ON THE BBGKY HIERARCHY
FOR HARD SPHERES

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§1. INTRODUCTION

A celebrated result of Oscar Lanford [10, 11] shows that the rescaled correlation functions associated with the motion of finitely many hard spheres approach solutions of the Boltzmann hierarchy in the Boltzmann-Grad limit, at least for sufficiently short times. The importance of this result is that it settles rigorously the long controversial discussion as to whether statistical mechanical irreversibility occurs in the Boltzmann-Grad limit or not, and establishes the relevance of the Boltzmann equation in statistical mechanics.

Lanford's proof begins with a form of the BBGKY hierarchy for hard spheres, and unfortunately the usual derivation of this hierarchy (see e.g. [4, 5]) is only formal. That there was a problem with the formal derivation was first pointed out by H. Spohn, who then attempted to provide a rigorous proof in the unpublished (and apparently incomplete) paper [13]. In this paper we present a proof of an integrated version of the hierarchy which would suffice for Lanford's results.

A few remarks are perhaps in order concerning the proof. In the usual derivation of the BBGKY hierachy for hard spheres, the collision integral arises from integration by parts, and it is not at all clear how the (presumably infinite) interparticle forces enter into it. In [6] H. Grad makes some relevant comments in this regard. In the proof presented here, it is made clear that the collision integral arises from collisions, or, more precisely, from discontinuities in the momenta in collisions.

After the research reported here was completed, the author learned of an independent and different derivation of the hierarchy by Illner and Pulvirenti [8]. Thanks to Marvin Shinbrot and Reinhard Illner for many helpful discussions during thew preparation of this paper.

§2 THE RESULT

Let Λ be a bounded domain with smooth boundary $\partial\Lambda$ contained in $S = \mathbb{R}^3$. We want to consider n identical hard spheres of diameter δ and (for convenience) unit mass moving in Λ. The position and momentum of the i^{th} particle is denoted by $x_i = (q_i, p_i)$. The phase space is the set $\hat{\pi}_n^{\delta}(\Lambda) = \{x = (x_1,\ldots,x_n) \in (\Lambda \times S)^n : |q_i - q| \geq \delta/2$ for all $q \in \partial\Lambda$, and $|q_i - q_j| \geq \delta$, $1 \leq i,j \leq n$, $i \neq j\}$ endowed with the Euclidean topology and (the restriction of) Lebesgue measure. We will use the notation x^n for a point $(x_1,\ldots,x_n)$ in $\hat{\pi}_n^{\delta}(\Lambda)$ to indicate the number of variables, which may vary.

The particles move according to the hard sphere dynamics; that is, each particle moves in a straight line until it encounters another particle or the boundary. In a collision with the boundary,the particles reflect elastically. In a collision between two particles having positions and momenta (q_1, p_1) and (q_2, p_2), the ingoing momenta p_1 and p_2 are abruptly changed to the outgoing momenta p_1' and p_2', where

$$
\begin{aligned}
p_1' &= p_1 - [\omega\cdot(p_1 - p_2)]\,\omega\,, \\
p_2' &= p_2 + [\omega\cdot(p_1 - p_2)]\,\omega\,,
\end{aligned}
\tag{2.1}
$$

and $\omega = (q_2 - q_1)/|q_2 - q_1|$ is the unit vector giving the direction between the centers (here and in the following we use single bars to denote the length of vectors in $\mathbb{R}^3$). After this interchange of momenta, the particles continue in rectilinial motion.

There are several problems with this definition of the dynamics. Firstly, it is not clear that, beginning with given initial conditions, we can find the positions and momenta of the particles at any later time, and indeed, multiple and grazing collisions can occur. But Alexander [1] has shown that these cases occur only on a set of measure zero in $\hat{\pi}_n^\delta(\Lambda)$ (see also [12]). Thus we may assume that there exists a measurable subset $\pi_n^\delta(\Lambda)$ of $\hat{\pi}_n^\delta(\Lambda)$ of full measure, and a group $\{T^t : \pi_n^\delta(\Lambda) \longrightarrow \pi_n^\delta(\Lambda) \ : \ t \in \mathbb{R}\}$ of measurable transformations, where $T^t(x)$ is obtained by evolving the initial conditions $x \in \pi_n^\delta(\Lambda)$ over a time t using the hard sphere dynamics.

The second and intrinsic problem is that the maps T^t, though measurable, are not even continuous, let alone differentiable. Due to the abrupt changes in momentum, the trajectories are only piecewise continuous. To be definite we impose the condition that the trajectories are continuous from the left, that is,

$$(2.2) \qquad \lim_{t \longrightarrow \tau - 0} T^t(x) = T^\tau(x).$$

In the ensuing discussion we will need to consider the set of points in $\pi_n^\delta(\Lambda)$ which involve collisions between particles. To this end we define the set

$$(2.3) \qquad K_{ij} = \{x \in \pi_n^\delta(\Lambda) \ : \ q_j = q_i + \delta\omega_{ij},\ \omega_{ij}\cdot(p_i - p_j) \geq 0, \quad \omega_{ij} = (q_j - q_i)/|q_j - q_i| \in S_2\}$$

(S_2 being the unit sphere in $\mathbb{R}^3$) consisting of those points representing an ingoing collision between the i^{th} and the j^{th} particles, and put

$$(2.4) \qquad K = \bigcup K_{ij} \ (1 \le i < j \le n).$$

K inherits its topology from $\pi_n^\delta(\Lambda)$, and on K there is defined the natural measure σ given by $d\sigma = \sum d\sigma_{ij}$ ($1 \le i < j \le n$), where $d\sigma_{ij}$ is the measure on K_{ij} given by

$$(2.5) \quad d\sigma_{ij} = \delta^2 \, dx_1 \ldots dx_{j-1} \, dx_{j+1} \ldots dx_n \, \omega_{ij} \cdot (p_j - p_i) \, d\omega_{ij} \, dp_j \ (1 \le i < j \le n)$$

where $d\omega_{ij}$ represents the measure on the unit sphere with center q_i in q_j space. Note that $K_{ij} \cap K_{ik} = \emptyset$ since there are no multiple collisions. In the same way we may define the set K^+, representing outgoing collisions by replacing the condition $\omega_{ij} \cdot (p_i - p_j) \ge 0$ by $\omega_{ij} \cdot (p_i - p_j) < 0$ in (2.3). There is a measure preserving map $\xi : K^+ \longrightarrow K$, where $\xi(y)$ is obtained from $y \in K^+$ by replacing outgoing by ingoing momenta using (2.1).

We now let $\Xi = \{ x \in \pi_n^\delta(\Lambda) : T^t(x) \in K$ for some $t > 0 \}$ denote the set of points which undergo a collision at some time $t > 0$. The set Ξ has a representation as a special flow with base K [2, 3, 12]. To do so, we define the function τ on K by

$$(2.6) \qquad \tau(y) = \min \{ t \in \mathbb{R}^+ : T^{-t}(y) \in K^+ \}.$$

The function τ is never zero and may take the value ∞, but elsewhere is continuous on K and hence measurable since we have ruled out grazing collisions for $x \in \pi_n^\delta(\Lambda)$. For any $x \in \Xi$, the set $\{t \in [0, \infty) : T^t(x) \in K \}$ has no finite point of accumulation since we have eliminated multiple collisions. We now let

$$(2.7) \qquad \tilde{\Xi} = \{ (s,y) : 0 \le s < \tau(y),\ y \in K \}.$$

$\tilde{\Xi}$ inherits a topology and measure from $K \times [0, \infty)$. Next we define a 1-1 bicontinuous measure-preserving map $\Phi : \Xi \longrightarrow \tilde{\Xi}$ by letting $\Phi(x) = (s, y)$, where $s = s(x) = \min \{t : T^t(x) = y \in K\}$. We can use the rules (2.1) to define a measurable map $S : K \longrightarrow K$ by the prescription $S(y) = \xi(T^{-\tau(y)})$. Then corresponding to the semigroup $\{T^{-t} : t \ge 0\}$ of maps on Ξ there is the semigroup $\{S^{-t} : t \ge 0\}$ of maps on $\tilde{\Xi}$ defined by

$$S^{-t}(s, y) = (t+s, y) \text{ for } 0 \le t < \tau(y) - s,$$

$$(2.8) \quad S^{-t}(y, s) = (t + s - \tau(y) - \dots - \tau(S^{n-1}y), S^n y)$$

$$\text{for } t(y) + \dots + t(S^{n-1}y) - s \le t < \tau(y) + \dots + \tau(S^n y) - s.$$

T^{-t} and S^{-t} are related by the equation $\Phi \circ T^{-t} = S^{-t} \circ \Phi$. We can replace integration of a function g on Ξ by integration of $\tilde{g} = g \circ \Phi^{-1}$ on $\tilde{\Xi}$. The collision integral will involve integration on $\tilde{\Xi}$.

Suppose now that f is a symmetric function in $L^1(\pi_n^\delta(\Lambda))$ (f is usually a probability density). The time evolved density $f(t)$ is defined by $f(t) = f \circ T^{-t}$. In the following we write $f(t)(x^n)$ as $f(t, x^n)$. The k^{th} correlation function of f is a function on $\pi_k^\delta(\Lambda)$ defined by

$$(2.9) \qquad \rho_k(x^k) = \frac{n!}{(n-k)!} \int f(x^n) dx_{k+1} \dots dx_n .$$

The correlation functions of $f(t)$ are denoted by $\rho_k(t)$, and we write $\rho_k(t)(x^k)$

$= \rho_k(t, x^k)$. If φ is a real- valued function on $\mathbb{R} \times \pi_k^\delta(\Lambda)$, we define the function $\varphi(t):\pi_k^\delta(\Lambda) \longrightarrow \mathbb{R}$ by $\varphi(t)(x^k) = \varphi(t, x^k)$, $t \in \mathbb{R}$. Finally, we use the notation $\rho_k(t)[\varphi(t)]$ for $\int\varphi(t)\ \rho_k(t)dx^k = \frac{n!}{(n-k)!}\int\varphi(t,x^k)f(t,x^n)dx^n$.

Our objective is to show that, for a certain class of smooth functions φ: $\mathbb{R} \times \pi_k^\delta(\Lambda) \longrightarrow \mathbb{R}$, the function $\rho_k(t)[\varphi(t)]$ is absolutely continuous and hence differentiable almost everywhere, and to obtain an equation for its derivative. The functions φ which will be allowed are specified in the following definition.

2.1 Definition

The set Φ_k consists of those functions $\varphi \in C^1(\mathbb{R} \times \pi_k^\delta(\Lambda))$ which are symmetric and for which

(i) $\varphi(t)$ has compact support in $\pi_k^\delta(\Lambda)$ for each $t \in \mathbb{R}$,

(ii) Whenever $|q_i - q_j| = \delta$, we have

$$\varphi(t,\dots,q_i,p_i,\dots,q_j,p_j,\dots,x_k) = \varphi(t,\dots,q_i,p_i',\dots,q_j,p_j,\dots,x_k)$$
$$= \varphi(t,\dots,q_i,p_i',\dots,q_j,p_j,\dots,x_k),$$

where p_i' is obtained from (2.1) with p_i and p_j replacing p_1 and p_2.

Note that any function φ which vanishes in a neighborhood of the collision set $\pi_k(\Lambda) - \pi_k^\delta(\Lambda)$ for all t satisfies condition 2.1(ii), and hence Φ_k is dense in $L^1(\pi_k^\delta(\Lambda))$.

Any function $\varphi \in \Phi_k$ can be regarded as a function on $\pi_n^\delta(\Lambda)$ which is independent of the variables x_m, $k+1 \leq m \leq n$. In the proof of the following theorem we use the fact that for $\varphi \in \Phi_k$,

$$(2.10) \qquad \int\varphi(t,x^k)f(t, x^n)dx_1\dots dx_n = \int\varphi(t,T^t(x^n))f(x^n)dx_1\dots dx_n.$$

Note that on the right-hand side we regard x^k as being a point in $\pi_n^\delta(\Lambda)$ by the obvious embedding, and the map T^t, though it acts only on x^k, involves the effects of all n particles. In the following, we denote $T^t(x)$ by $x(t) = (x_1(t),\ldots,x_n(t))$, where $x_i(t) = (q_i(t), p_i(t))$. Also we will denote $\varphi(t,T^t(x_n))$ by $\varphi(t, x_k(t))$, remembering that $x_k(t)$ depends on $x_1,\ldots,x_n$, as well as t.

To prove the theorem, we must assume that f is continuous along the trajectories defined by the T^t, i.e.,

$$(2.11) \qquad \lim_{t\to\tau} f(t,x^n) = f(\tau,x^n).$$

This assumption is used in the following way. Suppose that D is a compact subset of K such that $D \times [0,\tau] \subseteq \tilde{\Xi}$ for some $\tau \in \mathbb{R}^+$; we want to consider integrals of the function $\tilde{g} = g\circ\Phi^{-1}$ on $D \times [0,\tau]$, where g satisfies condition (2.9) and is integrable on $D \times [0,\tau]$. Since no collisions have taken place in the time interval $[0,\tau]$, i.e., $S^{-t}(y) \in K^+$ for $y \in D$ and $t \in [0,\tau]$, we see that the function $\tilde{g}(s.y) \in L^1([o,\tau] \times D)$ is continuous as a function of $s \in [0,\tau]$. By the Fubini theorem, $\tilde{g}(s,\cdot)$ is in $L^1(D)$ for a.e. $s \in [0,\tau]$, and $\int\tilde{g}(s,y)d\sigma(y)$ is continuous as a function of s, so we conclude that $\int\tilde{g}(0,y)d\sigma(y)$ is finite, and that

$$(2.12) \qquad \lim_{s\to 0} \int \tilde{g}(s,y)d\sigma(y) = \int \tilde{g}(0,y)d\sigma(y).$$

We are now ready to state the main result of this paper.

2.2 Theorem

Let $\tau \in \mathbb{R}^+$. For any $\varphi \in \Phi_k$ and $1 \le k < n$, we have

$$(2.13) \quad \rho_k(t)[\varphi(t)] = \rho_k(0)[\varphi(0)] + \int_0^\tau \{\rho_k(t)[H_k\varphi(t)] + C_{k+1}\rho_{k+1}(t)[\varphi(t)]\}\, dt,$$

where

$$(2.14) \quad H_k\varphi = \frac{\partial\varphi}{\partial t} + \sum_{i=1}^{k} p_i \frac{\partial\varphi}{\partial q_i},$$

$$(2.15) \quad C_{k+1}\rho_{k+1}[\varphi] = \delta^2 \sum_{i=1}^{s} \int_A \varphi(x^k) \{ \rho_{k+1}(t,\ldots,q_i,p_i{}',\ldots,q_i-d\omega,p) - \rho_{k+1}(t,x^k,q_i+d\omega,p)\}\, \omega\cdot(p_i-p)\, dx^k\, d\omega dp,$$

and $A = \{ (x^k,\omega,p) : \omega \in S_2,\ \omega\cdot(p_i-p) \ge 0\}$. The operator C_{k+1} is called the collision integral.

§3 THE PROOF:

To begin, we assume that the function f is supported in a compact set of the form $\Sigma_s = \Lambda \times S$, where $S = \{ p = (p_1,\ldots,p_n) : \sum p_i^2\ (1 \le i \le n) \le s$, s a positive real number $\}$; the proof will be completed by a limiting argument. Note that $\Sigma_s \cap \pi_n^\delta(\Lambda)$ is invariant under T^t since $\sum p_i^2$ is conserved under collisions..

The proof will use Keisler's Infinite Sum Theorem [7, 9], which in our case can be stated as follows: In order to show that (2.13) is true, we need only show that

$$(3.1) \quad d\rho_k(t)[\varphi(t)] = \rho_k(t+dt)[\varphi(t+dt)] - \rho_k(t)[\varphi(t)] \simeq dt\, \{\rho_k(t)[H_k\varphi(t)] + C_{k+1}\rho_{k+1}(t)[\varphi(t)]\}$$

for $t \ge 0$ in ${}^*\mathbb{R}$, where dt is a positive infinitesimal increment in time and $\simeq$

denotes the relation of being infinitesimally close [here and in the following we make no notational distinction between standard functions and their $*$ - transforms in the nonstandard model; the context will make clear what is meant].

Let $t \in {}^*\mathbb{R}^+$, and dt be an infinitesimal, positive increment in time. Using (2.11), we have

$$(3.2)\qquad \frac{(n-k)!}{n!}\, d\rho_k(t)[\varphi(t)] = \int[\varphi(t+dt,x^k(t+dt)) - \varphi(t,x^k(t))]f(x^n)dx^n.$$
$$= \int[\varphi(t+dt,x^n(t+dt)) - \varphi(t,x^n(t+dt))]f(x^n)dx^n$$
$$+\sum_{i=1}^{k} \int\{\varphi(t,x_1(t),\ldots,x_{i-1}(t),q_i(t+dt),p_i(t+dt),\ldots,x_k(t+dt))$$
$$-\varphi(t,x_1(t),\ldots,x_{i-1}(t),q_i(t),p_i(t+dt),\ldots,x_k(t+dt))\}f(x^n)dx^n$$
$$+\sum_{i=1}^{s} \int\{\varphi(t,x_1(t),\ldots,x_{i-1}(t),q_i(t),p_i(t+dt),\ldots,x_k(t+dt))$$
$$-\varphi(t,x_1(t),\ldots,x_{i-1}(t),q_i(t),p_i(t),\ldots,x_k(t+dt))\}f(x^n)dx^n.$$

Using the mean value theorem and the fact that $\varphi \in C^1$, we see that the first integral is infinitesimally close to

$$(3.3)\qquad dt \int \frac{\partial\varphi}{\partial t}(t,x^k(t))f(x^n)dx^n = dt \int \frac{\partial\varphi}{\partial t}(t,x^k)\; f(t,x^n)dx^n = dt\; \rho_k(t)[\frac{\partial\varphi}{\partial t}].$$

Next we consider the terms in the two sums in (3.2). We write each integral in the sums as the sum of integrals, depending on whether the i^{th} particle does or does not undergo a collision with the j^{th} particle in the time interval [t, t+dt]. More explicitly, we let $A_{ij}(t) = T^{-t}(A_{ij})$, where

$$(3.4)\qquad A_{ij} = \{\, x = (x_1,\ldots,x_n) \in \Sigma \cap \pi_n^{\delta}(\Lambda) \;:\; T^s(x) \in K_{ij},\ 0 \le s < dt\}.$$

$A_{ij}(t)$ is thus the set of initial conditions $x = (x_1,\ldots,x_n) \in \Sigma \cap \pi_n^{\delta}(\Lambda)$ for which there is a collision between the i^{th} and j^{th} particles in the time interval $[t, t+dt)$. It is important to note that the momenta p_i in $x_i = (q_i,p_i)$, $1 \le i \le n$ are all finite since $x \in \Sigma$, and the same is true of the momenta in $T^{-t}(x_1,\ldots,x_n)$. From this fact it follows that there is only one collision in the infinitesimal time interval $[t, t+dt)$. Because of the fact that there are no multiple collisions, it is easy to see that the A_{ij} are disjoint. We now let $A_i(t) = \bigcup A_{ij}(t)(i \le j \le n)$. It is important to remark at this point that we do not need to consider collisions with the boundary since φ has compact support.

On $A_i'(t) = [A_i(t)]'$ there are no collisions involving the i^{th} particle in $[t, t+dt)$. Thus we have $q_i(t+dt) - q_i(t) = p_i(t)dt$ and $p_i(t+dt) = p_i(t)$, and using the mean value theorem and the fact that $\varphi \in C^1$, we get

$$(3.5) \qquad \int_{A_i'(t)} [\varphi(\ldots,q_i(t+dt),p_i(t+dt),\ldots) - \\ -\varphi(\ldots,q_i(t),p_i(t+dt),\ldots)]\, f(x^n)\, dx^n \\ \simeq dt \int_{A_i'(t)} p_i \frac{\partial\varphi}{\partial q_i}(x^k(t))\, f(x^n)\, dx^n$$

On the other hand, since the dynamics is continuous in q_i and φ is continuous, $|\varphi(\ldots,q_i(t+dt),p_i(t+dt),\ldots) - \varphi(\ldots,q_i(t),p_i(t+dt),\ldots)| \le \eta$ where η is infinitesimal, and so

$$(3.6) \qquad \left| \int_{A_i(t)} [\varphi(\ldots,q_i(t+dt),P_i(t+dt),\ldots) - \\ - \varphi(\ldots,q_i(t),p_i(t+dt),\ldots)]\, f(x^n)\, dx^n \right|$$

$$\leq \eta \sum_{j=1}^{n} \int_{A_{ij}(t)} f(x^n)\, dx^n$$
$$= \eta \sum_{j=1}^{n} \int_{A_{ij}} f(t,x^n)\, dx^n.$$

Now note that

(3.7) $$\int_{A_{ij}} f^t(x^n)\, dx^n = \int_o^{dt} \int_{K_{ij}} \tilde{f}(S^{-t+s}(s,y)\, d\sigma(y)\, ds$$
$$= \int_o^{dt} \int_{K_{ij}} \tilde{f}^{t-s}(y,s)\, d\sigma(y) ds$$

where here and later we use the notation $\tilde{f}(t) = f(t)\circ\Phi^{-1}$. Using (2.12) and the mean-value theorem of the integral calculus, we see that the right-hand side of (3.7) is infinitesimally close to $\eta B dt$ for some positive constant B, and as a result the first sum in (3.2) is infinitesimally close to

(3.8) $$dt \sum_{i=1}^{k} \int p_i \frac{\partial\varphi}{\partial q_i}(t,x^k)\; f(t,x^n)\, dx^n = dt \sum_{i=1}^{k} \int p_i \frac{\partial\varphi}{\partial q_i}\, \rho_k(t)\, dx^k.$$

We now consider a typical term in the second sum in (3.2). First we note that $p_i(t+dt) = p_i(t)$ in $A_i'(t)$, so that

(3.9) $$\int_{A_i'(t)} [\varphi(\ldots,q_i(t),p_i(t+dt),\ldots) -$$
$$- \varphi(\ldots,q_i(t),p_i(t),\ldots)]\; f(x^n)\, dx^n = 0$$

Now in $A_{ij}(t)$, the i^{th} particle collides with the j^{th} particle in the time interval [t, t+dt). The (outgoing) velocity $p_i(t+dt)$ of the i^{th} particle after such a collision is given by

$$(3.10)\qquad p_i(t) - \left[\omega_{ij}(t+\alpha dt)\cdot[p_i(t) - p_j(t)]\right]\omega_{ij}(t+\alpha dt)$$
$$\simeq p_i(t) - \left[\omega_{ij}(t)\cdot[p_i(t) - p_j(t)\right]\omega_{ij}(t),$$
$$= p'_{ij}(t)$$

where $0 < \alpha < 1$ and $\omega_{ij}(t) = [q_j(t) - q_i(t)]/|q_j(t) - q_i(t)|$. Thus, using (2.1)(ii), we have

$$(3.11)\qquad \int \{\varphi(\ldots,q_i(t),p_i(t+dt),\ldots) -$$
$$-\varphi(\ldots,q_i(t),p_i(t),\ldots)\}\ f(x^n)\ dx^n$$
$$\simeq \sum_{j=k+1}^{n} \int_{A_{ij}(t)} \{\varphi.\ldots,q_i(t),p'_{ij}(t),\ldots) -$$
$$-\varphi(\ldots,x_i(t),\ldots)\}\ f(x^n)\ dx^n$$
$$\simeq \sum_{j=k+1}^{n} \int_{A_{ij}} \{\varphi(\ldots,q_i,p'_{ij},\ldots) -$$
$$-\varphi(\ldots,x_i,\ldots)\}\ f(t,x^n)dx^n$$
$$\simeq (n-k)\int_{A_{i,k+1}} \{\varphi(t,\ldots,q_i,p'_i,\ldots) - \varphi(t,x^k)\}\ f(t,x^n)\ dx^n,$$

the last by symmetry, where $p'_i = p'_{i,k+1}$. The last integral in (3.11) can again be treated as an integral over $\tilde{\Xi}$. Using the definition of the surface measure $d\sigma_{ij}$ and the continuity as before, we have

$$(3.12)\quad (n-k)\int_{A_{i,k+1}} \varphi(t,x^k)f(t,x^n)dx^n = (n-k)\int_{o}^{dt}\int_{K_{i,k+1}} \tilde{\varphi}(t)(\tau,y)\ \tilde{f}(t)(\tau,y)d\sigma_{i,k+1}d\tau$$
$$= (n-k)\ dt\ \delta^2 \int \omega\cdot(p_i-p)\ \varphi(t,x^k)\ f(t,x^n)\ dx^k\ dx_{k+2}\ldots dx_n d\omega\ dp$$
$$= dt\ \delta^2\ \frac{(n-k)!}{n!}\int_A \omega\cdot(p_i-p)\ \varphi(t,x^k)\ \rho_{k+1}(t,x^k,q_i+ d\omega,p)dx^k\ d\omega\ dp,$$

where the integration is over the set $A = \{(x^k, \omega, p) : \omega \in S_2, \omega\cdot(p_i - p) \geq \emptyset\}$. To deal with the integral $\int \varphi(t,\ldots,q_i,p_i',\ldots)\ f(t,x^n)\ dx^n$ over $A_{i,k+1}$ we first use the fact that $\omega\cdot(p_i-p) = -\omega\cdot(p_i'-p')$, and the above argument to obtain

$$(3.13)\qquad (n-k)\int \varphi(t,\ldots,q_i,p_i',\ldots)f(t,x^n)\ dx^n$$
$$\simeq -\,dt\ \delta^2\ \frac{(n-k)!}{n!}\int_A \omega\cdot(p_i'-p')\varphi(t,\ldots,q_i,p_i',\ldots)\rho_{k+1}(t,x^k,q_i+d\omega,p)dx^k\ d\omega dp.$$

Since the transformation (2.1) is orthogonal, and hence $dp_i dp = dp_i'\ dp'$, we may replace ω by $-\omega$, and we see that the last line in (3.13) is

$$(3.14)\qquad \simeq dt\ \delta^2\ \frac{(n-k)!}{n!}\int_A \omega\cdot(p_i-p)\ \varphi(x^k)\ \rho_{k+1}(t,\ldots,q_i,p_i',\ldots,q_i-d\omega,p)\ dx^k\ d\omega dp.$$

Putting all of the above together with Keisler's Infinite Sum Theorem, we obtain the desired result for functions of compact support.

To prove the result for general $f \in L^1(\pi_n^\delta(\Lambda))$ is now simply a matter of approximating f in L^1 by $f_s = f\chi_{\Sigma_s}$. The one point to notice in this connection is that the collision integral, when integrated over t, correspoonds to a full volume integral on $\tilde{\Xi}$, as we have seen from the calculations above.

9 REFERENCES

[1] Alexander, R. K.: The infinite hard sphere system, Ph. D. Thesis, Dept of

Mathematics, Univ. of California, Berkeley, 1975.

[2] Ambrose, W.: Representation of ergodic flows, Annals of Math., 42(1941), 723-739.

[3] Ambrose, W. and Kakutani, S.: Structure and continuity of measurable flows, Duke math. J. 9(1942), 25-42.

[4] Cercignani, C.: Mathematical Methods in Kinetic Theory, Plenum Press, New York, 1969.

[5] Cercignani, C.: On the Boltzmann equation for rigid spheres, Transport Theory and Stat. Phys. 2(1972), 211-225.

[6] Grad, H.: Principles of the kinetic theory of gases, in Handbuch der Physik, Vol. 12, p. 205, Springer, Berlin-gottingen-Heidelberg, 1958

[7] Hurd, A. E., and Loeb, P. A.: An Introduction to Nonstandard Real Analysis, Academic Press, New York, 1985.

[8] Illner, R., and Pulvirenti, M.: A derivation of the BBGKY hierarchy for hard sphere particle systems, preprint.

[9] Keisler, H. Jerome: Foundations of Infinitesimal Calculus, Prindle, Weber and Schmidt, Boston, 1976.

[10] Lanford, O. E.: Time evolution of large classical systems, Lecture Notes in Physics 38,1-111, Springer-Verlag, 1975.

[11] Lanford, O. E.: On a derivation of the Boltzmann equation, Asterisque 40 (1976), 117-139.

[12] Marchioro, C., Pellegrinotto, E., Presutti, M., and Pulvirenti, M.: On the dynamics of particles in a bounded region: a measure theoretical approach, J. Math. Phys. 17(1976) 647-652.

[13] Spohn, H.: On the integrated form of the BBGKY hierarchy for hard spheres, preprint.