ON THE CONSTRUCTION OF INDEPENDENCE COUNTEREXAMPLES

by

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DMS-632-IR **July 1993**

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A simple method of modifying probability to create independence counterexamples is given. This method can be used to construct counterexamples for finite collections of events and for infinite sequences of events. It is shown how to construct examples of countable collections of events where all equations in the (countable) family of "product rules" that define independence of the events are satisfied except for one, or, more generally, except for an arbitrarily specified subfamily.

KEY WORDS: Counterexamples; Dependence; Independence; Random events; Teaching of probability; Teaching of mathematical statistics.

1. INTRODUCTION

In probability and statistics, as well as all areas of mathematics, counterexamples are essential tools for understanding. Romano and Siegel (1986), and Stoyanov (1987), have

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assembled extensive collections of counterexamples in probability and statistics. The purpose of this note is to present a method of modifying probability structure that can be used to construct a variety of independence counterexamples for both finite and countably infinite collections of events.

Events $E_1, E_2, \ldots, E_n$ are said to be independent provided

$$P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) = P\left(E_{i_1}\right) P\left(E_{i_2}\right) \cdots P\left(E_{i_k}\right)$$

for every subcollection $\{E_{i_1}, E_{i_2}, \cdots, E_{i_k}\}$ of two or more E-events. Since there are $2^n - n - 1$ "product rules" in this definition, the task of verifying independence can be laborious when n is large. It is natural for students to wonder whether or not checking an appropriate subset of these equations would suffice. For example, perhaps pairwise independence would imply independence. Some version of S. Bernstein's classic counterexample showing that pairwise independence of three events does not imply independence is frequently given in textbooks (see, for example, Ash (1970), Feller (1968), Larsen and Marx (1985), Parzen (1960), and Ross (1988)). Crow (1967) has contributed an example showing that triplewise independence does not imply pairwise independence. Examples of n dependent events such that any $n - 1$ of them are independent have been given by Wong (1972). Krewski and Bickis (1984) show that there are no nontrivial finite collections of independent events that are exhaustive. Counterexamples covering all of the above mentioned aspects of independence plus several more are included in the collections of Romano and Siegel (1986), and Stoyanov (1987). That all $2^n - n - 1$ product rules are required for the definition of independence of n events is confirmed by a counterexample in Romano and Siegel (1986), credited to Richard M. Dudley, where all $2^n - n - 1$ product rules are satisfied except for one arbitrarily specified equation. Wang *et al.* (1993) have contributed a similar example.

Our method of modifying probability structure can be used to create examples where, for n events, all $2^n - n - 1$ product rules are satisfied except for an arbitrarily specified subcollection of product rules. Moreover, it can be extended to handle the case of an infinite sequence of events. That is, we will give a simple technique for constructing examples where the events $\{E_i\}_{i=1}^{\infty}$ satisfy all product rules of the form

$$P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) = P\left(E_{i_1}\right) P\left(E_{i_2}\right) \cdots P\left(E_{i_k}\right),$$

except for an arbitrarily specified subcollection of these product rules.

2. PROBABILITY MODIFICATION METHOD

Consider a sequence of k events $A_1, A_2, \cdots, A_k$; and let $\mathcal{A}$ denote the family of the 2^{k-1} events having the form $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}$, where $\{i_1, i_2, \cdots, i_r\}$ is a nonvoid subset of $\{1, 2, \cdots, k\}$. For a given event B let $\mathcal{P}$ denote the partition of B determined by

$A_1, A_2, \cdots, A_k$. That is, $\mathcal{P}$ is the family of the 2^k events of the form $A_1^{\delta_1} \cap A_2^{\delta_2} \cap \cdots \cap A_k^{\delta_k} \cap B$, where each $A_j^{\delta_j}$ is either A_j or A_j^c. Assume that every member of $\mathcal{P}$ has positive probability under the original probability structure. We will give a method for modifying the probability structure so that the probability of the event $A_1 \cap A_2 \cap \cdots \cap A_k$ is changed to a different positive value while not changing the probability of any other element of $\mathcal{A}$ or any event contained in B^c. Then we will show how this technique can be used to construct a variety of independence counterexamples.

We begin the probability modification procedure by fixing positive constant ε smaller than

$$\min\left\{P\left(A_1^{\delta_1} \cap \cdots \cap A_k^{\delta_k} \cap B\right):\ A_1^{\delta_1} \cap \cdots \cap A_k^{\delta_k} \cap B \in \mathcal{P}\right\}.$$

Now we alter the probability of each element in $\mathcal{P}$ according to the following scheme. The probability of $A_1^{\delta_1} \cap \cdots \cap A_k^{\delta_k} \cap B$ is changed by the amount $(-1)^j \varepsilon$ where j is the number of complemented A-events in its expression.

These modifications give a valid assignment of probability because all of the altered probabilities remain positive and the net change in the sum of probabilities over the (mutually exclusive) events that constitute $\mathcal{P}$ is

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j \varepsilon = (-1 + 1)^k \varepsilon = 0.$$

Obviously, no probabilities of events contained in B^c have been changed. Also, when we examine the modified probability of an arbitrary element $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}$ in $\mathcal{A}$ by expressing it as the sum of 2^{k-r} $\mathcal{P}$-event probabilities plus $P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r} \cap B^c\right)$, we find the probability has been changed by

$$\sum_{j=0}^{k-r} \binom{k-r}{j} (-1)^j \varepsilon = \begin{cases} 0 & \text{if } r < k, \\ \varepsilon & \text{if } r = k. \end{cases}$$

Thus, the probability of $A_1 \cap A_2 \cap \cdots \cap A_k$ has been increased by ε without changing any other $\mathcal{A}$-event probabilities.

3. CONSTRUCTION OF INDEPENDENCE COUNTEREXAMPLES FOR n EVENTS

To show that the definition of independence of n events involving $2^n - n - 1$ product rules cannot be simplified, we use our modification method to construct appropriate counterexamples. Starting with n independent events $E_1, E_2, \cdots, E_n$ such that $0 < P(E_i) < 1$ for $i = 1, 2, \cdots, n,$ we can modify the probability structure so that exactly one arbitrarily specified product rule fails. Suppose, for example, the equation to be invalidated is

$$P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) = P\left(E_{i_1}\right) P\left(E_{i_2}\right) \cdots P\left(E_{i_k}\right),$$

where $2 \le k \le n$ and $\left(i_1, \cdots, i_k, i_{k+1}, \cdots, i_n\right)$ is a permutation of $(1, 2, \cdots, n)$. This can be accomplished by setting

$$A_1 = E_{i_1}, \ A_2 = E_{i_2}, \cdots, A_k = E_{i_k}, \ B = E^c_{i_{k+1}} \cap E^c_{i_{k+2}} \cap \cdots \cap E^c_{i_n},$$

and applying the probability modification technique described in the previous section. Observing that every intersection of E-events that is not a member of $\mathcal{A}$ is contained in B^c, it is easy to see that the desired probability structure is obtained.

Note that the modified probability structure retains the necessary conditions to permit another application of the modification technique. Hence, through repeated modifications it is possible to construct independence counterexamples where any particular subcollection of product rules fail while the others remain valid.

4. CONSTRUCTION OF INDEPENDENCE COUNTEREXAMPLES FOR INFINITE SEQUENCES OF EVENTS

An infinite sequence of events is defined to be independent if every finite subcollection of these events is independent; hence, a countable infinity of product rules is required. Again the modification procedure can be used to construct counterexamples where a specific product rule or set of product rules fails while all other product rules are satisfied.

We begin with a sequence of independent events $\{E_i\}_{i=1}^{\infty}$ such that $0 < P(E_i) < 1$ for all i and $\sum_{i=1}^{\infty} P(E_i) < \infty$. For each n, let $\mathcal{C}_n$ denote the collection of 2^n mutually exclusive events of the form

$$E_1^{\delta_1} \cap E_2^{\delta_2} \cap \cdots E_n^{\delta_n} \cap E_{n+1}^c \cap E_{n+2}^c \cap \cdots,$$

where each $E_i^{\delta_i}$ is either E_i or E_i^c. Then $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a countable collection of events.

By the Borel-Cantelli lemma we know that

$$P(\text{infinitely many } E\text{-events occur}) = P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n\right) = 0.$$

Therefore,

$$1 = P(\text{only finitely many } E\text{-events occur}) = P\left(\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} E_n^c\right) = \lim_{m\to\infty} P\left(\bigcap_{n=m}^{\infty} E_n^c\right),$$

the last equality is by the continuity property of probability. Hence, each event in $\mathcal{C}$ has positive probability. Now suppose the product rule to be invalidated is

$$P\left(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}\right) = P\left(E_{i_1}\right) P\left(E_{i_2}\right) \cdots P\left(E_{i_k}\right),$$

where $k \geq 2$ and $i_1, i_2, \cdots, i_k$ are distinct positive integers. Letting K denote the set of all positive integers other than $i_1, i_2, \cdots, i_k$, the desired result is obtained by setting

$$A_1 = E_{i_1}, \;\; A_2 = E_{i_2}, \cdots, A_k = E_{i_k}, \;\; B = \bigcap_{i \in K} E_i^c,$$

and applying the probability modification method described in section 2. Since every finite intersection of E-events that is not a member of $\mathcal{A}$ is contained in B^c, a counterexample having the desired properties is obtained. Also, this modified probability structure satisfies the necessary conditions to allow another iteration of the modification technique; and through repeated iterations we can construct independence counterexamples where any particular finite subcollection of product rules fail while all others remain valid.

To invalidate a specified countable set of product rules, we pass to the limit with the epsilons chosen so that the sum of the absolute values of all of the individual changes is absolutely convergent. Then we are guaranteed that, in the limit, all of the $\mathcal{C}$-event probabilities remain well-defined, and the sum of probabilities over all (disjoint) members of $\mathcal{C}$ remains 1. Furthermore, each of the product rules that we changed remains changed in the limit and all others remain unchanged. For example, it is possible to construct a probability structure where only the k-fold product rules fail while all others hold.

5. EXAMPLE

We conclude this note by applying the modification procedure to construct an example where independence of E_1, E_2, E_3, E_4 fails only because $P(E_2 \cap E_4) \neq P(E_2)P(E_4)$. Beginning with E_1, E_2, E_3, E_4 independent such that $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{2}$,

we set $A_1 = E_2$, $A_2 = E_4$, $B = E_1^c \cap E_3^c$, and apply the probability modification procedure of section 2 with ε smaller than $\frac{1}{16}$. Then

$$P\left(E_1^c \cap E_2 \cap E_3^c \cap E_4\right) = P\left(A_1 \cap A_2 \cap B\right) = \frac{1}{16} + (-1)^0 \varepsilon = \frac{1}{16} + \varepsilon,$$

$$P\left(E_1^c \cap E_2^c \cap E_3^c \cap E_4\right) = P\left(A_1^c \cap A_2 \cap B\right) = \frac{1}{16} + (-1)^1 \varepsilon = \frac{1}{16} - \varepsilon,$$

$$P\left(E_1^c \cap E_2 \cap E_3^c \cap E_4^c\right) = P\left(A_1 \cap A_2^c \cap B\right) = \frac{1}{16} + (-1)^1 \varepsilon = \frac{1}{16} - \varepsilon,$$

$$P\left(E_1^c \cap E_2^c \cap E_3^c \cap E_4^c\right) = P\left(A_1^c \cap A_2^c \cap B\right) = \frac{1}{16} + (-1)^2 \varepsilon = \frac{1}{16} + \varepsilon,$$

and each of the remaining 12 events of the form $E_1^{\delta_1} \cap E_2^{\delta_2} \cap E_3^{\delta_3} \cap E_4^{\delta_4}$, being contained in B^c, has its probability unchanged at $\frac{1}{16}$. The modified probability structure yields

$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{2}$, and all $2^4 - 4 - 1$ product rules are satisfied, except

$$P(E_2 \cap E_4) = \frac{1}{4} + \varepsilon \neq P(E_2)P(E_4).$$

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