: Examples

Population Distribution	T_n	a_n	$b(\theta)$	F	Q_n
Normal $(\theta,1)$	$\bar{X}_n$	$1/\sqrt{n}$	1	Normal (0,1)	$Q_n(x;\theta)\equiv 0$
Normal $(0,\theta=\sigma^2)$	$\frac{1}{n}\sum_{i=1}^n X_i^2$	$1/\sqrt{n}$	$\sqrt{2}\,\theta$	Normal (0,1)	Satisfies Condition Q* by Esseen[1]
Exponential $(\theta=1/\lambda)$	$\bar{X}_n$	$1/\sqrt{n}$	θ	Normal (0,1)	Satisfies Condition Q* by Esseen[1]
Gamma: $G'(x)=x^{\theta-1}e^{-x}/\Gamma(\theta)$, $x\geq 0$	$\bar{X}_n$	$1/\sqrt{n}$	$\sqrt{\theta}$	Normal (0,1)	Satisfies Condition Q
Bernoulli (θ)	$\bar{X}_n$	$1/\sqrt{n}$	$\sqrt{\theta(1-\theta)}$	Normal (0,1)	Satisfies Condition Q
Poisson (θ)	$\bar{X}_n$	$1/\sqrt{n}$	$\sqrt{\theta}$	Normal (0,1)	Satisfies Condition Q
$G(x-\theta)$, where G is Exp(1)	$X_{(1)}$	$1/n$	1	Exp(1)	$Q_n(x;\theta)\equiv 0$
Uniform $[0,\theta]$	$X_{(n)}$	$1/n$	θ	$F(x)=e^x$, $x\leq 0$	$Q_n(x,\theta)=\begin{cases} n((1+x/n)^n-e^x), & -n\leq x\leq 0 \\ -ne^x, & x<-n \end{cases}$ Satisfies Condition Q*
Pareto: $G(x)=1-\theta/x$, $x\geq\theta>0$	$X_{(1)}$	$1/n$	θ	Exp(1)	$Q_n(x;\theta)=n(e^{-x}-(1+x/n)^{-n})$, $x\geq 0$ Satisfies Condition Q*
Laplace: $G(x-\theta)$, where $G'(x)=\frac{1}{2}e^{-\lvert x\rvert}$	$X_{([(n+1)/2])}$	$1/\sqrt{n}$	1	Normal (0,1)	Satisfies Condition Q* by Esseen[1]
$G(x-\theta)$, where G is Cauchy	$X_{([(n+1)/2])}$	$1/\sqrt{n}$	$\pi/2$	Normal (0,1)	Satisfies Condition Q* by Esseen[1]
Weibull: $G(x-\theta)$, where $G(x)=1-e^{-x^\alpha}$, $x\geq 0$	$X_{(1)}$	$n^{-\alpha}$	1	$F(x)=1-e^{-x^\alpha}$, $x\geq 0$	$Q_n(x;\theta)\equiv 0$

1. Under the column heading Q_n, Esseen refers to Esseen's Theorems (1944). See also Gnedenko and Kolmogorov (1968), pages 210 and 213.

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