LOCAL DERIVATIONS AND LOCAL AUTOMORPHISMS OF $\mathscr{B}(X)$

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Introduction. If $\mathscr{A}$ is a Banach algebra, we say that a linear transformation $\varphi : \mathscr{A} \longrightarrow \mathscr{A}$ is a *local derivation* (respectively, *local automorphism*) if for every $A \in \mathscr{A}$, there is a derivation (respectively, automorphism) $\theta_A : \mathscr{A} \longrightarrow \mathscr{A}$, depending on A, such that $\varphi(A) = \theta_A(A)$.

The purpose of this paper is to show that if $\mathscr{A} = \mathscr{B}(X)$, the algebra of all bounded linear operators on a Banach space X, then every local derivation of $\mathscr{A}$ is a derivation, and, provided that X is infinite–dimensional, every invertible local automorphism is an automorphism. Thus these types of linear transformations on $\mathscr{B}(X)$ are completely determined by their local actions.

The results stated above can be related to the concept of algebraic reflexivity of linear spaces of operators. This notion has appeared in different contexts [3,7,9]. We will adopt the terminology and notation of [7]. The concept extends naturally to include sets of operators which are not linear spaces. To describe it, we start by establishing some notation. Let V be a Banach space over the complex numbers. (In this paper, V will be the Banach space $\mathscr{B}(X)$.) Let $\mathscr{L}(V)$ denote the algebra of all linear transformations from V into itself and let $\mathscr{B}(V)$ denote the algebra of all bounded operators. If $\mathscr{S}$ is a subset of $\mathscr{B}(V)$, we write $\mathrm{ref}_a(\mathscr{S}) = \{T \in \mathscr{L}(V) \colon Tv \in \mathscr{S}v,\ v \in V\}$, where $\mathscr{S}v = \{Sv : S \in \mathscr{S}\}$. If $T \in \mathrm{ref}_a(\mathscr{S})$, then T is *locally in* $\mathscr{S}$ in the sense that for each $v \in V$, there exists a linear transformation $S_v \in \mathscr{S}$ such that $Tv = S_v v$. The set $\mathscr{S}$ is said to be *algebraically reflexive* if $\mathscr{S} = \mathrm{ref}_a(\mathscr{S})$.

Similarly, replacing $\mathscr{L}(V)$ by GL(V), the group of invertible elements of $\mathscr{L}(V)$ and replacing $\mathscr{B}(V)$ by the group of invertible operators $\mathscr{B}^{-1}(V)$, we may define a similar notion of algebraic reflexivity for subsets (especially subgroups) of $\mathscr{B}^{-1}(V)$ in GL(V).

The results of this paper imply that the space of all derivations $\mathrm{Der}(\mathscr{B}(X))$ is algebraically reflexive in $\mathscr{L}(\mathscr{B}(X))$ and that, for an infinite dimensional Banach space X, the group of automorphisms $\mathrm{Aut}(\mathscr{B}(X))$ is algebraically reflexive in $\mathrm{GL}(\mathscr{B}(X))$.

Since submitting this article, we have learned of the recent interesting article [11] by Richard Kadison in which the term "local derivation" is defined and used in precisely the same fashion as in our work. This suggests that the term is a natural one. There is some overlap between Kadison's results and ours, namely the case of continuous local derivations on $\mathscr{B}(H)$ for a Hilbert space H. He proves, among other things, that every continuous local derivation on a von Neumann algebra is a derivation. We thank Eric Christensen and Victor Kaftal for making us aware of Kadison's work.

We wish to thank John Phillips for bringing Johnson's paper [5] to our attention, and Steve Wright for organizing the Special Session on Derivations in which these results were presented.

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1. Derivations. A *derivation* on an algebra $\mathscr{A}$ is a linear map δ on $\mathscr{A}$ such that $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x,y \in \mathscr{A}$. The derivation δ is called *inner* if there exists an element $a \in \mathscr{A}$ such that $\delta(x) = ax - xa$ for all $x \in \mathscr{A}$. We start by giving a short proof of the well–known theorem which asserts that every derivation of $\mathscr{B}(X)$ is inner (see [5] and [6]).

First, we fix some notation. Given $x \in X$ and $f \in X^*$, we use the notation $x \otimes f$ for the operators $y \mapsto f(y)x$, $y \in X$. Every rank one operator in $\mathscr{B}(X)$ can, of course, be written in this fashion. We note that the adjoint of $x \otimes f$ is the operator $f \otimes x \in \mathscr{B}(X^*)$, where in the latter expression x is viewed as an element of X^{**}. The duality between X and X^* will be denoted by $\langle \cdot,\cdot \rangle$.

THEOREM 1.1. *Every derivation on* $\mathscr{B}(X)$ *is inner.*

PROOF. Let δ be a derivation on $\mathscr{B}(X)$. Let $y_0 \in X$, $f_0 \in X^*$ be chosen so that $f_0(y_0) = 1$. Define a map $A : X \longrightarrow X$ by

$$Ax = \delta(x \otimes f_0)y_0.$$

Then A is linear. Since $T \cdot (x \otimes f_0) = Tx \otimes f_0$, we have

$$\delta(Tx \otimes f_0) = T \cdot \delta(x \otimes f_0) + \delta(T) \cdot x \otimes f_0.$$

Applying both operators in this equation to y_0, we get $\delta(T)x = ATx - TAx$. This is true for every x, so $\delta(T) = AT - TA$.

We now prove that A is bounded (without assuming that δ is bounded). For every bounded operator T, the transformation $AT - TA$ is also bounded. On the other hand for every (bounded) rank one operator $R = u \otimes f$, the transformation AR is bounded, since $AR = Au \otimes f$. Therefore we must have that RA is bounded. We now apply the closed graph theorem to conclude that A is bounded. If $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$, then the boundedness of R and RA implies that $RAx_n \longrightarrow Ry$ and $RAx_n \longrightarrow RAx$. Thus $Ry = RAx$, *i.e.* $f(y)u = f(Ax)u$. This is true for every $u \in X$ and $f \in X^*$, so $y = Ax$. This shows that A has a closed graph and hence is bounded. ∎

We now prove the main theorem of this section.

THEOREM 1.2. *Let X be a Banach space and let φ be a linear transformation from $\mathscr{B}(X)$ into $\mathscr{B}(X)$ with the property that for each $T \in \mathscr{B}(X)$, there exist operators A_T and $B_T \in \mathscr{B}(X)$ such that $\varphi(T) = TA_T + B_TT$. Then there exist operators A and B in $\mathscr{B}(X)$, independent of the operator variable T, such that $\varphi(T) = TA + BT$ for every $T \in \mathscr{B}(X)$. Moreover, if $A_T = -B_T$ for every T, then $A = -B$ and so φ is a* *derivation.*

For clarity of exposition, we accomplish the proof in stages each of which is a refinement of its predecessor.

LEMMA 1. *For each* $f \in X^*$, *there is a functional* $g_f \in X^*$ *and a linear transformation* $B_f : X \longrightarrow X$ *such that*

$$(*) \qquad \varphi(x \otimes f) = x \otimes g_f + B_f x \otimes f \quad \textit{for every} \quad x \in X.$$

PROOF. Fix $f \in X^*$. If $f = 0$, we may take $g_f = 0$ and $B_f = 0$. So assume $f \neq 0$. Let x_0 be a nonzero vector in X. By the hypothesis on φ, there exist $g_0 \in X^*$ and $z_0 \in X$ such that

$$\varphi(x_0 \otimes f) = x_0 \otimes g_0 + z_0 \otimes f.$$

Now let $x \in X$ be arbitrary. We will show that there exists $z \in X$, (depending on x), such that

$$\varphi(x \otimes f) = x \otimes g_0 + z \otimes f.$$

This is obvious if $x = \lambda x_0$ for some scalar λ, so we assume that $\{x, x_0\}$ are linearly independent. Then, by the hypothesis on φ, there exist $g_1, g_2 \in X^*$ and $z_1, z_2 \in X$ such that

$$\varphi(x \otimes f) = x \otimes g_1 + z_1 \otimes f$$

and

(1) $$\varphi((x_0+x) \otimes f) = (x_0+x) \otimes g_2 + z_2 \otimes f.$$

But by linearity, also

(2) $$\varphi((x_0+x) \otimes f) = \varphi(x_0 \otimes f) + \varphi(x \otimes f)$$

$$= x_0 \otimes g_0 + x \otimes g_1 + (z_0+z_1) \otimes f.$$

Now let y be an arbitrary vector in X for which $f(y) = 0$, that is $y \in (f)_\perp$, the preannihilator of f. Equating (1) and (2), and applying each to the vector y, yields

$$g_2(y) \cdot (x_0+x) = g_0(y) \cdot x_0 + g_1(y) \cdot x,$$

and so

$$g_2(y) - g_0(y) = 0 \quad \text{and} \quad g_2(y) - g_1(y) = 0.$$

Thus $(g_1-g_0)(y) = 0$. Since y was an arbitrary member of $(f)_\perp$, it follows that $g_1 - g_0 = \lambda f$ for some scalar λ. Therefore equation (1) gives

$$\varphi(x \otimes f) = x \otimes (g_0+\lambda f) + z_1 \otimes f = x \otimes g_0 + z \otimes f,$$

where $z = \lambda x + z_1$, as desired.

We have established that there is a function $F : X \to X$, (depending on f), such that for each $x \in X$,

$$\varphi(x \otimes f) = x \otimes g_0 + F(x) \otimes f.$$

Let u be a vector in X for which $f(u) = 1$. Then $F(x) = [\varphi(x \otimes f) - x \otimes g_0](u)$. For fixed u, f, g_0 the right hand side of this equation is linear in the variable x. Thus $x \mapsto F(x)$ is a linear transformation. This establishes the lemma with $B_f = F$ and $g_f = g_0$. ∎

LEMMA 2. *If* f_1 *and* f_2 *are linearly independent elements of* X^* *and if* g_i, B_i, $i = 1,2$ *are chosen to satisfy the conclusion of Lemma* 1, *for* f_1 *and* f_2 *respectively, then* $B_1 - B_2$ *is a scalar multiple of the identity.*

PROOF. Clearly it suffices to show that for each $x \in X$, $(B_1 - B_2)x$ is a scalar multiple of x. Toward this end, fix $x \in X$. By Lemma 1, there exist $g_3 \in X^*$ and a linear transformation B_3 on X such that

$$(1) \qquad \varphi(x \otimes (f_1+f_2)) = x \otimes g_3 + B_3x \otimes (f_1+f_2).$$

But also we have,

$$(2) \qquad \varphi(x \otimes (f_1+f_2)) = \varphi(x \otimes f_1) + \varphi(x \otimes f_2)$$

$$= x \otimes g_1 + B_1x \otimes f_1 + x \otimes g_2 + B_2x \otimes f_2.$$

Equating (1) and (2) and taking of adjoints yields

$$(3) \qquad g_3 \otimes x + (f_1+f_2) \otimes B_3x = (g_1+g_2) \otimes x + f_1 \otimes B_1x + f_2 \otimes B_2x.$$

Now let h be an arbitrary functional in X^* for which $h(x) = 0$, that is, $h \in (x)^{\perp}$. By letting the two sides of equation (3) act on h, we obtain

$$h(B_3x)(f_1+f_2) = h(B_1x)f_1 + h(B_2x)f_2,$$

and the linear independence of $\{f_1,f_2\}$ implies that $h(B_1x-B_2x) = 0$. But h was an arbitrary element of $(x)^\perp$. Thus $(B_1-B_2)x$ is a scalar multiple of x, as desired. ∎

Next, we use Lemma 2 to prove that we can take the map B_f in Lemma 1 to be independent of f.

LEMMA 3. *There is a linear map* $B : X \to X$ *with the property that for each functional* $f \in X^*$ *there is a functional* $g_f \in X^*$ *such that*

$$(**) \qquad \varphi(x \otimes f) = x \otimes g_f + Bx \otimes f$$

for each $x \in X$.

PROOF. Fix a nonzero element $f_0 \in X^*$, and choose B_{f_0} and g_{f_0} satisfying condition (*) of Lemma 1. Let $B = B_{f_0}$. Therefore equation (**) holds for f_0, and also, by linearity, holds for scalar multiples of f_0. On the other hand, if $\{f,f_0\}$ are linearly independent, let g_f and B_f satisfy condition (*) of Lemma 1 for f. Then by Lemma 2, $B_f - B = \mu f$ for some scalar μ. So for each $x \in X$, we have

$$\varphi(x \otimes f) = x \otimes (g_f + \mu f) + Bx \otimes f$$

and condition (**) is satisfied for f, with g_f replaced by $g_f + \mu f$. ∎

LEMMA 4. *There exist linear maps* $B : X \to X$ *and* $C : X^* \to X^*$ *such that*

$$\varphi(x \otimes f) = x \otimes Cf + Bx \otimes f$$

for every $x \in X$ *and* $f \in X^*$.

PROOF. Fix a nonzero vector $x \in X$, and fix $h \in X^*$ with $h(x) = 1$. For each $f \in X^*$ the functional g_f in (**) of Lemma 3 is given by

$$g_f = [\varphi(x \otimes f) - Bx \otimes f]^*(h),$$

where B is as in (**). Therefore the map $f \mapsto g_f$ is linear. We call this map C. The proof is complete. ∎

LEMMA 5. *The maps* B *and* C *in Lemma* 4 *are bounded and* $C = A^*$ *for some* $A \in \mathscr{B}(X)$. *Consequently* $\varphi(R) = RA + BR$ *for every rank one operator* R. *Furthermore* $\varphi(1) = A + B$.

PROOF. Consider a rank one projection $P = x \otimes f$, so $f(x) = 1$. By Lemma 4, we have

$$\varphi(P) = Bx \otimes f + x \otimes Cf. \tag{1}$$

By the assumptions of Theorem 1.2, there exist bounded operators L and R such that $\varphi(1-P) = L(1-P) + (1-P)R$. Therefore $P\varphi(1-P)P = 0$, and so

$$P\varphi(P)P = P\varphi(1)P. \tag{2}$$

Using (1) we have that

$$P\varphi(P)P = \langle Bx,f\rangle P + \langle x,Cf\rangle P.$$

Furthermore $P\varphi(1)P = \langle \varphi(1)x,f\rangle$. So equation (2) yields

$$\langle Bx,f\rangle + \langle x,Cf\rangle = \langle \varphi(1)x,f\rangle \tag{3}$$

for all $x \in X$ and $f \in X^*$ which satisfy $f(x) = 1$. For arbitrary $x \in X$, $f \in X^*$ we can write f as a linear combination of two linear functionals f_1 and $f_2 \in X^*$ satisfying $f_1(x) = f_2(x) = 1$. Therefore equation (3) holds for all $x \in X$ and $f \in X^*$. Now, using the closed graph theorem, we get that B is bounded. Furthermore we get that $C = (\varphi(1)-B)^*$. Setting $A = \varphi(1) - B$, we get the desired conclusion. ∎

PROOF OF THEOREM 1.2. By Lemma 5, there exist A and $B \in \mathscr{B}(X)$ such that $\varphi(F) = FA + BF$ for every finite rank operator $F \in \mathscr{B}(X)$. Let ψ be defined by $\psi(T) = TA + BT$ for every $T \in \mathscr{B}(X)$, and let $\varphi_0 = \varphi - \psi$. We must show that $\varphi_0 = 0$. The transformation φ_0 satisfies the hypotheses of φ in the statement of Theorem 1.2, and has the additional property that $\varphi_0(T) = 0$ whenever T has finite rank.

Fix $T \in \mathscr{B}(X)$, and let $S = \varphi_0(T)$. Suppose $S \neq 0$, so there is a nonzero vector $x \in X$ with $Sx \neq 0$. Let $y = Sx$, and let P be the (bounded) projection from X onto $\text{span}\{x,y\}$ along any closed complement of $\text{span}\{x,y\}$ in X. The hypotheses on φ_0 imply that $P\varphi_0((I-P)T(I-P))P = 0$. However, $T - (I-P)T(I-P)$ has finite rank, so $\varphi_0(T) = \varphi_0((I-P)T(I-P))$. So $PSP = 0$, a contradiction, since x and y are in the range of P. Thus $S = 0$. We conclude that $\varphi_0 = 0$, as required. This shows that $\varphi(T) = \psi(T) = TA + BT$ for every $T \in \mathscr{B}(X)$.

To prove the last assertion of the theorem, we note that if $A_T = -B_T$ for $T = 1$ (in particular if $A_T = -B_T$ for every T), then $\varphi(1) = 0$ and by Lemma 5, $A + B = \varphi(1) = 0$, so $B = -A$ and φ is a derivation. ∎

2. Automorphisms. In this section, we consider the local automorphisms of $\mathscr{B}(X)$.

THEOREM 2.1. *Let* X *be an infinite–dimensional Banach space and let* φ *be a linear map from* $\mathscr{B}(X)$ *onto itself such that* φ *is a local automorphism. Then* φ *is an automorphism.*

PROOF. Every automorphism on $\mathscr{B}(X)$ is inner [1], so for every $T \in \mathscr{B}(X)$, there exists an invertible operator $A_T \in \mathscr{B}(X)$ such that $\varphi(T) = A_T \, T \, A_T^{-1}$. We wish to show that there exists an invertible $A \in \mathscr{B}(X)$, independent of T, such that $\varphi(T) = ATA^{-1}$ for every $T \in \mathscr{B}(X)$. The assumption on φ implies that it preserves the spectrum and so by [4], it either has the required form or the form $\varphi(T) = AT^*A^{-1}$ for a bounded invertible operator $A : X^* \to X$. We show that the latter case is not present when X is infinite dimensional. By a result of Banach [8; p. 4], there exists a sequence $\{y_n\}$ of unit vectors which forms a basis for the closure of its linear span Y. The associated coefficient functionals f_n (defined first on Y and then extended to X without increase in norm) are uniformly bounded and satisfy $f_n(y_m) = \delta_{nm}$ (see [8; pp. 1,2]). Now let $\{\lambda_n\}$ be a sequence in ℓ^1 such that $\lambda_n \neq 0$ for any n and let $S = \sum_{n=1}^{\infty} \lambda_n \, y_{n+1} \otimes f_n$. It is easy to see that $\ker S = \{x \in X : f_n(x) = 0 \text{ for all } n\}$ and that $S^2 = \sum_{n=1}^{\infty} \lambda_n \lambda_{n+1} y_{n+2} \otimes f_n$ and so $\ker S^2 = \ker S$. On the other hand $S^* = \sum_{n=1}^{\infty} \lambda_n \, f_n \otimes y_{n+1}$, hence $S^* f_2 = f_1$ and $S^* f_1 = 0$, and so $\ker(S^*) \neq \ker(S^{*2})$. This shows that there is no invertible linear map B_S such that $S^* = B_S \, S \, B_S^{-1}$. However if $\varphi(S) = AS^*A^{-1}$, then $S^* = A^{-1}\varphi(S)A$ and so $S^* = A^{-1}A_S S A_S^{-1} A$, a contradiction. ∎

For finite–dimensional spaces X, the situation is slightly different.

THEOREM 2.2. *Let* φ *be a linear map on* M_n, *the algebra of* $n \times n$ *complex matrices. Then* φ *is a local automorphism iff and only if* φ *is either an automorphism or an anti–automorphism, i.e.* φ *either takes the form* $\varphi(T) = ATA^{-1}$ *for all* $T \in M_n$, *or the form* $\varphi(T) = AT^tA^{-1}$ *for all* $T \in M_n$, *for some invertible* $A \in M_m$, *independent of* T.

PROOF. It is obvious that every local automorphism is injective and hence also surjective. The theorem now follows immediately from the proof of Theorem 2.1 together with the fact T^t is always similar to T. ∎

Instead of the group of automorphisms, we now consider the (larger) group generated by invertible left multiplications and invertible right multiplications.

THEOREM 2.3. *Let* $\varphi : \mathscr{B}(X) \to \mathscr{B}(X)$ *be a surjective linear map such that for every* $T \in \mathscr{B}(X)$, *there exist invertible operators* A_T *and* B_T *such that* $\varphi(T) = A_T\, T\, B_T$.
Then φ *takes one of the following forms*:
(1) $\varphi(T) = ATB$ *for some invertible operators* A *and* $B \in \mathscr{B}(X)$.
(2) $\varphi(T) = AT^*B$ *for some invertible operators* $A \in \mathscr{B}(X^*,X)$ *and* $B \in \mathscr{B}(X,X^*)$.
The second form is always present when X *is finite dimensional. It is not present if* X *has a basis or is not reflexive or is not isomorphic to* X^*.

PROOF. It follows from the hypotheses that φ is bijective and that it preserves invertibility, *i.e.* $\varphi(T)$ is invertible whenever T is. The main result in [10] asserts that such a map takes one of the forms (1) and (2). If X is finite dimensional, then T^* is always similar to T and so form (2) is always present. On the other hand if φ takes the form (2) then X must be isomorphic to X^* and since φ is surjective, X must be reflexive; furthermore $T^* = A^{-1}\varphi(T)B^{-1}$ and so for every T, there exist invertible operators $C_T \in \mathscr{B}(X,X^*)$, $D_T \in \mathscr{B}(X^*,X)$ such that $T^* = C_T\, T\, D_T$. Now if X has a basis $\{x_n\}$ with coefficient functionals $\{f_n\}$, the operator $S = \sum_{n=1}^{\infty} 2^{-n}\, x_{n+1} \otimes f_n$ is injective but $S^* f_1 = 0$, and so form (2) is not possible. ■

REMARK. We do not know of any infinite dimensional Banach space for which form (2) of Theorem 2.3 occurs.

3. A Counterexample. A linear transformation φ on $\mathscr{B}(X)$ is called an *elementary operator* if there exist $A_1,\cdots,A_m$, $B_1,\cdots,B_m \in \mathscr{B}(X)$ such that $\varphi(T) = \sum_{j=1}^{m} A_j\, T\, B_j$ for every $T \in \mathscr{B}(X)$. The results of the preceding two sections assert that certain subsets of the algebra of elementary operators are algebraically reflexive. We now show that the set of *all* elementary operators is not algebraically reflexive, *i.e.*, there exists a linear map φ on $\mathscr{B}(X)$ which is a local elementary operator, yet φ is not an elementary operator.

EXAMPLE. Let X be an infinite dimensional Banach space, let $x_0 \in X^*$ and $f_0 \in X^*$ be nonzero vectors, and let α be a nonzero linear functional on $\mathscr{B}(X)$. Define $\varphi : \mathscr{B}(X) \longrightarrow \mathscr{B}(X)$ by the equation $\varphi(T) = \alpha(T)\, x_0 \otimes f_0$.

First, we note that for every T, there exist operators A_T and B_T such that $\varphi(T) = A_T\, T\, B_T$. Choose $y_T \in X$ and $g_T \in X^*$ such that $\langle Ty_T, g_T\rangle = \alpha(T)$ and let $A_T = x_0 \otimes g_T$ and $B_T = y_T \otimes f_0$. Then $\varphi(T) = A_T\, T\, B_T$.

Next, we show that φ is not always an elementary operator. Indeed, we show that φ is an elementary operator only if α is implemented by a finite rank operator F via the equation $\alpha(T) = \mathrm{tr}(TF)$. To prove this assertion, assume that φ is an elementary operator and write φ in reduced form, *i.e.* $\varphi(T) = \sum_{n=1}^{m} A_n\, T\, B_n$ where both of $\{A_1,\cdots,A_m\}$ and $\{B_1,\cdots,B_m\}$ are linearly independent. Let P be a projection whose range is the pre–annihilator of f_0. Then $\varphi(T)P = 0$, *i.e.* $\sum_{n=1}^{m} A_n T B_n P = 0$ for all T. It follows from [2] that $B_n P = 0$ for all n. So every B_n annihilates the range of P, which is the pre–annihilator of f_0. Therefore B_n is a rank one operator of the form $y_n \otimes f_0$ for some $y_n \in X$. Similarly, we get that $A_n = x_0 \otimes g_n$ for some $g_n \in X^*$ and so $\sum_{n=1}^{\infty} A_n T B_n = \sum_{n=1}^{m} \langle Ty_n, g_n \rangle \, x_0 \otimes f_0$. Therefore $\alpha(T) = \sum_{n=1}^{m} \langle Ty_n, g_n \rangle = tr(TF)$ where $F = \sum_{n=1}^{m} y_n \otimes g_n$. This proves our assertion. Of course there are linear functionals on $\mathcal{B}(X)$ which are not implemented by a finite rank operator.

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